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Research article

Analyzing the structure of solutions for weakly singular integro-differential equations with partial derivatives

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Abstract: In this work, we analyze the approximate solution of a specific partial integro-differential equation (PIDE) with a weakly singular kernel using the spectral Tau method. It present a numerical solution procedure for this PIDE, which is transferred into a Volterra–Fredholm integral equation (VFIE), and the spectral method is performed on VFIE. In some illustrated examples, we show that the VFIE problem has high numerical stability with respect to the original form of the PIDE problem. For this aim, we apply the spectral Tau method in two cases, first for the problem in the form of VFIE and then also for the problem in the form of PIDE. The remarkable numerical results obtained from the VFIE problem form compared to those gained from the PIDE problem form show the efficiency of the proposal method. Also, we prove the convergence theorem of the numerical solution of the Tau method for the VFIE problem, and then it is generalized to the PIDE problem.

Keywords: integro-differential equations; weakly singular integral equations; Volterra–Fredholm integral equations; spectral method; convergence analysis
Mathematics Subject Classification: 35R09, 45A05, 65N35

1. Introduction

In some models of the physical and biological sciences, the impact of the systems' memory needs to be reflected, so that formulation by partial differential equations may not precisely model such situations. Then, for assimilating the memory effect in such systems, an integral term is added to the basic partial differential equation, which leads to a partial integro-differential equation (PIDE). Some examples of a parabolic PIDE occur in the study of the dynamics of nuclear reactors influenced by space-time [1, 2], the situation under control for a reaction-diffusion issue [3, 4], and the compression of poro-viscoelastic media [5] with the equation in the following form:

$$v_t(x,t) + av(x,t) - v_{xx}(x,t) = \sum_{k=1}^{N} b_k^2 c_k \int_0^t e^{-b_k(t-s)} v(x,s)ds, \quad (x,t) \in (0,1) \times (0,T], \tag{1.1}$$

where

$$v(x,0) = 0, \ 0 \le x \le 1, \quad v(0,t) = f_0(t), \ v_x(1,t) = 0, \ t \in (0,T],$$

such that b_k and c_k are dimensionless constants with $b_k > 0$ and $0 \le c_k \le 1$, and a is the effective stress (more details about Eq (1.1) can be found in [5]). In viscoelasticity [6, 7] and in some physical systems involving fluid flow [8–11], we deal with partial integro-differential equations with weakly singular kernels. As you know, Volterra integral equations with weakly singular kernels have solutions whose derivatives are unbounded at the left endpoint of the interval of integration, but the solutions of Volterra integro-differential equations with weakly singular kernels are slightly more regular [12].

The authors in the research works [13–18] concentrated on the PIDEs with a weakly singular kernel by using the Cubic B-spline least-square, Galerkin method with quadratic weight function, and Quintic B-spline collocation method. Now, we are motivated to focus on the partial integro-differential equation with a weakly singular kernel as follows:

$$V_t(x,t) + \lambda V_x(x,t) - \gamma V_{xx}(x,t) = \int_0^t H(t-s)V(x,s)ds + p(x,t), \ 0 \le x \le L, \ t \ge 0, \tag{1.2}$$

where λ and γ are positive constants that quantify the convection and diffusion processes, respectively. Also, $p(x,t)$ is a given function, and

$$H(t-s) = (t-s)^{-\alpha}, \ 0 < \alpha < 1,$$

subject to the initial and boundary conditions.

$$V(x,0) = g(x), \ 0 \le x \le L,$$

$$V(0,t) = f_0(t), \ V(L,t) = f_1(t), \ t \ge 0.$$

The outline of the paper is organized as follows: In Section 2, we describe a detailed transfer of the PIDE problem to the VFIE problem form, and also discuss the implementation of the spectral Tau schema for solving the VFIE is discussed in this section. The convergence of the numerical solution is analyzed in Section 3. In Section 4, the numerical results will be presented, and finally, the conclusion of the paper is brought up in Section 5.

2. Numerical scheme

We rewrite (1.2) as

$$V_{xx}(x,t) = \frac{1}{\gamma}V_t(x,t) + \frac{\lambda}{\gamma}V_x(x,t) - \frac{1}{\gamma}\int_0^t H(t-s)V(x,s)ds - \frac{1}{\gamma}p(x,t). \tag{2.1}$$

Let $V_{xx}(x,t) = \Phi(x,t)$, then

$$V_x(x,t) = V_x(0,t) + \int_0^x \Phi(\tau,t)d\tau, \tag{2.2}$$

and

$$\begin{aligned} V(x,t) &= V(0,t) + xV_x(0,t) + \int_0^x \int_0^\tau \Phi(t_2,t)dt_2 d\tau \\ &= V(0,t) + xV_x(0,t) + \int_0^x (x-\tau)\Phi(\tau,t)d\tau. \end{aligned} \tag{2.3}$$

Use $x = L$ in (2.3), and consider boundary conditions.

$$V_x(0,t) = \frac{1}{L}(f_1(t) - f_0(t)) - \frac{1}{L}\int_0^L (L-\tau)\Phi(\tau,t)d\tau. \tag{2.4}$$

By substituting (2.4) into (2.2) and (2.3), we have

$$V_x(x,t) = \frac{1}{L}(f_1(t) - f_0(t)) - \frac{1}{L}\int_0^L (L-\tau)\Phi(\tau,t)d\tau + \int_0^x \Phi(\tau,t)d\tau, \tag{2.5}$$

and

$$V(x,t) = \frac{L-x}{L}f_0(t) + \frac{x}{L}f_1(t) - \frac{x}{L}\int_0^L (L-\tau)\Phi(\tau,t)d\tau + \int_0^x (x-\tau)\Phi(\tau,t)d\tau. \tag{2.6}$$

A derivative from both sides of (2.6) with respect to t; yields

$$V_t(x,t) = \frac{L-x}{L}f_0'(t) + \frac{x}{L}f_1'(t) - \frac{x}{L}\int_0^L (L-\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau + \int_0^x (x-\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau. \tag{2.7}$$

