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Research article

New stochastic solitary solutions for the modified Korteweg-de Vries equation with stochastic term/random variable coefficients

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Abstract: In this research, we are examining the stochastic modified Korteweg-de Vries (SMKdV) equation forced in the Itô sense by multiplicative noise. We use an appropriate transformation to convert the SMKdV equation to another MKdV equation with random variable coefficients (MKdV-RVCs). We use the generalizing Riccati equation mapping and Jacobi elliptic functions methods in order to acquire new trigonometric, hyperbolic, and rational solutions for MKdV-RVCs. After that, we can get the solutions to the SMKdV equation. To our knowledge, this is the first time we have assumed that the solution of the wave equation for the SMKdV equation is stochastic, since all earlier research assumed that it was deterministic. Furthermore, we provide different graphic representations to show the influence of multiplicative noise on the exact solutions of the SMKdV equation.

Keywords: modified KdV equation; random variable coefficients; stability by noise; Jacobi elliptic functions method
Mathematics Subject Classification: 60H10, 60H15, 35Q51, 35A20, 83C15

1. Introduction

The Modified Korteweg-de Vries (MKdV) equation [1] is a nonlinear partial differential equation used to explain dispersive waves in shallow water. It is an extension of the original KdV equation that has qualities such as soliton solutions, integrability, and wave breaking, making it a useful model in many domains of physics and mathematics. The MKdV equation has several applications, involving the investigation of wave propagation in plasma [2], the dynamics of traffic flow [3], and fluid mechanics [4]. Additionally, it is applied in the field of nonlinear optics to describe pulses composed of multiple optical cycles [5]. Therefore, numerous authors have tackled exact solutions for (1.1) using various approaches, including bifurcation [6], the Riccati equation method [7], (G'/G)-expansion [8], the Exp-function method [9], the first integral method [10], and the tanh method [11]. While, Mohammed et al. [12] obtained the exact solutions of SMKdV Eq (1.1) by using the mapping method.

On the other hand, random fluctuations in the MKdV equation play an important role in the dynamics of nonlinear dispersive waves. They arise from uncertainties, or noise in the initial conditions or from external forces and can lead to deviations from the deterministic predictions of the equation. The study of these fluctuations has both theoretical and practical effects, helping us to get a better understanding of wave statistical behavior in complex systems as well as improve wave dynamics prediction and control in real-world applications. Various approaches, such as stochastic analysis and numerical simulations, have been developed to study and model these fluctuations, providing valuable insights into their effects on wave propagation.

In this study, we look at the stochastic MKdV (SMKdV) equation induced in the Itô sense by multiplicative noise as follows:

$$\mathcal{Y}_t + a\mathcal{Y}^2\mathcal{Y}_x + b\mathcal{Y}_{xxx} = \sigma\mathcal{Y}\mathcal{B}_t, \tag{1.1}$$

where $\mathcal{Y}$ represents the wave amplitude, t is time, and x is position. The nonlinear term, $a\mathcal{Y}^2\mathcal{Y}_x$, describes the self-interaction of the wave, while the dispersive term, $b\mathcal{Y}_{xxx}$, accounts for the effect of higher-order dispersion. a and b are constants, $\mathcal{B}(t)$ is the Brownian motion, $\mathcal{B}_t = \frac{\partial\mathcal{B}}{\partial t}$ and σ is the noise strength.

In all previous studies, all authors assumed the solutions to wave equations for some stochastic differential equations were deterministic, such as the Sasa-Satsuma equation [13], the nonlinear Schrödinger equation [14], the Davey-Stewartson equations [15], the potential Yu-Toda-Sasa-Fukuyama equation [16], the Burgers' equation [17], the Jimbo-Miwa equation [18], the coupled stochastic Korteweg-de Vries equations [19], the modified Benjamin-Bona-Mahony equation [20], and the Fokas system [21]. While we consider in this study that the wave equation is stochastic and its solutions are also stochastic.

Our goal in this paper is to find the exact stochastic solutions to the SMKdV Eq (1.1). To achieve this goal, we convert the SMKdV equation to another MKdV equation with random variable coefficients (MKdVE-RVCs) by using a suitable transformation. After that, we get the exact solutions for MKdVE-RVCs by using the generalizing Riccati equation mapping method (GREM-method) and the Jacobi elliptic method (JEF-method). To our knowledge, this is the first time we have assumed that the solution of the wave equation for the SMKdV equation is stochastic, since all earlier research assumed that it was deterministic. Finally, by using the used transformation, we can acquire the stochastic solutions of SMKdV. These acquired solutions are crucial in understanding several difficult physical processes due to the importance of SMKdV Eq (1.1) in fluid dynamics, nonlinear optics, and plasma physics. In order to see the influence of stochastic terms, we provide some figures by utilizing MATLAB tools.

Here is how the rest of the paper is organized: In Section 2, we derive MKdVE-RVCs from SMKdV Eq (1.1) and by utilizing the GREM and JEF-methods to find the exact solutions of MKdVE-RVCs. In Section 3, we acquire the solutions of SMKdV Eq (1.1). In Section 4, we discuss the results that we obtained. Finally, we present the conclusions of this paper.

