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Research article

A new class of hybrid contractions with higher-order iterative Kirk's method for reckoning fixed points

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Abstract: The concept of contraction mappings plays a significant role in mathematics, particularly in the study of fixed points and the existence of solutions for various equations. In this study, we described two types of enriched contractions: enriched F-contraction and enriched F'-contraction associated with u-fold averaged mapping, which are involved with Kirk's iterative technique with order u. The contractions extracted from this paper generalized and unified many previously common super contractions. Furthermore, u-fold averaged mappings can be seen as a more general form of both averaged mappings and double averaged mappings. Moreover, we demonstrated that the u-fold averaged mapping with enriched contractions has a unique fixed point. Our work examined the necessary conditions for the u-fold averaged mapping and weak enriched contractions to have equal sets of fixed points. Additionally, we illustrated that an appropriate Kirk's iterative algorithm can effectively approximate a fixed point of a u-fold averaged mapping as well as the two enriched contractions. Also, we delved into the well-posedness, limit shadowing property, and Ulam-Hyers stability of the u-fold averaged mapping. Furthermore, we established necessary conditions that guaranteed the periodic point property for each of the illustrated strengthened contractions. To underscore the generality of our findings, we presented several examples that aligned with comparable results found in the existing literature.

Keywords: Kirk's iterative technique; strengthened F-contraction; evaluation metrics; u-fold averaged mapping; mathematical operators; fixed point technique
Mathematics Subject Classification: 47H09, 47H10, 54H25

Nomenclature

NEM→non-expansive mapping	FP→fixed point
ECM→enriched contraction mapping	MS→metric space
EKC→enriched Kannan contraction	BS→Banach space
ECC→enriched Chatterjea contraction	BC→Banach contraction
ECRRC→enriched Ćirić-Reich-Rus-contraction	PO→Picard operator
EIKC→enriched interpolative Kannan-contraction	KI→Kirk's iteration
EICRRC→enriched interpolative Ćirić-Reich-Rus-contraction	AM→averaged mapping
KIS→Krasnoselskii iterative scheme	NS→normed space
HEF-C→hybrid enriched F-contraction	UH→Ulam-Hyers
HEF'-C→hybrid enriched F'-contraction	PPP→periodic point property

1. Introduction and basic facts

One of the most helpful methods for studying nonlinear equations, whether they be differential, integral, or algebraic equations, is the contraction mapping principle. The idea is based on the fixed point (FP) theorem, which states that every contraction mapping of a complete metric space (MS) to itself will have a single FP. This FP can be found as the limit of an iteration scheme made up of repeated images under the mapping of any arbitrary beginning point in the space. Since it is a constructive FP theorem, the FP can be computed numerically using it.

Assume that Θ is a nonempty set of a Banach space (BS) Ω. A mapping $\Im : \Theta \to \Theta$ is called a non-expansive mapping (NEM), if for all $\omega, \theta \in \Theta$, the inequality below holds:

$$\left\|\Im\omega - \Im\theta\right\| \leq \|\omega - \theta\|.$$

The FP of $\Im$ is an element $\omega^* \in \Theta$, which satisfies an operator equation $\Im\omega^* = \omega^*$. The set of FPs of the mapping $\Im$ is denoted by $Fix(\Im)$. Let $\omega_0 \in \Omega$ be an arbitrary point, and the forward orbit of ω_0 is denoted by $O(\Im, \omega_0, \infty)$, and is described as the set $\{\omega_0, \Im^m(\omega_0) : m \geq 1\}$. The set $\{\omega, \Im(\omega), \cdots, \Im^m(\omega)\}$ will be described as $O\{\omega, \Im, m\}$. The m^{th} iterate of the mapping $\Im$ is described as $\Im^m = \Im^{m-1} \circ \Im$, $m \geq 1$, $\Im^0 = I$ (where I is the identity mapping on Ω).

If $Fix(\Im) = \{\omega^*\}$ and $O(\Im, \omega_0, \infty) \to \omega^*$ as $m \to \infty$, then the mapping $\Im$ is called a Picard operator (PO). Moreover, if there is a constant $\rho \in [0, 1)$ such that

$$d(\Im\omega, \Im\theta) \leq \rho d(\omega, \theta),$$

for all ω and θ belonging to a complete MS Ω, then the mapping $\Im : \Omega \to \Omega$ is known as a Banach contraction (BC) mapping. Clearly, the BC mapping converts to NEM if $\rho = 1$. As the limiting situation of BC mappings, one can consider the NEMs. A BC mapping's m^{th} iterates are referred to as Picard's iterates. Any BC mapping constructed on a complete MS (Ω, d) is a PO, as per the BC principle [1]. Furthermore, Picard's iterates can approximate the FP of the mapping $\Im$ for each $\omega_0 \in \Omega$, but an NEM $\Im$ does not produce a forward orbit that converges to $\Im$'s FP. In other words, if $\Im : \Theta \to \Theta$ is an NEM, then $\Im$ may not have an FP, may have more than one FP, or may even have a unique FP; in contrast, the forward orbit created by a NEM will not converge to its FP. Therefore, other approximation methods are required in order to estimate the FPs of NEMs. Additionally, a complex geometric structure of the underlying spaces is necessary for the FPs of NEMs to exist. Due to these factors, one of the main and most active subfields of nonlinear analytic research is the study of NEMs.

Banach fixed-point theorem provides a powerful tool for establishing the existence and uniqueness of fixed points in metric spaces, which has implications in optimization, inverse problems, and other mathematical contexts, for more details, see [2–9]

Exact averaged iterations of the form $\omega_{m+1} = g(\omega_m, \Im\omega_{m+1})$, $m \geq 1$ have been used by numerous writers. One well-known method is to create an averaged mapping (AM): If $\Im_\vartheta = (1-\vartheta)I + \vartheta\Im$, then an operator $\Im_\vartheta$ associated with $\Im$ and identity mapping I is an AM for a given operator $\Im$ on Ω and $\vartheta \in (0,1)$. This concept was first used in [10], when it was demonstrated that the forward orbit caused by $\Im_\vartheta$ converges to an FP of $\Im$ under specific circumstances. The initial noteworthy outcome in this regard was acquired by Krasnoselskii [11]. In the event that Θ represents a closed convex subset of a uniformly convex BS and $\Im$ is an NEM on Θ into a compact subset of Θ, then the forward orbit of any ω in Θ for $\vartheta = 0.5$ converges to an FP of $\Im$. Schaefer [12] demonstrated the aforementioned outcomes for an arbitrary $\vartheta \in (0,1)$. The same result was then presented by Edelstein [13] in the context of a strictly convex BS, which is a broader concept than a uniformly convex BS. It is obvious that Picard's iteration method is generalized by Krasnoselskii's iteration.

In 1971, Kirk [14] created a significant iteration technique called Kirk's iteration (KI) scheme, which is defined by

$$\omega_m = \kappa_0\omega_{m-1} + \kappa_1\Im\omega_{m-1} + \kappa_2\Im^2\omega_{m-1} + \cdots + \kappa_u\Im^u\omega_{m-1},$$

where $\omega_0 \in \Theta$, $\kappa_0 > 0$, and for $j = 1, 2, \cdots u$, $\kappa_j \geq 0$ with $\sum_{j=1}^{u}\kappa_j = 1$.

KI method, in fact, is a forward orbit of the mapping $\mho : \Theta \to \Theta$ [14] described by

$$\mho = \kappa_0 I + \kappa_1\Im + \kappa_2\Im^2 + \cdots + \kappa_u\Im^u, \tag{1.1}$$

where $\kappa_0 > 0$, and for $j = 1, 2, \cdots u$, $\kappa_j \geq 0$ with $\sum_{j=1}^{u}\kappa_j = 1$. Undoubtedly, the mapping $\mho$ is a generalization of the AM $\Im_\vartheta$.

Kirk demonstrated that, under certain appropriate conditions, the set of FPs of the mapping $\mho$ corresponds with $Fix(\Im)$, and that the KI method converges to the FP of $\Im$:

Theorem 1.1. *[14] Assume that Θ is a convex subset of a BS Ω, and $\Im : \Omega \to \Omega$ is a NEM. If $\mho : \Omega \to \Omega$ is a mapping described as in (1.1), then $\mho(\omega) = \omega$ if $\Im(\omega) = \omega$.*

The concept of enriched contractive mappings (ECMs) was recently introduced by Berinde and Păcurar [15]. Let Ω be a BS, and the mapping $\Im : \Omega \to \Omega$ is said to be ECM if there are $\tau \geq 0$ and $\sigma \in [0, \tau+1)$ in order that

$$\left\|\tau(\omega-\theta) + \Im\omega - \Im\theta\right\| \leq \sigma\|\omega-\theta\|, \text{ for all } \omega, \theta \in \Omega.$$

They established the existence of an FP of an ECM, which may be roughly represented using a suitable Krasnoselskii iterative scheme (KIS). To be more precise, the sequence $\{\Im_\vartheta^m\omega_0\}$ can approximate the FP of $\Im$, which is also an FP of the AM $\Im_\vartheta$ with $\vartheta \in (0,1]$ for each $\omega_0 \in \Omega$.

Theorem 1.2. *[15] Assume that $\Im : \Omega \to \Omega$ is an ECM defined on a BS Ω. Then $\left|Fix(\Im)\right| = 1$, and there is $\vartheta \in (0,1]$ such that the KIS $\{\omega_m\}$ iterated by*

$$\omega_m = (1-\vartheta)\omega_{m-1} + \vartheta\Im\omega_{m-1}, \textit{ for all } \omega_0 \in \Omega, \textit{ and } m \geq 0$$

converges to a unique FP of $\Im$.