Substituting (2.5), (2.6), and (2.7) into (2.1), we obtain

$$\begin{aligned} \Phi(x,t) = &-\frac{1}{\gamma}p(x,t) + \frac{\lambda}{\gamma L}(f_1(t) - f_0(t)) + \frac{L-x}{\gamma L}f_0'(t) + \frac{x}{\gamma L}f_1'(t) - \frac{L-x}{\gamma L}\int_0^t H(t-s)f_0(s)ds \\ &- \frac{x}{\gamma L}\int_0^t H(t-s)f_1(s)ds - \frac{\lambda}{\gamma L}\int_0^L (L-\tau)\Phi(\tau,t)d\tau \\ &+ \frac{x}{\gamma L}\int_0^L\int_0^t (L-\tau)H(t-s)\Phi(\tau,s)dsd\tau - \frac{x}{\gamma L}\int_0^L (L-\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau \\ &+ \frac{\lambda}{\gamma}\int_0^x \Phi(\tau,t)d\tau - \frac{1}{\gamma}\int_0^x\int_0^t (x-\tau)H(t-s)\Phi(\tau,s)dsd\tau + \frac{1}{\gamma}\int_0^x (x-\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau. \end{aligned}$$

Consequently, the following Volterra–Fredholm integral equation with a weakly singular kernel is obtained:

$$\begin{aligned} \Phi(x,t) = \hat{p}(x,t) &+ \int_0^L K_1(x,\tau)\Phi(\tau,t)d\tau + \int_0^L\int_0^t K_2(x,\tau)H(t-s)\Phi(\tau,s)dsd\tau \\ &+ \int_0^L K_3(x,\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau + \int_0^x K_4(x,\tau)\Phi(\tau,t)d\tau \\ &+ \int_0^x\int_0^t K_5(x,\tau)H(t-s)\Phi(\tau,s)dsd\tau + \int_0^x K_6(x,\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \hat{p}(x,t) = &-\tfrac{1}{\gamma}p(x,t) + \tfrac{\lambda}{\gamma L}(f_1(t) - f_0(t)) + \tfrac{L-x}{\gamma L}f_0'(t) + \tfrac{x}{\gamma L}f_1'(t) \\ &-\tfrac{L-x}{\gamma L}\int_0^t H(t-s)f_0(t)ds - \frac{x}{\gamma L}\int_0^t H(t-s)f_1(s)ds, \end{aligned}$$

$$K_1(x,t) = -\frac{\lambda}{\gamma L}(L-t), \quad K_2(x,t) = \frac{x}{\gamma L}(L-t),$$
$$K_3(x,t) = -\frac{x}{\gamma L}(L-t), \quad K_4(x,t) = \frac{\lambda}{\gamma},$$
$$K_5(x,t) = -\frac{1}{\gamma}(x-t), \qquad K_6(x,t) = \frac{1}{\gamma}(x-t).$$

Now, we consider the approximate solution $\Phi_N(x,t) \in \Omega_N$ of Eq (2.8) as:

$$\Phi_N(x,t) = \sum_{k=0}^{N} c_k(t)\phi_k(x), \tag{2.9}$$

where $\phi_k(x)$ is an orthogonal polynomial taken from the space

$$\Omega_N = span\{\phi_i(x) \ | \left\langle \phi_i(x), \phi_j(x) \right\rangle_w = \int_a^b \phi_i(x)\phi_j(x)w(x)dx = 0, i \neq j\}.$$

Note that $\phi_k(x)$ can be included the classical orthogonal polynomials, consisting of the Jacobi polynomials, ultraspherical polynomials as a subclass of Jacobi polynomials which include the Legendre, Chebyshev and Gegenbaue polynomials. We now require that the residual

$$\begin{aligned} R_N(x,t) = &\sum_{k=0}^{N} c_k(t)\phi_k(x) - \hat{p}(x,t) - \sum_{k=0}^{N} c_k(t)(\int_0^L K_1(x,t)\phi_k(t)dt + \int_0^x K_4(x,t)\phi_k(t)dt) \\ &+ c_k'(t)(\int_0^L K_3(x,t)\phi_k(t)dt + \int_0^x K_6(x,t)\phi_k(t)dt) \\ &- \sum_{k=0}^{N}(\int_0^t H(t-s)c_k(s)ds)(\int_0^L K_2(x,t)\phi_k(t)dt + \int_0^x K_5(x,t)\phi_k(t)dt), \end{aligned} \tag{2.10}$$

is orthogonal to Ω_N. This procedure yields

$$\begin{aligned} c_l(t)\left\langle \phi_l(x), \phi_l(x) \right\rangle_w = &\left\langle \hat{p}(x,t), \phi_l(x) \right\rangle_w + \sum_{k=0}^{N} c_k(t)\lambda_{kl} + c_k'(t)\eta_{kl} \\ &+ \sum_{k=0}^{N} \varepsilon_{kl} \int_0^t H(t-s)c_k(s)ds, \quad l = 0, \cdots, N, \end{aligned} \tag{2.11}$$

where

$$\lambda_{kl} = \left\langle \int_0^L K_1(x,t)\phi_k(t)dt + \int_0^x K_4(x,t)\phi_k(t)dt, \phi_l(x) \right\rangle_w,$$

$$\eta_{kl} = \left\langle \int_0^L K_3(x,t)\phi_k(t)dt + \int_0^x K_6(x,t)\phi_k(t)dt, \phi_l(x) \right\rangle_w,$$

$$\varepsilon_{kl} = \left\langle \int_0^L K_2(x,t)\phi_k(t)dt + \int_0^x K_5(x,t)\phi_k(t)dt, \phi_l(x) \right\rangle_w.$$