2. MKdV equation with RVCs and its solutions

In this section, we obtain the MKdV equation with random variable coefficients (MKdVE-RVCs). By using the transformation

$$\mathcal{Y}(t,x) = \mathcal{Z}(t,x)e^{\sigma\mathcal{B}(t)}, \tag{2.1}$$

and the Itô derivatives rule, one can get the MKdVE-RVCs as follows

$$\mathcal{Z}_t + b\mathcal{Z}_{xxx} + A(t)\mathcal{Z}^2\mathcal{Z}_x + \frac{1}{2}\sigma^2\mathcal{Z} = 0, \tag{2.2}$$

where $\mathcal{Z}$ is a stochastic real function and $A(t) = ae^{2\sigma\mathcal{B}(t)}$.

2.1. GREM-method

Here, we utilize the GREM-method stated in [22]. To find the solutions of the MKdVE-RVCs (2.2), assuming the solutions of Eq (2.2) have the form

$$\mathcal{Z}(t,x) = \sum_{k=0}^{M} \alpha_k(t)\mathcal{X}^k(\xi), \quad \xi = kx + \int_0^t \lambda(s)ds, \tag{2.3}$$

where

$$\mathcal{X}' = s\mathcal{X}^2 + r\mathcal{X} + p. \tag{2.4}$$

By balancing $\mathcal{Z}'''$ with $\mathcal{Z}^2\mathcal{Z}'$, we can calculate the value of M as follows:

$$M = 1.$$

Rewriting Eq (2.3) as

$$\mathcal{Z}(t,x) = \alpha_0(t) + \alpha_1(t)\mathcal{X}(\xi). \tag{2.5}$$

Differentiating Eq (2.5) with regards to t and x, we get

$$\begin{aligned}
\mathcal{Z}_t(t,x) &= (\dot{\alpha}_0 + p\alpha_1\lambda) + (\dot{\alpha}_1 + \alpha_1 r\lambda)\mathcal{X} + s\lambda\alpha_1\mathcal{X}^2,\\
\mathcal{Z}_x(t,x) &= k[s\alpha_1\mathcal{X}^2 + r\alpha_1\mathcal{X} + p\alpha_1],\\
\mathcal{Z}_{xxx}(t,x) &= k^3[6s^3\alpha_1\mathcal{X}^4 + 12rs^2\alpha_1\mathcal{X}^3 + (7r^2s\alpha_1 + 8ps^2\alpha_1)\mathcal{X}^2\\
&\quad +(r^3\alpha_1 + 8rps\alpha_1)\mathcal{X} + (pr^2\alpha_1 + 2p^2s\alpha_1)],\\
\mathcal{Z}^2\mathcal{Z}_x &= k[(s\alpha_1^3)\mathcal{X}^4 + (2s\alpha_0\alpha_1^2 + r\alpha_1^3)\mathcal{X}^3 + (s\alpha_0^2\alpha_1 + 2r\alpha_0\alpha_1^2 + p\alpha_1^3)\mathcal{X}^2\\
&\quad +(r\alpha_0^2\alpha_1 + 2p\alpha_0\alpha_1^2)\mathcal{X} + p\alpha_0^2\alpha_1].
\end{aligned} \tag{2.6}$$

Substituting Eqs (2.5) and (2.6) into Eq (2.2), we obtain a polynomial of degree 4 in $\mathcal{X}$ as follows

$$[6bs^3k^3\alpha_1 + Aks\alpha_1^3]\mathcal{X}^4 + [12rbk^3s^2\alpha_1 + 2kAs\alpha_0\alpha_1^2 + rkA\alpha_1^3]\mathcal{X}^3$$

$$+[s\lambda\alpha_1 + 7r^2bk^3s\alpha_1 + 8pbk^3s^2\alpha_1 + skA\alpha_0^2\alpha_1 + 2rkA\alpha_0\alpha_1^2 + pkA\alpha_1^3]\mathcal{X}^2$$

$$[\dot{\alpha}_1 + \alpha_1 r\lambda + bk^3r^3\alpha_1 + 8rpsbk^3\alpha_1 + rkA\alpha_0^2\alpha_1 + 2pkA\alpha_0\alpha_1^2 + \frac{1}{2}\sigma^2\alpha_1]\mathcal{X}$$

$$+[\dot{\alpha}_0 + p\alpha_1\lambda + pbk^3r^2\alpha_1 + 2bk^3p^2s\alpha_1 + pkA\alpha_0^2\alpha_1 + \frac{1}{2}\sigma^2\alpha_0] = 0.$$