It is important to note that only the displacements $\left\|\Im\omega - \Im\theta\right\|$ and $\|\omega - \theta\|$ are included in the enriched contraction mapping that Berinde and Păcurar [15] presented. Nonetheless, for every two distinct points $\omega, \theta \in \Omega$, there are four more displacements linked to a self-mapping $\Im$, denoted by $\left\|\omega - \Im\omega\right\|$, $\left\|\theta - \Im\theta\right\|$, $\left\|\omega - \Im\theta\right\|$, and $\left\|\theta - \Im\omega\right\|$. More than one displacement is involved in a number of well-known contraction mappings. For more details, see [16–27]. The authors in [28] have proposed the concept of weak ECMs, which are an extension of AMs known as double AMs. Assume that $\kappa_1 > 0$, $\kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 = 1$ and $\Im : \Omega \to \Omega$ is a mapping defined on a BS Ω. Double AM $\Im_{\kappa_1,\kappa_2}$ is a mapping related to I, $\Im$, and $\Im^2$ and is described as

$$\Im_{\kappa_1,\kappa_2} = (1 - \kappa_1 - \kappa_2)I + \kappa_1\Im + \kappa_2\Im^2.$$

Clearly, $\Im_{\kappa_1,\kappa_2}$ is more general than $\Im_\vartheta$ ($\Im_\vartheta = \Im_{\kappa_1,0}$). Additionally, the mapping $\mho$ in [14] of order $u = 2$ is a specific instance of the double AM $\Im_{\kappa_1,\kappa_2}$. A given mapping $\Im : \Omega \to \Omega$ on a BS Ω is said to be a weak ECM if there are $\tau, \widetilde{\tau} \geq 0$ and $\ell \in [0, \tau + \widetilde{\tau} + 1)$ such that

$$\left\|\tau(\omega - \theta) + \Im\omega - \Im\theta + \widetilde{\tau}\left(\Im^2\omega - \Im^2\theta\right)\right\| \leq \ell\,\|\omega - \theta\|, \text{ for all } \omega, \theta \in \Omega.$$

According to the findings of [28], for every self-mapping $\Im$ on a closed convex subset of a BS that satisfies the weak ECM, there exist $\kappa_1 > 0$, $\kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 \in (0, 1]$ such that $\Im_{\kappa_1,\kappa_2}$ has a unique FP that can be approximated by a suitable KIS. We make reference to the next paragraph. Their theorem was formulated as follows:

Theorem 1.3. *[28] Assume that $(\Omega, \|.\|)$ is a BS, Θ is a closed convex subset of Ω and $\Im : \Theta \to \Theta$ is a weak ECM. Then, there are $\kappa_1 > 0$, $\kappa_2 \geq 0$ with $\kappa_1 + \kappa_2 \in (0, 1]$ such that the assertions below hold*

(1) $\left|Fix\left(\Im_{\kappa_1,\kappa_2}\right)\right| = 1$;
(2) For any $\omega_0 \in \Theta$, the iterated sequence $\{\omega_m\} \subset \Theta$ generated by

$$\omega_m = (1 - \kappa_1 - \kappa_2)\omega_{m-1} + \kappa_1\Im\omega_{m-1} + \kappa_2\Im^2\omega_{m-1}, \textit{ for } m \in \mathbb{N}$$

converges to a unique FP of $\Im_{\kappa_1,\kappa_2}$.

KIS of order u, which is produced by a generalized ECM, appears to be a good way to unify the FP results that have been described. This unification has two components: KIS of order more than two is examined, and ECMs are generalized such that the many ECMs that currently exist are inferred as specific examples.

So, in this article, two types of enriched contractions related to KIS with order u are described in this paper: hybrid enriched F-contraction and hybrid enriched F'-contraction connected with u-fold AM. The contractions taken from this paper unify and generalize a lot of super contractions that were previously widespread. Additionally, one may consider u-fold ANs to be a more universal version of double and AMs. Furthermore, we prove the existence of a unique FP for the u-fold AM with enriched contractions. We investigate what requirements must be met in order for the weak hybrid ECMs and the u-fold AM to have identical sets of FPs. In addition, we demonstrate how a suitable KIS can efficiently approximate both the FP and the average of a u-fold mapping.

2. Hybrid enriched contraction mappings

We begin by introducing two mapping families. Assume that F is the class of all mappings $\hbar : \mathbb{R}_+^4 \to \mathbb{R}_+$ that meet the requirements listed below:

($\hbar_1$) In every argument, $\hbar$ is continuous;
($\hbar_2$) there is $\zeta \in [0,1)$ such that if $\varkappa < \hbar(\varrho,\varkappa,\varrho,\varrho+\varkappa)$ or $\varkappa < \hbar(\varrho,\varkappa,\varrho,\varkappa)$ or $\varkappa < \hbar(\varkappa,\varrho,\varrho,\varkappa)$ or $\varkappa < \hbar(\varkappa,\varrho,\varrho,\varrho)$, then for all $\varkappa,\varrho \in \mathbb{R}_+$, $\varkappa \leq \zeta\varrho$;
($\hbar_3$) $\vartheta\hbar(\varkappa,\varrho,\xi,\upsilon) \leq \hbar(\vartheta\varkappa,\vartheta\varrho,\vartheta\xi,\vartheta\upsilon)$, for $\vartheta > 0$ and for all $\varkappa,\varrho,\xi,\upsilon \in \mathbb{R}_+$;
($\hbar_4$) if $\upsilon \leq \upsilon'$, then $\hbar(\varkappa,\varrho,\xi,\upsilon) \leq \hbar(\varkappa,\varrho,\xi,\upsilon')$ for all $\varkappa,\varrho,\xi,\upsilon,\upsilon' \in \mathbb{R}_+$.

To demonstrate that the family F is nonempty, we now provide some examples.

Example 2.1. It is simple to confirm that the mappings shown below are a part of class F:

(i) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa\max\{\varkappa+\varrho,\varrho+\xi,\xi+\upsilon,\varkappa+\upsilon\}$, where $\kappa \in [0,\frac{1}{2})$;
(ii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \max\kappa\{\varkappa,\varrho,\xi,\upsilon\}$, where $\kappa \in [0,1)$;
(iii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \max\kappa\{\varrho,\xi,\upsilon\}$, where $\kappa \in [0,1)$;
(iv) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa\varkappa$, where $\kappa \in [0,1)$;
(v) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa(\varkappa+\varrho)$, where $\kappa \in [0,\frac{1}{2})$;
(vi) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa(\xi+\upsilon)$, where $\kappa \in [0,\frac{1}{2})$;
(vii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \xi^{\kappa}\upsilon^{1-\kappa}$, where $\kappa \in (0,1)$;
(viii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \varkappa^{\kappa_1}\varrho^{\kappa_2}\xi^{\kappa_3}\upsilon^{1-\kappa_1-\kappa_2-\kappa_3}$, where $\kappa_1,\kappa_2,\kappa_3 \in (0,1)$ with $\kappa_1+\kappa_2+\kappa_3 < 1$;
(ix) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa_1\varkappa+\kappa_2\varrho+\kappa_3\xi+\kappa_4\upsilon$, where $\kappa_1,\kappa_2,\kappa_3,\kappa_4 \in [0,1)$ with $\kappa_1+\kappa_2+\kappa_3+\kappa_4 = 1$.

Assume that F' is the class of all mappings $\hbar : \mathbb{R}_+^4 \to \mathbb{R}_+$ that meet the requirements listed below:

($\hbar'_1$) In every argument, $\hbar$ is continuous;
($\hbar'_2$) there is $\zeta \in [0,1)$ such that if $\varkappa < \hbar(\varrho,\varrho+\varkappa,0,\varrho+\varkappa)$, or $\varkappa < \hbar(\varrho,\varkappa,\varkappa,\varkappa)$, or $\varkappa < \hbar(\varrho,0,0,\varkappa+\varrho)$, or $\varkappa < \hbar(\varkappa,\varrho,\varrho,\varrho)$, then for all $\varkappa,\varrho \in \mathbb{R}_+$, $\varkappa \leq \zeta\varrho$;
($\hbar'_3$) $\vartheta\hbar(\varkappa,\varrho,\xi,\upsilon) \leq \hbar(\vartheta\varkappa,\vartheta\varrho,\vartheta\xi,\vartheta\upsilon)$, for $\vartheta > 0$ and for all $\varkappa,\varrho,\xi,\upsilon \in \mathbb{R}_+$;
($\hbar'_4$) if $\upsilon \leq \upsilon'$, then $\hbar(\varkappa,\varrho,\xi,\upsilon) \leq \hbar(\varkappa,\varrho,\xi,\upsilon')$ for all $\varkappa,\varrho,\xi,\upsilon,\upsilon' \in \mathbb{R}_+$;
($\hbar'_5$) if $\varkappa \leq \hbar(\varkappa,\varkappa,\varkappa,\varkappa)$, then $\varkappa = 0$.

To illustrate that the family F' is nonempty, we consider the following examples:

Example 2.2. It is simple to confirm that the mappings shown below are a part of the family F':

(i) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa\max\{\varkappa+\varrho,\varrho+\xi,\xi+\upsilon,\varkappa+\upsilon\}$, where $\kappa \in [0,\frac{1}{2})$;
(ii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa\varkappa$, where $\kappa \in [0,1)$;
(iii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa(\varrho+\xi)$, where $\kappa \in [0,\frac{1}{2})$;
(iv) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa(\xi+\upsilon)$, where $\kappa \in [0,\frac{1}{2})$;
(vi) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa(\varkappa+\varrho+\xi+\upsilon)$, where $\kappa \in [0,\frac{1}{3})$;
(vii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa(\varkappa\varrho\xi\upsilon)^{\frac{1}{4}}$, where $\kappa \in [0,1)$;
(viii) $\hbar(\varkappa,\varrho,\xi,\upsilon) = \kappa\sqrt{\varkappa\varrho}$, where $\kappa \in [0,\frac{1}{3})$.