Define $\mathbf{D} = diag(\langle\phi_0(x), \phi_0(x)\rangle_w, \cdots, \langle\phi_N(x), \phi_N(x)\rangle_w), \mathbf{\Upsilon} = \{\lambda_{kl}\}_{k,l=0}^N, \mathbf{\Lambda} = \{\eta_{kl}\}_{k,l=0}^N, \mathbf{P} = \{\varepsilon_{kl}\}_{k,l=0}^N$ and $\hat{\mathbf{P}}(t) = (\langle\hat{p}(x,t), \phi_0(x)\rangle_w, \cdots, \langle\hat{p}(x,t), \phi_N(x)\rangle_w)^T$. Let $\bar{\mathbf{c}} = (c_0, \ldots, c_N)^T$, then we have a system of Volterra integro-differential equations with weakly singular kernels as follows:

$$\mathbf{\Lambda}^T \bar{\mathbf{c}}'(t) = (\mathbf{D} - \mathbf{\Upsilon}^T)\bar{\mathbf{c}}(t) - \hat{\mathbf{P}}(t) - \mathbf{P}^T \int_0^t H(t-s)\bar{\mathbf{c}}(s)ds, \tag{2.12}$$

with the initial conditions

$$c_l(0) = \frac{\langle g''(x), \phi_l(x)\rangle_w}{\langle\phi_l(x), \phi_l(x)\rangle_w}, \quad l = 0, \cdots, N.$$

For the numerical solution of the problem (2.12), we now take into consideration the piecewise polynomial collocation method [12] within the interval $[0, \tilde{T}]$. For a given integer $N_1 \geq 2$ and a real value $r \geq 1$, we define the mesh

$$I_h^r = \{t_n = (\frac{n}{N_1})^r \tilde{T}, \ \ n = 0 \cdots, N_1\}.$$

If $r > 1$, I_h^r is a graded mesh on $[0, \tilde{T}]$ with a grading exponent of r if $r > 1$. When $r = 1$, the mesh is considered uniform. Additionally, let $h_n = t_{n+1} - t_n$ be the stepsize and Θ be determined as follows:

$$\Theta = \{t_{nj} = t_n + q_j h_n : \ \ 0 \leq q_1 < q_2 < ... < q_m \leq 1, \quad 0 \leq n \leq N_1 - 1\}.$$

The collocation solution $\bar{\mathbf{c}}_h \in S_m^0(I_h^r)$ (S_m^0 is the piecewise polynomial space with degree $m \geq 0$) for (2.12)by the collocation equation, which defines

$$\mathbf{\Lambda}^T \bar{\mathbf{c}}_h'(t) = (\mathbf{D} - \mathbf{\Upsilon}^T)\bar{\mathbf{c}}_h(t) - \hat{\mathbf{P}}(t) - \mathbf{P}^T \int_0^t H(t-s)\bar{\mathbf{c}}_h(s)ds, \ \ t \in \Theta, \ \ \bar{\mathbf{c}}_h(0) = \bar{\mathbf{c}}(0) = \bar{\mathbf{c}}_0. \tag{2.13}$$

Since $\bar{\mathbf{c}}_h'|_{(t_n, t_{n+1}]} \in \Pi_{m-1}$ (Π_m is the space occupied by polynomials with real coefficients that have a degree not greater than m), for $\varepsilon \in [0, 1]$, the following equations hold:

$$\bar{\mathbf{c}}_h'(t_n + \varepsilon h_n) = \sum_{j=1}^m L_j(\varepsilon)\bar{\mathbf{C}}_{nj}, \quad \bar{\mathbf{C}}_{nj} = \bar{\mathbf{c}}_h'(t_n + q_j h_n), \tag{2.14}$$

where

$$L_j(\varepsilon) = \prod_{k=1, k\neq j}^m \frac{(\varepsilon - q_k)}{(q_j - q_k)}, \quad j = 1, ..., m. \tag{2.15}$$

Assume that $\bar{\mathbf{c}}_n = \bar{\mathbf{c}}_h(t_n)$ and $\alpha_j(\varepsilon) = \int_0^\varepsilon L_j(s)ds, \ \ j = 1, \ldots, m$, then

$$\bar{\mathbf{c}}_h(t_n + \varepsilon h_n) = \bar{\mathbf{c}}_n + h_n \sum_{j=1}^m \alpha_j(\varepsilon)\bar{\mathbf{C}}_{nj}, \quad \varepsilon \in [0, 1]. \tag{2.16}$$

Substituting (2.14) and (2.16) into collocation Eq (2.13) leads to

$$\mathbf{\Lambda}^T \bar{\mathbf{C}}_{ni} = (\mathbf{D} - \mathbf{\Upsilon}^T)(\bar{\mathbf{c}}_n + h_n \sum_{j=1}^m \alpha_j(q_i)\bar{\mathbf{C}}_{nj}) - \hat{\mathbf{P}}(t_{ni})$$

$$
\begin{aligned}
&-\mathbf{P}^T\Big(\sum_{l=0}^{n-1} h_l\Big(\int_0^1 H(t_{ni}-(t_l+sh_l))ds\Big)\bar{\mathbf{c}}_l+\sum_{l=0}^{n-1} h_l^2\sum_{j=1}^{m}\Big(\int_0^1 H(t_{ni}-(t_l+sh_l))\alpha_j(s)ds\Big)\bar{\mathbf{C}}_{lj}\\
&+h_n\Big(\int_0^{q_i} H(t_{ni}-(t_n+sh_n))ds\Big)\bar{\mathbf{c}}_n+h_n^2\sum_{j=1}^{m}\Big(\int_0^{q_i} H(t_{ni}-(t_n+sh_n))\alpha_j(s)ds\Big)\bar{\mathbf{C}}_{nj}\Big),\\
&n=0,\dots,N_1-1,\ \ i=1,\dots,m. \qquad (2.17)
\end{aligned}
$$

Consequently, we will find the numerical solution of the differential equation problem (2.12) for arbitrary $\varepsilon\in[0,1]$ by inserting $\bar{\mathbf{C}}_{nj}$ as the solution of the linear system (2.17) into (2.16).

Algorithm 1. Coding algorithm for PIDE with VFIE form (2.8).

Step 1. Input N, N_1, m, $\tilde{T}$ and $q_j,\ j=1,\dots,m$.
Step 2. Input the values of matrices $\mathbf{\Lambda}$, $\mathbf{D}$, $\mathbf{\Upsilon}$, $\hat{\mathbf{P}}(t)$, $\mathbf{P}$.
Step 3. Compute $L_j(\varepsilon)$ in (2.15).
Step 4. Compute $\alpha_j(\varepsilon)=\int_0^{\varepsilon} L_j(s)ds,\ \ j=1,\dots,m$.
Step 5. Solve the system of (2.17).
Step 6. Put c_k's, which outputs from step 5, in Eq (2.9).
Step 7. Put $\Phi_N(x,t)$, which outputs from step 6, in Eq (2.6).
Step 8. Output $V_N(x,t)$.