Equating each coefficient of $\mathcal{X}^k$ to zero, we have

$$6bs^3k^3\alpha_1 + Aks\alpha_1^3 = 0,$$

$$12rbk^3s^2\alpha_1 + 2kAs\alpha_0\alpha_1^2 + rkA\alpha_1^3 = 0,$$

$$s\lambda\alpha_1 + 7r^2bk^3s\alpha_1 + 8pbk^3s^2\alpha_1 + skA\alpha_0^2\alpha_1 + 2rkA\alpha_0\alpha_1^2 + pkA\alpha_1^3 = 0,$$

$$\dot{\alpha}_1 + \alpha_1 r\lambda + bk^3r^3\alpha_1 + 8rpsbk^3\alpha_1 + rkA\alpha_0^2\alpha_1 + 2pkA\alpha_0\alpha_1^2 + \frac{1}{2}\sigma^2\alpha_1 = 0,$$

and

$$\dot{\alpha}_0 + p\alpha_1\lambda + pbk^3r^2\alpha_1 + 2bk^3p^2s\alpha_1 + pkA\alpha_0^2\alpha_1 + \frac{1}{2}\sigma^2\alpha_0 = 0.$$

We solve these equations to get

$$\alpha_0(t) = r = 0, \quad \alpha_1 = \ell e^{-\frac{1}{2}\sigma^2 t}, \quad b = \frac{-A\ell^2}{6k^2s^2}e^{-\sigma^2 t}, \text{ and } \lambda(t) = \frac{akp\ell^2}{3s}e^{2\sigma\mathcal{B}(t)-\sigma^2 t}, \tag{2.7}$$

where ℓ is a constant. Hence, by utilizing Eqs (2.5) and (2.7), the solutions of MKdVE-RVCs (2.2) are

$$\mathcal{Z}(t,x) = \ell\mathcal{X}(\xi)e^{-\frac{1}{2}\sigma^2 t}, \ \xi = kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau. \tag{2.8}$$

To find $\mathcal{X}$, there are three various cases for the solutions of Eq (2.4) based on p and s as follows:

Case 1. If $ps > 0$, then Eq (2.4) has the solutions:

$$\mathcal{X}_1(\xi) = \sqrt{\frac{p}{s}}\tan\left(\sqrt{ps}\xi\right),$$

$$\mathcal{X}_2(\xi) = -\sqrt{\frac{p}{s}}\cot\left(\sqrt{ps}\xi\right),$$

$$\mathcal{X}_3(\xi) = \sqrt{\frac{p}{s}}\left(\tan(\sqrt{4ps}\xi) \pm \sec(\sqrt{4ps}\xi)\right),$$

$$\mathcal{X}_4(\xi) = -\sqrt{\frac{p}{s}}\left(\cot(\sqrt{4ps}\xi) \pm \csc(\sqrt{4ps}\xi)\right),$$

$$\mathcal{X}_5(\xi) = \frac{1}{2}\sqrt{\frac{p}{s}}\left(\tan(\frac{1}{2}\sqrt{ps}\xi) - \cot(\frac{1}{2}\sqrt{ps}\xi)\right).$$

Then, MKdVE-RVCs (2.2) has the trigonometric function solutions:

$$\mathcal{Z}_1(t,x) = \ell\sqrt{\frac{p}{s}}\left(\tan(\sqrt{ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\right)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.9}$$

$$\mathcal{Z}_2(t,x) = -\ell\sqrt{\frac{p}{s}}\left(\cot(\sqrt{ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\right)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.10}$$

$$\begin{aligned}\mathcal{Z}_3(t,x) &= \ell\sqrt{\frac{p}{s}}\Big(\tan(\sqrt{4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &\pm\sec(\sqrt{4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\mathcal{Z}_4(t,x) &= -\ell\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &\pm\csc(\sqrt{4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{2.12}$$

$$\begin{aligned}\mathcal{Z}_5(t,x) &= \ell\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &-\cot(\frac{1}{2}\sqrt{ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t}.\end{aligned} \tag{2.13}$$

Case 2. If $ps < 0$, then Eq (2.4) has the solutions:

$$\mathcal{X}_6(\xi) = -\sqrt{\frac{-p}{s}}\tanh\Big(\sqrt{-ps}\xi\Big),$$

$$\mathcal{X}_7(\xi) = -\sqrt{\frac{-p}{s}}\coth\Big(\sqrt{-ps}\xi\Big),$$

$$\mathcal{X}_8(\xi) = -\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-4ps}\xi)\pm\text{csch}(\sqrt{-4ps}\xi)\Big),$$

$$\mathcal{X}_9(\xi) = \frac{-1}{2}\sqrt{\frac{-p}{s}}\Big(\tanh(\frac{1}{2}\sqrt{-ps}\xi)+\coth(\frac{1}{2}\sqrt{-ps}\xi)\Big).$$

Then, MKdVE-RVCs (2.2) has the hyperbolic function solution:

$$\mathcal{Z}_6(t,x) = -\ell\sqrt{\frac{-p}{s}}\Big(\tanh(\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.14}$$

$$\mathcal{Z}_7(t,x) = -\ell\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-ps}\xi(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.15}$$

$$\begin{aligned}\mathcal{Z}_8(t,x) &= -\ell\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &\pm\text{csch}(\sqrt{-4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{2.16}$$

$$\mathcal{Z}_9(t,x) = -\frac{\ell}{2}\sqrt{\frac{-p}{s}}\Big(\tanh(\frac{1}{2}\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))$$

$$+\coth(\frac{1}{2}\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{-\frac{1}{2}\sigma^2 t}. \tag{2.17}$$