Here, we provide the u-fold AM using the mapping $\mho$ [14].

Definition 2.1. Let Ω be a BS, Θ be a nonempty subset of Ω, and $\Im : \Omega \to \Omega$ is a given mapping. Describe the mapping $\widehat{\Im} : \Theta \to \Theta$ associated with $\Im$ as

$$\widehat{\Im} = (1 - \kappa_1 - \kappa_2 - \kappa_3 - \cdots - \kappa_u) I + \kappa_1 \Im + \kappa_2 \Im^2 + \kappa_3 \Im^3 + \cdots + \kappa_u \Im^u,$$

where $\kappa_j > 0$, $\sum_{j=1}^{u} \kappa_j \in (0, 1]$, $u \geq 4$, $u \in \mathbb{N}$. We say that the mapping $\Im'$ is u-fold AM.

Now, let us provide two concepts of ECMs.

Definition 2.2. Suppose that $(\Omega, \|.\|)$ is a normed space (NS). We say that the mapping $\Im : \Omega \to \Omega$ is a hybrid enriched F-contraction (HEF-C) if there is $\hbar \in F$ in order that for all $\omega, \theta \in \Omega$, $b_j \in (0, \infty)$, $j = 1, 2, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, we get

$$\begin{aligned} &\left\| b_1 (\omega - \theta) + \Im\omega - \Im\theta + b_2 \left(\Im^2\omega - \Im^2\theta\right) + b_3 \left(\Im^3\omega - \Im^3\theta\right) + \cdots + b_u \left(\Im^u\omega - \Im^u\theta\right) \right\| \\ \leq\ & \hbar \Bigg(\left(\sum_{j=1}^{u} b_j + 1 \right) \|\omega - \theta\|, \\ & \left\| (\omega - \Im\omega) + b_2 \left(\omega - \Im^2\omega\right) + b_3 \left(\omega - \Im^3\omega\right) + \cdots + b_u \left(\omega - \Im^u\omega\right) \right\|, \\ & \left\| (\theta - \Im\theta) + b_2 \left(\theta - \Im^2\theta\right) + b_3 \left(\theta - \Im^3\theta\right) + \cdots + b_u \left(\theta - \Im^u\theta\right) \right\|, \\ & \left\| (\theta - \Im\omega) + b_2 \left(\theta - \Im^2\omega\right) + b_3 \left(\theta - \Im^3\omega\right) + \cdots + b_u \left(\theta - \Im^u\omega\right) \right\| \Bigg). \end{aligned} \tag{2.1}$$

Definition 2.3. Let $(\Omega, \|.\|)$ be an NS. We say that the mapping $\Im : \Omega \to \Omega$ is a hybrid enriched F'-contraction (HEF'-C) if there is $\hbar \in F'$ in order that for all $\omega, \theta \in \Omega$, $b_j \in (0, \infty)$, $j = 1, 2, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, we get

$$\begin{aligned} &\left\| b_1 (\omega - \theta) + \Im\omega - \Im\theta + b_2 \left(\Im^2\omega - \Im^2\theta\right) + b_3 \left(\Im^3\omega - \Im^3\theta\right) + \cdots + b_u \left(\Im^u\omega - \Im^u\theta\right) \right\| \\ \leq\ & \hbar \Bigg(\left(\sum_{j=1}^{u} b_j + 1 \right) \|\omega - \theta\|, \\ & \left\| \left(\sum_{j=1}^{u} b_j + 1 \right) (\theta - \omega) + (\omega - \Im\omega) + b_2 \left(\omega - \Im^2\omega\right) + b_3 \left(\omega - \Im^3\omega\right) + \cdots + b_u \left(\omega - \Im^u\omega\right) \right\|, \\ & \left\| \left(\sum_{j=1}^{u} b_j + 1 \right) (\omega - \theta) + (\theta - \Im\theta) + b_2 \left(\theta - \Im^2\theta\right) + b_3 \left(\theta - \Im^3\theta\right) + \cdots + b_u \left(\theta - \Im^u\theta\right) \right\|, \\ & \left\| (\theta - \Im\omega) + b_2 \left(\theta - \Im^2\omega\right) + b_3 \left(\theta - \Im^3\omega\right) + \cdots + b_u \left(\theta - \Im^u\omega\right) \right\| \Bigg). \end{aligned} \tag{2.2}$$

The definitions above are supported by the following examples:

Example 2.3. Assume that $\Omega = \mathbb{R}$ is a usual NS and $\Im : [0, \infty) \to [0, \infty)$ is a given mapping described as $\Im\omega = \frac{\omega}{3}$ for $\omega \in [0, \infty)$. It is clear for $b_j = \frac{1}{3}$, $j = 1, 2, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, and $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa\varkappa$, $\kappa = \frac{6}{7} \in [0, 1)$ that $\Im$ is an HEF-C mapping. In fact, Definition 2.2 indicates that

$$\left\| b_1 (\omega - \theta) + \Im\omega - \Im\theta + b_2 \left(\Im^2\omega - \Im^2\theta\right) + b_3 \left(\Im^3\omega - \Im^3\theta\right) + \cdots + b_u \left(\Im^u\omega - \Im^u\theta\right) \right\|$$

$$
\begin{aligned}
&= \left\|\frac{1}{3}(\omega-\theta)+\left(\frac{\omega}{3}-\frac{\theta}{3}\right)+\frac{1}{3}\left(\frac{\omega}{9}-\frac{\theta}{9}\right)+\frac{1}{3}\left(\frac{\omega}{27}-\frac{\theta}{27}\right)+\cdots+\frac{1}{3^u}\left(\frac{\omega}{3^u}-\frac{\theta}{3^u}\right)\right\| \\
&= \left\|\frac{1}{3}(\omega-\theta)+\frac{1}{3}(\omega-\theta)+\frac{1}{27}(\omega-\theta)+\frac{1}{81}(\omega-\theta)+\cdots+\frac{1}{3^{u+1}}(\omega-\theta)\right\| \\
&\leq 2\|\omega-\theta\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hbar\Bigg(\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\|,\left\|(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)+b_3\left(\omega-\Im^3\omega\right)+\cdots+b_u\left(\omega-\Im^u\omega\right)\right\|, \\
&\left\|(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)+b_3\left(\theta-\Im^3\theta\right)+\cdots+b_u\left(\theta-\Im^u\theta\right)\right\|, \\
&\left\|(\theta-\Im\omega)+b_2\left(\theta-\Im^2\omega\right)+b_3\left(\theta-\Im^3\omega\right)+\cdots+b_u\left(\theta-\Im^u\omega\right)\right\|\Bigg) \\
= \; &\kappa\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\| \\
= \; &\frac{6}{7}\left(1+\frac{u}{3}\right)\|\omega-\theta\| \\
\geq \; &\frac{6}{7}\left(1+\frac{4}{3}\right)\|\omega-\theta\| \\
= \; &2\|\omega-\theta\|.
\end{aligned}
$$

Hence, the inequality (2.1) holds. Therefore, $\Im$ is an HEF-C mapping and $\Im$ has a unique FP $0 \in [0,\infty)$.

Example 2.4. Assume that $\Omega = \mathbb{R}$ is a usual NS and $\Im : [0,\infty) \to [0,\infty)$ is a given mapping described as $\Im\omega = 1-\frac{\omega}{3}$ for $\omega \in [0,\infty)$. It is clear for $b_j = \frac{1}{3^j}$, $j = 1,2,3,\cdots,u$, $u \geq 4$, $u \in \mathbb{N}$, and $\hbar(\varkappa,\varrho,\xi,\upsilon) = \varkappa$ that $\Im$ is an HEF-C mapping. In fact, Definition 2.2 indicates that

$$
\begin{aligned}
&\left\|b_1(\omega-\theta)+\Im\omega-\Im\theta+b_2\left(\Im^2\omega-\Im^2\theta\right)+b_3\left(\Im^3\omega-\Im^3\theta\right)+\cdots+b_u\left(\Im^u\omega-\Im^u\theta\right)\right\| \\
= \; &\left\|\frac{1}{3}(\omega-\theta)+\frac{1}{3}(\theta-\omega)+\frac{1}{9}\left(\frac{\omega}{9}-\frac{\theta}{9}\right)+\frac{1}{27}\left(\frac{\theta}{27}-\frac{\omega}{27}\right)+\cdots+\frac{1}{3^u}(-1)^u\left(\frac{\omega}{3^u}-\frac{\theta}{3^u}\right)\right\| \\
\leq \; &\sum_{j=1}^{u}\left(\frac{1}{3^j}\right)\|\omega-\theta\|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\hbar\Bigg(\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\|,\left\|(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)+b_3\left(\omega-\Im^3\omega\right)+\cdots+b_u\left(\omega-\Im^u\omega\right)\right\|, \\
&\left\|(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)+b_3\left(\theta-\Im^3\theta\right)+\cdots+b_u\left(\theta-\Im^u\theta\right)\right\|, \\
&\left\|(\theta-\Im\omega)+b_2\left(\theta-\Im^2\omega\right)+b_3\left(\theta-\Im^3\omega\right)+\cdots+b_u\left(\theta-\Im^u\omega\right)\right\|\Bigg) \\
= \; &\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^u}\right)\|\omega-\theta\| \\
&\geq \left(\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^u}\right)\|\omega-\theta\| \\
&= \sum_{j=1}^{u}\left(\frac{1}{3^j}\right)\|\omega-\theta\|.
\end{aligned}
$$

Hence, the inequality (2.1) is true and $\mathfrak{I}$ is an HEF-C mapping. Here, $\mathfrak{I}$ has a unique FP $\frac{3}{4} \in [0,\infty)$.

