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Research article

Novel categorical relations between $\mathcal{L}$-fuzzy co-topologies and $\mathcal{L}$-fuzzy ideals

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Abstract: The goal of this paper is to construct novel relationships among $\mathcal{L}$-fuzzy ideal, $\mathcal{L}$-fuzzy co-topological, and $\mathcal{L}$-fuzzy pre-proximity spaces in complete residuated lattices. We illustrate and prove four functors between the categories of those spaces and finally, we give examples.

Keywords: complete residuated lattice; $\mathcal{L}$-fuzzy ideal; $\mathcal{L}$-fuzzy co-topological space; $\mathcal{L}$-fuzzy pre-proximity; functors
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1. Introduction

Primitively, Ward and Dilworth [1] introduced a structure of truth value in many valued logics which gave a hand to Bělohlávek [2] to use fuzzy relations with truth values in modeling intelligent systems with insufficient and vacuous information. Then, Höhle and Šostak [3] used various algebraic structures (quantales, cqm, MV-algebra) of truth values to give the concepts of $\mathcal{L}$-fuzzy topologies. Later, in the works [3–6], various attitudes toward studying mathematics in addition to logic and $\mathcal{L}$-fuzzy topologies were introduced by these algebraic structures.

In 1977, the idea of filters in $\mathcal{I}^X$ for $\mathcal{I} = [0, 1]$ as a unit interval of the real line was developed by Lowen [7]. He called it pre-filters and discussed the convergence in fuzzy topological spaces. Then, in 1999, Burton et al. [8] introduced the concept of generalized filters as a mapping from 2^X to I. Subsequently Höhle and Šostak developed the notion of L-filters [3]. Recently in 2013, Jäger [9] introduced the stratified LM-filters using stratification mapping, where $\mathcal{L}$ and M are frames. The dual of smooth filters [10] is the concept of smooth ideal as a mapping from I^X to I, and, were introduced by Ramadan et al. in [11]. It has developed in many directions, such as $\mathcal{L}$-fuzzy filters [12], fuzzy ideals [13], $\mathcal{L}$-filters [14], fuzzy filters [15], soft closure spaces [16], hyperlattice [17], fuzzy sets [18].

In this paper, we identify $\mathcal{L}$-fuzzy co-topological spaces and $\mathcal{L}$-fuzzy pre-proximity spaces induced by $\mathcal{L}$-fuzzy (prime) ideals and study categorical interrelations among $\mathcal{L}$-fuzzy (prime) ideal spaces, $\mathcal{L}$-fuzzy co-topological spaces, and $\mathcal{L}$-fuzzy pre-proximity spaces. The study obtains four novel functors among the categories of $\mathcal{L}$-fuzzy (prime) ideal spaces, $\mathcal{L}$-fuzzy co-topological spaces, and $\mathcal{L}$-fuzzy pre-proximity spaces.

2. Preliminaries

Definition 1. [1, 18] A complete residuated lattice is an algebra $(\mathcal{L}, \wedge, \vee, \odot, \to, \triangledown, \vartriangle)$ that fulfils the next terms:

(CRL1) $\mathcal{L}$ is a complete lattice denoted by $(\mathcal{L}, \leq, \vee, \wedge, \vartriangle, \triangledown)$ with the greatest (least) elements $\vartriangle$ ($\triangledown$) resp.

(CRL2) $\mathcal{L}$ with $\odot$ and $\vartriangle$ forms a commutative monoid.

(CRL3) For all $a, b, c \in \mathcal{L}$, we have $a \odot b \leq c$ iff $a \leq b \to c$.

In the upcoming proofs, we presume that $(\mathcal{L}, \leq, \odot, ^*)$ is a complete residuated lattice accompanied by * as an order reversing involution such that for each $x \in L$,

$$a \oplus b = (a^* \odot b^*)^*, \quad a^* = a \to \triangledown, \quad (a^*)^* = a.$$

Finally, $\mathcal{L}$ has the idempotence property if $a \odot a = a$ for all $a \in \mathcal{L}$.

Some essential operations on $\mathcal{L}$-fuzzy sets and lattice elements are given in the next lemma, and they were previously proposed in many papers [1, 5, 18].

Lemma 1. For a complete residuated lattice $\mathcal{L}$ accompanied by order reversing involution * and for each $a, b, c, a_j, b_j, d \in \mathcal{L}$, $j \in \Gamma$, we have the next operations:

(1) $a \to b = \bigvee\{c : c \odot a \leq b\}$;

(2) $\vartriangle \to a = a$, $\triangledown \odot a = \triangledown$ and $a \leq b$ iff $a \to b = \vartriangle$;

(3) If $b \leq c$, then $a \odot b \leq a \odot c, a \oplus b \leq a \oplus c, a \to b \leq a \to c$ and $c \to a \leq b \to a$;

(4) $(\bigwedge_{j\in\Gamma} a_j)^* = \bigvee_{j\in\Gamma} a_j^*, (\bigvee_{j\in\Gamma} a_j)^* = \bigwedge_{j\in\Gamma} a_j^*$;

(5) $a \odot (\bigvee_{j\in\Gamma} b_j) = \bigvee_{j\in\Gamma}(a \odot b_j)$ and $(\bigwedge_{j\in\Gamma} a_j) \oplus b = \bigwedge_{j\in\Gamma}(a_j \oplus b)$;

(6) $\bigvee_{j\in\Gamma} a_j \to \bigvee_{j\in\Gamma} b_j \geq \bigwedge_{j\in\Gamma}(a_j \to b_j)$, $\bigwedge_{j\in\Gamma} a_j \to \bigwedge_{j\in\Gamma} b_j \geq \bigwedge_{j\in\Gamma}(a_j \to b_j)$;

(7) $(a \odot b) \odot (c \oplus d) \leq (a \odot c) \oplus (b \odot d)$;

(8) $(a \oplus c) \odot (b \oplus d) \leq (a \oplus b) \oplus (c \odot d)$.

A map p: $X \to \mathcal{L}$ is called $\mathcal{L}$-subset on a set X [19]. The collection of all L-subsets on X is denoted by $\mathcal{L}^X$. For the $\mathcal{L}$-subset p and q, we define $(p \to q)$, $\vartriangle_a$, $\vartriangle_a^*$ and $(p \odot q) \in \mathcal{L}^X$ by

$$(p \to q)(a) = p(a) \to q(a),$$
$$(p \odot q)(a) = p(a) \odot q(a),$$
$$\vartriangle_a(b) = \begin{cases} \vartriangle, & \text{if } b = a, \\ \triangledown, & \text{otherwise,} \end{cases}$$

$$\triangle_a^*(b) = \begin{cases} \nabla, & \text{if } b = a, \\ \triangle, & \text{otherwise.} \end{cases}$$

Lemma 2. [2, 4, 20] Let $\mathcal{X}$ be a nonempty set. Define a binary mapping $\mathcal{S}$: $\mathcal{L}^{\mathcal{X}} \times \mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ for the degree of subsethood of $p, q \in \mathcal{L}^{\mathcal{X}}$ by

$$\mathcal{S}(p, q) = \bigwedge_{a \in \mathcal{X}} (p(a) \to q(a)).$$

Hence, for all $r, s, p_j, q_j \in \mathcal{L}^{\mathcal{X}}$, $j \in \Gamma$, the next conditions apply:

(SH1) $\mathcal{S}(p, q) = \triangle \Leftrightarrow p \leq q$;

(SH2) $p \leq q \Rightarrow \mathcal{S}(p, r) \geq \mathcal{S}(q, r)$ and $\mathcal{S}(r, p) \leq \mathcal{S}(r, q)$;

(SH3) $\mathcal{S}(p, q) \odot \mathcal{S}(r, s) \leq \mathcal{S}(p \odot r, q \odot s)$;

(SH4) $\mathcal{S}(p, q) \odot \mathcal{S}(r, s) \leq \mathcal{S}(p \oplus r, q \oplus s)$;

(SH5)

$$\bigwedge_{j \in \Gamma} \mathcal{S}(p_j, q_j) \leq \mathcal{S}(\bigvee_{j \in \Gamma} p_j, \bigvee_{j \in \Gamma} q_j)$$

and

$$\bigwedge_{j \in \Gamma} \mathcal{S}(p_j, q_j) \leq \mathcal{S}(\bigwedge_{j \in \Gamma} p_j, \bigwedge_{j \in \Gamma} q_j).$$

Definition 2. [21] If C is a category and $\mathcal{W}$: $C \to Set$ is a faithful functor, then the pair $(C, \mathcal{W})$ is a concrete category. For every C-object $\mathcal{X}, \mathcal{W}(\mathcal{X})$ is the underlying set of $\mathcal{X}$. Hence, all objects in a concrete category can be taken as structured sets.

Shortly in this paper, we take C for $(C, \mathcal{W})$ if the concrete functor is clear.

A concrete functor $\mathcal{H}$: $\mathcal{E} \to \mathcal{K}$ is a functor between two concrete categories $(\mathcal{E}, \mathcal{U})$ and $(\mathcal{K}, \mathcal{V})$ with $\mathcal{U} = \mathcal{V} \circ \mathcal{H}$, where $\mathcal{H}$ modifies the structures on the underlying sets. Thus, to define a concrete functor $\mathcal{H}$: $\mathcal{E} \to \mathcal{K}$, we satisfy the next two conditions:

(1) We appoint to each $\mathcal{E}$-object $\mathcal{X}$, a $\mathcal{K}$-object $\mathcal{H}(\mathcal{X})$ in which

$$\mathcal{V}(\mathcal{G}(\mathcal{X})) = \mathcal{U}(\mathcal{X}).$$

(2) We confirm that if a function ψ: $\mathcal{U}(\mathcal{X}) \to \mathcal{U}(\mathcal{Y})$ is a $\mathcal{E}$-morphism for $\mathcal{X} \to \mathcal{Y}$ then it is also $\mathcal{K}$-morphism for $\mathcal{H}(\mathcal{X}) \to \mathcal{H}(\mathcal{Y})$.

Definition 3. [5, 18, 20] An $\mathcal{L}$-fuzzy co-topological space $(\mathcal{X}, \mathcal{F})$ is a mapping $\mathcal{F}$: $\mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ on a nonempty set $\mathcal{X}$ that fulfills the next conditions for each $p, q \in \mathcal{L}^{\mathcal{X}}$:

(CTP1) $\mathcal{F}(\nabla_{\mathcal{X}}) = \mathcal{F}(\triangle_{\mathcal{X}}) = \triangle$;

(CTP2) $\mathcal{F}(p \oplus q) \geq \mathcal{F}(p) \odot \mathcal{F}(q)$;

(CTP3) $\mathcal{F}(\bigwedge_{j \in \Gamma} p_j) \geq \bigwedge_{j \in \Gamma} \mathcal{F}(p_j)$ for every $\{p_j : j \in \Gamma\} \subseteq \mathcal{L}^{\mathcal{X}}$.

An $\mathcal{L}$-fuzzy co-topological space $(\mathcal{X}, \mathcal{F})$ is:

(AL) Alexandrov if $\mathcal{F}(\bigvee_{j \in \Gamma} p_j) \geq \bigwedge_{j \in \Gamma} \mathcal{F}(p_j)$ for every $\{p_j : j \in \Gamma\} \subseteq \mathcal{L}^{\mathcal{X}}$;

(SP) separated if $\mathcal{F}(\triangle_a^*) = \triangle$ for all $a \in \mathcal{X}$.

We define the $\mathcal{LF}$-continuous map ψ: $\mathcal{X} \to \mathcal{Y}$ for two $\mathcal{L}$-fuzzy co-topological spaces $(\mathcal{X}, \mathcal{F}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{F}_{\mathcal{Y}})$ by

$$\mathcal{F}_{\mathcal{Y}}(p) \leq \mathcal{F}_{\mathcal{X}}(\psi^{\leftarrow}(p))$$

for each $p \in \mathcal{L}^{\mathcal{Y}}$.

The category of $\mathcal{L}$-fuzzy co-topological spaces with $\mathcal{LF}$-continuous maps as morphisms is denoted by **LF-CTP**.

Definition 4. [11, 13] An $\mathcal{L}$-fuzzy ideal space $(\mathcal{X}, \mathcal{I})$ is a mapping $\mathcal{I}$: $\mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ on a nonempty set $\mathcal{X}$ fulfils the next conditions for all $p, q \in \mathcal{L}^{\mathcal{X}}$:

(ID1) $\mathcal{I}(\triangledown_{\mathcal{X}}) = \vartriangle$;

(ID2) $p \leq q \Rightarrow \mathcal{I}(p) \geq \mathcal{I}(q)$;

(ID3) $\mathcal{I}(p \oplus q) \geq \mathcal{I}(p) \odot \mathcal{I}(q)$.

An $\mathcal{L}$-fuzzy ideal space $(\mathcal{X}, \mathcal{I})$ is called:

(AL) Alexandrov if $\mathcal{I}(\bigvee_{j\in\Gamma} p_j) \geq \bigwedge_{j\in\Gamma} \mathcal{I}(p_j)$ for all $\{p_j : j \in \Gamma\} \subseteq \mathcal{L}^{\mathcal{X}}$;

(SP) separated if $\mathcal{I}(\vartriangle_a^*) = \vartriangle$ for all $a \in \mathcal{X}$.

We define the $\mathcal{LF}$-ideal map ψ: $\mathcal{X} \to \mathcal{Y}$ for two $\mathcal{L}$-fuzzy ideal spaces $(\mathcal{X}, \mathcal{I}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ by

$$\mathcal{I}_{\mathcal{Y}}(p) \leq \mathcal{I}_{\mathcal{X}}(\psi^{\leftarrow}(p))$$

for each $p \in \mathcal{L}^{\mathcal{Y}}$.

The category of $\mathcal{L}$-fuzzy ideal spaces with $\mathcal{LF}$-ideal maps as morphisms is denoted by **LF-I**.

Remark 1. In addition to the above axioms, if

(ID4) $\mathcal{I}(\vartriangle_{\mathcal{X}}) = \triangledown$.

Then, $(\mathcal{X}, \mathcal{I})$ is an $\mathcal{L}$-fuzzy prime ideal space.