Case 3. If $p=0$, and $s\neq 0$, then the solution of Eq (2.4) is

$$\mathcal{X}_{10}(\xi)=\frac{-1}{s\xi}.$$

Hence, the MKdVE-RVCs (2.2) have the rational function solution:

$$\mathcal{Z}_{10}(t,x)=\Big(\frac{-\ell}{(skx+\frac{akp\ell^2}{3}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau)}\Big)e^{-\frac{1}{2}\sigma^2 t}. \tag{2.18}$$

2.2. JEF-method

We use here the JEF-method stated in [23]. Supposing the solutions of MKdVE-RVCs (2.2), with $N=1$, have the form

$$\mathcal{Z}(t,x)=a_0(t)+a_1(t)J(\eta), \tag{2.19}$$

where $J(\eta)$ is one of the following elliptic functions: $sn(\omega\eta,\check{n})$, $cn(\omega\eta,\check{n})$, or $dn(\omega\eta,\check{n})$. Differentiating Eq (2.19) with respect to t, x, and y, we get

$$\begin{aligned}\mathcal{Z}_t &= \dot{a}_0+\dot{a}_1 J+\omega\lambda a_1 J',\ \ \mathcal{Z}_x=\omega k a_1 J',\\ \mathcal{Z}_{xx} &= k^2 a_1(B_1 J+B_2 J^3),\ \mathcal{Z}_{xxx}=k^3 a_1(B_1+2B_2J^2)J',\\ \mathcal{Z}_x\mathcal{Z}^2 &= \omega k a_1(a_1^2J^2+2a_0a_1J+a_0^2)J',\end{aligned} \tag{2.20}$$

where B_1 and B_2 are constants depending on ω, $\check{n}$, and they will be defined later. Plugging Eqs (2.19) and (2.20) into MKdVE-RVCs (2.2). After that, by putting each coefficient of $J'J^k$ equal to zero, we acquire

$$\begin{aligned}J^0 &: \quad \dot{a}_0+\frac{1}{2}\sigma^2 a_0=0,\\ J^1 &: \quad \dot{a}_1+\frac{1}{2}\sigma^2 a_1=0,\\ J^0J' &: \quad \omega a_1[\lambda+bk^3B_1+ka_0^2A(t)]=0,\\ JJ' &: \quad 2\omega k a_0 a_1^2 A(t)=0,\end{aligned}$$

and

$$J^2J':\quad 2b\omega k^3 a_1 B_2+\omega k a_1^3 A(t)=0.$$

We solve these equations to get

$$a_0(t)=0,\ \ a_1=\ell e^{-\frac{1}{2}\sigma^2 t},\ b=\frac{-\ell^2A(t)}{2k^2B_2}e^{-\sigma^2 t},\ \ \lambda(t)=\frac{\ell^2kB_1}{2B_2}A(t)e^{-\sigma^2 t},$$

where ℓ is a constant. Hence, the solution of the MKdVE-RVCs (2.2) is

$$\mathcal{Z}(t,x)=\ell J(\eta),\ \eta=kx+\frac{a\ell^2kB_1}{2B_2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau. \tag{2.21}$$

Let us now define $J(\eta)$ as follows:
Set 1. If $J(\eta) = sn(\omega\eta, \check{n})$, then Eq (2.21) takes the form

$$\mathcal{Z}(t,x) = \ell\Big(sn(k\omega x + \frac{a\omega\ell^2 kB_1}{2B_2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau, \check{n})\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.22}$$

where

$$B_1 = -\omega^2(1+\check{n}^2) \text{ and } B_2 = 2\omega^2\check{n}^2.$$

Set 2. If $J(\eta) = cn(\omega\eta, \check{n})$, then Eq (2.21) takes the form

$$\mathcal{Z}(t,x) = \ell\Big(cn(k\omega x + \frac{a\omega\ell^2 kB_1}{2B_2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau, \check{n})\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.23}$$

where

$$B_1 = \omega^2(1-2\check{n}^2) \text{ and } B_2 = -2\omega^2\check{n}^2.$$

Set 3. If $J(\eta) = dn(\omega\eta, \check{n})$, then Eq (2.21) takes the form

$$\mathcal{Z}(t,x) = \ell\Big(dn(k\omega x + \frac{a\omega\ell^2 kB_1}{2B_2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau, \check{n})\Big)e^{-\frac{1}{2}\sigma^2 t}, \tag{2.24}$$

where

$$B_1 = \omega^2(2-\check{n}^2) \text{ and } B_2 = -2\omega^2.$$

3. Exact solutions of stochastic MKdV equation

Now, we can use the solutions of MKdVE-RVCs (2.2) that we obtained in the previous section to get the solutions of SMKdV Eq (1.1) as follows:

3.1. GREM-method

Substituting Eqs (2.9)–(2.18) into Eq (2.1), we get the solutions of SMKdV Eq (1.1) as:

$$\mathcal{Y}_1(t,x) = \ell\sqrt{\frac{p}{s}}\Big(\tan(\sqrt{ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}, \tag{3.1}$$

$$\mathcal{Y}_2(t,x) = -\ell\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}, \tag{3.2}$$

$$\begin{aligned}\mathcal{Y}_3(t,x) &= \ell\sqrt{\frac{p}{s}}\Big(\tan(\sqrt{4ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau)) \\ &\quad \pm\sec(\sqrt{4ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{3.3}$$

$$\mathcal{Y}_4(t,x) = -\ell\sqrt{\frac{p}{s}}\Big(\cot(\sqrt{4ps}(kx + \frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))$$

$$\pm\csc(\sqrt{4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}, \tag{3.4}$$

$$\begin{aligned}\mathcal{Y}_5(t,x) &= \ell\sqrt{\frac{p}{s}}\Big(\tan(\frac{1}{2}\sqrt{ps}\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &\quad -\cot(\frac{1}{2}\sqrt{ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{3.5}$$

for $ps > 0$,

$$\mathcal{Y}_6(t,x) = -\ell\sqrt{\frac{-p}{s}}\Big(\tanh(\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}, \tag{3.6}$$

$$\mathcal{Y}_7(t,x) = -\ell\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-ps}\xi(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}, \tag{3.7}$$

$$\begin{aligned}\mathcal{Y}_8(t,x) &= -\ell\sqrt{\frac{-p}{s}}\Big(\coth(\sqrt{-4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &\quad \pm\mathrm{csch}(\sqrt{-4ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{Y}_9(t,x) &= -\frac{\ell}{2}\sqrt{\frac{-p}{s}}\Big(\tanh(\frac{1}{2}\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\\ &\quad +\coth(\frac{1}{2}\sqrt{-ps}(kx+\frac{akp\ell^2}{3s}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau))\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t},\end{aligned} \tag{3.9}$$

for $ps < 0$, and

$$\mathcal{Y}_{10}(t,x) = \Big(\frac{-\ell}{(skx+\frac{akp\ell^2}{3}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau)}\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2 t}, \tag{3.10}$$

for $p = 0$ and $s \neq 0$.

Remark 1. *Putting $p = -1$, $s = a$, $\ell_1 = -\sqrt{c}$, $k = \sqrt{\frac{-c}{2b}}$, and $\sigma = 0$ (i.e., no noise) in Eqs (3.1) and (3.2) we have the results stated in [11] as follows:*

$$\mathcal{Y}(t,x) = \sqrt{\frac{c}{a}}\tanh(\sqrt{\frac{-c}{2b}}(x-ct))\Big),$$

and

$$\mathcal{Y}(t,x) = \sqrt{\frac{c}{a}}\coth(\sqrt{\frac{-c}{2b}}(x-ct))\Big).$$

3.2. JEF-method

Substituting Eqs (2.22)–(2.24) into Eq (2.1), we have the SMKdV Eq (1.1):

$$\mathcal{Y}(t,x)=\ell\Big(sn(k\omega x-\frac{a\omega\ell^2k(1+\check{n}^2)}{4\check{n}^2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau,\check{n})\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2t}, \tag{3.11}$$

$$\mathcal{Y}(t,x)=\ell\Big(cn(k\omega x+\frac{a\omega\ell^2k(1-2\check{n}^2)}{4\check{n}^2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau,\check{n})\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2t}, \tag{3.12}$$

and

$$\mathcal{Y}(t,x)=\ell\Big(dn(k\omega x-\frac{a\omega\ell^2k(2-\check{n}^2)}{4}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau,\check{n})\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2t}. \tag{3.13}$$

If $\check{n}\to 1$, then the Eqs (3.11)–(3.13) become

$$\mathcal{Y}(t,x)=\ell\Big(\tanh(k\omega x-\frac{a\omega\ell^2k}{2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau,\check{n})\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2t}, \tag{3.14}$$

and

$$\mathcal{Y}(t,x)=\ell\Big(\mathrm{sech}(k\omega x-\frac{a\omega\ell^2k}{2}\int_0^t e^{2\sigma\mathcal{B}(\tau)-\sigma^2\tau}d\tau,\check{n})\Big)e^{\sigma\mathcal{B}(t)-\frac{1}{2}\sigma^2t}. \tag{3.15}$$

