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Research article

Efficient solutions for time fractional Sawada-Kotera, Ito, and Kaup-Kupershmidt equations using an analytical technique

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Abstract: We focused on the analytical solution of strong nonlinearity and complicated time-fractional evolution equations, including the Sawada-Kotera equation, Ito equation, and Kaup-Kupershmidt equation, using an effective and accurate method known as the Aboodh residual power series method (ARPSM) in the framework of the Caputo operator. Therefore, the Caputo operator and the ARPSM are practical for figuring out a linear or nonlinear system with a fractional derivative. This technique was effectively proposed to obtain a set of analytical solutions for various types of fractional differential equations. The derived solutions enabled us to understand the mechanisms behind the propagation and generation of numerous nonlinear phenomena observed in diverse scientific domains, including plasma physics, fluid physics, and optical fibers. The fractional property also revealed some ambiguity that may be observed in many natural phenomena, and this is one of the most important distinguishing factors between fractional differential equations and non-fractional ones. We also helped clarify fractional calculus in nonlinear dynamics, motivating researchers to work in mathematical physics.

Keywords: fractional PDE; fractional Sawada-Kotera equation; fractional Ito equation; fractional Kaup-Kupershmidt equation; Aboodh residual power series method; Caputo operator; solitary waves
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1. Introduction

Fractional calculus (FC) deals with the derivatives and integrals of fractional orders [1, 2]. It can be used to find many complicated objects' essential properties and memory effects [3]. Converting classical derivatives and integrals into non-integer order has been used in several recent FC applications to study the dynamics of large-scale physical processes [4–8]. Signal processing, mathematical biology, flow models, relaxation, and viscoelasticity are some engineering and physical study fields that utilize it. The study of nonlinear processes is fundamental to many academic and technical disciplines. These include thermodynamics, chemical kinetics, computational biology, quantum mechanics, fluid dynamics, nonlinear spectroscopy, and solid-state physics [9, 10]. These higher-order nonlinear partial differential equations (PDEs) establish the concept of nonlinearity concept. Numerous high-order evolution equations exist, encompassing both nonlinear and dispersion combinations. One notable example is the family of Kawahara equations [11–16], which have gained significant traction in recent years due to their practical applications. This family holds considerable significance and utility in elucidating various nonlinear phenomena within physical and engineering systems. They are instrumental in comprehending the mechanisms underlying the propagation and generation of nonlinear waves in diverse plasma systems. Primary events are explained by the nonlinear concepts of all physical systems [17, 18]. The studies cover diverse areas such as credit rating algorithms, digital integrators, control systems for autonomous underwater vehicles (AUVs), stabilization of high-order systems, and analysis of stochastic linear complementarity problems [19–21]. These works contribute to various fields including finance, industrial electronics, autonomous systems, control theory, mathematics, and cybernetics [22–24]. These studies not only contribute to the theoretical understanding of complex systems but also offer practical solutions applicable to real-world scenarios. Moreover, they demonstrate the interdisciplinary nature of modern research, where concepts from mathematics, engineering, and computer science converge to address contemporary challenges and drive innovation across various domains [25, 26].

The renowned dispersive classical Kaup-Kupershmidt equation [27] was first proposed by Kaup in 1980 and revised by Kupershmidt in 1994 [28]. This work focuses on the time-fractional modified Kaup-Kupershmidt (KK) equation. By applying the fractional-order Kaup-Kupershmidt equation, we analyze nonlinear dispersive waves and how capillary gravity waves behave. The nonlinear fifth-order evolution equation is as follows:

$$D^{p}_{\Omega}\phi(\delta,\Omega) - \frac{\partial^5\phi(\delta,\Omega)}{\partial\delta^5} - 5\phi(\delta,\Omega)\frac{\partial^3\phi(\delta,\Omega)}{\partial\delta^3} - \frac{25}{2}\frac{\partial\phi(\delta,\Omega)}{\partial\delta}\frac{\partial^2\phi(\delta,\Omega)}{\partial\delta^2} - 5\phi^2(\delta,\Omega)\frac{\partial\phi(\delta,\Omega)}{\partial\delta} = 0, \quad (1.1)$$

where $0 < p \leq 1$.

Studying the traditional Kaup-Kupershmidt equation has received a lot of attention recently. There are four distinct methods that, when used separately, may provide solitary and soliton wave solutions to generic nonlinear evolution equations. Soliton solutions were created by Ablowitz and Clarkson using the inverse scattering method to study physically relevant nonlinear equations. Tam and Hu followed Hirota's method and used Mathematica to get the same result. As an example of an integrable Henon-Heiles system, Musette and Verhoeven published the fifth-order Kaup-Kupershmidt equation.

Consider the following fifth-order equation of Sawada-Kotera and Ito as follows:

$$D_{\Omega}^{p}\phi(\delta,\Omega)+\frac{\partial^5\phi(\delta,\Omega)}{\partial\delta^5}+15\phi(\delta,\Omega)\frac{\partial^3\phi(\delta,\Omega)}{\partial\delta^3}+15\frac{\partial\phi(\delta,\Omega)}{\partial\delta}\frac{\partial^2\phi(\delta,\Omega)}{\partial\delta^2}+45\phi^2(\delta,\Omega)\frac{\partial\phi(\delta,\Omega)}{\partial\delta}=0, \quad (1.2)$$

and

$$D_{\Omega}^{p}\phi(\delta,\Omega)+\frac{\partial^5\phi(\delta,\Omega)}{\partial\delta^5}+3\phi(\delta,\Omega)\frac{\partial^3\phi(\delta,\Omega)}{\partial\delta^3}+6\frac{\partial\phi(\delta,\Omega)}{\partial\delta}\frac{\partial^2\phi(\delta,\Omega)}{\partial\delta^2}+2\phi^2(\delta,\Omega)\frac{\partial\phi(\delta,\Omega)}{\partial\delta}=0. \quad (1.3)$$

The FC has been widely used in many fields during the last 30 years, including physics, fluid dynamics, chemical physics, electrical networks, control theory of dynamical systems, and many more. Numerous academics in this area are now preoccupied with pursuing precise strategies for solving the resulting nonlinear model, which includes fractional order.

Many researchers have applied different analytical and numerical methods, such as the Adomian Decomposition Method (ADM) [29], the Variational Iteration Method (VIM) [30, 31], and the Homotopy Perturbation Method (HPM) [32–34]. One strong analytical technique for solving nonlinear differential equations is the Homotopy Analysis Method (HAM) [35]. The q-Homotopy Analysis Method, a modified HAM, was recently presented in [36]. Many more techniques are used to accurately and precisely solve higher order nonlinear PDEs [37, 38]. Analytical solutions of the time-fractional Sawada-Kotera and time-fractional Ito equations utilizing the Aboodh residual power series method (ARPSM) have not been attempted as far as we are concerned.

Omar Abu Arqub established RPSM in 2013 [39]. The Taylor series and the residual error function are combined to generate it. An infinite convergence series [40] provides the solution for DEs. In response to numerous differential equations (DEs), including Boussinesq DEs, fuzzy DEs, KdV Burger's equation, and many others, modern RPSM algorithms have been devised to generate approximation solutions that are both efficient and accurate [41–47].

A novel methodology was devised to address fractional-order differential equations (FODEs) by integrating two highly effective techniques. The methods above comprise the categories that are comprised of the Sumudu transform in conjunction with the following: the Shehu transformation and the Adomian decomposition method; the Laplace transform with RPSM [48]; the natural transform [49]; and the homotopy perturbation approach [50]. For further details on integrating the two methods, please refer to [51–53]. In this study, we will employ an innovative combination technique known as the ARPSM, which will allow us to obtain both precise and approximative solutions for time-fractional partial differential equations (PDEs).

In this paper, we use the ARPSM to solve the Kaup-Kupershmidt equation, Ito equation, and Sawada-Kotera equation of time fractional. We compare the approximate solution via ARPSM of these equations with the exact solution. Furthermore, the graphical illustration for different fractional orders is compared and contrasted with the exact solution.

2. Foundations

Definition 2.1. *[54] Let us assume that $\phi(\delta,\Omega)$ is an exponentially ordered piecewise continuous function. The Aboodh transform (AT) may be described as follows, assuming $\tau \geq 0$ for $\phi(\delta,\Omega)$.*

$$A[\phi(\delta,\Omega)]=\Psi(\delta,\xi)=\frac{1}{\xi}\int_0^{\infty}\phi(\delta,\Omega)e^{-\Omega\xi}d\Omega, \quad r_1\leq\xi\leq r_2.$$

In particular, the Aboodh inverse transform (AIT) is defined as follows:

$$A^{-1}[\Psi(\delta,\xi)] = \phi(\delta,\Omega) = \frac{1}{2\pi i}\int_{u-i\infty}^{u+i\infty} \Psi(\delta,\Omega)\xi e^{\Omega\xi} d\Omega,$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_p) \in \mathbb{R}$ and $p \in \mathbb{N}$.