The category of $\mathcal{L}$-fuzzy prime ideal spaces with $\mathcal{LF}$-ideal maps as morphisms is denoted by **LF-PI**.

3. The functors between $\mathcal{L}$-fuzzy co-topological and $\mathcal{L}$-fuzzy (prime) ideal spaces

The following two theorems give a functor from **LF-PI** to **LF-CTP**.

Theorem 1. Given $(\mathcal{X}, \mathcal{I})$ as an $\mathcal{L}$-fuzzy prime ideal space, we define $\mathcal{F}^{\mathcal{I}}$: $\mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ by

$$\mathcal{F}^{\mathcal{I}}(p) = \bigwedge_{a\in\mathcal{X}} p(a) \oplus p^*(a) \odot \mathcal{I}(p).$$

Then,

(1) $(\mathcal{X}, \mathcal{F}^{\mathcal{I}})$ is an $\mathcal{L}$-fuzzy co-topological space.

(2) Let

$$\bigwedge_{j\in\Gamma}(a_j \odot b_j) = \bigwedge_{j\in\Gamma} a_j \odot \bigwedge_{j\in\Gamma} b_j, \quad \forall\, a_j, b_j \in \mathcal{L},$$

then $\mathcal{F}^{\mathcal{I}}$ is Alexandrov if $\mathcal{I}$ is so.

(3) $\mathcal{F}^{\mathcal{I}}$ is separated if $\mathcal{I}$ is so.

Proof. (1) (CTP1)

$$\mathcal{F}^{\mathcal{I}}(\nabla_{\mathcal{X}}) = \bigwedge_{a\in\mathcal{X}} \nabla_{\mathcal{X}}(a)\oplus \vartriangle_{\mathcal{X}} (a) \odot \mathcal{I}(\nabla_{\mathcal{X}}) = \vartriangle$$

and

$$\mathcal{F}^{\mathcal{I}}(\vartriangle_{\mathcal{X}}) = \bigwedge_{a\in\mathcal{X}} \vartriangle_{\mathcal{X}} (a) \oplus \nabla_{\mathcal{X}}(a) \odot \mathcal{I}(\vartriangle_{\mathcal{X}}) = \vartriangle .$$

(CTP2) For $p, q \in \mathcal{L}^{\mathcal{X}}$, we have

$$\begin{aligned}
\mathcal{F}^{\mathcal{I}}(p) \odot \mathcal{F}^{\mathcal{I}}(q) &= \Big(\bigwedge_{a\in\mathcal{X}} p(a) \oplus p^*(a) \odot \mathcal{I}(p)\Big) \odot \Big(\bigwedge_{a\in\mathcal{X}} q(a) \oplus q^*(a) \odot \mathcal{I}(q)\Big) \\
&\leq \bigwedge_{a\in\mathcal{X}} \Big(p(a) \oplus p^*(a) \odot \mathcal{I}(p)\Big) \odot \Big(q(a) \oplus q^*(a) \odot \mathcal{I}(q)\Big) \\
&\leq \bigwedge_{a\in\mathcal{X}} (p(a) \oplus q(a)) \oplus (p^*(a) \odot \mathcal{I}(p) \odot q^*(a) \odot \mathcal{I}(q)) \\
&\leq \bigwedge_{a\in\mathcal{X}} (p \oplus q)(a) \oplus (p \oplus q)^*(a) \odot \mathcal{I}(p \oplus q) \\
&= \mathcal{F}^{\mathcal{I}}(p \oplus q).
\end{aligned}$$

(CTP3) For each family $\{p_j : j \in \Gamma\}$, we have

$$\begin{aligned}
\mathcal{F}^{\mathcal{I}}(\bigwedge_{j\in\Gamma} p_j) &= \bigwedge_{a\in\mathcal{X}} (\bigwedge_{j\in\Gamma} p_j)(a) \oplus (\bigvee_{j\in\Gamma} p_j^*)(a) \odot \mathcal{I}(\bigwedge_{j\in\Gamma} p_j) \\
&= \bigwedge_{a\in\mathcal{X}} \bigwedge_{j\in\Gamma} p_j(a) \oplus (\bigvee_{j\in\Gamma} p_j^*(a) \odot \mathcal{I}(\bigwedge_{j\in\Gamma} p_j)) \\
&\geq \bigwedge_{a\in\mathcal{X}} \bigwedge_{j\in\Gamma} p_j(a) \oplus (\bigvee_{j\in\Gamma} p_j^*(a) \odot \mathcal{I}(p_j)) \\
&\geq \bigwedge_{j\in\Gamma} \bigwedge_{a\in\mathcal{X}} p_j(a) \oplus p_j^*(a) \odot \mathcal{I}(p_j) \\
&= \bigwedge_{j\in\Gamma} \mathcal{F}^{\mathcal{I}}(p_j).
\end{aligned}$$

Thus, $(\mathcal{X}, \mathcal{F}^{\mathcal{I}})$ is an $\mathcal{L}$-fuzzy co-topological space.
(2) For each family $\{p_j : j \in \Gamma\}$, we have

$$\begin{aligned}
\bigwedge_{j\in\Gamma} \mathcal{F}^{\mathcal{I}}(p_j) &= \bigwedge_{j\in\Gamma} \bigwedge_{a\in\mathcal{X}} p_j(a) \oplus p_j^*(a) \odot \mathcal{I}(p_j) \\
&= \bigwedge_{a\in\mathcal{X}} (\bigwedge_{j\in\Gamma} p_j)(a) \oplus \bigwedge_{j\in\Gamma} (p_j^*(a) \odot \mathcal{I}(p_j)) \\
&= \bigwedge_{a\in\mathcal{X}} (\bigwedge_{j\in\Gamma} p_j)(a) \oplus (\bigwedge_{j\in\Gamma} p_j^*(a) \odot \bigwedge_{j\in\Gamma} \mathcal{I}(p_j)) \\
&\leq \bigwedge_{a\in\mathcal{X}} (\bigvee_{j\in\Gamma} p_j)(a) \oplus (\bigvee_{j\in\Gamma} p_j)^*(a) \odot \mathcal{I}(\bigvee_{j\in\Gamma} p_j) \\
&= \mathcal{F}^{\mathcal{I}}(\bigvee_{j\in\Gamma} p_j).
\end{aligned}$$

(3)

$$\begin{aligned}
\mathcal{F}^{\mathcal{I}}(\triangle_a^*) &= \bigwedge_{b\in\mathcal{X}} \triangle_a^*(b) \oplus \triangle_a(b) \odot \mathcal{I}(\triangle_a^*) \\
&= (\triangle_a^*(a) \oplus \triangle_a(a) \odot \mathcal{I}(\triangle_a^*)) \odot \bigwedge_{b\in\mathcal{X}, b\neq a} (\triangle_a^*(b) \oplus \triangle_a(b) \odot \mathcal{I}(\triangle_a^*)) \\
&= (\triangledown \oplus \triangle \odot \triangle) \odot \bigwedge_{b\in\mathcal{X}, b\neq a} (\triangle \oplus \triangledown \odot \triangle) \\
&= \triangle .
\end{aligned}$$

□

Theorem 2. Let ψ: $\mathcal{X} \to \mathcal{Y}$ be an $\mathcal{L}F$-ideal map for $(\mathcal{X}, \mathcal{I}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ two $\mathcal{L}$-fuzzy prime ideal spaces, then ψ: $(\mathcal{X}, \mathcal{F}^{\mathcal{I}_{\mathcal{X}}}) \to (\mathcal{Y}, \mathcal{F}^{\mathcal{I}_{\mathcal{Y}}})$ is an $\mathcal{L}F$-continuous map.