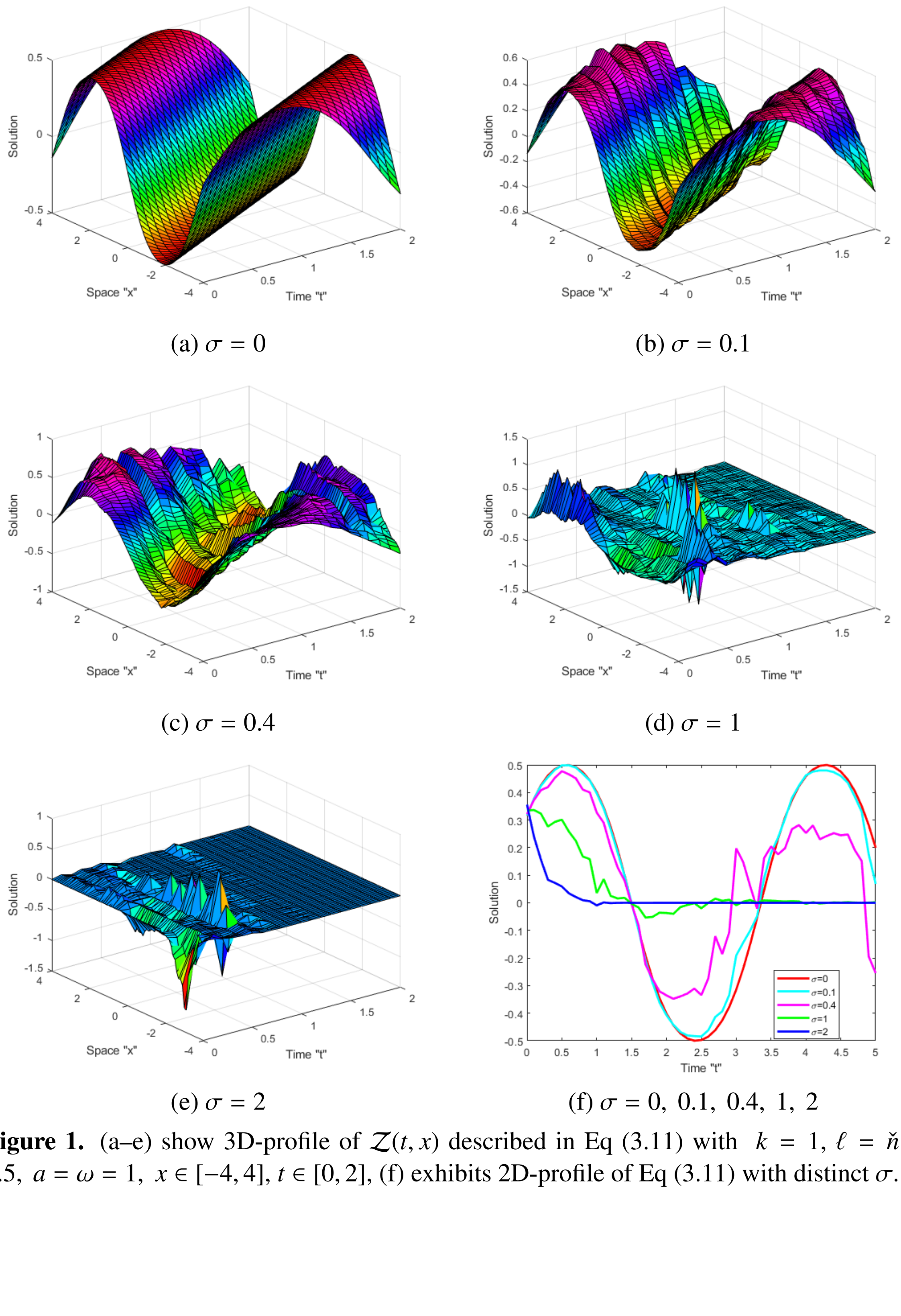
4. Discussion and impacts of noise

4.1. Discussion

Here, we obtained the solutions of the SMKdV Eq (1.1). We used the METF and JEF methods, which produced a wide range of solutions, including optical trigonometric solutions (3.1)–(3.5), optical hyperbolic solutions (3.6)–(3.9), optical rational solution (3.10), and optical elliptic solutions (3.11)–(3.13). Optical solutions are an effective tool for studying the behavior of solutions to the modified KdV equations, as they provide a unique viewpoint on wave dynamics and interactions in complex systems. Furthermore, optical solutions enable researchers to look at the stability and nonlinear dynamics of solutions to MKdV equations. Using optical solutions, researchers may investigate the system's nonlinear effects, such as wave breaking and soliton formation, which are important for understanding the solutions' long-term behavior. By investigating the stability of optical systems, researchers may gain a better understanding of the system's behavior and make more accurate predictions about its future evolution.

4.2. Impacts of noise

The impact of multiplicative noise on the exact solution of SMKdV Eq (1.1) is examined in this section. For our knowledge, the key difference between the solutions provided here and the ones acquired in [12] is the amplitude function $\mathcal{Z}(t,x)$. Here, $\mathcal{Z}(t,x)$ is a stochastic function, while $\mathcal{Z}(t,x)$ is an assumed deterministic function in [12]. Numerous numerical simulations of various solutions with different intensities of noise are shown. Figures 1–3 display the solutions $\mathcal{Z}(t,x)$ described in Eqs (3.11), (3.14), and (3.15), respectively, for various amplitudes of noise σ as follows:

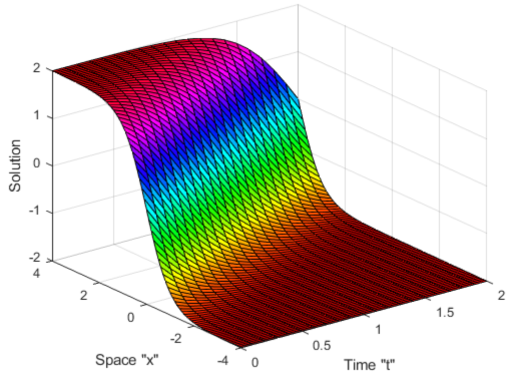


(a) $\sigma = 0$ (b) $\sigma = 0.1$

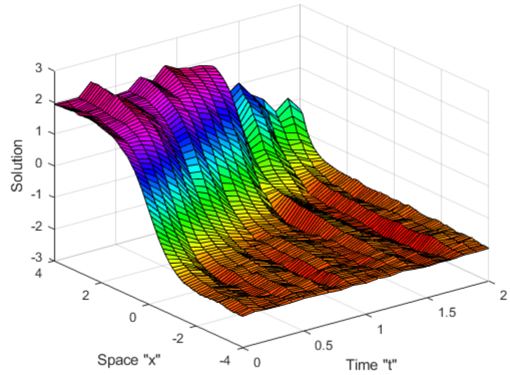
(c) $\sigma = 0.4$ (d) $\sigma = 1$

(e) $\sigma = 2$ (f) $\sigma = 0,\ 0.1,\ 0.4,\ 1,\ 2$

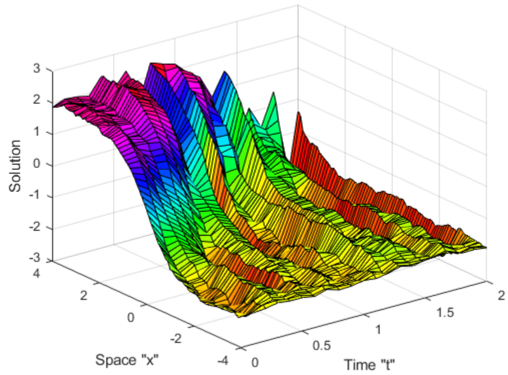
Figure 1. (a–e) show 3D-profile of $\mathcal{Z}(t, x)$ described in Eq (3.11) with $k = 1,\ \ell = \check{n} = 0.5,\ a = \omega = 1,\ x \in [-4, 4],\ t \in [0, 2]$, (f) exhibits 2D-profile of Eq (3.11) with distinct σ.



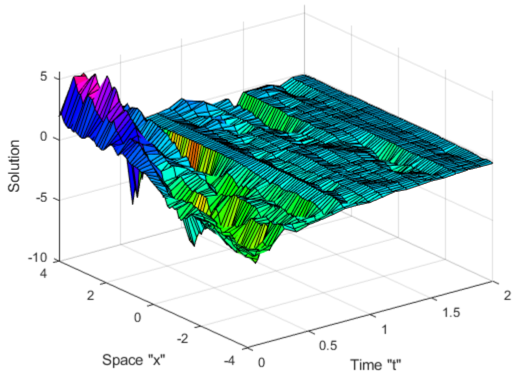
(a) $\sigma = 0$



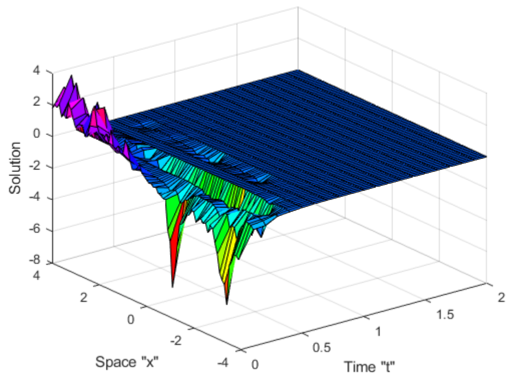
(b) $\sigma = 0.1$



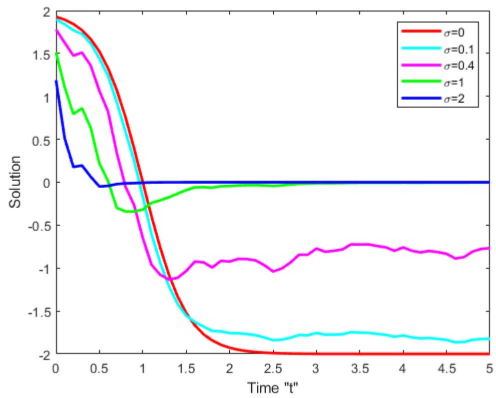
(c) $\sigma = 0.4$



(d) $\sigma = 1$



(e) $\sigma = 2$



(f) $\sigma = 0,\ 0.1,\ 0.4,\ 1,\ 2$

Figure 2. (a–e) display 3D-shape of $\mathcal{Z}(t, x)$ described in Eq (3.14) with $\ell = \omega = k = 1,\ a = 1,\ x \in [-4, 4]$, and $t \in [0, 2]$, (f) exhibits 2D-shape of Eq (3.14) with different σ.