Lemma 2.1. *[55,56] $\phi_1(\delta,\Omega)$ and $\phi_2(\delta,\Omega)$ are two functions. They are considered to be exponentially ordered and piecewise continuous on $[0,\infty[$. Let $A[\phi_1(\delta,\Omega)] = \Psi_1(\delta,\Omega), A[\phi_2(\delta,\Omega)] = \Psi_2(\delta,\Omega)$ and χ_1,χ_2 are constants. As a result, the following features are true:*

(1) $A[\chi_1\phi_1(\delta,\Omega) + \chi_2\phi_2(\delta,\Omega)] = \chi_1\Psi_1(\delta,\xi) + \chi_2\Psi_2(\delta,\Omega)$,
(2) $A^{-1}[\chi_1\Psi_1(\delta,\Omega) + \chi_2\Psi_2(\delta,\Omega)] = \chi_1\phi_1(\delta,\xi) + \chi_2\phi_2(\delta,\Omega)$,
(3) $A[J_\Omega^p\phi(\delta,\Omega)] = \frac{\Psi(\delta,\xi)}{\xi^p}$,
(4) $A[D_\Omega^p\phi(\delta,\Omega)] = \xi^p\Psi(\delta,\xi) - \sum_{K=0}^{r-1}\frac{\phi^K(\delta,0)}{\xi^{K-p+2}}, r-1 < p \le r,\ r \in \mathbb{N}$.

Definition 2.2. *[57] The fractional derivative of $\phi(\delta,\Omega)$ is defined in terms of order p, according to the Caputo.*

$$D_\Omega^p\phi(\delta,\Omega) = J_\Omega^{m-p}\phi^{(m)}(\delta,\Omega),\ r \ge 0,\ m-1 < p \le m,$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_p) \in \mathbb{R}^p$ and $m, p \in R$, J_Ω^{m-p} is the R-L integral of $\phi(\delta,\Omega)$.

Definition 2.3. *[58] The structure of the power series representation reads*

$$\sum_{r=0}^{\infty} \hbar_r(\delta)(\Omega-\Omega_0)^{rp} = \hbar_0(\Omega-\Omega_0)^0 + \hbar_1(\Omega-\Omega_0)^p + \hbar_2(\Omega-\Omega_0)^{2p} + \cdots,$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Multiple fractional power series (MFPS) are series about Ω_0 in which Ω is variable and the series coefficients are $\hbar_r(\delta)'s$.

Lemma 2.2. *Let us assume that $\phi(\delta,\Omega)$ is the exponential order function. The AT in this instance is represented by the equation $A[\phi(\delta,\Omega)] = \Psi(\delta,\xi)$. Hence,*

$$A[D_\Omega^{rp}\phi(\delta,\Omega)] = \xi^{rp}\Psi(\delta,\xi) - \sum_{j=0}^{r-1}\xi^{p(r-j)-2}D_\Omega^{jp}\phi(\delta,0), 0 < p \le 1, \tag{2.1}$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$ and $D_\Omega^{rp} = D_\Omega^p.D_\Omega^p.\cdots.D_\Omega^p(r - times)$.

Proof. We prove Eq (2.2) via induction. Using $r = 1$ in Eq (2.2) yields the following outcomes:

$$A[D_\Omega^{2p}\phi(\delta,\Omega)] = \xi^{2p}\Psi(\delta,\xi) - \xi^{2p-2}\phi(\delta,0) - \xi^{p-2}D_\Omega^p\phi(\delta,0).$$

Lemma 2.1(4) states that for $r = 1$, Eq (2.2) is valid. Upon putting $r = 2$ in Eq (2.2), we get

$$A[D_r^{2p}\phi(\delta,\Omega)] = \xi^{2p}\Psi(\delta,\xi) - \xi^{2p-2}\phi(\delta,0) - \xi^{p-2}D_\Omega^p\phi(\delta,0). \tag{2.2}$$

Left-hand side (L.H.S.) of Eq (2.2) allows us to infer

$$L.H.S = A[D_\Omega^{2p}\phi(\delta,\Omega)]. \tag{2.3}$$

The expressions of Eq (2.3) can be written in the following form:

$$L.H.S = A[D_{\Omega}^{p}(D_{\Omega}^{p}\phi(\delta,\Omega))]. \tag{2.4}$$

Assume

$$z(\delta,\Omega) = D_{\Omega}^{p}\phi(\delta,\Omega). \tag{2.5}$$

Thus, Eq (2.4) becomes

$$L.H.S = A[D_{\Omega}^{p}z(\delta,\Omega)]. \tag{2.6}$$

Equation (2.6) is changed as a consequence of using the Caputo derivative.

$$L.H.S = A[J^{1-p}z^{'}(\delta,\Omega)]. \tag{2.7}$$

The R-L integral for AT, which is given in Eq (2.7), may be used to deduce the following:

$$L.H.S = \frac{A[z^{'}(\delta,\Omega)]}{\xi^{1-p}}. \tag{2.8}$$

The present form of Eq (2.8) is obtained by using the differential characteristic of the AT.

$$L.H.S = \xi^{p}Z(\delta,\xi) - \frac{z(\delta,0)}{\xi^{2-p}}, \tag{2.9}$$

From Eq (2.5), we obtain:

$$Z(\delta,\xi) = \xi^{p}\Psi(\delta,\xi) - \frac{\phi(\delta,0)}{\xi^{2-p}},$$

where $A[z(\delta,\Omega)] = Z(\delta,\xi)$. Therefore, Eq (2.9) is transformed to

$$L.H.S = \xi^{2p}\Psi(\delta,\xi) - \frac{\phi(\delta,0)}{\xi^{2-2p}} - \frac{D_{\Omega}^{p}\phi(\delta,0)}{\xi^{2-p}}. \tag{2.10}$$

At $r = K$, Eqs (2.2) and (2.10) become compatible. Assuming that Eq (2.2) is valid for $r = K$. Consequently, we put $r = K$ in Eq (2.2):

$$A[D_{\Omega}^{Kp}\phi(\delta,\Omega)] = \xi^{Kp}\Psi(\delta,\xi) - \sum_{j=0}^{K-1}\xi^{p(K-j)-2}D_{\Omega}^{jp}D_{\Omega}^{jp}\phi(\delta,0),\ 0 < p \leq 1. \tag{2.11}$$

Illustrating Eq (2.2) for the value of $r = K + 1$ is the next step. Equation (2.2) allows us to write

$$A[D_{\Omega}^{(K+1)p}\phi(\delta,\Omega)] = \xi^{(K+1)p}\Psi(\delta,\xi) - \sum_{j=0}^{K}\xi^{p((K+1)-j)-2}D_{\Omega}^{jp}\phi(\delta,0). \tag{2.12}$$

Examining L.H.S of Eq (2.12)'s which give us the following result

$$L.H.S = A[D_{\Omega}^{Kp}(D_{\Omega}^{Kp})]. \tag{2.13}$$

Let

$$D_{\Omega}^{Kp} = g(\delta,\Omega).$$

By Eq (2.13), we drive

$$L.H.S = A[D_{\Omega}^{p} g(\delta, \Omega)]. \tag{2.14}$$

The following outcomes are obtained by using the Caputo derivative and R-L integral to Eq (2.14).

$$L.H.S = \xi^{p} A[D_{\Omega}^{Kp} \phi(\delta, \Omega)] - \frac{g(\delta, 0)}{\xi^{2-p}}. \tag{2.15}$$

We utilize Eq (2.11) to get Eq (2.15).

$$L.H.S = \xi^{rp} \Psi(\delta, \xi) - \sum_{j=0}^{r-1} \xi^{p(r-j)-2} D_{\Omega}^{jp} \phi(\delta, 0), \tag{2.16}$$

Furthermore, using Eq (2.16), the following result is achieved.

$$L.H.S = A[D_{\Omega}^{rp} \phi(\delta, 0)].$$

Equation (2.2) holds for $r = K + 1$. By mathematical induction, Eq (2.2) is thus valid for all positive integers. □

The following lemma provides a new insight into multiple fractional Taylor's series (MFTS). The ATIM, which will be covered in more detail later on, will benefit from this formula.

Lemma 2.3. *Assume that the exponential order function to be $\phi(\delta, \Omega)$. In $A[\phi(\delta, \Omega)] = \Psi(\xi, \delta)$ is the AT for $\phi(\delta, \Omega)$. The AT MFTS representation is as follows:*

$$\Psi(\delta, \xi) = \sum_{r=0}^{\infty} \frac{\hbar_r(\delta)}{\xi^{rp+2}}, \xi > 0, \tag{2.17}$$

where, $\delta = (s_1, \delta_2, \cdots, \delta_p) \in \mathbb{R}^p,\ p \in \mathbb{N}$.