Proof. For any $p \in \mathcal{L}^{\mathcal{Y}}$, we have

$$\begin{aligned}
\mathcal{F}^{\mathcal{I}_{\mathcal{X}}}(\psi^{\leftarrow}(p)) &= \bigwedge_{a\in\mathcal{X}} \psi^{\leftarrow}(p)(a) \oplus \psi^{\leftarrow}(p^*)(a) \odot \mathcal{I}_{\mathcal{X}}(\psi^{\leftarrow}(p)) \\
&\geq \bigwedge_{a\in\mathcal{X}} p(\psi(a)) \oplus p^*(\psi(a)) \odot \mathcal{I}_{\mathcal{Y}}(p) \\
&\geq \bigwedge_{b\in\mathcal{Y}} p(b) \oplus p^*(b) \odot \mathcal{I}_{\mathcal{Y}}(p) \\
&= \mathcal{F}^{\mathcal{I}_{\mathcal{Y}}}(p).
\end{aligned}$$

□

Corollary 1. Υ: **LF-PI** $\to$ **LF-CTP** is a concrete functor defined by

$$\Upsilon(X, \mathcal{I}_X) = (X, \mathcal{F}^{\mathcal{I}_X}),\ \Upsilon(\varphi) = \varphi.$$

Further, the following two theorems give a rise to another functor from **LF-PI** to **LF-CTP**.

Theorem 3. Given $(\mathcal{X}, \mathcal{I})$ as an $\mathcal{L}$-fuzzy prime ideal space, we define $\mathcal{F}_1^{\mathcal{I}}$: $\mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ by

$$\mathcal{F}_1^{\mathcal{I}}(p) = \mathcal{S}(p^*, p^* \odot \mathcal{I}(p)).$$

Then,

(1) $(\mathcal{X}, \mathcal{F}_1^{\mathcal{I}})$ is an $\mathcal{L}$-fuzzy co-topological space;

(2) $\mathcal{F}_1^{\mathcal{I}}$ is separated if $\mathcal{I}$ is so;

(3) Let

$$\bigwedge_{j\in\Gamma}(a_j \odot b_j) = \bigwedge_{j\in\Gamma} a_j \odot \bigwedge_{j\in\Gamma} b_j \ \forall\ a_j, b_j \in \mathcal{L},$$

then $\mathcal{F}_1^{\mathcal{I}}$ is Alexandrov if $\mathcal{I}$ is so.

Proof. (1) (CTP1)

$$\mathcal{F}_1^{\mathcal{I}}(\triangledown_{\mathcal{X}}) = \mathcal{S}(\vartriangle_{\mathcal{X}}, \vartriangle_{\mathcal{X}} \odot \mathcal{I}(\triangledown_{\mathcal{X}})) = \mathcal{S}(\vartriangle_{\mathcal{X}}, \vartriangle_{\mathcal{X}}) = \vartriangle$$

and

$$\mathcal{F}_1^{\mathcal{I}}(\vartriangle_{\mathcal{X}}) = \mathcal{S}(\triangledown_{\mathcal{X}}, \triangledown_{\mathcal{X}} \odot \mathcal{I}(\vartriangle_{\mathcal{X}})) = \mathcal{S}(\triangledown_{\mathcal{X}}, \triangledown_{\mathcal{X}}) = \vartriangle .$$

(CTP2) For $p, q \in \mathcal{L}^{\mathcal{X}}$, we have

$$\begin{aligned} \mathcal{F}_1^{\mathcal{I}}(p) \odot \mathcal{F}_1^{\mathcal{I}}(q) &= \mathcal{S}(p^*, p^* \odot \mathcal{I}(p)) \odot \mathcal{S}(q^*, q^* \odot \mathcal{I}(q)) \\ &\leq \mathcal{S}(p^* \odot q^*, \mathcal{I}(p) \odot \mathcal{I}(q) \odot (p^* \odot q^*)) \\ &\leq \mathcal{S}((p \oplus q)^*, \mathcal{I}(p \oplus q) \odot (p \oplus q)^*) \\ &= \mathcal{F}_1^{\mathcal{I}}(p \oplus q). \end{aligned}$$

(CTP3) For each family $\{p_j : j \in \Gamma\}$, we have

$$\begin{aligned} \mathcal{F}_1^{\mathcal{I}}(\bigwedge_{j\in\Gamma} p_j) &= \mathcal{S}(\bigvee_{j\in\Gamma} p_j^*, \bigvee_{j\in\Gamma} p_j^* \odot \mathcal{I}(\bigwedge_{j\in\Gamma} p_j)) \\ &= \mathcal{S}(\bigvee_{j\in\Gamma} p_j^*, \bigvee_{j\in\Gamma} (p_j^* \odot \mathcal{I}(\bigwedge_{j\in\Gamma} p_j))) \\ &\geq \mathcal{S}(\bigvee_{j\in\Gamma} p_j^*, \bigvee_{j\in\Gamma} (p_j^* \odot \mathcal{I}(p_j))) \\ &\geq \bigwedge_{j\in\Gamma} \mathcal{S}(p_j^*, p_j^* \odot \mathcal{I}(p_j)) \\ &= \bigwedge_{j\in\Gamma} \mathcal{F}_1^{\mathcal{I}}(p_j). \end{aligned}$$

Hence, $(\mathcal{X}, \mathcal{F}_1^{\mathcal{I}})$ is an $\mathcal{L}$-fuzzy co-topological space.
(2)

$$\mathcal{F}_1^{\mathcal{I}}(\vartriangle_a^*) = \mathcal{S}(\vartriangle_a, \vartriangle_a \odot \mathcal{I}(\vartriangle_a^*)) = \mathcal{S}(\vartriangle_a, \vartriangle_a \odot \vartriangle) = \vartriangle .$$

(3) For each family $\{p_j : j \in \Gamma\}$, we have

$$\begin{aligned} \bigwedge_{j\in\Gamma} \mathcal{F}_1^{\mathcal{I}}(p_j) &= \bigwedge_{j\in\Gamma} \mathcal{S}(p_j, p_j^* \odot \mathcal{I}(p_j)) \\ &\leq \mathcal{S}(\bigwedge_{j\in\Gamma} p_j^*, \bigwedge_{j\in\Gamma} (p_j^* \odot \mathcal{I}(p_j))) \\ &= \mathcal{S}(\bigwedge_{j\in\Gamma} p_j^*, \bigwedge_{j\in\Gamma} p_i^* \odot \bigwedge_{j\in\Gamma} \mathcal{I}(p_j)) \\ &= \mathcal{S}((\bigvee_{j\in\Gamma} p_j)^*, (\bigvee_{j\in\Gamma} p_j)^* \odot \bigwedge_{j\in\Gamma} \mathcal{I}(p_j)) \\ &\leq \mathcal{S}((\bigvee_{j\in\Gamma} p_j)^*, (\bigvee_{j\in\Gamma} p_j)^* \odot \mathcal{I}(\bigvee_{j\in\Gamma} p_j)) \\ &= \mathcal{F}_1^{\mathcal{I}}(\bigvee_{j\in\Gamma} p_j). \end{aligned}$$

□

Theorem 4. Let ψ: $\mathcal{X} \to \mathcal{Y}$ be an $\mathcal{L}F$-prime ideal map for $(\mathcal{X}, \mathcal{I}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ two $\mathcal{L}$-fuzzy prime ideal spaces, then ψ: $(\mathcal{X}, \mathcal{F}_1^{\mathcal{I}_{\mathcal{X}}}) \to (\mathcal{Y}, \mathcal{F}_1^{\mathcal{I}_{\mathcal{Y}}})$ is an $\mathcal{L}F$-continuous map.