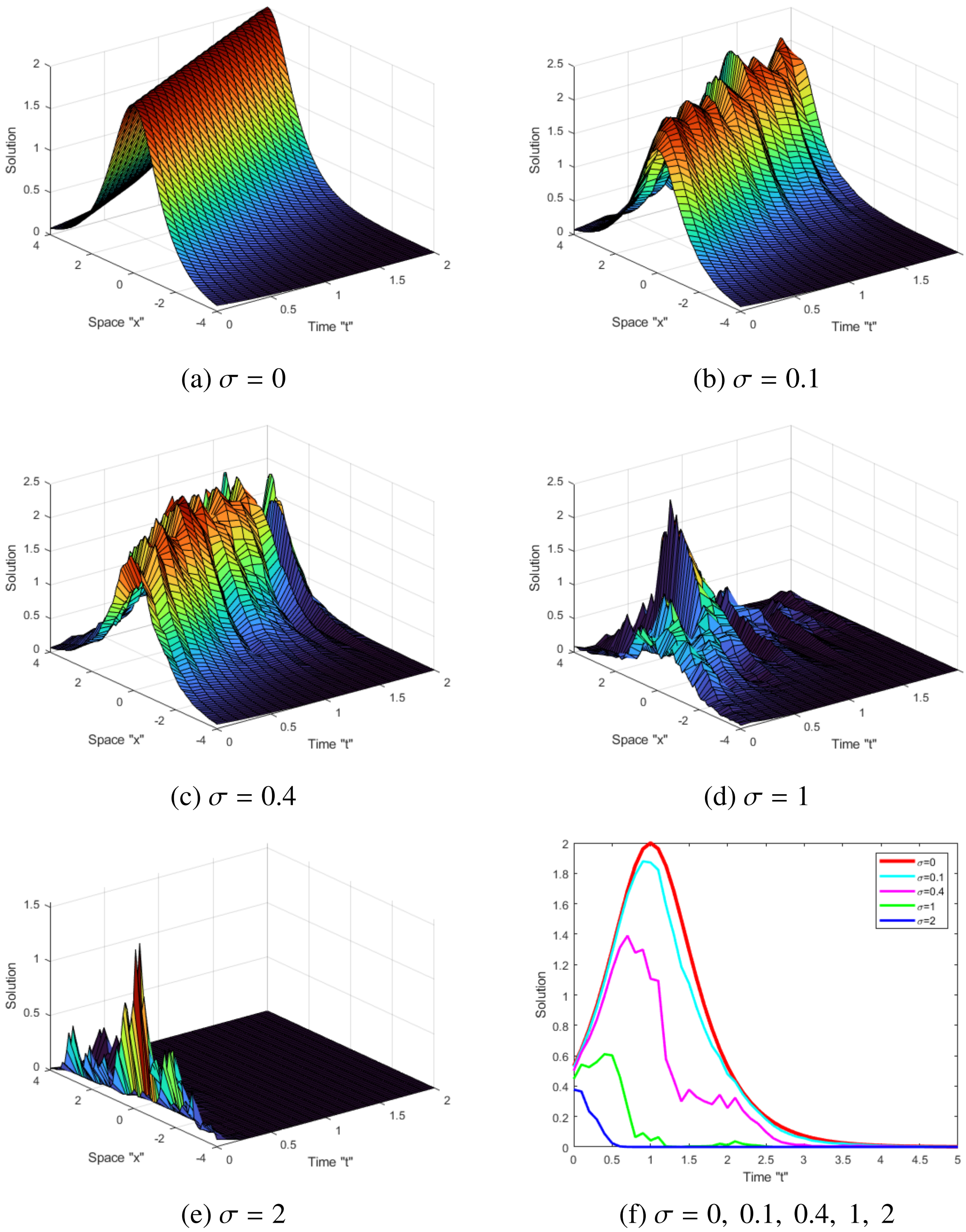


Figure 3. (a–e) show 3D-shape of $\mathcal{Z}(t, x)$ described in Eq (3.15) with $\ell = \omega = k = 1,\ a = 1,\ x \in [-4, 4]$, and $t \in [0, 2]$, (f) exhibits 2D-shape of Eq (3.15) with various σ.

Figures 1–3 show that when noise is ignored (i.e., $\sigma = 0$), numerous types of solutions emerge, including optical periodic solutions, optical singular solutions, optical kink solutions, and so on. When noise is introduced at $\sigma = 0.1,\ 0.4,\ 1,\ 2$, the surface flattens after some transit patterns. This result shows how multiplicative Brownian motion affects the SMKdV Eq (1.1) solutions, stabilizing them around zero.

5. Conclusions

In this paper, we looked at the SMKdV Eq (1.1) driven by multiplicative noise in the Itô sense. By using appropriate transformations, we converted the SMKdV equation to another MKdV equation with random variable coefficients (MKdV-RVCs) (2.2). Using the GREM-method and the JEF-method, we obtained a new stochastic exact solutions for MKdV-RVCs in the form of trigonometric, hyperbolic, and rational functions. After that, we acquired the obtained solutions of SMKdV (1.1). Moreover, we generated some previous solutions, such as the results reported in [11]. Because of the importance of MKdV equation used in fluid dynamics, nonlinear optics, and plasma physics, the acquired solutions are crucial in understanding several difficult physical processes. Finally, some graphics were included to demonstrate the effect of the stochastic term on the stochastic exact solutions of the SMKdV equation.

Author contributions

Wael W. Mohammed: Conceptualization, Methodology, Software, Formal analysis, Writing-original draft; Farah M. Al-Askar: Conceptualization, Formal analysis, Writing-original draft. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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