Proof. Let's examine the fractional order expression for Taylor's series:

$$\phi(\delta, \Omega) = \hbar_0(\delta) + \hbar_1(\delta) \frac{\Omega^p}{\Gamma[p+1]} + +\hbar_2(\delta) \frac{\Omega^{2p}}{\Gamma[2p+1]} + \cdots \tag{2.18}$$

The following equality is obtained by applying the AT to Eq (2.18):

$$A\left[\phi(\delta, \Omega)\right] = A\left[\hbar_0(\delta)\right] + A\left[\hbar_1(\delta) \frac{\Omega^p}{\Gamma[p+1]}\right] + A\left[\hbar_1(\delta) \frac{\Omega^{2p}}{\Gamma[2p+1]}\right] + \cdots$$

The AT's features are used to do this.

$$A\left[\phi(\delta, \Omega)\right] = \hbar_0(\delta) \frac{1}{\xi^2} + \hbar_1(\delta) \frac{\Gamma[p+1]}{\Gamma[p+1]} \frac{1}{\xi^{p+2}} + \hbar_2(\delta) \frac{\Gamma[2p+1]}{\Gamma[2p+1]} \frac{1}{\xi^{2p+2}} \cdots$$

Consequently, we get (2.17), which is an AT special instance of Taylor's series. □

Lemma 2.4. *The MFPS may be represented as $A[\phi(\delta, \Omega)] = \Psi(\delta, \xi)$ using the new form of Taylor's series (2.17).*

$$\hbar_0(\delta) = \lim_{\xi \to \infty} \xi^2 \Psi(\delta, \xi) = \phi(\delta, 0). \tag{2.19}$$

Proof. The following is determined using Taylor's series in its newly modified form:

$$\hbar_0(\delta) = \xi^2\Psi(\delta,\xi) - \frac{\hbar_1(\delta)}{\xi^p} - \frac{\hbar_2(\delta)}{\xi^{2p}} - \cdots \tag{2.20}$$

By using $\lim_{\xi\to\infty}$ to Eq (2.19) and performing a quick calculation, the required result, (2.20), may be achieved. □

Theorem 2.5. *In MFPS notation, the function $A[\phi(\delta,\Omega)] = \Psi(\delta,\xi)$ may be written as follows:*

$$\Psi(\delta,\xi) = \sum_{0}^{\infty} \frac{\hbar_r(\delta)}{\xi^{rp+2}},\ \xi > 0,$$

where $\delta = (\delta_1, \delta_2, \cdots, \delta_p) \in \mathbb{R}^p$ and $p \in \mathbb{N}$. Then we have

$$\hbar_r(\delta) = D_r^{\xi p}\phi(\delta,0),$$

where, $D_\xi^{rp} = D_\xi^p.D_\xi^p.\cdots.D_\xi^p(r - times)$.

Proof. The following is the new Taylor's series:

$$\hbar_1(\delta) = \xi^{p+2}\Psi(\delta,\xi) - \xi^p\hbar_0(\delta) - \frac{\hbar_2(\delta)}{\xi^p} - \frac{\hbar_3(\delta)}{\xi^{2p}} - \cdots \tag{2.21}$$

$\lim_{\xi\to\infty}$, is applied to Eq (2.21), we get

$$\hbar_1(\delta) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\delta,\xi) - \xi^p\hbar_0(\delta)) - \lim_{\xi\to\infty}\frac{\hbar_2(\delta)}{\xi^p} - \lim_{\xi\to\infty}\frac{\hbar_3(\delta)}{\xi^{2p}} - \cdots$$

We get the following equivalence by taking the limit:

$$\hbar_1(\delta) = \lim_{\xi\to\infty}(\xi^{p+2}\Psi(\delta,\xi) - \xi^p\hbar_0(\delta)). \tag{2.22}$$

Upon using Lemma 2.2 to solve Eq (2.22), the following outcome is obtained:

$$\hbar_1(\delta) = \lim_{\xi\to\infty}(\xi^2 A[D_\xi^p\phi(\delta,\xi)](\xi)). \tag{2.23}$$

Furthermore, Lemma 2.3 is applied to Eq (2.23) to convert it into the following form:

$$\hbar_1(\delta) = D_\xi^p\phi(\delta,0).$$

Once again use the new form of Taylor's series and assuming limit $\xi \to \infty$, we get

$$\hbar_2(\delta) = \xi^{2p+2}\Psi(\delta,\xi) - \xi^{2p}\hbar_0(\delta) - \xi^p\hbar_1(\delta) - \frac{\hbar_3(\delta)}{\xi^p} - \cdots$$

Lemma 2.3 yields the following outcome:

$$\hbar_2(\delta) = \lim_{\xi\to\infty}\xi^2(\xi^{2p}\Psi(\delta,\xi) - \xi^{2p-2}\hbar_0(\delta) - \xi^{p-2}\hbar_1(\delta)). \tag{2.24}$$

Lemmas 2.2 and 2.4 are used to translate Eq (2.24) into the following form:

$$\hbar_2(\delta) = D_\xi^{2p}\phi(\delta, 0).$$

The following outcomes are obtained by using the same procedure on the new Taylor's series:

$$\hbar_3(\delta) = \lim_{\xi\to\infty} \xi^2(A[D_\xi^{2p}\phi(\delta, p)](\xi)).$$

Using Lemma 2.4, the final equation is obtained

$$\hbar_3(\delta) = D_\xi^{3p}\phi(\delta, 0).$$

In general

$$\hbar_r(\delta) = D_\xi^{rp}\phi(\delta, 0).$$

Consequently, the proof ends here. □

The conditions under which Taylor's series in its modified form will converge are explained and established in the following theorem.

Theorem 2.6. *Lemma 2.3 provides the formula for multiple fractional Taylor's, which may be expressed as follows:* $A[\phi(\delta,\Omega)] = \Psi(\delta, \xi)$. *When* $|\xi^a A[D_\Omega^{(K+1)p}\phi(\delta,\Omega)]| \leq T$, *for all* $0 < \xi \leq s$ *and* $0 < p \leq 1$, *the residual* $R_K(\delta, \xi)$ *of the new MFTS is satisfied with the following inequality:*

$$|R_K(\delta, \xi)| \leq \frac{T}{\xi^{(K=1)p+2}},\ 0 < \xi \leq s.$$

Proof. Assume $A[D_\Omega^{rp}\phi(\delta, \Omega)](\xi)$ is defined on $0 < \xi \leq s$ for $r = 0, 1, 2, \cdots, K + 1$. Proceed with the assumption that $|\xi^2 A[D_{\Omega^{K+1}}\phi(\delta, tau)]| \leq T,\ on\ 0 < \xi \leq s$. Establish the relation below using the new Taylor's series:

$$R_K(\delta, \xi) = \Psi(\delta, \xi) - \sum_{r=0}^{K} \frac{\hbar_r(\delta)}{\xi^{rp+2}}. \tag{2.25}$$

Theorem 2.5 is used to transform Eq (2.25) into the following form:

$$R_K(\delta, \xi) = \Psi(\delta, \xi) - \sum_{r=0}^{K} \frac{D_\Omega^{rp}\phi(\delta, 0)}{\xi^{rp+2}}. \tag{2.26}$$

Simply multiply $\xi^{(K+1)a+2}$ on both sides to solve Eq (2.26).

$$\xi^{(K+1)p+2}R_K(\delta, \xi) = \xi^2(\xi^{(K+1)p}\Psi(\delta, \xi) - \sum_{r=0}^{K} \xi^{(K+1-r)p-2}D_\Omega^{rp}\phi(\delta, 0)). \tag{2.27}$$

Applying Lemma 2.2 to Eq (2.27) yields

$$\xi^{(K+1)p+2}R_K(\delta, \xi) = \xi^2 A[D_\Omega^{(K+1)p}\phi(\delta, \Omega)]. \tag{2.28}$$

Taking absolute of Eq (2.28) yields

$$|\xi^{(K+1)p+2}R_K(\delta, \xi)| = |\xi^2 A[D_\Omega^{(K+1)p}\phi(\delta, \Omega)]|. \tag{2.29}$$

The following outcome may be obtained by applying the conditions given in Eq (2.29):

$$\frac{-T}{\xi^{(K+1)p+2}} \leq R_K(\delta, \xi) \leq \frac{T}{\xi^{(K+1)p+2}}. \tag{2.30}$$

To get the required outcome, we use Eq (2.30)

$$|R_K(\delta, \xi)| \leq \frac{T}{\xi^{(K+1)p+2}}.$$

As a result, we develop novel requirements for series convergence. □

3. Methodology

3.1. The ARPSM technique for solve time-fractional PDEs

This section discusses the ARPSM collection of rules that we used to develop our overall model solution.

Step 1. When we simplify the general equation, we get

$$D_{\Omega}^{qp}\phi(\delta, \Omega) + \vartheta(\delta)N(\phi) - \zeta(\delta, \phi) = 0. \tag{3.1}$$

Step 2. Applying AT to Eq (3.1) yields:

$$A[D_{\Omega}^{qp}\phi(\delta, \Omega) + \vartheta(\delta)N(\phi) - \zeta(\delta, \phi)] = 0. \tag{3.2}$$

Lemma 2.2 will be used to convert Eq (3.2) into the following form:

$$\Psi(\delta, s) = \sum_{j=0}^{q-1} \frac{D_{\Omega}^{j}\phi(\delta, 0)}{s^{qp+2}} - \frac{\vartheta(\delta)Y(s)}{s^{qp}} + \frac{F(\delta, s)}{s^{qp}}, \tag{3.3}$$

where, $A[\zeta(\delta, \phi)] = F(\delta, s), A[N(\phi)] = Y(s)$.