Proof. For all $p \in \mathcal{L}^{\mathcal{Y}}$ and by Lemma 1(3), we have

$$\begin{aligned}\mathcal{F}_1^{\mathcal{I}_{\mathcal{X}}}(\psi^{\leftarrow}(p)) &= \mathcal{S}(\psi^{\leftarrow}(p^*), \psi^{\leftarrow}(p^*) \odot \mathcal{I}_{\mathcal{X}}(\psi^{\leftarrow}(p))) \\ &= \bigwedge_{a\in\mathcal{X}} (p^*(\psi(a)) \to (p^*(\psi(a)) \odot \mathcal{I}_{\mathcal{X}}(\psi^{\leftarrow}(p)))) \\ &\geq \bigwedge_{b\in\mathcal{Y}} (p^*(b) \to (p^*(b) \odot \mathcal{I}_{\mathcal{X}}(\psi^{\leftarrow}(p)))) \\ &\geq \bigwedge_{b\in\mathcal{Y}} (p^*(b) \to (p^*(b) \odot \mathcal{I}_{\mathcal{Y}}(p))) \\ &= \mathcal{S}(p^*, p^* \odot \mathcal{I}_{\mathcal{Y}}(p)) \\ &= \mathcal{F}_1^{\mathcal{I}_{\mathcal{Y}}}(p).\end{aligned}$$

□

Corollary 2. Ω: **LF-PI** → **LF-CTP** is a concrete functor.

Finally, the following two theorems provide yet another functor from **LF-I** to **LF-CTP**.

Theorem 5. Given $(\mathcal{X}, \mathcal{I})$ as an $\mathcal{L}$-fuzzy ideal space, we define $\mathcal{F}_2^{\mathcal{I}}$: $\mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ by

$$\mathcal{F}_2^{\mathcal{I}}(p) = \begin{cases} \mathcal{I}(p), & \text{if } p \neq \triangle_{\mathcal{X}}, \\ \triangle, & \text{if } p = \triangle_{\mathcal{X}}\ . \end{cases}$$

Then,

(1) $(\mathcal{X}, \mathcal{F}_2^{\mathcal{I}})$ is an $\mathcal{L}$-fuzzy co-topological space;

(2) $\mathcal{F}_2^{\mathcal{I}}$ is separated (Alexandrov) if $\mathcal{I}$ is so respectively.

Proof. (1) (CTP1) By definition, we have:

$$\mathcal{F}_2^{\mathcal{I}}(\triangle_{\mathcal{X}}) = \triangle$$

and

$$\mathcal{F}_2^{\mathcal{I}}(\nabla_{\mathcal{X}}) = \mathcal{I}(\nabla_{\mathcal{X}}) = \triangle\ .$$

(CTP2) For any $p, q \in \mathcal{L}^{\mathcal{X}}$, we have:

Case 1. If $p \oplus q = \triangle_{\mathcal{X}}$, then

$$\mathcal{F}_2^{\mathcal{I}}(p \oplus q) = \triangle \geq \mathcal{F}_2^{\mathcal{I}}(p) \odot \mathcal{F}_2^{\mathcal{I}}(q).$$

Case 2. If $p \oplus q \neq \triangle_{\mathcal{X}}$, then $p \neq \triangle_{\mathcal{X}}$ and $q \neq \triangle_{\mathcal{X}}$. So,

$$\mathcal{F}_2^{\mathcal{I}}(p \oplus q) = \mathcal{I}(p \oplus q) \geq \mathcal{I}(p) \odot \mathcal{I}(q) = \mathcal{F}_2^{\mathcal{I}}(p) \odot \mathcal{F}_2^{\mathcal{I}}(q).$$

(CTP3) For each family $\{p_j : j \in \Gamma\}$, we have:

Case 1. If

$$\bigwedge_{j\in\Gamma} p_j = \vartriangle_{\mathcal{X}},$$

then $p_j = \vartriangle_{\mathcal{X}}$, $j \in \Gamma$. So,

$$\mathcal{F}_2^{\mathcal{I}}(\bigwedge_{j\in\Gamma} p_j) = \vartriangle \geq \bigwedge_{j\in\Gamma} \mathcal{F}_2^{\mathcal{I}}(p_j).$$

Case 2. If

$$\bigwedge_{j\in\Gamma} p_j \neq \vartriangle_{\mathcal{X}},$$

then $p_{j_0} \neq \vartriangle_{\mathcal{X}}$ for some $j_0 \in \Gamma$. So,

$$\bigwedge_{j\in\Gamma} \mathcal{F}_2^{\mathcal{I}}(p_j) \leq \mathcal{I}(p_{j_0}) \leq \mathcal{I}(\bigwedge_{j\in\Gamma} p_j) = \mathcal{F}_2^{\mathcal{I}}(\bigwedge_{j\in\Gamma} p_j).$$

Hence, $(\mathcal{X}, \mathcal{F}_2^{\mathcal{I}})$ is an $\mathcal{L}$-fuzzy co-topological space.

(2) (SP) $\mathcal{F}_2^{\mathcal{I}}(\vartriangle_a^*) = \mathcal{I}(\vartriangle_a^*) = \vartriangle$.

(AL) For each family $\{p_j : j \in \Gamma\}$, we have:

Case 1. If

$$\bigvee_{j\in\Gamma} p_j = \vartriangle_{\mathcal{X}},$$

then

$$\mathcal{F}_2^{\mathcal{I}}(\bigvee_{j\in\Gamma} p_j) = \vartriangle \geq \bigwedge_{j\in\Gamma} \mathcal{F}_2^{\mathcal{I}}(p_j).$$

Case 2. If

$$\bigvee_{j\in\Gamma} p_j \neq \vartriangle_{\mathcal{X}},$$

then $p_j \neq \vartriangle_{\mathcal{X}}$ for each $j \in \Gamma$. So,

$$\mathcal{F}_2^{\mathcal{I}}(\bigvee_{j\in\Gamma} p_j) = \mathcal{I}(\bigvee_{j\in\Gamma} p_j) \geq \bigwedge_{j\in\Gamma} \mathcal{I}(p_j) = \bigwedge_{j\in\Gamma} \mathcal{F}_2^{\mathcal{I}}(p_j).$$

□

Theorem 6. Let ψ: $\mathcal{X} \to \mathcal{Y}$ be an $\mathcal{L}F$-ideal map for $(\mathcal{X}, \mathcal{I}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ two $\mathcal{L}$-fuzzy ideal spaces, then ψ: $(\mathcal{X}, \mathcal{F}_2^{\mathcal{I}_{\mathcal{X}}}) \to (\mathcal{Y}, \mathcal{F}_2^{\mathcal{I}_{\mathcal{Y}}})$ is an $\mathcal{L}F$-continuous map.