3. Convergence analysis

We require the following lemmas to demonstrate the error estimate: In the beginning, we view Gronwall's inequality as

Theorem 3.1. *[19] Let $u(t), v(t), h(t,r)$, and $H(t,r,x)$ be nonnegative functions for $t\geqslant r\geqslant x\geqslant a$, and c_1, c_2, and c_3 be nonnegative constants not all zero. If*

$$
u(t)\leqslant c_1+c_2\int_a^t\Big[v(s)u(s)ds+\int_a^s h(s,r)u(r)dr\Big]ds+c_3\int_a^t\int_a^s\int_a^r H(s,r,x)u(x)dxdrds,
$$

then for $t\geqslant a$,

$$
u(t)\leqslant c_1 exp\Big\{c_2\int_a^s\Big[v(s)ds+\int_a^s h(s,r)dr\Big]ds+c_3\int_a^t\int_a^s\int_a^r H(s,r,x)dxdrds\Big\}.
$$

The following result can be obtained directly, by considering $H(s,r,x)=0$, from Theorem 3.1.

Lemma 3.2. *Suppose $u(t)$ is a nonnegative function satisfying the following inequality:*

$$
u(t)\le c+\int_{t_0}^t k(t,s)u(s)ds+\int_{t_0}^t\int_{t_0}^s \hat{k}(t,\sigma)(s-\sigma)^{-\alpha}u(\sigma)d\sigma ds, \qquad (3.1)
$$

where two functions of $k(t,s)$ and $\hat{k}(t,\sigma)$, as $c>0$ and $t\geqslant s\geqslant\sigma\ge t_0>0$, are nonnegative, then

$$
\begin{aligned}
u(t)\le &c\, exp\Big\{\int_{t_0}^t k(s,s)ds+\int_{t_0}^t\int_{t_0}^{\tau}\frac{\partial k(\tau,s)}{\partial\tau}dsd\tau+\int_{t_0}^t\int_{t_0}^{\tau}\hat{k}(\tau,\sigma)(\tau-\sigma)^{-\alpha}d\sigma d\tau\\
&+\int_{t_0}^t\int_{t_0}^{\tau}\int_{t_0}^{s}\frac{\partial\hat{k}(\tau,\sigma)}{\partial\tau}(s-\sigma)^{-\alpha}d\sigma dsd\tau\Big\}.
\end{aligned}
$$

Proof. According to the same procedure in the proof of Theorem 3.1 from [19], let the right, hand side of (3.1) be denoted by $B(t)$. Then $B(s) \leqslant B(t)$ for $s \leqslant t$ since all the terms are nonnegative. We have

$$\frac{B'(t)}{B(t)} = \frac{k(t,t)u(t) + \int_{t_0}^{t} \frac{\partial k(t,s)}{\partial t} u(s)ds + \int_{t_0}^{t} \hat{k}(t,\sigma)(t-\sigma)^{-\alpha} u(\sigma)d\sigma + \int_{t_0}^{t}\int_{t_0}^{s} \frac{\partial \hat{k}(t,\sigma)}{\partial t}(s-\sigma)^{-\alpha} u(\sigma)d\sigma ds}{B(t)}$$

$$\leqslant k(t,t) + \int_{t_0}^{t} \frac{\partial k(t,s)}{\partial t} ds + \int_{t_0}^{t} \hat{k}(t,\sigma)(t-\sigma)^{-\alpha} d\sigma + \int_{t_0}^{t}\int_{t_0}^{s} \frac{\partial \hat{k}(t,\sigma)}{\partial t}(s-\sigma)^{-\alpha} d\sigma ds.$$

Integration from t_0 to t yields

$$\log B(t) - \log c \leqslant \int_{t_0}^{t} k(s,s)ds + \int_{t_0}^{t}\int_{t_0}^{\tau} \frac{\partial k(\tau,s)}{\partial \tau} ds d\tau + \int_{t_0}^{t}\int_{t_0}^{\tau} \hat{k}(\tau,\sigma)(\tau-\sigma)^{-\alpha} d\sigma d\tau$$

$$+ \int_{t_0}^{t}\int_{t_0}^{\tau}\int_{t_0}^{s} \frac{\partial \hat{k}(\tau,\sigma)}{\partial \tau}(s-\sigma)^{-\alpha} d\sigma ds d\tau.$$

Writing this in terms of $B(t)$ and using $u(t) \leq B(t)$ completes the proof. □

Lemma 3.3. *[20] The Sobolev space $W_w^m(\Omega)$ is the set of all functions $\phi(\mathbf{x})(\mathbf{x} = (x_1 \cdots, x_p))$ on $\Omega = (0,1)^p$,for $p = 1, 2$, which the functions $\phi(\mathbf{x})$ and its weak derivatives to order m are in $L_w^2(\Omega)$, let $\mathbb{P}_N(\Lambda)$ be the space of all polynomials with degrees not exceeding N on Ω. Denote by P_N the orthogonal projective operator from $L_w^2(\Omega)$ on to $\mathbb{P}_N(\Omega)$. For all $\phi \in W_w^m(\Omega), m \geq 1$, the following estimate holds*

$$\|\phi - P_N\phi\|_{L_w^2(\Omega)} \leq CN^{-m}|\phi|_{W_w^{m,N}(\Omega)}, \tag{3.2}$$

where the semi-norm $|.|$ is defined as

$$|\phi|_{W_w^{m,N}(\Omega)} = \left(\sum_{j=min(m,N+1)}^{m} \sum_{i=1}^{p} \|D_i^j \phi\|_{L_w^2(\Omega)}^2 \right)^{1/2},$$

such that $\alpha = (\alpha_1, \cdots, \alpha_p)$ is a nonnegative multi-index with $D^\alpha \phi = \frac{\partial^{\alpha_1+\cdots+\alpha_p}\phi}{\partial x_1^{\alpha_1}\cdots\partial x_p^{\alpha_p}}$. The following estimate extends (3.2) to higher-order Sobolev norms in those situations where the truncation error of the derivatives is significant:

$$\|\phi - P_N\phi\|_{W_w^r(\Lambda)} \leq CN^{2r-\frac{1}{2}-m}|\phi|_{W_w^{m,N}(\Lambda)}, \tag{3.3}$$

for any r such that $1 \leq r \leq m$.