Step 3. The expression for the solution to Eq (3.3) reads:

$$\Psi(\delta, s) = \sum_{r=0}^{\infty} \frac{\hbar_r(\delta)}{s^{rp+2}}, \ s > 0.$$

Step 4. The following steps are necessary:

$$\hbar_0(\delta) = \lim_{s\to\infty} s^2\Psi(\delta, s) = \phi(\delta, 0).$$

Applying Theorem 2.6 yields the following result:

$$\hbar_1(\delta) = D_{\Omega}^{p}\phi(\delta, 0),$$

$$\hbar_2(\delta) = D_{\Omega}^{2p}\phi(\delta, 0),$$

$$\vdots$$

$$\hbar_w(\delta) = D_\Omega^{wp}\phi(\delta, 0).$$

Step 5. The following formula may be used to get the $\Psi(\delta, s)$ series after the K^{th} truncation:

$$\Psi_K(\delta, s) = \sum_{r=0}^{K} \frac{\hbar_r(\delta)}{s^{rp+2}}, \ s > 0,$$

$$\Psi_K(\delta, s) = \frac{\hbar_0(\delta)}{s^2} + \frac{\hbar_1(\delta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\delta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\delta)}{s^{rp+2}}.$$

Step 6. The Aboodh residual function (ARF) from (3.3) and the Kth-truncated ARF must be taken into account separately for the following results:

$$ARes(\delta, s) = \Psi(\delta, s) - \sum_{j=0}^{q-1} \frac{D_\Omega^j\phi(\delta, 0)}{s^{jp+2}} + \frac{\vartheta(\delta)Y(s)}{s^{jp}} - \frac{F(\delta, s)}{s^{jp}},$$

and

$$ARes_K(\delta, s) = \Psi_K(\delta, s) - \sum_{j=0}^{q-1} \frac{D_\Omega^j\phi(\delta, 0)}{s^{jp+2}} + \frac{\vartheta(\delta)Y(s)}{s^{jp}} - \frac{F(\delta, s)}{s^{jp}}. \tag{3.4}$$

Step 7. Rather than using its expansion form, $\Psi_K(\delta, s)$ should be substituted in Eq (3.4)

$$\begin{aligned} ARes_K(\delta, s) =& \Big(\frac{\hbar_0(\delta)}{s^2} + \frac{\hbar_1(\delta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\delta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\delta)}{s^{rp+2}}\Big) \\ & - \sum_{j=0}^{q-1} \frac{D_\Omega^j\phi(\delta, 0)}{s^{jp+2}} + \frac{\vartheta(\delta)Y(s)}{s^{jp}} - \frac{F(\delta, s)}{s^{jp}}. \end{aligned} \tag{3.5}$$

Step 8. Multiply each side of Eq (3.5) by s^{Kp+2} to get the following outcome

$$\begin{aligned} s^{Kp+2}ARes_K(\delta, s) =& s^{Kp+2}\Big(\frac{\hbar_0(\delta)}{s^2} + \frac{\hbar_1(\delta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\delta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\delta)}{s^{rp+2}} \\ & - \sum_{j=0}^{q-1} \frac{D_\Omega^j\phi(\delta, 0)}{s^{jp+2}} + \frac{\vartheta(\delta)Y(s)}{s^{jp}} - \frac{F(\delta, s)}{s^{jp}}\Big). \end{aligned} \tag{3.6}$$

Step 9. Assuming that $\lim_{s\to\infty}$, we can calculate Eq (3.6) to derive the following:

$$\begin{aligned} \lim_{s\to\infty} s^{Kp+2}ARes_K(\delta, s) = \lim_{s\to\infty} & s^{Kp+2}\Big(\frac{\hbar_0(\delta)}{s^2} + \frac{\hbar_1(\delta)}{s^{p+2}} + \cdots + \frac{\hbar_w(\delta)}{s^{wp+2}} + \sum_{r=w+1}^{K} \frac{\hbar_r(\delta)}{s^{rp+2}} \\ & - \sum_{j=0}^{q-1} \frac{D_\Omega^j\phi(\delta, 0)}{s^{jp+2}} + \frac{\vartheta(\delta)Y(s)}{s^{jp}} - \frac{F(\delta, s)}{s^{jp}}\Big). \end{aligned}$$

Step 10. The value of $\hbar_K(\delta)$ may be obtained by solving the provided equation

$$\lim_{s\to\infty}(s^{Kp+2}ARes_K(\delta, s)) = 0,$$

where $K = w+1, w+2, \cdots$

Step 11. To obtain the K-approximate solution of Eq (3.3), substitute the values of $\hbar_K(\delta)$ with a $\Psi(\delta, s)$ series that has been truncated.

Step 12. To get the K-approximation solution Solve $\Psi_K(\delta, s)$ using the AIT and obtain the required function $\phi_K(\delta, \Omega)$.

3.1.1. Problem 1

Consider the time-fractional Sawada-Kotera equation:

$$D^p_\Omega \phi(\delta,\Omega) + \frac{\partial^5\phi(\delta,\Omega)}{\partial\delta^5} + 15\phi(\delta,\Omega)\frac{\partial^3\phi(\delta,\Omega)}{\partial\delta^3} + 15\frac{\partial\phi(\delta,\Omega)}{\partial\delta}\frac{\partial^2\phi(\delta,\Omega)}{\partial\delta^2} + 45\phi^2(\delta,\Omega)\frac{\partial\phi(\delta,\Omega)}{\partial\delta} = 0, \quad (3.7)$$

with the following IC:

$$\phi(\delta, 0) = 2k^2\text{sech}^2(k(\delta - a)). \quad (3.8)$$

Remember that the exact solution of Eq (3.7) for integer-order reads

$$\phi(\delta,\Omega) = 2k^2\text{sech}^2\left(k\left(-a - 16k^4\Omega + \delta\right)\right). \quad (3.9)$$

Using Eq (3.8) and applying AT to Eq (3.7) yields

$$\begin{aligned}&\phi(\delta, s) - \frac{2k^2\text{sech}^2(k(\delta-a))}{s^2} + \frac{1}{s^p}\Big[\frac{\partial^5\phi(\delta,s)}{\partial\delta^5}\Big] + \frac{15}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi(\delta,s)\times\frac{\partial^3A_\Omega^{-1}\phi(\delta,s)}{\partial\delta^3}\Big]\\&+\frac{15}{s^p}A_\Omega\Big[\frac{\partial A_\Omega^{-1}\phi(\delta,s)}{\partial\delta}\times A_\Omega^{-1}\frac{\partial^2\phi(\delta,s)}{\partial\delta^2}\Big] + \frac{45}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi^2(\delta,s)\times\frac{\partial A_\Omega^{-1}\phi(\delta,s)}{\partial\delta}\Big] = 0.\end{aligned} \quad (3.10)$$

Thus, the following are the k^{th}-truncated term series:

$$\phi(\delta, s) = \frac{2k^2\text{sech}^2(k(\delta-a))}{s^2} + \sum_{r=1}^{k}\frac{f_r(\delta,s)}{s^{rp+1}}, \quad r = 1,2,3,4,\cdots \quad (3.11)$$

The ARF is given by:

$$\begin{aligned}A_\Omega Res(\delta, s) = {}&\phi(\delta, s) - \frac{2k^2\text{sech}^2(k(\delta-a))}{s^2} + \frac{1}{s^p}\Big[\frac{\partial^5\phi(\delta,s)}{\partial\delta^5}\Big] + \frac{15}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi(\delta,s)\times\frac{\partial^3A_\Omega^{-1}\phi(\delta,s)}{\partial\delta^3}\Big]\\&+\frac{15}{s^p}A_\Omega\Big[\frac{\partial A_\Omega^{-1}\phi(\delta,s)}{\partial\delta}\times A_\Omega^{-1}\frac{\partial^2\phi(\delta,s)}{\partial\delta^2}\Big] + \frac{45}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi^2(\delta,s)\times\frac{\partial A_\Omega^{-1}\phi(\delta,s)}{\partial\delta}\Big] = 0,\end{aligned} \quad (3.12)$$

and the k^{th}-LRFs as:

$$\begin{aligned}A_\Omega Res_k(\delta, s) = {}&\phi_k(\delta, s) - \frac{2k^2\text{sech}^2(k(\delta-a))}{s^2} + \frac{1}{s^p}\Big[\frac{\partial^5\phi_k(\delta,s)}{\partial\delta^5}\Big] + \frac{15}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi_k(\delta,s)\times\frac{\partial^3A_\Omega^{-1}\phi_k(\delta,s)}{\partial\delta^3}\Big]\\&+\frac{15}{s^p}A_\Omega\Big[\frac{\partial A_\Omega^{-1}\phi_k(\delta,s)}{\partial\delta}\times A_\Omega^{-1}\frac{\partial^2\phi_k(\delta,s)}{\partial\delta^2}\Big] + \frac{45}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi_k^2(\delta,s)\times\frac{\partial A_\Omega^{-1}\phi_k(\delta,s)}{\partial\delta}\Big] = 0,\end{aligned} \quad (3.13)$$

Finding the values of $f_r(\delta, s)$ for $r = 1, 2, 3, ...$ need some computation. Follow this procedure, take the r^{th}-ARF Eq (3.13) and substitute it for the r^{th}-truncated series Eq (3.11), and we solve the relation $\lim_{s\to\infty}(s^{rp+1})$ by multiplying the resultant equation by s^{rp+1}. $A_\Omega Res_{\phi,r}(\delta, s)) = 0$, and $r = 1, 2, 3, \cdots$. A few of the terms that we obtain are as follows:

$$f_1(\delta, s) = 64k^7 \tanh(k(\delta - a))\text{sech}^2(k(\delta - a)), \tag{3.14}$$

$$f_2(\delta, s) = 1024k^{12}(\cosh(2k(\delta - a)) - 2)\text{sech}^4(k(\delta - a)), \tag{3.15}$$

and so on.