Proof. For any $p \in \mathcal{L}^{\mathcal{Y}}$, we have

Case 1. If $\psi^{\leftarrow}(p) = \vartriangle_{\mathcal{X}}$, then

$$\mathcal{F}_2^{\mathcal{I}_{\mathcal{X}}}(\psi^{\leftarrow}(p)) = \vartriangle \geq \mathcal{F}_2^{\mathcal{I}_{\mathcal{Y}}}(p).$$

Case 2. If $\psi^{\leftarrow}(p) \neq \vartriangle_{\mathcal{X}}$, then $p \neq \vartriangle_{\mathcal{Y}}$. So,

$$\mathcal{F}_2^{\mathcal{I}_{\mathcal{X}}}(\psi^{\leftarrow}(p)) = \mathcal{I}_{\mathcal{X}}(\psi^{\leftarrow}(p)) \geq \mathcal{I}_{\mathcal{Y}}(p) = \mathcal{F}_2^{\mathcal{I}_{\mathcal{Y}}}(p).$$

□

Corollary 3. Δ: **LF-I** $\to$ **LF-CTP** is a concrete functor.

Example 1. Let $\mathcal{X} = \{a\}$ be a single set and

$$\mathcal{L} = \{\triangledown, x, y, z, w, \vartriangle\}$$

be a lattice whose Hasse diagram is given by Figure 1. Simple calculations show $(\mathcal{L}, \vee, \wedge, \odot, \to, \triangledown, \vartriangle)$ is a regular residuated lattice in which the commutative operation $\odot$ is given by Table 1, and the operation "$\to$" is given by

$$a \to b = \bigvee \{c \in \mathcal{L} \mid a \odot c \le b\}$$

for any $a, b \in \mathcal{L}$. Then,

$$\mathcal{L}^{\mathcal{X}} = \{\underline{\triangledown},\ \underline{x}, \underline{y}, \underline{z}, \underline{w}, \underline{\vartriangle}\},\ \ \underline{\triangledown}^* = \underline{\vartriangle},\ \ \underline{\vartriangle}^* = \underline{\triangledown},\ \ \underline{x}^* = \underline{w},\ \ \underline{w}^* = \underline{x},\ \ \underline{y}^* = \underline{z},\ \ \underline{z}^* = \underline{y}.$$

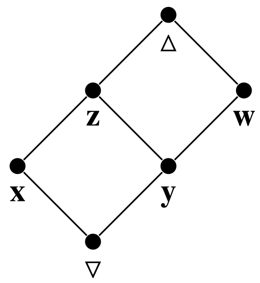

Figure 1. Hasse diagram of $\mathcal{L}$.

Table 1. Cayley table for $\odot$ of $\mathcal{L}$.

$\odot$	$\triangledown$	x	y	z	w	$\vartriangle$
$\triangledown$	$\triangledown$	$\triangledown$	$\triangledown$	$\triangledown$	$\triangledown$	$\triangledown$
x	$\triangledown$	x	$\triangledown$	x	$\triangledown$	x
y	$\triangledown$	$\triangledown$	$\triangledown$	$\triangledown$	y	y
z	$\triangledown$	x	$\triangledown$	x	y	z
w	$\triangledown$	$\triangledown$	y	y	w	w
$\vartriangle$	$\triangledown$	x	y	z	w	$\vartriangle$

We define the mapping $\mathcal{I}$: $\mathcal{L}^{\mathcal{X}} \to L$ by

$$\mathcal{I}(p) = \begin{cases} \vartriangle, & if\ \ p = \underline{\triangledown}, \\ z, & if\ \ p = \underline{x}, \\ y, & if\ \ p = \underline{y}, \underline{z}, \\ \triangledown, & otherwise. \end{cases}$$

Then, $(\mathcal{X}, \mathcal{I})$ is an $\mathcal{L}$-fuzzy prime ideal space. By Theorem 1(1), we obtain an $\mathcal{L}$-fuzzy co-topology

$\mathcal{F}^{\mathcal{I}}\colon \mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ on $\mathcal{X}$ by

$$\mathcal{F}^{\mathcal{I}}(p) = \begin{cases} z, & if \;\; p = \underline{x}, \underline{z}, \\ y, & if \;\; p = \underline{y}, \\ w, & if \;\; p = \underline{w}, \\ \triangle, & otherwise. \end{cases}$$

By Theorem 3(1), we obtain an $\mathcal{L}$-fuzzy co-topology $\mathcal{F}_1^{\mathcal{I}}\colon \mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ on $\mathcal{X}$ by

$$\mathcal{F}_1^{\mathcal{I}}(p) = \begin{cases} z, & if \;\; p = \underline{x}, \underline{z}, \\ y, & if \;\; p = \underline{y}, \\ w, & if \;\; p = \underline{w}, \\ \triangle, & otherwise. \end{cases}$$

By Theorem 5(1), we obtain an $\mathcal{L}$-fuzzy co-topology $\mathcal{F}_2^{\mathcal{I}}\colon \mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ on $\mathcal{X}$ by

$$\mathcal{F}_2^{\mathcal{I}}(p) = \begin{cases} z, & if \;\; p = \underline{x}, \\ y, & if \;\; p = \underline{y}, \underline{z}, \\ \triangledown, & if \;\; p = \underline{w}, \\ \triangle, & otherwise. \end{cases}$$

4. The relationships between $\mathcal{L}$-fuzzy pre-proximity and $\mathcal{L}$-fuzzy ideal spaces

In this section, we give a relationship between $\mathcal{L}$-fuzzy pre-proximity spaces [22, 23] and $\mathcal{L}$-fuzzy ideal spaces. In addition, we find and prove the functor between **LF-I** and **LF-PRX**.

Definition 5. An $\mathcal{L}$-fuzzy pre-proximity on $\mathcal{X}$ is a mapping $\delta\colon\ \mathcal{L}^{\mathcal{X}} \times \mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ such that for all $p, q, p_1, p_2, q_1, q_2 \in \mathcal{L}^{\mathcal{X}}$, we have

(PX1) $\delta(p, \triangledown_{\mathcal{X}}) = \triangledown$;

(PX2)

$$\delta(p, q) \geq \bigvee_{a \in \mathcal{X}} p(a) \odot q(a);$$

(PX3) If $p_1 \leq p_2$ and $q_1 \leq q_2$, then $\delta(p_1, q_1) \leq \delta(p_2, q_2)$;

(PX4) $\delta(p_1 \odot p_2, q_1 \oplus q_2) \leq \delta(p_1, q_1) \oplus \delta(p_2, q_2)$.