Example 2.5. Suppose that $\Omega = \mathbb{R}$ is a usual NS, $\Lambda = \left[-1, \frac{-1}{3}\right] \cup \left[1, \frac{1}{3}\right] \subseteq \Omega$, and $\mathfrak{I} : \Lambda \to \Lambda$ is a given mapping given by

$$
\mathfrak{I}\omega = \begin{cases} -\omega, & \text{if } \omega \in \left[-1, \frac{-1}{3}\right], \\ 1-\omega, & \text{if } \omega \in \left[\frac{1}{3}, 1\right]. \end{cases}
$$

Then, for $b_j = 1,\ j = 1,2,3,\cdots,u,\ u \geq 4,\ u \in \mathbb{N}$, and $\hbar(\varkappa,\varrho,\xi,\upsilon) = \frac{1}{5}(\xi+\upsilon)$, the mapping $\mathfrak{I}$ is an HEF'-C.

To illustrate this, without loss of the generality, we consider $\omega,\theta \in \Lambda$ with $\omega \leq \theta$. We have the following cases:

Case 1. For each $\omega,\theta \in \left[-1, \frac{-1}{3}\right]$ or $\omega,\theta \in \left[\frac{1}{3}, 1\right]$, the Definition 2.3 implies that

$$
\begin{aligned}
&\left\|b_1(\omega-\theta)+\mathfrak{I}\omega-\mathfrak{I}\theta+b_2\left(\mathfrak{I}^2\omega-\mathfrak{I}^2\theta\right)+b_3\left(\mathfrak{I}^3\omega-\mathfrak{I}^3\theta\right)+\cdots+b_u\left(\mathfrak{I}^u\omega-\mathfrak{I}^u\theta\right)\right\| \\
&= \|(\omega-\theta)+(\theta-\omega)+(\omega-\theta)+(\theta-\omega)+\cdots+(-1)^u(\omega-\theta)\| \\
&= \begin{cases} 0, & \text{if } u \text{ is odd}, \\ \|\omega-\theta\|, & \text{if } u \text{ is even}, \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\hbar\Bigg(\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\|, \\
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\theta-\omega)+(\omega-\mathfrak{I}\omega)+b_2\left(\omega-\mathfrak{I}^2\omega\right)+b_3\left(\omega-\mathfrak{I}^3\omega\right)+\cdots+b_u\left(\omega-\mathfrak{I}^u\omega\right)\right\|, \\
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\omega-\theta)+(\theta-\mathfrak{I}\theta)+b_2\left(\theta-\mathfrak{I}^2\theta\right)+b_3\left(\theta-\mathfrak{I}^3\theta\right)+\cdots+b_u\left(\theta-\mathfrak{I}^u\theta\right)\right\|, \\
&\left\|(\theta-\mathfrak{I}\omega)+b_2\left(\theta-\mathfrak{I}^2\omega\right)+b_3\left(\theta-\mathfrak{I}^3\omega\right)+\cdots+b_u\left(\theta-\mathfrak{I}^u\omega\right)\right\|\Bigg) \\
&= \begin{cases} \frac{1}{5}\left[\left\|(u+1)(\theta-\omega)+(u+1)\omega-\frac{u+1}{2}\right\|+\left\|(u+1)(\omega-\theta)+(u+1)\theta-\frac{u+1}{2}\right\|\right] & \text{if } u \text{ is odd}, \\ \frac{1}{5}\left[\left\|(u+1)(\theta-\omega)+u\omega-\frac{u}{2}\right\|+\left\|(u+1)(\omega-\theta)+u\theta-\frac{u}{2}\right\|\right] & \text{if } u \text{ is even}, \end{cases} \\
&\geq \begin{cases} \frac{(u+1)}{5}\|\theta-\omega\| & \text{if } u \text{ is odd}, \\ \frac{(u+2)}{5}\|\theta-\omega\| & \text{if } u \text{ is even}, \end{cases} \\
&\geq \begin{cases} \|\theta-\omega\| & \text{if } u \text{ is odd}, \\ \frac{6}{5}\|\theta-\omega\| & \text{if } u \text{ is even}. \end{cases}
\end{aligned}
$$

Case 2. For all $\omega \in \left[-1, \frac{-1}{3}\right]$ or $\theta \in \left[1, \frac{1}{3}\right]$, we get

$$\begin{aligned}
&\left\|b_1(\omega-\theta)+\Im\omega-\Im\theta+b_2\left(\Im^2\omega-\Im^2\theta\right)+b_3\left(\Im^3\omega-\Im^3\theta\right)+\cdots+b_u\left(\Im^u\omega-\Im^u\theta\right)\right\| \\
=\ &\|(\omega-\theta)+(\theta-\omega-1)+(1+\omega-\theta)+(\theta-\omega-1)+\cdots+(-1)^u(1+\omega-\theta)\| \\
=\ &\begin{cases} 1, & \text{if } u \text{ is odd,} \\ \|\omega-\theta\|, & \text{if } u \text{ is even,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\hbar\Bigg(\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\|, \\
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\theta-\omega)+(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)+b_3\left(\omega-\Im^3\omega\right)+\cdots+b_u\left(\omega-\Im^u\omega\right)\right\|, \\
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\omega-\theta)+(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)+b_3\left(\theta-\Im^3\theta\right)+\cdots+b_u\left(\theta-\Im^u\theta\right)\right\|, \\
&\left\|(\theta-\Im\omega)+b_2\left(\theta-\Im^2\omega\right)+b_3\left(\theta-\Im^3\omega\right)+\cdots+b_u\left(\theta-\Im^u\omega\right)\right\|\Bigg) \\
=\ &\begin{cases} \frac{1}{5}\left[\left\|(u+1)(\theta-\omega)+(u+1)\omega-\frac{u+1}{2}\right\|+\left\|(u+1)(\omega-\theta)+(u+1)\theta-\frac{u+1}{2}\right\|\right] & \text{if } u \text{ is odd,} \\ \frac{1}{5}\left[\left\|(u+1)(\theta-\omega)+u\omega-\frac{u}{2}\right\|+\left\|(u+1)(\omega-\theta)+u\theta-\frac{u}{2}\right\|\right] & \text{if } u \text{ is even,} \end{cases} \\
\geq\ &\begin{cases} \frac{(u+1)}{5}\|\theta-\omega\| & \text{if } u \text{ is odd,} \\ \frac{(u+2)}{5}\|\theta-\omega\| & \text{if } u \text{ is even,} \end{cases} \\
\geq\ &\begin{cases} \|\theta-\omega\| & \text{if } u \text{ is odd,} \\ \frac{6}{5}\|\theta-\omega\| & \text{if } u \text{ is even.} \end{cases}
\end{aligned}$$

Verifying the conditions in the aforementioned cases confirms the validity of (2.2). Therefore, $\Im$ qualifies as an HEF'-C and $\frac{1}{2} \in \left[\frac{1}{3}, 1\right]$ is a unique FP of $\Im$.

Remark 2.1. The weak ECM in [28] is obtained if we select $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa\varkappa$, $0 \leq \kappa < 1$, and $b_j = 0$, for $j = 3, 4, \cdots, u$ in Definition 2.2 or 2.3.

Hence, through the selection of suitable functions $\hbar$ and values b_j (for $j = 1, 2, \cdots, u$), we can derive modified weak enriched variants of the traditional contractions discussed, which, as far as we are aware, have not been explored previously.

Definition 2.4. Describe $\hbar \in F$ as $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa(\varrho + \xi)$, $0 \leq \kappa < \frac{1}{2}$ and $b_j = 0$, for $j = 3, 4, \cdots, u$ in Definition 2.2. Then, the mapping $\Im$ is called an enriched Kannan-contraction (EKC), that is, there are $b_1, b_2 > 0$ and $0 \leq \kappa < \frac{1}{2}$ such that

$$\begin{aligned}
&\left\|b_1(\omega-\theta)+\Im\omega-\Im\theta+b_2\left(\Im^2\omega-\Im^2\theta\right)\right\| \\
\leq\ &\kappa\left[\left\|(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)\right\|+\left\|(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)\right\|\right],
\end{aligned}$$

for all $\omega, \theta \in \Omega$.

Definition 2.5. Describe $\hbar \in F$ as $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa(\varrho + \xi)$, $0 \leq \kappa < \frac{1}{2}$ and $b_j = 0$, for $j = 3, 4, \cdots, u$ in Definition 2.3. Then, the mapping $\Im$ is called an enriched Chatterjea-contraction (ECC), that is, there are $b_1, b_2 > 0$ and $0 \leq \kappa < \frac{1}{2}$ such that

$$\begin{aligned} & \left\| b_1(\omega - \theta) + \Im\omega - \Im\theta + b_2\left(\Im^2\omega - \Im^2\theta\right) \right\| \\ \leq \ & \kappa \Big[\left\| (1 + b_1 + b_2)(\theta - \omega) + (\omega - \Im\omega)) + b_2\left(\omega - \Im^2\omega\right) \right\| \\ & + \left\| (1 + b_1 + b_2)(\omega - \theta) + (\theta - \Im\theta)) + b_2\left(\theta - \Im^2\theta\right) \right\| \Big] \end{aligned}$$

for all $\omega, \theta \in \Omega$.