In the following, we scrutinize the error estimate of the numerical Tau method, which is proposed in Section 2, equipped with the L_w^2 weighted norm. As you have seen, we consider the approximate solution $\Phi_N(x,t) = \sum_{k=0}^{N} c_k(t)\phi_k(x)$ of Eq (2.8) such that for finding $\bar{\mathbf{c}} = (c_0, \ldots, c_N)^T$, we get a system of Volterra integro-differential equations with weakly singular kernels, and the approximate solution of this system is obtained by the piecewise polynomial collocation method. These two approximate methods (first with respect to x and then with respect to t) are completely different. Then we can not investigate convergence analysis simultaneously. In this position, we find $\|V - V_N\|_{L_w^2(0,1)} \to 0$ with respect to x and do not consider the effect of approximation with respect to t. In other words, t is assumed to be a constant value in this situation.

Theorem 3.4. *Let $V(x,t)$ as a sufficiently smooth function be the exact solution of the partial differential equation (1.2). Also, let $V_N(x,t)$ be the numerical solution of $V(x,t)$ by the spectral Tau method, which is defined in the previous section. Then, for all sufficiently large N, we have*

$$||V - V_N||_{L^2_w(0,1)} \to 0.$$

Proof. Without loss of generality, we let $L = 1$. Then Eq (2.1) can be considered in the below form:

$$\begin{cases} V_{xx}(x,t) = \frac{1}{\gamma}V_t(x,t) + \frac{\lambda}{\gamma}V_x(x,t) - \frac{1}{\gamma}\int_0^t H(t-s)V(x,s)ds - \frac{1}{\gamma}p(x,t), \\ V_t(x,0) = g(x,0), \\ H(t-s) = (t-s)^{-\alpha}, \ 0 < \alpha < 1, \\ V(0,t) = f_0(t), \ V(1,t) = f_1(t), \ t \geq 0. \end{cases} \tag{3.4}$$

We use the similar procedure in Section 2, then we gain

$$V(x,t) = (1-x)f_0(t) + xf_1(t) - x\int_0^1 (1-\tau)\Phi(\tau,t)d\tau + \int_0^x (x-\tau)\Phi(\tau,t)d\tau, \tag{3.5}$$

and

$$\begin{aligned} \Phi(x,t) = \hat{p}(x,t) &+ \int_0^1 K_1(x,\tau)\Phi(\tau,t)d\tau + \int_0^1\int_0^t K_2(x,\tau)H(t-s)\Phi(\tau,s)dsd\tau \\ &+ \int_0^1 K_3(x,\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau + \int_0^x K_4(x,\tau)\Phi(\tau,t)d\tau \\ &+ \int_0^x\int_0^t K_5(x,\tau)H(t-s)\Phi(\tau,s)dsd\tau + \int_0^x K_6(x,\tau)\frac{\partial\Phi(\tau,t)}{\partial t}d\tau. \end{aligned} \tag{3.6}$$

Also,

$$V_N(x,t) = (1-x)f_0(t) + xf_1(t) - x\int_0^1 (1-\tau)\Phi_N(\tau,t)d\tau + \int_0^x (x-\tau)\Phi_N(\tau,t)d\tau, \tag{3.7}$$

and

$$\begin{aligned} \Phi_N(x,t) = \hat{p}_N(x,t) &+ \int_0^1 K_{1.N}(x,\tau)\Phi_N(\tau,t)d\tau + \int_0^1\int_0^t K_{2,N}(x,\tau)H(t-s)\Phi_N(\tau,s)dsd\tau \\ &+ \int_0^1 K_{3,N}(x,\tau)\frac{\partial\Phi_N(\tau,t)}{\partial t}d\tau + \int_0^x K_{4,N}(x,\tau)\Phi_N(\tau,t)dsd\tau \\ &+ \int_0^x\int_0^t K_{5,N}(x,\tau)H(t-s)\Phi_N(\tau,s)dsd\tau + \int_0^x K_{6,N}(x,\tau)\frac{\partial\Phi_N(\tau,t)}{\partial t}d\tau. \end{aligned} \tag{3.8}$$

Subtracting (3.6) from (3.8) yields

$$\begin{aligned} e(x,t) = &\int_0^1 K_1(x,\tau)e(\tau,t)d\tau + \int_0^1\int_0^t K_2(x,\tau)H(t-s)e(\tau,s)dsd\tau \\ &+ \int_0^x K_4(x,\tau)e(\tau,t)d\tau + \int_0^x\int_0^t K_5(x,\tau)H(t-s)e(\tau,s)dsd\tau \\ &+ e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7, \end{aligned} \tag{3.9}$$

such that $e(x,t) = \Phi(x,t) - \Phi_N(x,t)$ and

$$
\begin{aligned}
e_1 &= \hat{p}(x,t) - \hat{p}_N(x,t),\\
e_2 &= \int_0^1 (K_1(x,\tau) - K_{1,N}(x,\tau))\Phi_N(\tau,t)d\tau,\\
e_3 &= \int_0^1 \int_0^t (K_2(x,\tau) - K_{2,N}(x,\tau))H(t-s)\Phi_N(\tau,s)dsd\tau,\\
e_4 &= \int_0^1 K_{3,N}(x,\tau)\Big(\tfrac{\partial\Phi(\tau,t)}{\partial t} - \tfrac{\partial\Phi_N(\tau,t)}{\partial t}\Big)d\tau + \int_0^1 (K_3(x,\tau) - K_{3,N}(x,\tau))\tfrac{\partial\Phi(\tau,t)}{\partial t}d\tau,\\
e_5 &= \int_0^x (K_4(x,\tau) - K_{4,N}(x,\tau))\Phi_N(\tau,t)d\tau,\\
e_6 &= \int_0^x \int_0^t (K_5(x,\tau) - K_{5,N}(x,\tau))H(t-s)\Phi_N(\tau,s)dsd\tau,\\
e_7 &= \int_0^x K_{6,N}(x,\tau)\Big(\tfrac{\partial\Phi(\tau,t)}{\partial t} - \tfrac{\partial\Phi_N(\tau,t)}{\partial t}\Big)d\tau + \int_0^x (K_6(x,\tau) - K_{6,N}(x,\tau))\tfrac{\partial\Phi(\tau,t)}{\partial t}d\tau.
\end{aligned}
$$