Put $f_r(\delta, s)$, for $r = 1, 2, 3, \cdots$, in Eq (3.11):

$$\begin{aligned}\phi(\delta, s) = &\frac{2k^2\text{sech}^2(k(\delta - a))}{s^2} + \frac{64k^7 \tanh(k(\delta - a))\text{sech}^2(k(\delta - a))}{s^{p+1}} \\ &+ \frac{1024k^{12}(\cosh(2k(\delta - a)) - 2)\text{sech}^4(k(\delta - a))}{s^{2p+1}} + \cdots\end{aligned} \tag{3.16}$$

Applying AIT on the above equation, we finally obtain the following approximation:

$$\begin{aligned}\phi(\delta, \Omega) = &2k^2\text{sech}^2(k(\delta - a)) + \frac{64k^7\Omega^p \tanh(k(\delta - a))\text{sech}^2(k(\delta - a))}{\Gamma(p + 1)} \\ &+ \frac{1024k^{12}\Omega^{2p}(\cosh(2k(\delta - a)) - 2)\text{sech}^4(k(\delta - a))}{\Gamma(2p + 1)} + \cdots\end{aligned} \tag{3.17}$$

Initially, we conduct a numerical and graphical comparison between the deduced approximations (3.17) and the exact solution (3.9) for the integer case (specifically, for $p = 1$). This comparison aims to assess the accuracy and efficiency of these approximations, thereby confirming the effectiveness of the used approach and its stability across various parameters associated with the problem being investigated. Figure 1 compares the approximation (3.17) for $p = 1$ and the exact solution (3.9) for the integer case. It is clear from this figure that there is complete harmony between the two solutions, which enhances the accuracy and efficiency of the deduced approximations. This, in turn, prompts us to study the effect of the fractional parameter on the behavior of these approximations, and we are convinced of the accuracy of the obtained results. In order to investigate the impact of the fractional parameter on the characteristics of the obtained approximations, which subsequently characterize the solitary waves, we conducted an analysis of this approximation, as depicted in Figure 2. Based on this figure, it is evident that the fractional parameter significantly influences the behavior and characteristics of the solitary waves. Consequently, the fractional parameter has unveiled previously unknown aspects of solitary waves, so exerting a substantial influence on elucidating numerous nonlinear phenomena, encompassing practical applications, natural phenomena, and space observations.

Table 1 offers a sufficient qualitative comparison of the derived approximations using ARPSM for different values of the fractional parameter. Therefore, the combined results of the figures and table assist in discussing whether the ARPSM effectively solves the Sawada-Kotera equation under various conditions, which may provide valuable feedback for further studies and results.

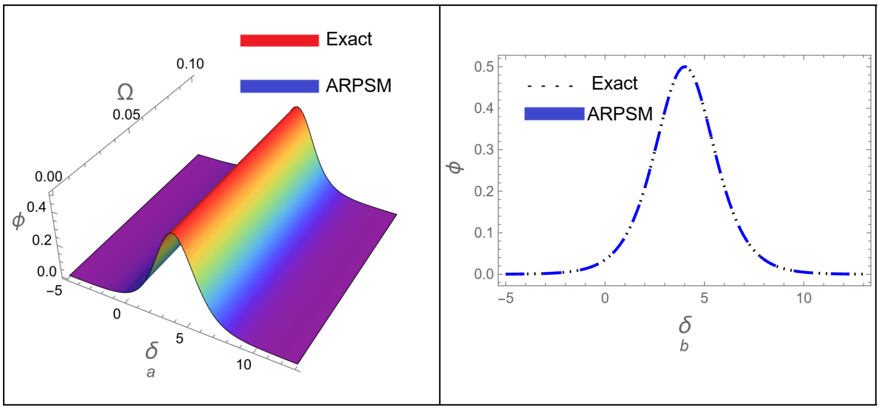

Figure 1. Comparing the approximation (3.17) and the exact solution (3.9) to the problem (3.7) for the integer case, i.e., for $p = 1$: (a) three-dimensional graphic for the approximation (3.17) and the exact solution (3.9) and (b) two-dimensional graphic for the approximation (3.17) and the exact solution (3.9) at $\Omega = 0.01$. Here, $k = 0.5$ and $a = 4$.

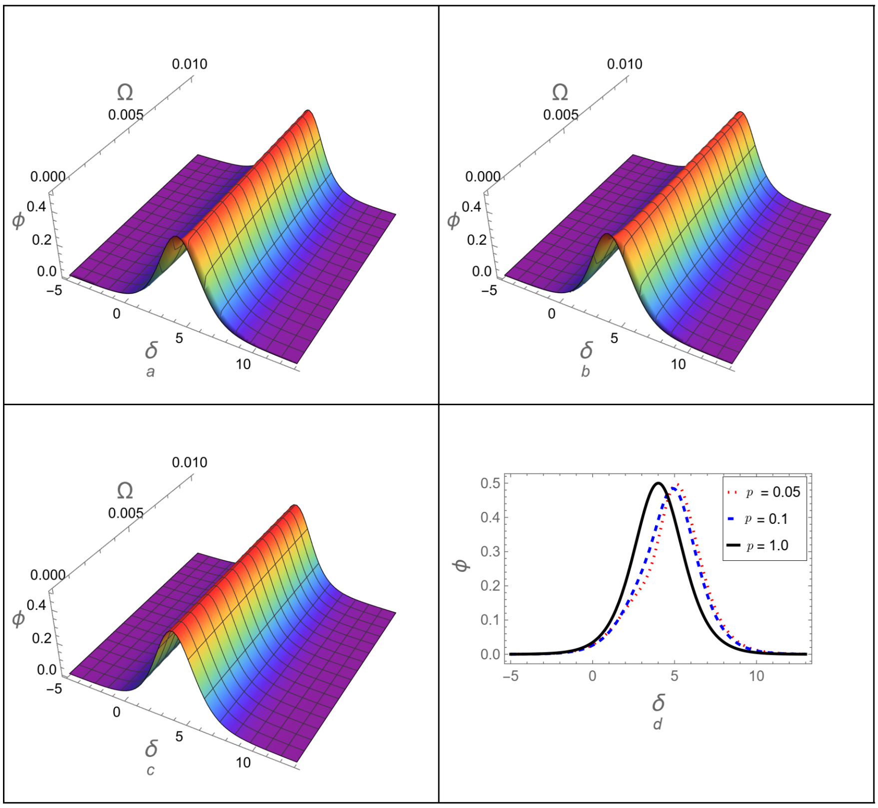

Figure 2. The approximation (3.17) is plotted vs the fractional parameter p: (a) three-dimensional graphic for $p = 0.05$, (b) three-dimensional graphic for $p = 0.1$, (c) three-dimensional graphic for $p = 1$, and (d) two-dimensional graphic for various values of p at $\Omega = 0.01$. Here, $k = 0.5$ and $a = 4$.

Table 1. The impact of fractional parameter on the approximation (3.17) for $\Omega = 0.1$ and $k = 0.2$.

δ	$ARPSM_{p=0.54}$	$ARPSM_{p=0.74}$	$ARPSM_{p=1.00}$	$Exact$	$Error_{p=1.00}$
1.0	0.0568245	0.0568641	0.0568949	0.0568949	6.206861×10^{-12}
1.1	0.0580446	0.0580839	0.0581145	0.0581145	6.418809×10^{-12}
1.2	0.0592564	0.0592954	0.0593258	0.0593258	6.613508×10^{-12}
1.3	0.0604577	0.0604963	0.0605265	0.0605265	6.788715×10^{-12}
1.4	0.0616461	0.0616842	0.0617140	0.0617140	6.942293×10^{-12}
1.5	0.0628190	0.0628567	0.0628860	0.0628860	7.071995×10^{-12}
1.6	0.0639742	0.0640112	0.0640400	0.0640400	7.175801×10^{-12}
1.7	0.0651088	0.0651452	0.0651735	0.0651735	7.251588×10^{-12}
1.8	0.0662205	0.0662561	0.0662838	0.0662838	7.297551×10^{-12}
1.9	0.0673066	0.0673413	0.0673683	0.0673683	7.311873×10^{-12}
2.0	0.0683645	0.0683982	0.0684245	0.0684245	7.293124×10^{-12}

3.1.2. Problem 2

Consider the Ito time-fractional equation:

$$D^{p}_{\Omega}\phi(\delta,\Omega) + \frac{\partial^5\phi(\delta,\Omega)}{\partial\delta^5} + 3\phi(\delta,\Omega)\frac{\partial^3\phi(\delta,\Omega)}{\partial\delta^3} + 6\frac{\partial\phi(\delta,\Omega)}{\partial\delta}\frac{\partial^2\phi(\delta,\Omega)}{\partial\delta^2} + 2\phi^2(\delta,\Omega)\frac{\partial\phi(\delta,\Omega)}{\partial\delta} = 0, \quad (3.18)$$

with the following IC:

$$\phi(\delta,0) = 20k^2 - 30k^2\tanh^2(k\delta). \quad (3.19)$$

Remember that the exact solution of problem (3.18) for integer-order reads

$$\phi(\delta,\Omega) = 20k^2 - 30k^2\tanh^2\left(k\delta - 96k^4\Omega\right). \quad (3.20)$$

Using Eq (3.19) and applying AT to Eq (3.18) yields

$$\begin{aligned}\phi(\delta,s) &- \frac{20k^2 - 30k^2\tanh^2(k\delta)}{s^2} + \frac{1}{s^p}\Big[\frac{\partial^5\phi(\delta,s)}{\partial\delta^5}\Big] + \frac{3}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi(\delta,s)\times\frac{\partial^3A_\Omega^{-1}\phi(\delta,s)}{\partial\delta^3}\Big]\\ &+\frac{6}{s^p}A_\Omega\Big[A_\Omega^{-1}\frac{\partial\phi(\delta,s)}{\partial\delta}\times\frac{\partial^2A_\Omega^{-1}\phi(\delta,s)}{\partial\delta^2}\Big] + \frac{2}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi^2(\delta,s)\times\frac{\partial A_\Omega^{-1}\phi(\delta,s)}{\partial\delta}\Big] = 0,\end{aligned} \quad (3.21)$$