An $\mathcal{L}$-fuzzy pre-proximity space $(\mathcal{X}, \delta)$ is called:

(SP) separated if $\delta(\triangle_a, \triangle_a^*) = \delta(\triangle_a^*, \triangle_a) = \triangledown$;

(AL) Alexandrov if

$$\delta(p, \bigvee_{j \in \Gamma} q_j) \leq \bigvee_{j \in \Gamma} \delta(p, q_j)$$

for all $\{p_j, q_j : j \in \Gamma\} \subseteq \mathcal{L}^{\mathcal{X}}$.

We define the $\mathcal{LF}$-proximity map $\psi\colon\ \mathcal{X} \to \mathcal{Y}$ between two $\mathcal{L}$-fuzzy pre-proximity spaces $(\mathcal{X}, \delta_{\mathcal{X}})$ and $(\mathcal{Y}, \delta_{\mathcal{Y}})$ by

$$\delta_{\mathcal{X}}(\psi^{\leftarrow}(p), \psi^{\leftarrow}(q)) \leq \delta_{\mathcal{Y}}(p, q)$$

for all $p, q \in \mathcal{L}^{\mathcal{Y}}$.

The category of $\mathcal{L}$-fuzzy pre-proximity spaces with $\mathcal{L}F$-proximity maps is denoted by **LF-PRX**.

Theorem 7. Given $(\mathcal{X}, \delta)$ an $\mathcal{L}$-fuzzy pre-proximity space with idempotent $\mathcal{L}$. We define a mapping $\mathcal{I}_r^\delta \colon L^X \longrightarrow \mathcal{L}$ by $\mathcal{I}_r^\delta(p) = \delta^*(r, p)$ for all $r \in \mathcal{L}^X$. Then, $\mathcal{I}_r^\delta$ is $\mathcal{L}$-fuzzy ideal on $\mathcal{X}$.

Proof. (ID1) $\mathcal{I}_r^\delta(\triangledown_{\mathcal{X}}) = \delta^*(r, \triangledown_{\mathcal{X}}) = \vartriangle$.

(ID2) Let $p \leq r$, then $\mathcal{I}_r^\delta(q) = \delta^*(r, p) \geq \delta^*(r, q) = \mathcal{I}_r^\delta(q)$.

(ID3) $\mathcal{I}_r^\delta(p \oplus q) = \delta^*(r, p \oplus q) \geq \delta^*(r, p) \odot \delta^*(r, q) = \mathcal{I}_r^\delta(p) \odot \mathcal{I}_r^\delta(q)$. □

Now, let $\Pi(\mathcal{X})$ be the family of all $\mathcal{L}$-fuzzy ideals and $\mathcal{P}(\mathcal{X})$ be the family of all $\mathcal{L}$-fuzzy pre-proximities on $\mathcal{X}$.

Theorem 8. Let $\mathcal{L}$ be idempotent and $\mathcal{G} \colon \mathcal{P}(\mathcal{X}) \times \Pi(\mathcal{X}) \to \Pi(\mathcal{X})$ be a mapping defined for all $p \in \mathcal{L}^X$ by

$$\mathcal{G}(\delta, \mathcal{I})(p) = \bigvee_{q \in \mathcal{L}^X} \delta^*(q, p) \odot \mathcal{I}(p).$$

Then, we have the next results:

(1) $\mathcal{G}(\delta, \mathcal{I}) \in \Pi(\mathcal{X})$;

(2) $\mathcal{G}(\delta, \mathcal{I}_r^\delta) = \mathcal{I}_r^\delta$ for all $r \in \mathcal{L}^X$.

Proof. (1) (ID1)

$$\mathcal{G}(\delta, \mathcal{I})(\triangledown_{\mathcal{X}}) = \bigvee_{q \in L^X} \delta^*(q, \triangledown_{\mathcal{X}}) \odot \mathcal{I}(\triangledown_{\mathcal{X}}) = \vartriangle .$$

(ID2) Let $s \in \mathcal{L}^X$ and $p \leq s$, then

$$\mathcal{G}(\delta, \mathcal{I})(s) = \bigvee_{q \in \mathcal{L}^X} \delta^*(q, s) \odot \mathcal{I}(s) \leq \bigvee_{q \in \mathcal{L}^X} \delta^*(q, p) \odot \mathcal{I}(p) = \mathcal{G}(\delta, \mathcal{I})(p).$$

(ID3)

$$\begin{aligned}
\mathcal{G}(\delta, \mathcal{I})(p \oplus s) &= \bigvee_{q \in L^X} \delta^*(q, p \oplus s) \odot \mathcal{I}(p \oplus s) \\
&\geq \bigvee_{q \in L^X} (\delta^*(q, p) \odot \delta^*(q, s)) \odot (\mathcal{I}(p) \odot \mathcal{I}(s)) \\
&= (\bigvee_{q \in L^X} \delta^*(q, p) \odot \mathcal{I}(p)) \odot (\bigvee_{q \in L^X} \delta^*(q, s) \odot \mathcal{I}(s)) \\
&= \mathcal{G}(\delta, \mathcal{I})(p) \odot \mathcal{G}(\delta, \mathcal{I})(s).
\end{aligned}$$

(2) $\mathcal{G}(\delta, \mathcal{I}_r^\delta)(p) = \bigvee_{q \in L^X} \delta^*(q, p) \odot \mathcal{I}_r^\delta(p) \leq \vartriangle \odot \mathcal{I}_r^\delta(p) = \mathcal{I}_r^\delta(p)$.

Conversely,

$$\mathcal{G}(\delta, \mathcal{I}_r^\delta)(p) = \bigvee_{q \in L^X} \delta^*(q, p) \odot \mathcal{I}_r^\delta(p) = \bigvee_{q \in L^X} \delta^*(q, p) \odot \delta^*(r, p) \geq \delta^*(r, p) \odot \delta^*(r, p) = \delta^*(r, p) = \mathcal{I}_r^\delta(p).$$

Hence, $\mathcal{G}(\delta, \mathcal{I}_r^\delta) = \mathcal{I}_r^\delta$. □

Theorem 9. Given $(\mathcal{X}, \mathcal{I})$ as an $\mathcal{L}$-fuzzy ideal space such that $\mathcal{I}(q) \leq q^*(a)$ for each $a \in \mathcal{X}$ and $q \in \mathcal{L}^\mathcal{X}$. Define a mapping $\delta^\mathcal{I}\colon \mathcal{L}^\mathcal{X} \times \mathcal{L}^\mathcal{X} \to \mathcal{L}$ by

$$\delta^\mathcal{I}(p, q) = \bigvee_{a \in \mathcal{X}} p(a) \odot \mathcal{I}^*(q).$$

Then, $(\mathcal{X}, \delta^\mathcal{I})$ is an $\mathcal{L}$-fuzzy pre-proximity space. Moreover, $\delta^\mathcal{I}$ is separated (Alexandrov) if $\mathcal{I}$ is so, respectively.