Using Gronwall's inequality from Theorem 3.1 and applying the generalized Hardy's inequality from [21], and also using Lemma 3.2 by considering $H(t-s) = (t-s)^{-\alpha}$ for (3.9), yields

$$
\|e\|_{L^2_w(0,1)} \le C\|e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7\|_{L^2_w(0,1)}. \tag{3.10}
$$

Using the inequality (3.2) from Lemma 3.3,

$$
\|e_1\|_{L^2_w(0,1)} \le CN^{-m}|\hat{p}|_{W^{m,N}_w(0,1)}.
$$

It follows from generalized Hardy's inequality and the inequality (3.2) that

$$
\begin{aligned}
\|e_2\|_{L^2_w(0,1)} &\le C\|K_1 - K_{1,N}\|_{L^2_w(0,1)}\|\Phi_N\|_{L^2_w(0,1)} \le CN^{-m}|K_1|_{W^{m,N}_w(0,1)}\big(\|\Phi\|_{L^2_w(0,1)} + \|e\|_{L^2_w(0,1)}\big),\\
\|e_3\|_{L^2_w(0,1)} &\le C\|K_2 - K_{2,N}\|_{L^2_w(0,1)}\|\Phi_N\|_{L^2_w(0,1)} \le CN^{-m}|K_2|_{W^{m,N}_w(0,1)}\big(\|\Phi\|_{L^2_w(0,1)} + \|e\|_{L^2_w(0,1)}\big).
\end{aligned}
$$

Also,

$$
\|e_4\|_{L^2_w(0,1)} \le C\Big(\|K_{3,N}\|_{L^2_w(0,1)}\|\frac{\partial\Phi}{\partial t} - \frac{\partial\Phi_N}{\partial t}\|_{L^2_w(0,1)} + \|K_3 - K_{3,N}\|_{L^2_w(0,1)}\|\frac{\partial\Phi}{\partial t}\|_{L^2_w(0,1)}\Big),
$$

using the inequality (3.3) for $r = 1$

$$
\|\frac{\partial\Phi}{\partial t} - \frac{\partial\Phi_N}{\partial t}\|_{L^2_w(0,1)} \le \|\Phi - \Phi_N\|_{W^1_w(0,1)} \le CN^{\frac{3}{2}-m}|\Phi|_{W^{m,N}_w(0,1)},
$$

then

$$
\|e_4\|_{L^2_w(0,1)} \le C\Big(N^{\frac{3}{2}-m}\|K_{3,N}\|_{L^2_w(0,1)}|\Phi|_{W^{m,N}_w(0,1)} + N^{-m}|K_3|_{W^{m,N}_w(0,1)}\|\frac{\partial\Phi}{\partial t}\|_{L^2_w(0,1)}\Big),
$$

and similarly

$$
\begin{aligned}
\|e_5\|_{L^2_w(0,1)} &\le \|K_4 - K_{4,N}\|_{L^2_w(0,1)}\|\Phi_N\|_{L^2_w(0,1)} \le CN^{-m}|K_4|_{W^{m,N}_w(0,1)}\big(\|\Phi\|_{L^2_w(0,1)} + \|e\|_{L^2_w(0,1)}\big),\\
\|e_6\|_{L^2_w(0,1)} &\le \|K_5 - K_{5,N}\|_{L^2_w(0,1)}\|\Phi_N\|_{L^2_w(0,1)} \le CN^{-m}|K_5|_{W^{m,N}_w(0,1)}\big(\|\Phi\|_{L^2_w(0,1)} + \|e\|_{L^2_w(0,1)}\big),\\
\|e_7\|_{L^2_w(0,1)} &\le C\Big(N^{\frac{3}{2}-m}\|K_{6,N}\|_{L^2_w(0,1)}|\Phi|_{W^{m,N}_w(0,1)} + N^{-m}|K_6|_{W^{m,N}_w(0,1)}\|\frac{\partial\Phi}{\partial t}\|_{L^2_w(0,1)}\Big).
\end{aligned}
$$

Now, considering (3.10), for $N \to \infty$, we have $\Phi_N \to \Phi$, then the desired result is obtained. □

4. Numerical examples

In this section, we apply the numerical procedure introduced in Section 2 to solve three examples. Also, the PDE problems are solved directly, and by comparing the deduced results, it would be clarified that the integral form of the PIDE problem has high numerical stability. All codes were written by Mathematica 11 on ASUS Laptop, Processor: Intel(R) Core(TM) i7-1065G7 CPU @ 1.30GHz 1.50 GHz, 8.00 GB. The shifted Legendre polynomials into the interval $[0,1]$ are used as orthogonal basis functions, and $q_1 = \dfrac{1}{2}$ and $q_2 = 1$ on the interval $[0,1]$ are selected as collocation parameters. Also, we consider $N1 = 10$ with uniform mesh, $h_n = \frac{1}{N1}$, and $t = t_n + \varepsilon h$ with $\varepsilon = 1$ for $n = 0, \cdots, N1$. Tables 1–6 show the maximum and L^2 errors in two cases; the converted integral and direct forms with $m = 2, N_1 = 10$. Also, Figures 1–6 display plot of the error function in two cases; the converted integral and direct forms for different values of N.