Thus, the following are the k^{th}-truncated term series:

$$\phi(\delta,s) = \frac{20k^2 - 30k^2\tanh^2(k\delta)}{s^2} + \sum_{r=1}^{k}\frac{f_r(\delta,s)}{s^{rp+1}}, \quad r = 1,2,3,4,\cdots \quad (3.22)$$

The ARF reads:

$$\begin{aligned}A_\Omega Res(\delta,s) = \phi(\delta,s) &- \frac{20k^2 - 30k^2\tanh^2(k\delta)}{s^2} + \frac{1}{s^p}\Big[\frac{\partial^5\phi(\delta,s)}{\partial\delta^5}\Big] + \frac{3}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi(\delta,s)\times\frac{\partial^3A_\Omega^{-1}\phi(\delta,s)}{\partial\delta^3}\Big]\\ &+\frac{6}{s^p}A_\Omega\Big[A_\Omega^{-1}\frac{\partial\phi(\delta,s)}{\partial\delta}\times\frac{\partial^2A_\Omega^{-1}\phi(\delta,s)}{\partial\delta^2}\Big] + \frac{2}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi^2(\delta,s)\times\frac{\partial A_\Omega^{-1}\phi(\delta,s)}{\partial\delta}\Big] = 0,\end{aligned} \quad (3.23)$$

and the k^{th}-LRFs as:

$$A_\Omega Res_k(\delta,s) = \phi_k(\delta,s) - \frac{20k^2 - 30k^2\tanh^2(k\delta)}{s^2} + \frac{1}{s^p}\Big[\frac{\partial^5\phi_k(\delta,s)}{\partial\delta^5}\Big] + \frac{3}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi_k(\delta,s)\times\frac{\partial^3 A_\Omega^{-1}\phi_k(\delta,s)}{\partial\delta^3}\Big]$$
$$+\frac{6}{s^p}A_\Omega\Big[A_\Omega^{-1}\frac{\partial\phi_k(\delta,s)}{\partial\delta}\times\frac{\partial^2 A_\Omega^{-1}\phi_k(\delta,s)}{\partial\delta^2}\Big] + \frac{2}{s^p}A_\Omega\Big[A_\Omega^{-1}\phi_k^2(\delta,s)\times\frac{\partial A_\Omega^{-1}\phi_k(\delta,s)}{\partial\delta}\Big] = 0, \tag{3.24}$$

Finding the values of $f_r(\delta,s)$ for $r = 1,2,3,...$ need some computation. Follow this procedure, take the r^{th}-Aboodh residual function Eq (3.24) and substitute it for the r^{th}-truncated series Eq (3.22), and we solve the relation $\lim_{s\to\infty}(s^{rp+1})$ by multiplying the resultant equation by s^{rp+1}. $A_\Omega Res_{\phi,r}(\delta,s)) = 0$, and $r = 1,2,3,\cdots$. A few of the terms that we obtain are as follows:

$$f_1(\delta,s) = 64k^7\tanh(k(\delta-a))\text{sech}^2(k(\delta-a)), \tag{3.25}$$

$$f_2(\delta,s) = 1024k^{12}(\cosh(2k(\delta-a))-2)\text{sech}^4(k(\delta-a)), \tag{3.26}$$

and so on.

Put $f_r(\delta,s)$, for $r = 1,2,3,\cdots$, in Eq (3.22):

$$\phi(\delta,s) = \frac{2k^2\text{sech}^2(k(\delta-a))}{s^2} + \frac{64k^7\tanh(k(\delta-a))\text{sech}^2(k(\delta-a))}{s^{p+1}} + \frac{1024k^{12}(\cosh(2k(\delta-a))-2)\text{sech}^4(k(\delta-a))}{s^{2p+1}} + \cdots \tag{3.27}$$

Applying AIT to the above equation yields the following approximation

$$\phi(\delta,\Omega) = 2k^2\text{sech}^2(k(\delta-a)) + \frac{64k^7\Omega^p\tanh(k(\delta-a))\text{sech}^2(k(\delta-a))}{\Gamma(p+1)} + \frac{1024k^{12}t^{2p}(\cosh(2k(\delta-a))-2)\text{sech}^4(k(\delta-a))}{\Gamma(2p+1)} + \cdots \tag{3.28}$$

For the integer case, i.e., for $p = 1$, Figure 3 presents a comparison between the exact solution (3.20) of problem 2 and the derived approximation (3.28). It is observed that there is excellent agreement between the two solutions, which enhances the high accuracy of the derived approximations. The general behavior of the approximation (3.28) versus various values for the fractional parameter is depicted in Figure 4, which includes both two and three-dimensional representations. Furthermore, these demonstrations in Figure 4 will provide a comprehensive understanding of how the variable p influences the dynamics of the derivedÂ approximations. Table 2 adequately presents the qualitative comparison of the obtained approximations using ARPSM for various values of the fractional parameter. Subjection to the fractional order, a lower and a higher value, brings the shape and properties of the solution, influencing the change of behavior. The exact graph modeling of the analytical solutions is critical in determining the errors and variability analyzed using such models. Hence, the collective findings from Figures 3 and 4 and Table 2 aid in evaluating the efficacy of the ARPSM in solving the Ito time-fractional equation across different scenarios, thereby offering valuable insights for future research and outcomes.

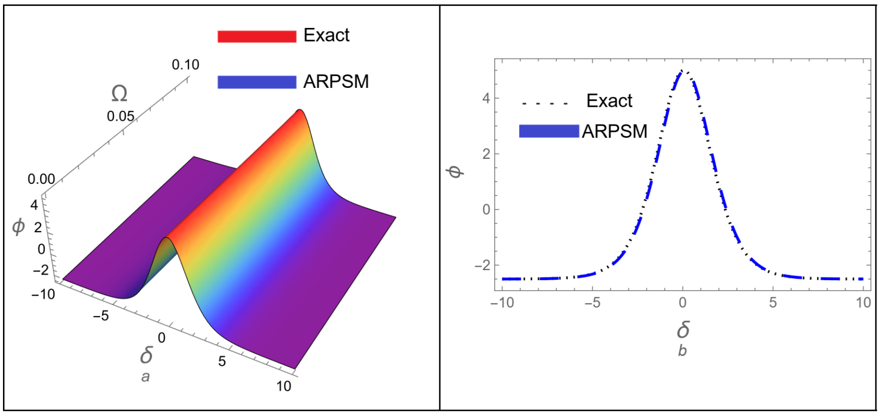

Figure 3. Comparing the approximation (3.28) and the exact solution (3.20) to the problem (3.18) for the integer case, i.e., for $p = 1$: (a) three-dimensional graphic for the approximation (3.28) and the exact solution (3.20) and (b) two-dimensional graphic for the approximation (3.28) and the exact solution (3.20) at $\Omega = 0.01$. Here, $k = 0.5$.

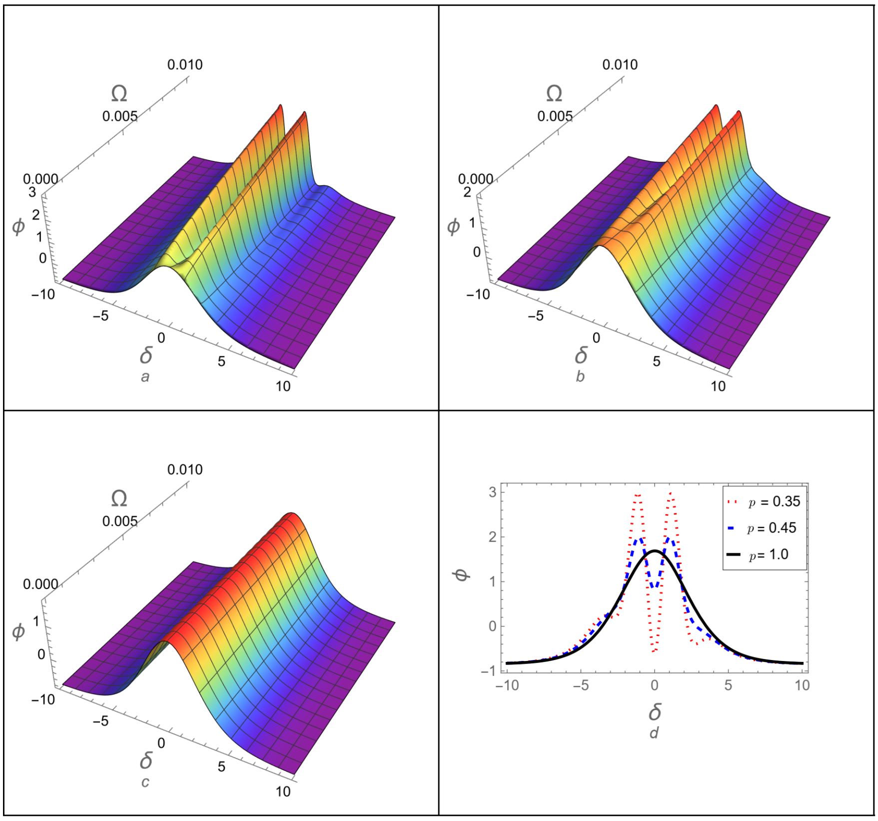

Figure 4. The approximation (3.28) is plotted vs the fractional parameter p: (a) three-dimensional graphic for $p = 0.35$, (b) three-dimensional graphic for $p = 0.45$, (c) three-dimensional graphic for $p = 1$, and (d) two-dimensional graphic for various values of p at $\Omega = 0.01$. Here, $k = 0.5$.