Definition 2.6. Describe $\hbar \in F$ as $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa\varkappa + \mu(\varrho + \xi)$, $\kappa, \mu \geq 0$ with $\kappa + 2\mu < 1$. Set $b_j = 0$, for $j = 3, 4, \cdots, u$ in Definition 2.2. Then, the mapping $\Im$ is called an enriched Ćirić-Reich-Rus-contraction (ECRRC), that is, there are $b_1, b_2 > 0$ and $\kappa, \mu \geq 0$ with $\kappa + 2\mu < 1$ such that

$$\begin{aligned} & \left\| b_1(\omega - \theta) + \Im\omega - \Im\theta + b_2\left(\Im^2\omega - \Im^2\theta\right) \right\| \\ \leq \ & \kappa\|\omega - \theta\| + \mu\Big[\left\| (\omega - \Im\omega)) + b_2\left(\omega - \Im^2\omega\right) \right\| + \left\| (\theta - \Im\theta)) + b_2\left(\theta - \Im^2\theta\right) \right\| \Big] \end{aligned}$$

for all $\omega, \theta \in \Omega$.

Definition 2.7. Describe $\hbar \in F$ as $\hbar(\varkappa, \varrho, \xi, \upsilon) = \varrho^{\kappa}\xi^{1-\kappa}$, $0 < \kappa < 1$ and put $b_j = 0$, for $j = 3, 4, \cdots, u$ in Definition 2.2. Then, the mapping $\Im$ is called an enriched interpolative Kannan-contraction (EIKC), that is, there are $b_1, b_2 > 0$ and $0 < \kappa < 1$ such that

$$\begin{aligned} & \left\| b_1(\omega - \theta) + \Im\omega - \Im\theta + b_2\left(\Im^2\omega - \Im^2\theta\right) \right\| \\ \leq \ & \left\| (\omega - \Im\omega)) + b_2\left(\omega - \Im^2\omega\right) \right\|^{\kappa} \left\| (\theta - \Im\theta)) + b_2\left(\theta - \Im^2\theta\right) \right\|^{1-\kappa} \end{aligned}$$

for all $\omega, \theta \in \Omega$.

Definition 2.8. Describe $\hbar \in F$ as $\hbar(\varkappa, \varrho, \xi, \upsilon) = \varkappa^{\kappa}\varrho^{\mu}\xi^{1-\kappa-\mu}$, $0 < \kappa, \mu < 1$ and put $b_j = 0$, for $j = 3, 4, \cdots, u$ in Definition 2.2. Then, the mapping $\Im$ is called an enriched interpolative Ćirić–Reich–Rus-contraction (EICRRC), that is, there are $b_1, b_2 > 0$ and $0 < \kappa + \mu < 1$ with $\kappa + 2\mu < 1$ such that

$$\begin{aligned} & \left\| b_1(\omega - \theta) + \Im\omega - \Im\theta + b_2\left(\Im^2\omega - \Im^2\theta\right) \right\| \\ \leq \ & \|\omega - \theta\|^{\kappa} \left\| (\omega - \Im\omega)) + b_2\left(\omega - \Im^2\omega\right) \right\|^{\mu} \left\| (\theta - \Im\theta)) + b_2\left(\theta - \Im^2\theta\right) \right\|^{1-\kappa-\mu} \end{aligned}$$

for all $\omega, \theta \in \Omega$.

Remark 2.2. When b_2 is set to 0 in Definitions 2.2 and 2.4–2.8, we derive enriched adaptations of the ECM introduced by Berinde [15], Kanan [29], EIKC [30], and EICRRCs, respectively.

Let's revisit the definitions of well-posedness, the limit shadowing property of a mapping, and the Ulam-Hyers (UH) stability concerning the FP equation.

Assume that $\Im : \Omega \to \Omega$ is a mapping on an MS (Ω, d).

Definition 2.9. The FP issue $Fix(\Im)$ is deemed well-posed when $\Im$ possesses a unique FP ω^*, and for any sequence $\{\omega_m\}$ in Ω where $\lim_{m\to\infty} d(\omega_m, \Im\omega_m) = 0$, it follows that $\lim_{m\to\infty} d(\omega_m, \omega^*) = 0$.

Definition 2.10. The FP challenge $Fix(\Im)$ is considered to exhibit the limit shadowing property in Ω if for any sequence Ω where $\lim_{m\to\infty} d(\omega_m, \Im\omega_m) = 0$, there exists $\varphi \in \Omega$ such that $\lim_{m\to\infty}(\Im\omega_m, \omega_m) = 0$.

Definition 2.11. The FP equation $\omega = \Im\omega$ demonstrates UH stability if there exists a constant $\delta > 0$ such that for every $\epsilon > 0$ and each $\varpi^* \in \Omega$ where $d(\varpi^*, \Im\varpi^*) \leq \epsilon$, there exists $\omega^* \in \Omega$ satisfying $\Im\omega^* = \omega^*$ and $d(\omega^*, \varpi^*) \leq \delta\epsilon$.

We begin with the outcome concerning the existence and uniqueness of an FP for a u-fold AM associated with these two categories of hybrid enriched contractions within a BS context.

Theorem 2.1. *Let Ω be a BS and $\Im : \Omega \to \Omega$ be an HEF-C mapping. Then, there are $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \in (0, 1]$ such that the assertions below are true:*

(i) The m-fold AM $\widehat{\Im}$ associated with $\Im$ owns a unique FP;
(ii) KI described as $\omega_m = \widehat{\Im}\omega_{m-1}$, for any $\omega_0 \in \Omega$, i.e., for $m \in \mathbb{N}$, the sequence $\{\omega_m\}$ defined by

$$\begin{aligned}\omega_m &= (1 - \kappa_1 - \kappa_2 - \kappa_3 - \cdots - \kappa_u)\,\omega_{m-1} + \kappa_1\Im\omega_{m-1} + \kappa_2\Im^2\omega_{m-1} \\ &\quad + \kappa_3\Im^3\omega_{m-1} + \cdots + \kappa_u\Im^u\omega_{m-1}\end{aligned}$$

converges to a unique FP of $\widehat{\Im}$.

Proof. Since $\Im$ is an HEF-C, there exist $b_j \geq 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$ fulfilling the inequality (2.1). Consider $\kappa_1 = \frac{1}{\sum_{j=1}^{u} b_j + 1} > 0$ and $\kappa_s = \frac{b_s}{\sum_{j=1}^{u} b_j + 1} \geq 0$, $s = 2, 3, \cdots, u$. Then, the inequality (2.1) can be written as

$$\begin{aligned}&\left\|\left(\frac{1 - \kappa_2 - \kappa_3 - \cdots - \kappa_u}{\kappa_1} - 1\right)(\omega - \theta) + \Im\omega - \Im\theta + \frac{\kappa_2}{\kappa_1}\left(\Im^2\omega - \Im^2\theta\right)\right. \\ &\left. + \frac{\kappa_3}{\kappa_1}\left(\Im^3\omega - \Im^3\theta\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\Im^u\omega - \Im^u\theta\right)\right\| \\ \leq\ & \hbar\left(\frac{1}{\kappa_1}\|\omega - \theta\|,\right. \\ &\left\|(\omega - \Im\omega) + \frac{\kappa_2}{\kappa_1}\left(\omega - \Im^2\omega\right) + \frac{\kappa_3}{\kappa_1}\left(\omega - \Im^3\omega\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\omega - \Im^u\omega\right)\right\|, \\ &\left\|(\theta - \Im\theta) + \frac{\kappa_2}{\kappa_1}\left(\theta - \Im^2\theta\right) + \frac{\kappa_3}{\kappa_1}\left(\theta - \Im^3\theta\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\theta - \Im^u\theta\right)\right\|, \\ &\left.\left\|(\theta - \Im\omega) + \frac{\kappa_2}{\kappa_1}\left(\theta - \Im^2\omega\right) + \frac{\kappa_3}{\kappa_1}\left(\theta - \Im^3\omega\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\theta - \Im^u\omega\right)\right\|\right),\end{aligned}$$

for $\omega, \theta \in \Omega$. Because $\kappa_1 > 0$ and $(\hbar_3)$ holds, we have

$$\begin{aligned}&\left\|(1 - \kappa_1 - \kappa_2 - \kappa_3 - \cdots - \kappa_u)(\omega - \theta) + \kappa_1(\Im\omega - \Im\theta) + \kappa_2\left(\Im^2\omega - \Im^2\theta\right)\right. \\ &\left. + \kappa_3\left(\Im^3\omega - \Im^3\theta\right) + \cdots + \kappa_u\left(\Im^u\omega - \Im^u\theta\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq \kappa_1 \hbar \left(\frac{1}{\kappa_1} \|\omega - \theta\|, \right. \\
&\quad \left\| (\omega - \Im\omega) + \frac{\kappa_2}{\kappa_1}\left(\omega - \Im^2\omega\right) + \frac{\kappa_3}{\kappa_1}\left(\omega - \Im^3\omega\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\omega - \Im^u\omega\right) \right\|, \\
&\quad \left\| (\theta - \Im\theta) + \frac{\kappa_2}{\kappa_1}\left(\theta - \Im^2\theta\right) + \frac{\kappa_3}{\kappa_1}\left(\theta - \Im^3\theta\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\theta - \Im^u\theta\right) \right\|, \\
&\quad \left. \left\| (\theta - \Im\theta) + \frac{\kappa_2}{\kappa_1}\left(\theta - \Im^2\theta\right) + \frac{\kappa_3}{\kappa_1}\left(\theta - \Im^3\theta\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\theta - \Im^u\theta\right) \right\| \right) \\
&\leq \hbar \left(\|\omega - \theta\|, \right. \\
&\quad \left\| \kappa_1 (\omega - \Im\omega) + \kappa_2\left(\omega - \Im^2\omega\right) + \kappa_3\left(\omega - \Im^3\omega\right) + \cdots + \kappa_u\left(\omega - \Im^u\omega\right) \right\|, \\
&\quad \left\| \kappa_1 (\theta - \Im\theta) + \kappa_2\left(\theta - \Im^2\theta\right) + \kappa_3\left(\theta - \Im^3\theta\right) + \cdots + \kappa_u\left(\theta - \Im^u\theta\right) \right\|, \\
&\quad \left. \left\| \kappa_1 (\theta - \Im\omega) + \kappa_2\left(\theta - \Im^2\omega\right) + \kappa_3\left(\theta - \Im^3\omega\right) + \cdots + \kappa_u\left(\theta - \Im^u\omega\right) \right\| \right).
\end{aligned}
$$