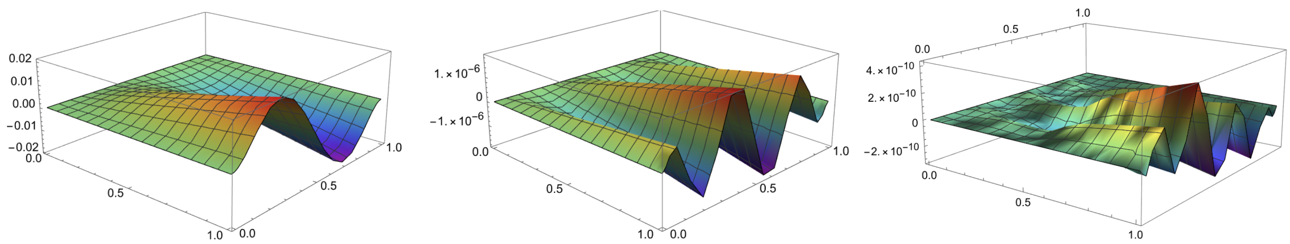
Example 4.1.

$$V(x,t) = xt^3 + t\cos(\pi x),\ V(x,0) = 0,\ \ H(t-s) = (t-s)^{-\alpha},\ \ 0 < \alpha < 1,$$
$$V(0,t) = t,\ \ V(1,t) = t^3 + t,\ \ t \geq 0.\ \lambda = 0.05,\ \ \gamma = 0.4,\ \ \alpha = \frac{1}{2},$$

$p(x,t)$ is such that the exact solution is $V(x,t)$.

Table 1. Max and L^2 errors in Example 4.1 for the converted integral form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0,1)$ error
2	2.05×10^{-2}	1.35×10^{-3}
4	2.61×10^{-4}	1.52×10^{-6}
6	2.10×10^{-6}	1.18×10^{-7}
8	1.14×10^{-8}	6.31×10^{-10}
10	4.29×10^{-10}	2.03×10^{-11}

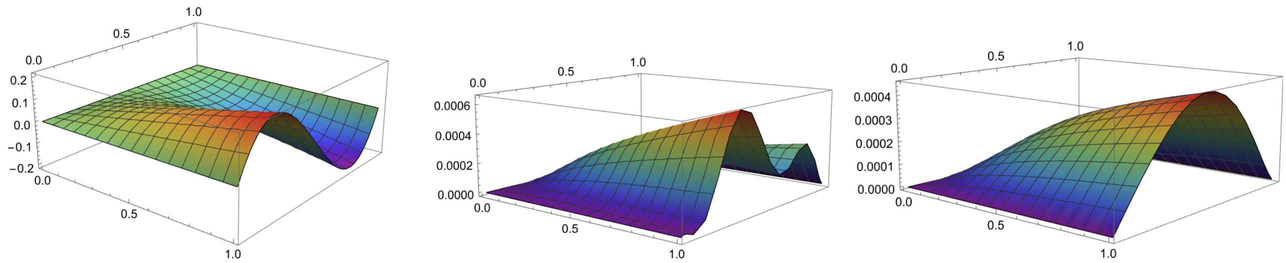


(a) N=2 **(b)** N=6 **(c)** N=10

Figure 1. Plot of error function in Example 4.1 for the converted integral form.

Table 2. Max and L^2 errors in Example 4.1 for the direct form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0,1)$ error
2	2.10×10^{-1}	1.50×10^{-1}
4	2.00×10^{-2}	1.31×10^{-2}
6	6.51×10^{-3}	3.51×10^{-3}
8	4.55×10^{-4}	3.19×10^{-4}
10	4.56×10^{-5}	2.11×10^{-5}



(a) N=2 **(b)** N=6 **(c)** N=10

Figure 2. Plot of error function in Example 4.1 for the direct form.

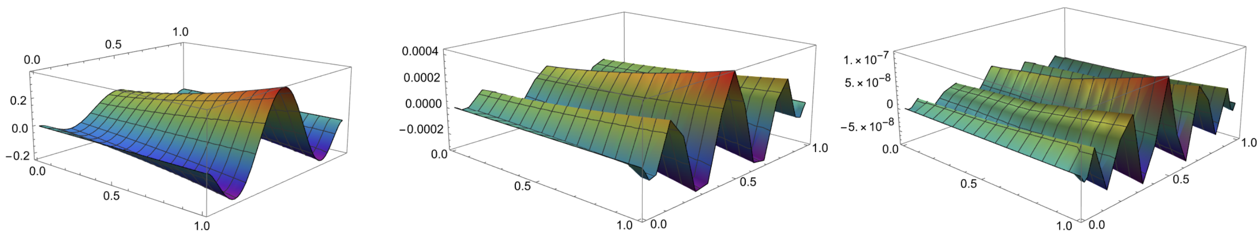
Example 4.2.

$$V(x,t) = 2(t^2 + t + 1)\sin^2(\pi x),\ \ V(0,t) = 0,\ \ V(1,t) = 0,\ \ t \geq 0,$$

$$V(x,0) = 2\sin^2(\pi x),\ \ H(t-s) = (t-s)^{-\alpha},\ \ 0 < \alpha < 1,\ \ \lambda = 0.05,\ \ \gamma = 0.4,\ \ \alpha = \frac{1}{2}.$$

Table 3. Max and L^2 errors in Example 4.2 for the converted integral form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0,1)$ error
2	3.80×10^{-1}	2.05×10^{-2}
4	1.50×10^{-2}	8.50×10^{-4}
6	4.24×10^{-4}	2.30×10^{-5}
8	7.90×10^{-6}	4.27×10^{-7}
10	1.20×10^{-7}	5.85×10^{-9}

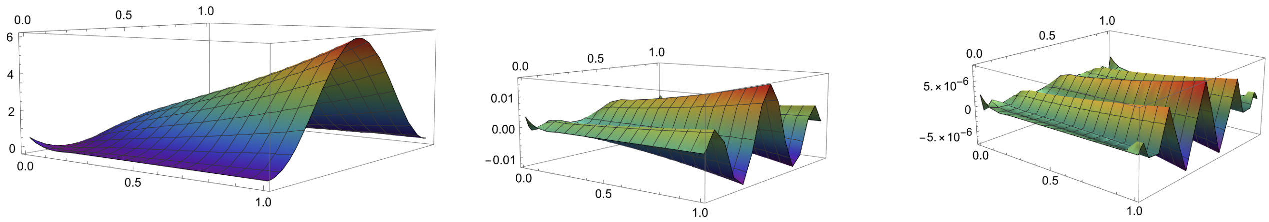


(a) N=2 **(b)** N=6 **(c)** N=10

Figure 3. Plot of error function in Example 4.2 for the converted integral form.