Table 2. The impact of fractional parameter on the approximation (3.28) for $\Omega = 0.01$ and $k = 0.1$.

δ	$ARPSM_{p=0.54}$	$ARPSM_{p=0.74}$	$ARPSM_{p=1.00}$	$Exact$	$Error_{p=1.00}$
1.0	0.197025	0.197022	0.197020	0.197026	5.112834×10^{-6}
1.1	0.196405	0.196401	0.196400	0.196405	5.608732×10^{-6}
1.2	0.195727	0.195724	0.195722	0.195728	6.100225×10^{-6}
1.3	0.194993	0.194989	0.194987	0.194994	6.586942×10^{-6}
1.4	0.194203	0.194199	0.194197	0.194204	7.068523×10^{-6}
1.5	0.193358	0.193353	0.193351	0.193358	7.544614×10^{-6}
1.6	0.192458	0.192452	0.192450	0.192458	8.014870×10^{-6}
1.7	0.191503	0.191498	0.191495	0.191504	8.478957×10^{-6}
1.8	0.190495	0.190490	0.190487	0.190496	8.936550×10^{-6}
1.9	0.189435	0.189429	0.189426	0.189436	9.387335×10^{-6}
2.0	0.188323	0.188317	0.188314	0.188324	9.831007×10^{-6}

3.1.3. Problem 3

Consider the nonlinear time-fractional Kaup-Kupershmidt equation:

$$D_{\Omega}^{p}\phi(\delta,\Omega)-\frac{\partial^5\phi(\delta,\Omega)}{\partial\delta^5}-5\phi(\delta,\Omega)\frac{\partial^3\phi(\delta,\Omega)}{\partial\delta^3}-\frac{25}{2}\frac{\partial\phi(\delta,\Omega)}{\partial\delta}\frac{\partial^2\phi(\delta,\Omega)}{\partial\delta^2}-5\phi^2(\delta,\Omega)\frac{\partial\phi(\delta,\Omega)}{\partial\delta}=0, \quad (3.29)$$

with the following IC:

$$\phi(\delta,0)=\frac{24k^2}{e^{k\delta}+1}-\frac{24k^2}{\left(e^{k\delta}+1\right)^2}-2k^2. \quad (3.30)$$

Remember that the exact solution of problem (3.30) for integer-order reads

$$\phi(\delta,\Omega)=\frac{24k^2}{e^{11k^2\Omega+k\delta}+1}-\frac{24k^2}{\left(e^{11k^2\Omega+k\delta}+1\right)^2}-2k^2. \quad (3.31)$$

Using Eq (3.30) and applying AT to Eq (3.29) yields

$$\begin{aligned}&\phi(\delta,s)-\frac{\frac{24k^2}{e^{k\delta}+1}-\frac{24k^2}{\left(e^{k\delta}+1\right)^2}-2k^2}{s^2}-\frac{1}{s^p}\Big[\frac{\partial^5\phi(\delta,s)}{\partial\delta^5}\Big]-\frac{5}{s^p}A_{\Omega}\Big[A_{\Omega}^{-1}\phi(\delta,s)\times\frac{\partial^3A_{\Omega}^{-1}\phi(\delta,s)}{\partial\delta^3}\Big]\\&-\frac{25}{2}\frac{1}{s^p}A_{\Omega}\Big[A_{\Omega}^{-1}\frac{\partial\phi(\delta,s)}{\partial\delta}\times\frac{\partial^2A_{\Omega}^{-1}\phi(\delta,s)}{\partial\delta^2}\Big]-\frac{5}{s^p}A_{\Omega}\Big[A_{\Omega}^{-1}\phi^2(\delta,s)\times\frac{\partial A_{\Omega}^{-1}\phi(\delta,s)}{\partial\delta}\Big]=0,\end{aligned} \quad (3.32)$$

Thus, the following are the k^{th}-truncated term series:

$$\phi(\delta,s)=\frac{\frac{24k^2}{e^{k\delta}+1}-\frac{24k^2}{\left(e^{k\delta}+1\right)^2}-2k^2}{s^2}+\sum_{r=1}^{k}\frac{f_r(\delta,s)}{s^{rp+1}},\quad r=1,2,3,4,\cdots \quad (3.33)$$

The ARF is as follows:

$$A_\Omega Res(\delta, s) = \phi(\delta, s) - \frac{\frac{24k^2}{e^{k\delta}+1} - \frac{24k^2}{\left(e^{k\delta}+1\right)^2} - 2k^2}{s^2} - \frac{1}{s^p}\Big[\frac{\partial^5 \phi(\delta, s)}{\partial \delta^5}\Big] - \frac{5}{s^p} A_\Omega\Big[A_\Omega^{-1}\phi(\delta, s) \times \frac{\partial^3 A_\Omega^{-1}\phi(\delta, s)}{\partial \delta^3}\Big]$$
$$- \frac{25}{2}\frac{1}{s^p} A_\Omega\Big[A_\Omega^{-1}\frac{\partial \phi(\delta, s)}{\partial \delta} \times \frac{\partial^2 A_\Omega^{-1}\phi(\delta, s)}{\partial \delta^2}\Big] - \frac{5}{s^p} A_\Omega\Big[A_\Omega^{-1}\phi^2(\delta, s) \times \frac{\partial A_\Omega^{-1}\phi(\delta, s)}{\partial \delta}\Big] = 0, \tag{3.34}$$

and the k^{th}-LRFs as:

$$A_\Omega Res_k(\delta, s) = \phi_k(\delta, s) - \frac{\frac{24k^2}{e^{k\delta}+1} - \frac{24k^2}{\left(e^{k\delta}+1\right)^2} - 2k^2}{s^2} - \frac{1}{s^p}\Big[\frac{\partial^5 \phi_k(\delta, s)}{\partial \delta^5}\Big] - \frac{5}{s^p} A_\Omega\Big[A_\Omega^{-1}\phi_k(\delta, s) \times \frac{\partial^3 A_\Omega^{-1}\phi_k(\delta, s)}{\partial \delta^3}\Big]$$
$$- \frac{25}{2}\frac{1}{s^p} A_\Omega\Big[A_\Omega^{-1}\frac{\partial \phi_k(\delta, s)}{\partial \delta} \times \frac{\partial^2 A_\Omega^{-1}\phi_k(\delta, s)}{\partial \delta^2}\Big] - \frac{5}{s^p} A_\Omega\Big[A_\Omega^{-1}\phi_k^2(\delta, s) \times \frac{\partial A_\Omega^{-1}\phi_k(\delta, s)}{\partial \delta}\Big] = 0, \tag{3.35}$$

Finding the values of $f_r(\delta, s)$ for $r = 1, 2, 3, ...$ need some computation. Follow this procedure, take the r^{th}-Aboodh residual function Eq (3.35) and substitute it for the r^{th}-truncated series Eq (3.33), and we solve the relation $\lim_{s\to\infty}(s^{rp+1})$ by multiplying the resultant equation by s^{rp+1}. $A_\Omega Res_{\phi,r}(\delta, s)) = 0$, and $r = 1, 2, 3, \cdots$. A few of the terms that we obtain are as follows:

$$f_1(\delta, s) = -\Big(72k^7 e^{k\delta}\Big(-49e^{k\delta} + 194e^{2k\delta} - 194e^{3k\delta} + 49e^{4k\delta} + 3e^{5k\delta} - 3\Big)\Big)/\Big(e^{k\delta} + 1\Big)^7, \tag{3.36}$$

$$f_2(\delta, s) = \Big(216k^{12} e^{k\delta}\Big(-684e^{k\delta} + 78693e^{2k\delta} - 1183952e^{3k\delta} + 5210386e^{4k\delta} - 8383752e^{5k\delta}$$
$$+ 5210386e^{6k\delta} - 1183952e^{7k\delta} + 78693e^{8k\delta} - 684e^{9k\delta} + 9e^{10k\delta} + 9\Big)\Big)/\Big(e^{k\delta} + 1\Big)^{12}, \tag{3.37}$$

and so on.