This coupled with Definition 2.1, signifies that for $\omega, \theta \in \Omega$,

$$
\left\|\widehat{\Im}\omega - \widehat{\Im}\theta\right\| \leq \hbar\left(\|\omega - \theta\|, \left\|\omega - \widehat{\Im}\omega\right\|, \left\|\theta - \widehat{\Im}\theta\right\|, \left\|\theta - \widehat{\Im}\omega\right\|\right). \tag{2.3}
$$

Assume that $\omega_0 \in \Omega$ is an arbitrary element and describe the sequence $\{\omega_m\}_{m\in\mathbb{N}}$ as $\omega_m = \widehat{\Im}^m\omega_0$ for $m \geq 1$. Setting $\omega = \omega_m$ and $\theta = \omega_{m-1}$ in (2.3), and using ($\hbar_4$), one can write

$$
\begin{aligned}
\|\omega_{m+1} - \omega_m\| &\leq \hbar\left(\|\omega_m - \omega_{m-1}\|, \|\omega_m - \omega_{m+1}\|, \|\omega_{m-1} - \omega_m\|, \|\omega_{m-1} - \omega_{m+1}\|\right) \\
&\leq \hbar\begin{pmatrix} \|\omega_m - \omega_{m-1}\|, \|\omega_m - \omega_{m+1}\|, \|\omega_{m-1} - \omega_m\|, \\ \|\omega_{m-1} - \omega_m\| + \|\omega_m - \omega_{m+1}\| \end{pmatrix}.
\end{aligned}
$$

By the condition ($\hbar_2$), there is $\zeta \in [0, 1)$ such that

$$
\|\omega_{m+1} - \omega_m\| \leq \zeta \|\omega_m - \omega_{m-1}\|.
$$

Through iterating this procedure, we deduce that

$$
\|\omega_{m+1} - \omega_m\| \leq \zeta^m \|\omega_1 - \omega_0\|.
$$

Next, for $j, m \geq 1$, one has

$$
\begin{aligned}
\left\|\omega_{m+j} - \omega_m\right\| &\leq \left\|\omega_{m+j} - \omega_{m+j-1}\right\| + \left\|\omega_{m+j-1} - \omega_{m+j-2}\right\| + \cdots + \|\omega_{m+1} - \omega_m\| \\
&\leq \left(\zeta^{m+j-1} + \zeta^{m+j-2} + \cdots + \zeta^m\right)\|\omega_1 - \omega_0\| \\
&= \frac{\zeta^m\left(1 - \zeta^j\right)}{1 - \zeta}\|\omega_1 - \omega_0\|,
\end{aligned}
$$

which implies that the sequence $\{\omega_m\}$ is a Cauchy sequence in Ω. Thus, there is $\omega^* \in \Omega$ such that $\lim_{m\to\infty} \omega_m = \omega^*$.

Now, setting $\omega = \omega^*$ and $\theta = \omega_m$ in (2.3), we can write

$$
\left\|\widehat{\Im}\omega^* - \widehat{\Im}\omega_m\right\| \leq \hbar\left(\|\omega^* - \omega_m\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \left\|\omega_m - \widehat{\Im}\omega_m\right\|, \left\|\omega_m - \widehat{\Im}\omega^*\right\|\right). \tag{2.4}
$$

Letting $m \to \infty$ in (2.4), we have

$$\left\|\widehat{\Im}\omega^* - \omega^*\right\| \leq \hbar\left(\|\omega^* - \omega^*\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \|\omega^* - \omega^*\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|\right).$$

From the conditions ($\hbar_1$) and ($\hbar_2$), we have

$$\begin{aligned}\left\|\widehat{\Im}\omega^* - \omega^*\right\| &\leq \hbar\left(\|\omega^* - \omega^*\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \|\omega^* - \omega^*\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|\right)\\ &\leq \zeta\|\omega^* - \omega^*\| = 0.\end{aligned}$$

Thus, $\widehat{\Im}\omega^* = \omega^*$. For the uniqueness, assume that η_1 and η_2 are distinct FPs of $\widehat{\Im}$. Putting $\omega = \eta_1$ and $\theta = \eta_2$ in (2.3), we get

$$\begin{aligned}\|\eta_1 - \eta_2\| &= \left\|\widehat{\Im}\eta_1 - \widehat{\Im}\eta_2\right\|\\ &\leq \hbar\left(\|\eta_1 - \eta_2\|, \left\|\eta_1 - \Im'\eta_1\right\|, \left\|\eta_2 - \Im'\eta_2\right\|, \left\|\eta_2 - \Im'\eta_1\right\|\right)\\ &= \hbar\left(\|\eta_1 - \eta_2\|, \|\eta_1 - \eta_1\|, \|\eta_2 - \eta_2\|, \|\eta_2 - \eta_1\|\right)\\ &= \hbar\left(\|\eta_1 - \eta_2\|, 0, 0, \|\eta_2 - \eta_1\|\right)\\ &\leq \zeta.0 = 0,\end{aligned}$$

which implies that $\eta_1 = \eta_2$. This completes the proof. □

Theorem 2.2. *Let Ω be a BS and $\Im : \Omega \to \Omega$ be an HEF′-C. Then, there exist $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \kappa_j \in (0, 1]$ such that the following assertions hold:*

(i) The m-fold AM $\widehat{\Im}$ associated with $\Im$ possesses a unique FP;
(ii) KI defined by $\omega_m = \widehat{\Im}\omega_{m-1}$, for any $\omega_0 \in \Omega$, converges to a unique FP of $\widehat{\Im}$.

Proof. Because $\Im$ is an HEF'-C, there exist $b_j \geq 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$ justifying the inequality (2.2). Assume that $\kappa_1 = \frac{1}{\sum_{j=1}^{u} b_j + 1} > 0$ and $\kappa_s = \frac{b_s}{\sum_{j=1}^{u} b_j + 1} \geq 0$, $s = 2, 3, \cdots, u$. Then, the inequality (2.1) takes the form

$$\begin{aligned}&\left\|\left(\frac{1 - \kappa_2 - \kappa_3 - \cdots - \kappa_u}{\kappa_1} - 1\right)(\omega - \theta) + \Im\omega - \Im\theta + \frac{\kappa_2}{\kappa_1}\left(\Im^2\omega - \Im^2\theta\right)\right.\\ &\left. + \frac{\kappa_3}{\kappa_1}\left(\Im^3\omega - \Im^3\theta\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\Im^u\omega - \Im^u\theta\right)\right\|\\ \leq\ & \hbar\left(\frac{1}{\kappa_1}\|\omega - \theta\|,\right.\\ &\left\|\frac{1}{\kappa_1}(\theta - \omega) + (\omega - \Im\omega) + \frac{\kappa_2}{\kappa_1}\left(\omega - \Im^2\omega\right) + \frac{\kappa_3}{\kappa_1}\left(\omega - \Im^3\omega\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\omega - \Im^u\omega\right)\right\|,\\ &\left\|\frac{1}{\kappa_1}(\omega - \theta) + (\theta - \Im\theta) + \frac{\kappa_2}{\kappa_1}\left(\theta - \Im^2\theta\right) + \frac{\kappa_3}{\kappa_1}\left(\theta - \Im^3\theta\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\theta - \Im^u\theta\right)\right\|,\\ &\left.\left\|(\theta - \Im\omega) + \frac{\kappa_2}{\kappa_1}\left(\theta - \Im^2\omega\right) + \frac{\kappa_3}{\kappa_1}\left(\theta - \Im^3\omega\right) + \cdots + \frac{\kappa_u}{\kappa_1}\left(\theta - \Im^u\omega\right)\right\|\right),\end{aligned}$$

for $\omega, \theta \in \Omega$. As $\kappa_1 > 0$ and $(\hbar_3')$ holds, we get

$$\begin{aligned}
&\left\|(1-\kappa_1-\kappa_2-\kappa_3-\cdots-\kappa_u)(\omega-\theta)+\kappa_1(\Im\omega-\Im)\theta+\kappa_2\left(\Im^2\omega-\Im^2\theta\right)\right. \\
&\left.+\kappa_3\left(\Im^3\omega-\Im^3\theta\right)+\cdots+\kappa_u\left(\Im^u\omega-\Im^u\theta\right)\right\| \\
\leq\ & \kappa_1\hbar\left(\frac{1}{\kappa_1}\|\omega-\theta\|,\right. \\
&\left\|\frac{1}{\kappa_1}(\theta-\omega)+(\omega-\Im\omega)+\frac{\kappa_2}{\kappa_1}\left(\omega-\Im^2\omega\right)+\frac{\kappa_3}{\kappa_1}\left(\omega-\Im^3\omega\right)+\cdots+\frac{\kappa_u}{\kappa_1}(\omega-\Im^u\omega)\right\|, \\
&\left\|\frac{1}{\kappa_1}(\omega-\theta)+(\theta-\Im\theta)+\frac{\kappa_2}{\kappa_1}\left(\theta-\Im^2\theta\right)+\frac{\kappa_3}{\kappa_1}\left(\theta-\Im^3\theta\right)+\cdots+\frac{\kappa_u}{\kappa_1}(\theta-\Im^u\theta)\right\|, \\
&\left.\left\|(\theta-\Im\omega)+\frac{\kappa_2}{\kappa_1}\left(\theta-\Im^2\omega\right)+\frac{\kappa_3}{\kappa_1}\left(\theta-\Im^3\omega\right)+\cdots+\frac{\kappa_u}{\kappa_1}(\theta-\Im^u\omega)\right\|\right), \\
\leq\ & \hbar\left(\|\omega-\theta\|,\right. \\
&\left\|(\theta-\omega)+\kappa_1(\omega-\Im\omega)+\kappa_2\left(\omega-\Im^2\omega\right)+\kappa_3\left(\omega-\Im^3\omega\right)+\cdots+\kappa_u(\omega-\Im^u\omega)\right\|, \\
&\left\|(\omega-\theta)+\kappa_1(\theta-\Im\theta)+\kappa_2\left(\theta-\Im^2\theta\right)+\kappa_3\left(\theta-\Im^3\theta\right)+\cdots+\kappa_u(\theta-\Im^u\theta)\right\|, \\
&\left.\left\|\kappa_1(\theta-\Im\omega)+\kappa_2\left(\theta-\Im^2\omega\right)+\kappa_3\left(\theta-\Im^3\omega\right)+\cdots+\kappa_u(\theta-\Im^u\omega)\right\|\right).
\end{aligned}$$