Proof. (PX1) Since $\mathcal{I}(\nabla_\mathcal{X}) =\triangle$, then we have

$$\delta^\mathcal{I}(p, \nabla_\mathcal{X}) = \bigvee_{a \in \mathcal{X}} p(a) \odot \mathcal{I}^*(\nabla_\mathcal{X}) = \nabla.$$

(PX2) Since $\mathcal{I}(q) \leq q^*(a)$, then

$$\delta^\mathcal{I}(p, q) = \bigvee_{a \in \mathcal{X}} p(a) \odot \mathcal{I}^*(q) \geq \bigvee_{a \in \mathcal{X}} p(a) \odot q(a).$$

(PX3) Let $p_1 \leq p_2$ and $q_1 \leq q_2$, then we have

$$\delta^\mathcal{I}(p_1, q_1) = \bigvee_{a \in \mathcal{X}} p_1(a) \odot \mathcal{I}^*(q_1) \leq \bigvee_{a \in \mathcal{X}} p_2(a) \odot \mathcal{I}^*(q_2) = \delta^\mathcal{I}(p_2, q_2).$$

(PX4) For all $p_1, p_2, q_1, q_2 \in \mathcal{L}^\mathcal{X}$ and by Lemma 1(8), we have

$$\begin{aligned}
\delta^\mathcal{I}(p_1 \odot p_2, q_1 \oplus q_2) &= \bigvee_{a \in \mathcal{X}} (p_1 \odot p_2)(a) \odot \mathcal{I}^*(q_1 \oplus q_2) \\
&\leq \bigvee_{a \in \mathcal{X}} (p_1(a) \odot p_2(a)) \odot (\mathcal{I}^*(q_1) \oplus \mathcal{I}^*(q_2)) \\
&\leq \bigvee_{a \in \mathcal{X}} (p_1(a) \odot \mathcal{I}^*(q_1)) \oplus (p_2(a) \odot \mathcal{I}^*(q_2)) \\
&\leq (\bigvee_{a \in \mathcal{X}} p_1(a) \odot \mathcal{I}^*(q_1)) \oplus (\bigvee_{a \in \mathcal{X}} p_2(a) \odot \mathcal{I}^*(q_2)) \\
&= \delta^\mathcal{I}(p_1, q_1) \oplus \delta^\mathcal{I}(p_2, q_2).
\end{aligned}$$

Other properties can be proved easily. □

Example 2. (1) If we define $\mathcal{I}^1\colon \mathcal{L}^\mathcal{X} \to \mathcal{L}$ as

$$\mathcal{I}^1(p) = \bigwedge_{a \in \mathcal{X}} p^*(a),$$

then $(\mathcal{X}, \mathcal{I}^1)$ is an Alexandrov $\mathcal{L}$-fuzzy ideal space by simple calculations. But, $\mathcal{I}^1$ is not separated since

$$\mathcal{I}^1(\triangle_{a^*}) = \bigwedge_{b \in \mathcal{X}} \triangle_a(b) =\triangle_a(a) \wedge \bigwedge_{b \neq a} \triangle_a(b) = \nabla.$$

By Theorem 9, we have

$$\delta^{\mathcal{I}^1}(p, q) = \bigvee_{a \in \mathcal{X}} p(a) \odot (\mathcal{I}^1)^*(q) = \bigvee_{a \in \mathcal{X}} p(a) \odot \bigvee_{b \in \mathcal{X}} q(b).$$

(2) We define $\mathcal{I}^2$: $\mathcal{L}^{\mathcal{X}} \to \mathcal{L}$ by

$$\mathcal{I}^2(p) = p^*(a),$$

then $(\mathcal{X}, \mathcal{I}^2)$ is an Alexandrov $\mathcal{L}$-fuzzy ideal space simply. But, $\mathcal{I}^2$ is not separated since for all $b \in \mathcal{X}$, we have

$$\mathcal{I}^2(\Delta_{b^*}) = \Delta_b\ (a) = \begin{cases} \triangle, & \text{if } a = b, \\ \triangledown, & \text{otherwise.} \end{cases}$$

By Theorem 9, we have

$$\delta^{\mathcal{I}^2}(p, q) = \bigvee_{a \in \mathcal{X}} p(a) \odot (\mathcal{I}^2)^*(q) = \bigvee_{a \in ^*} p(a) \odot q(a).$$

Theorem 10. Let ψ: $\mathcal{X} \to \mathcal{Y}$ be an $\mathcal{L}F$-ideal map for $(\mathcal{X}, \mathcal{I}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{I}_{\mathcal{Y}})$ two $\mathcal{L}$-fuzzy ideal spaces, then ψ: $(\mathcal{X}, \delta_{\mathcal{I}_{\mathcal{X}}}) \to (\mathcal{Y}, \delta_{\mathcal{I}_{\mathcal{Y}}})$ is an $\mathcal{L}F$-proximity map.

Proof. For all $p, q \in \mathcal{L}^{\mathcal{Y}}$, we have

$$\begin{aligned} \delta_{\mathcal{I}_{\mathcal{X}}}(\psi^{\leftarrow}(p), \psi^{\leftarrow}(q)) &= \bigvee_{a \in \mathcal{X}} \psi^{\leftarrow}(p)(a) \odot \mathcal{I}^*(\psi^{\leftarrow}(q)) \\ &\leq \bigvee_{a \in \mathcal{X}} p(\psi(a)) \odot \mathcal{I}^*_{\mathcal{Y}}(q) \\ &\leq \bigvee_{b \in \mathcal{Y}} p(b) \odot \mathcal{I}^*_{\mathcal{Y}}(q) \\ &= \delta_{\mathcal{I}_{\mathcal{Y}}}(p, q). \end{aligned}$$

□

Corollary 4. Υ: **LF-I** → **LF-PRX** is a concrete functor.

5. Conclusions

This paper has established novel categorical relationships between $\mathcal{L}$-fuzzy ideal spaces, $\mathcal{L}$-fuzzy co-topological spaces, and $\mathcal{L}$-fuzzy pre-proximity spaces in complete residuated lattices. The main contributions are:

(1) Four new functors were introduced between the categories **LF-PI**, **LF-CTP**, and **LF-PRX** of $\mathcal{L}$-fuzzy prime ideal spaces, $\mathcal{L}$-fuzzy co-topological spaces, and $\mathcal{L}$-fuzzy pre-proximity spaces, respectively.

(2) Theorems proving that $\mathcal{L}$-fuzzy prime ideal spaces can be converted into $\mathcal{L}$-fuzzy co-topological spaces via three distinct functors Υ, Ω, and Δ. Important properties like separation and Alexandrov are preserved.

(3) Theorems showing $\mathcal{L}$-fuzzy pre-proximity spaces can be constructed from $\mathcal{L}$-fuzzy ideal spaces via the functor Υ. Key properties again carry over under mild conditions.

(4) Theorems demonstrating reverse relationships, building $\mathcal{L}$-fuzzy ideal spaces from $\mathcal{L}$-fuzzy pre-proximities, and recovering the original $\mathcal{L}$-fuzzy pre-proximity via the mapping $\mathcal{G}$.

(5) The categorical perspective yields new insight into the intrinsic connections between these different structures fundamental to fuzzy mathematics. The functors provide mathematical machinery to translate between ideals, topologies, and proximities in a fuzzy setting. The results and examples lay the groundwork for further categorical research related to fuzzy mathematical concepts.

Author contributions

Ahmed Ramadan: ideas, states, proofs, first draft, and revision; Anwar Fawakhreh, states, proofs, and edition; Enas Elkordy: states, proofs, edition, submission, and revision of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors clarify that there is no conflicts of interest.

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