Put $f_r(\delta, s)$, for $r = 1, 2, 3, \cdots$, in Eq (3.33):

$$\phi(\delta, s) = \Big(\frac{24k^2}{e^{k\delta} + 1} - \frac{24k^2}{(e^{k\delta} + 1)^2} - 2k^2\Big)/\Big(s^2\Big) - \Big(72k^7 e^{k\delta}\Big(-49e^{k\delta} + 194e^{2k\delta} - 194e^{3k\delta} + 49e^{4k\delta} + 3e^{5k\delta}$$
$$- 3\Big)\Big)/\Big(\Big(e^{k\delta} + 1\Big)^7 s^{p+1}\Big) + \Big(216k^{12} e^{k\delta}\Big(-684e^{k\delta} + 78693e^{2k\delta} - 1183952e^{3k\delta} + 5210386e^{4k\delta} - 8383752e^{5k\delta}$$
$$+ 5210386e^{6k\delta} - 1183952e^{7k\delta} + 78693e^{8k\delta} - 684e^{9k\delta} + 9e^{10k\delta} + 9\Big)\Big)/\Big(\Big(e^{k\delta} + 1\Big)^{12} s^{2p+1}\Big) + \cdots \tag{3.38}$$

Applying AIT on the above equation, we finally get the following approximation:

$$\phi(\delta, \Omega) = \Big(\frac{24k^2}{e^{k\delta} + 1} - \frac{24k^2}{(e^{k\delta} + 1)^2} - 2k^2\Big) - \Big(72\Omega^p k^7 e^{k\delta}\Big(-49e^{k\delta} + 194e^{2k\delta} - 194e^{3k\delta} + 49e^{4k\delta} + 3e^{5k\delta}$$
$$- 3\Big)\Big)/\Big(\Big(e^{k\delta} + 1\Big)^7 \Gamma(p + 1)\Big) + \Big(216\Omega^{2p} k^{12} e^{k\delta}\Big(-684e^{k\delta} + 78693e^{2k\delta} - 1183952e^{3k\delta} + 5210386e^{4k\delta}$$
$$- 8383752e^{5k\delta} + 5210386e^{6k\delta} - 1183952e^{7k\delta} + 78693e^{8k\delta} - 684e^{9k\delta} + 9e^{10k\delta}$$
$$+ 9\Big)\Big)/\Big(\Big(e^{k\delta} + 1\Big)^{12} \Gamma(2p + 1)\Big) + \cdots \tag{3.39}$$

Figure 5 compares the derived approximation (3.39) for the nonlinear time-fractional Kaup-Kupershmidt issue at p=1 and its exact solution (3.31) for the integer case. The visual representation presented above depicts the analytical solutions graphically and offers valuable insights into the system's behavior being investigated. Furthermore, the numerical analysis of the approximation (3.39) is conducted against the fractional parameter, as depicted in Figure 6. This analysis helps identify the system's analytical approximation behavior and understand the mechanism of the nonlinear phenomena described by this approximation. Table 3 shows a statistical analysis of several fractional values for the approximation (3.39). The statistical analysis of various fractional values for the approximation (3.39) is presented in Table 3. Quantitative analysis can determine convergence accuracy and which method is more suitable for solving such problems.

Table 3. The impact of fractional parameter on the approximation (3.39) for $\Omega = 0.01$.

δ	$ARPSM_{p=0.54}$	$ARPSM_{p=0.74}$	$ARPSM_{p=1.00}$	$Exact$	$Error_{p=1.00}$
1.0	0.0398503	0.0398503	0.0398503	0.0398469	3.308740×10^{-6}
1.1	0.0398189	0.0398189	0.0398189	0.0398152	3.635238×10^{-6}
1.2	0.0397845	0.0397845	0.0397845	0.0397806	3.961010×10^{-6}
1.3	0.0397472	0.0397472	0.0397472	0.0397429	4.285991×10^{-6}
1.4	0.0397070	0.0397070	0.0397070	0.0397024	4.610117×10^{-6}
1.5	0.0396638	0.0396638	0.0396638	0.0396588	4.933324×10^{-6}
1.6	0.0396177	0.0396176	0.0396176	0.0396124	5.255549×10^{-6}
1.7	0.0395686	0.0395686	0.0395686	0.0395630	5.576729×10^{-6}
1.8	0.0395166	0.0395166	0.0395166	0.0395107	5.896801×10^{-6}
1.9	0.0394618	0.0394618	0.0394617	0.0394555	6.215704×10^{-6}
2.0	0.0394040	0.0394040	0.0394040	0.0393974	6.533376×10^{-6}

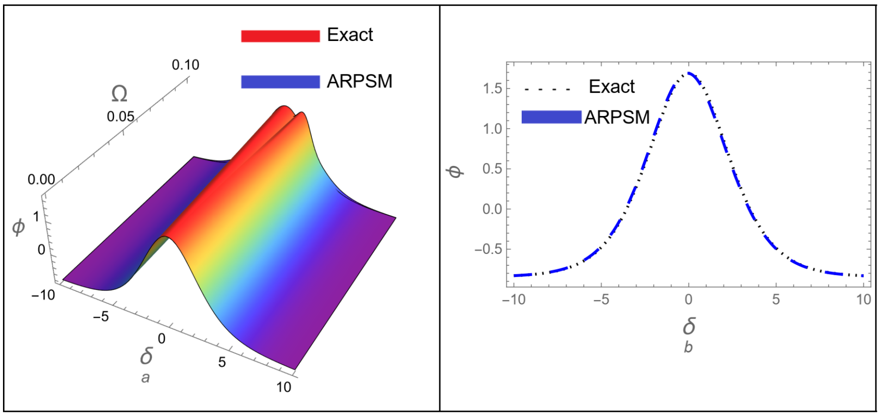

Figure 5. Comparing the approximation (3.39) and the exact solution (3.31) to the problem (3.29) for the integer case, i.e., for $p = 1$: (a) three-dimensional graphic for the approximation (3.39) and the exact solution (3.31) and (b) two-dimensional graphic for the approximation (3.39) and the exact solution (3.31) at $\Omega = 0.01$. Here, $k = 0.65$.

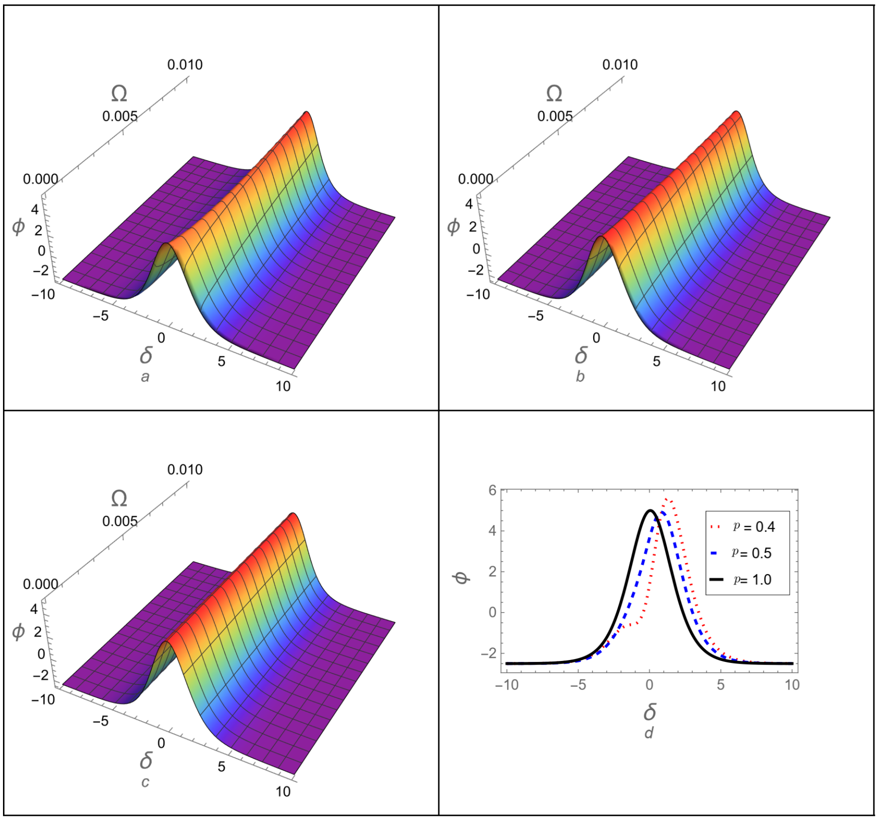

Figure 6. The approximation (3.39) is plotted vs the fractional parameter p: (a) three-dimensional graphic for $p = 0.35$, (b) three-dimensional graphic for $p = 0.45$, (c) three-dimensional graphic for $p = 1$, and (d) two-dimensional graphic for various values of p at $\Omega = 0.01$. Here, $k = 0.65$.

4. Conclusions

In conclusion, the utilization of the Aboodh residual power series method, in conjunction with the Caputo operator, presented a viable and pragmatic methodology for examining and resolving different types of time-fractional differential equations, including the Sawada-Kotera equation, Ito equation, and Kaup-Kupershmidt equation. The study showcased the potential application of the recommended analytical procedures in obtaining analytical solutions for fractional differential equations and generating traveling wave solutions. It was found that the suggested approaches can be applied to a range of more complicated and higher-order nonlinearity systems. The analysis and discussion of all generated approximations for the three given problems yielded significant findings. These approximations were compared to the exact solutions for the integer case, a crucial step in validating their accuracy. The analytical findings demonstrated a high level of concurrence between the estimations and the precise solutions, thereby underscoring the importance of the suggested approach in examining diverse fractional evolution equations. Therefore, these methods have successfully solved and analyzed the most complicated problems, enhancing their bright future in modeling many of the most complex issues related to natural, engineering, and physical phenomena.

The findings of this work contribute to the expansion of understanding regarding fractional order partial differential equations (PDEs) and present practical methodologies for analytical applications in any model that incorporates fractional derivatives. Significantly, this study highlights the pressing need to employ novel methods within fractional calculus approaches to enhance comprehension of the domain of nonlinear dynamics and mathematical physics.

5. Future works

The results of our study have not only showcased the precision, effectiveness, and adaptability of the methodologies used but also hold the potential to revolutionize the way we approach the most intricate problems. These methodologies can be effectively used for analyzing and solving several issues related to nonlinear plasma physics and fluid mechanics. Therefore, the proposed methodology can be utilized to investigate the impact of fractional parameters on the propagation characteristics of solitary waves, shock waves, and rogue waves, which are described by various evolution equations, including Korteweg-De Vries-type equations [59–61], Burgers equations [62, 63], and nonlinear Schrodinger-type equations [64, 65] in their fractional forms.

Authors contributions

Humaira Yasmin and Aljawhara H. Almuqrin: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review & editing. All authors contributed equally and approved the final version of the current manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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