This coupled with Definition 2.1, signifies that for $\omega, \theta \in \Omega$,

$$\left\|\widehat{\Im}\omega-\widehat{\Im}\theta\right\| \leq \hbar\left(\|\omega-\theta\|,\left\|\theta-\widehat{\Im}\omega\right\|,\left\|\omega-\widehat{\Im}\theta\right\|,\left\|\theta-\widehat{\Im}\omega\right\|\right). \tag{2.5}$$

Let $\omega_0 \in \Omega$ be an arbitrary element and define the sequence $\{\omega_m\}_{m\in\mathbb{N}}$ as $\omega_m = \widehat{\Im}^m\omega_0$ for $m \geq 1$. Putting $\omega = \omega_m$ and $\theta = \omega_{m-1}$ in (2.5), and using $(\hbar_4')$, we can write

$$\begin{aligned}
\|\omega_{m+1}-\omega_m\| &\leq \hbar\left(\|\omega_m-\omega_{m-1}\|,\|\omega_{m-1}-\omega_{m+1}\|,\|\omega_m-\omega_m\|,\|\omega_{m-1}-\omega_{m+1}\|\right) \\
&\leq \hbar\begin{pmatrix} \|\omega_m-\omega_{m-1}\|,\|\omega_{m-1}-\omega_m\|+\|\omega_m-\omega_{m+1}\|,0, \\ \|\omega_{m-1}-\omega_m\|+\|\omega_m-\omega_{m+1}\| \end{pmatrix}.
\end{aligned}$$

By the condition $(\hbar_2')$, there is $\zeta \in [0,1)$ such that

$$\|\omega_{m+1}-\omega_m\| \leq \zeta\|\omega_{m-1}-\omega_m\|.$$

Repeating this process, we have

$$\|\omega_{m+1}-\omega_m\| \leq \zeta^m\|\omega_1-\omega_0\|.$$

Next, for $j, m \geq 1$, one has

$$\begin{aligned}
\left\|\omega_{m+j}-\omega_m\right\| &\leq \left\|\omega_{m+j}-\omega_{m+j-1}\right\|+\left\|\omega_{m+j-1}-\omega_{m+j-2}\right\|+\cdots+\|\omega_{m+1}-\omega_m\| \\
&\leq \left(\zeta^{m+j-1}+\zeta^{m+j-2}+\cdots+\zeta^m\right)\|\omega_1-\omega_0\| \\
&= \frac{\zeta^m\left(1-\zeta^j\right)}{1-\zeta}\|\omega_1-\omega_0\|,
\end{aligned}$$

which implies that the sequence $\{\omega_m\}$ is a Cauchy sequence in Ω. Thus, there is $\omega^* \in \Omega$ such that $\lim_{m\to\infty} \omega_m = \omega^*$.

Now, setting $\omega = \omega^*$ and $\theta = \omega_m$ in (2.5), we can write

$$\left\|\widehat{\Im}\omega^* - \widehat{\Im}\omega_m\right\| \le \hbar\left(\|\omega^* - \omega_m\|, \left\|\omega_m - \widehat{\Im}\omega^*\right\|, \left\|\omega_m - \widehat{\Im}\omega^*\right\|, \left\|\omega_m - \widehat{\Im}\omega^*\right\|\right). \tag{2.6}$$

When $m \to \infty$ in (2.6), we have

$$\left\|\widehat{\Im}\omega^* - \omega^*\right\| \le \hbar\left(\|\omega^* - \omega^*\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|\right).$$

From the conditions ($\hbar_1'$) and ($\hbar_2'$), we get

$$\begin{aligned}\left\|\widehat{\Im}\omega^* - \omega^*\right\| &\le \hbar\left(\|\omega^* - \omega^*\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|, \left\|\omega^* - \widehat{\Im}\omega^*\right\|\right)\\ &\le \zeta\,\|\omega^* - \omega^*\| = 0.\end{aligned}$$

and $\widehat{\Im}\omega^* = \omega^*$.

Finally, assume that η_1 and η_2 are distinct FPs of $\Im'$. Putting $\omega = \eta_1$ and $\theta = \eta_2$ in (2.5), we get

$$\begin{aligned}\|\eta_1 - \eta_2\| &= \left\|\widehat{\Im}\eta_1 - \widehat{\Im}\eta_2\right\|\\ &\le \hbar\left(\|\eta_1 - \eta_2\|, \left\|\eta_2 - \Im'\eta_1\right\|, \left\|\eta_1 - \Im'\eta_2\right\|, \left\|\eta_2 - \Im'\eta_1\right\|\right)\\ &= \hbar\left(\|\eta_1 - \eta_2\|, \|\eta_2 - \eta_1\|, \|\eta_1 - \eta_2\|, \|\eta_2 - \eta_1\|\right)\end{aligned}$$

By ($\hbar_5'$), we deduce that $\|\eta_1 - \eta_2\| = 0$. Thus, $\widehat{\Im}$ has a unique FP. □

Remark 2.3. In Theorems 2.1 and 2.2, if we take $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa\varkappa$, $\kappa \in [0, 1)$, and $b_j = 0, j = 3, 4, \cdots u$, we have Theorem 2.3 in [28].

Corollary 2.1. *Let Ω be a BS and $\Im : \Omega \to \Omega$ is an EKC (or ECRRC, ECC, EIKC, EICRRC). Then there are $\kappa_1, \kappa_2 > 0$ with $\kappa_1 + \kappa_2 \in (0, 1]$ such that the assertions below are true:*

(i) The 2–fold AM $\Im_{\kappa_1,\kappa_2}$ owns a unique FP;

(ii) KI $\{\omega_m\}$ defined by $\omega_m = \Im_{\kappa_1,\kappa_2}\omega_{m-1}$, for any $\omega_0 \in \Omega$, that is, the sequence $\{\omega_m\}$ described as

$$\omega_m = (1 - \kappa_1 - \kappa_2)\,\omega_{m-1} + \kappa_1\Im\omega_{m-1} + \kappa_2\Im^2\omega_{m-1},\ m \in \mathbb{N}$$

converges to a unique FP of $\Im_{\kappa_1,\kappa_2}$.

Proof. The proof can be simplified as follows:

- Choosing $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa(\varrho + \xi)$, where $\kappa \in [0, \frac{1}{2})$ in Theorem 2.1, we have the FP theorems for EKC.
- Selecting $\hbar(\varkappa, \varrho, \xi, \upsilon) = \rho\varkappa + \sigma(\varrho + \xi)$, $\rho, \sigma \in [0, 1)$ with $\rho + 2\sigma < 1$ in Theorem 2.1, we have the FP theorems for ECRRC.
- Taking $\hbar(\varkappa, \varrho, \xi, \upsilon) = \varkappa^{\kappa}\xi^{1-\kappa}$, where $\kappa \in (0, 1)$ in Theorem 2.1, we have the FP theorems for EIKC.
- Putting $\hbar(\varkappa, \varrho, \xi, \upsilon) = \varkappa^{\kappa}\varrho^{\lambda}\xi^{1-\kappa-\lambda}$, where $\kappa, \lambda \in (0, 1)$ with $\kappa + \lambda < 1$ in Theorem 2.1, we have the FP theorems for EICRRC.

- Setting $\hbar(\varkappa, \varrho, \xi, \upsilon) = \kappa(\varrho + \xi)$, where $\kappa \in [0, \frac{1}{2})$ in Theorem 2.2, we have the FP theorems for ECC.

□

Remark 2.4. In Corollary 2.1, if we put $\kappa_2 = 0$, we get the FP theorems corresponding to EKC, ECC, ECRRC, EIKC, and EICRRC in [30–32].

Next, we require the subsequent definitions and notations:

Definition 2.12. [33] Assume that $(\Omega, \|.\|)$ is an NS and $\Im : \Omega \to \Omega$ is a given mapping. The diameter of a set B, represented as $\phi[B]$, is described as $\{\sup \|\omega - \theta\| : \omega, \theta \in B\}$, where B is a bounded subset of Ω.

An NS $(\Omega, \|.\|)$ is termed as $\Im-$orbital BS if every Cauchy sequence within $Q(\Im, \omega, \infty)$ for a given $\omega \in \Omega$ converges in Ω.

We will now demonstrate the lemmas below for the category of HEF-Cs (or HEF'-Cs).