Table 4. Max and L^2 errors in Example 4.2 for the direct form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0, 1)$ error
2	6.10×10^{-1}	3.74×10^{-1}
4	3.76×10^{-1}	2.00×10^{-2}
6	1.57×10^{-2}	8.50×10^{-4}
8	4.20×10^{-4}	2.30×10^{-5}
10	7.90×10^{-6}	4.27×10^{-7}



(a) N=2 **(b)** N=6 **(c)** N=10

Figure 4. Plot of error function in Example 4.2 for the direct form.

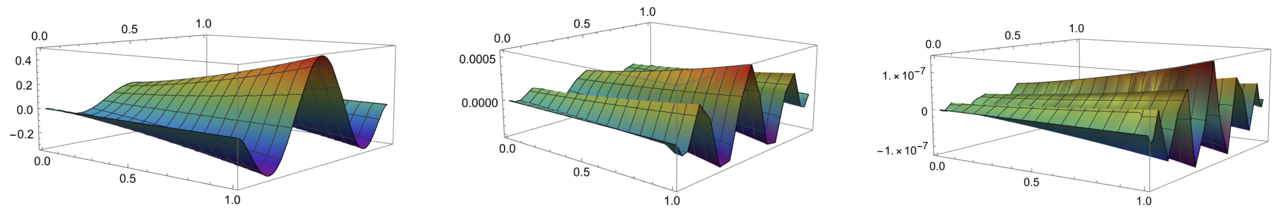
Example 4.3.

$$V(x, t) = (t + 1)^2(1 - Cos(2\pi x) + 2\pi^2 x(1 - x)),\ \ V(x, 0) = (1 - Cos(2\pi x) + 2\pi^2 x(1 - x)),$$

$$H(t - s) = (t - s)^{-\alpha},\ \ 0 < \alpha < 1,\ \ V(0, t) = 0,\ \ V(1, t) = 0,\ \ t \geq 0.\ \lambda = 0.05,\ \ \gamma = 0.4,\ \ \alpha = \frac{1}{3}.$$

Table 5. Max and L^2 errors in Example 4.3 for the converted integral form with $m = 2, N_1 = 10$.

N	Max error	$L^2(0, 1)$ error
2	4.83×10^{-2}	2.69×10^{-2}
4	2.08×10^{-2}	1.13×10^{-3}
6	5.65×10^{-4}	3.06×10^{-5}
8	1.05×10^{-5}	5.69×10^{-7}
10	1.48×10^{-7}	7.65×10^{-9}

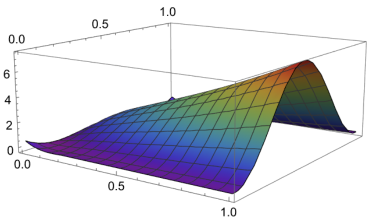


(a) N=2 **(b)** N=6 **(c)** N=10

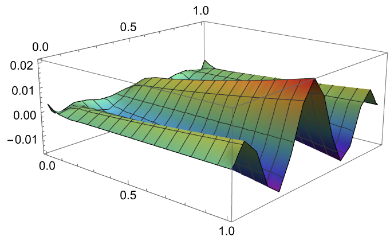
Figure 5. Plot of error function in Example 4.3 for the converted integral form.

Table 6. Max and L^2 errors in Example 4.3 for the direct form with $m = 2, N_1 = 10$.

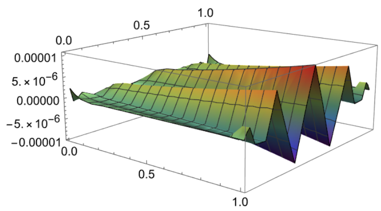
N	Max error	$L^2(0,1)$ error
2	6.74×10^{-1}	4.01×10^{-1}
4	4.73×10^{-1}	2.67×10^{-2}
6	2.08×10^{-2}	1.13×10^{-3}
8	5.65×10^{-4}	3.06×10^{-5}
10	1.05×10^{-5}	5.69×10^{-7}



(a) N=2



(b) N=6



(c) N=10

Figure 6. Plot of error function in Example 4.3 for the direct form.

Remark 4.4. *When you want to find the approximate solution of PIDE directly, you need to know the derivatives of the approximation. Using the relation* (3.3) *(truncation error of the derivatives) for the first or second derivatives, we can let $r = 1$ or $r = 2$ and write*

$$\|\phi' - (P_N\phi)'\|_{L^2_w(\Omega)} \leq CN^{\frac{3}{2}-m}|\phi|_{W^{m,N}_w(\Lambda)},$$
$$\|\phi'' - (P_N\phi)''\|_{L^2_w(\Omega)} \leq CN^{\frac{7}{2}-m}|\phi|_{W^{m,N}_w(\Lambda)},$$

whereas by the the relation (3.2)*, we have*

$$\|\phi - P_N\phi\|_{L^2_w(\Omega)} \leq CN^{-m}|\phi|_{W^{m,N}_w(\Omega)}.$$

Comparing these obtained relations, we observe that for the truncation error of the derivatives, the order of convergence has worsened. Due to the elimination of these derivatives in VFIE form, the reported errors in this form are better than the PIDE form and show good numerical stability compared to that.

5. Conclusions

In this paper, in order to study the numerical solution of a partial integro-differential equation with a weakly singular kernel (PIDE), we first transferred this equation into a Volterra–Fredholm integral equation (VFIE), then applied the Tau method based on orthogonal polynomials in two cases. In the first case, we performed the Tau method for the numerical solution of the problem in VFIE form, and in the second case, the Tau method was used for the PIDE form. The convergence of the numerical solution has been analyzed. Through the remarkable numerical results, we have shown that the VFIE problem has high numerical stability with respect to the original form of the PIDE problem.

Author contributions

Mr Ahmed wrote some part of the text and prepared the results. Dr Pishbin wrote the code in mathematica software, reviewed and edited the contents. Dr Shokri wrote part of the text about the convergence analysis of the method and edited the contents. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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