Lemma 2.1. *Let $(\Omega, \|.\|)$ be an NS and $\Im : \Omega \to \Omega$ be an HEF-C mapping (or HEF′-C mapping). Assume that the following statements hold:*

(S) *For each HEF-C, there is $\delta \in [0, 1)$ such that*

$$
\begin{aligned}
&\hbar\Bigg(\left(\sum_{j=1}^{u} b_j + 1\right)\|\omega - \theta\|, \\
&\left\|(\omega - \Im\omega) + b_2\left(\omega - \Im^2\omega\right) + b_3\left(\omega - \Im^3\omega\right) + \cdots + b_u\left(\omega - \Im^u\omega\right)\right\|, \\
&\left\|(\theta - \Im\theta) + b_2\left(\theta - \Im^2\theta\right) + b_3\left(\theta - \Im^3\theta\right) + \cdots + b_u\left(\theta - \Im^u\theta\right)\right\|, \\
&\left\|(\theta - \Im\omega) + b_2\left(\theta - \Im^2\omega\right) + b_3\left(\theta - \Im^3\omega\right) + \cdots + b_u\left(\theta - \Im^u\omega\right)\right\|\Bigg) \\
\leq\ & \delta \max\Bigg\{\left(\sum_{j=1}^{u} b_j + 1\right)\|\omega - \theta\|, \\
&\left\|(\omega - \Im\omega) + b_2\left(\omega - \Im^2\omega\right) + b_3\left(\omega - \Im^3\omega\right) + \cdots + b_u\left(\omega - \Im^u\omega\right)\right\|, \\
&\left\|(\theta - \Im\theta) + b_2\left(\theta - \Im^2\theta\right) + b_3\left(\theta - \Im^3\theta\right) + \cdots + b_u\left(\theta - \Im^u\theta\right)\right\|, \\
&\left\|\left(\sum_{j=1}^{u} b_j + 1\right)(\theta - \omega) + (\omega - \Im\omega) + b_2\left(\omega - \Im^2\omega\right) + b_3\left(\omega - \Im^3\omega\right) + \cdots + b_u\left(\omega - \Im^u\omega\right)\right\|, \\
&\left\|\left(\sum_{j=1}^{u} b_j + 1\right)(\omega - \theta) + (\theta - \Im\theta) + b_2\left(\theta - \Im^2\theta\right) + b_3\left(\theta - \Im^3\theta\right) + \cdots + b_u\left(\theta - \Im^u\theta\right)\right\|\Bigg\}
\end{aligned}
$$

or

(S') *for each HEF-C, there is $\delta \in [0, 1)$ such that*

$$
\hbar\Bigg(\left(\sum_{j=1}^{u} b_j + 1\right)\|\omega - \theta\|,
$$

$$
\begin{aligned}
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\theta-\omega)+(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)+b_3\left(\omega-\Im^3\omega\right)+\cdots+b_u\left(\omega-\Im^u\omega\right)\right\|,\\
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\omega-\theta)+(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)+b_3\left(\theta-\Im^3\theta\right)+\cdots+b_u\left(\theta-\Im^u\theta\right)\right\|,\\
&\left\|(\theta-\Im\omega)+b_2\left(\theta-\Im^2\omega\right)+b_3\left(\theta-\Im^3\omega\right)+\cdots+b_u\left(\theta-\Im^u\omega\right)\right\|\Bigg)\\
\leq\ & \delta\max\left\{\left(\sum_{j=1}^{u} b_j+1\right)\|\omega-\theta\|,\right.\\
&\left\|(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)+b_3\left(\omega-\Im^3\omega\right)+\cdots+b_u\left(\omega-\Im^u\omega\right)\right\|,\\
&\left\|(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)+b_3\left(\theta-\Im^3\theta\right)+\cdots+b_u\left(\theta-\Im^u\theta\right)\right\|,\\
&\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\theta-\omega)+(\omega-\Im\omega)+b_2\left(\omega-\Im^2\omega\right)+b_3\left(\omega-\Im^3\omega\right)+\cdots+b_u\left(\omega-\Im^u\omega\right)\right\|,\\
&\left.\left\|\left(\sum_{j=1}^{u} b_j+1\right)(\omega-\theta)+(\theta-\Im\theta)+b_2\left(\theta-\Im^2\theta\right)+b_3\left(\theta-\Im^3\theta\right)+\cdots+b_u\left(\theta-\Im^u\theta\right)\right\|\right\}
\end{aligned}
$$

for all $\omega, \theta \in \Omega$, $b_j \in (0, \infty)$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$.

Then, there exist $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, *with* $\sum_{j=1}^{u} \kappa_j \in (0, 1]$ *so that for each* $\omega \in \Omega$ *and for all* $r, l \in \{1, 2, 3, \cdots, m\}$ *for a positive integer m, we have*

$$\left\|\widehat{\Im}^r\omega - \widehat{\Im}^l\omega\right\| \leq \delta\phi\left[Q\left(\Im', \omega, m\right)\right],$$

where $\widehat{\Im}$ *is the u-fold AM linked to an HEF-C (or HEF′-C).*

Proof. As $\Im$ is a HEF-C, there is $b_j \in (0, \infty)$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, fulfilling the inequality (1.1). Consider $\kappa_1 = \frac{1}{\sum_{j=1}^{u} b_j+1} > 0$ and $\kappa_s = \frac{b_s}{\sum_{j=1}^{u} b_j+1} \geq 0$, $s = 2, 3, \cdots, u$. Then, the inequality (2.1) takes the form

$$
\begin{aligned}
&\left\|\left(\frac{1-\kappa_2-\kappa_3-\cdots-\kappa_u}{\kappa_1}-1\right)(\omega-\theta)+\Im\omega-\Im\theta+\frac{\kappa_2}{\kappa_1}\left(\Im^2\omega-\Im^2\theta\right)\right.\\
&\left.+\frac{\kappa_3}{\kappa_1}\left(\Im^3\omega-\Im^3\theta\right)+\cdots+\frac{\kappa_u}{\kappa_1}\left(\Im^u\omega-\Im^u\theta\right)\right\|\\
\leq\ & \hbar\left(\frac{1}{\kappa_1}\|\omega-\theta\|,\right.\\
&\left\|(\omega-\Im\omega)+\frac{\kappa_2}{\kappa_1}\left(\omega-\Im^2\omega\right)+\frac{\kappa_3}{\kappa_1}\left(\omega-\Im^3\omega\right)+\cdots+\frac{\kappa_u}{\kappa_1}\left(\omega-\Im^u\omega\right)\right\|,\\
&\left\|(\theta-\Im\theta)+\frac{\kappa_2}{\kappa_1}\left(\theta-\Im^2\theta\right)+\frac{\kappa_3}{\kappa_1}\left(\theta-\Im^3\theta\right)+\cdots+\frac{\kappa_u}{\kappa_1}\left(\theta-\Im^u\theta\right)\right\|,\\
&\left.\left\|(\theta-\Im\omega)+\frac{\kappa_2}{\kappa_1}\left(\theta-\Im^2\omega\right)+\frac{\kappa_3}{\kappa_1}\left(\theta-\Im^3\omega\right)+\cdots+\frac{\kappa_u}{\kappa_1}\left(\theta-\Im^u\omega\right)\right\|\right).
\end{aligned}
$$

With the help of Assertion (S), the above inequality reduces to

$$\begin{aligned}\left\|\widehat{\Im}\omega-\widehat{\Im}\theta\right\| &\leq \hbar\left(\|\omega-\theta\|,\left\|\omega-\widehat{\Im}\omega\right\|,\left\|\theta-\widehat{\Im}\theta\right\|,\left\|\theta-\widehat{\Im}\omega\right\|\right)\\ &\leq c\max\left\{\|\omega-\theta\|,\left\|\omega-\widehat{\Im}\omega\right\|,\left\|\theta-\widehat{\Im}\theta\right\|,\left\|\theta-\widehat{\Im}\omega\right\|,\left\|\omega-\widehat{\Im}\theta\right\|\right\}. \end{aligned} \tag{2.7}$$

For a fixed positive integer m, assume that $\omega\in\Omega$ is an arbitrary point. From (2.7), we get

$$\begin{aligned}\left\|\widehat{\Im}^r\omega-\widehat{\Im}^l\omega\right\| &= \left\|\widehat{\Im}\widehat{\Im}^{r-1}\omega-\widehat{\Im}\widehat{\Im}^{l-1}\omega\right\|\\ &\leq c\max\left\{\begin{array}{c}\left\|\widehat{\Im}^{r-1}\omega-\widehat{\Im}^{l-1}\omega\right\|,\left\|\widehat{\Im}^{r-1}\omega-\widehat{\Im}^{r}\omega\right\|,\left\|\widehat{\Im}^{l-1}\omega-\widehat{\Im}^{l}\omega\right\|,\\ \left\|\widehat{\Im}^{l-1}\omega-\widehat{\Im}^{r}\omega\right\|,\left\|\widehat{\Im}^{r-1}\omega-\widehat{\Im}^{l}\omega\right\|\end{array}\right\},\end{aligned}$$

which yields

$$\left\|\widehat{\Im}^r\omega-\widehat{\Im}^l\omega\right\|\leq\delta\phi\left[Q(\widehat{\Im},\omega,m)\right].$$

A comparable conclusion for HEF'-C with Assertion (S') can be reached by employing reasoning akin to the ones mentioned earlier. □

Remark 2.5. Based on Lemma 2.1, if $\Im$ is an HEF-C (or HEF'-C) and $\omega\in\Omega$, then for any positive integer m, there exists $s\leq m$ such that

$$\left\|\omega-\widehat{\Im}^s\omega\right\|=\phi\left[Q(\widehat{\Im},\omega,m)\right].$$

Lemma 2.2. *Let* $(\Omega,\|.\|)$ *be an NS and* $\Im:\Omega\to\Omega$ *be an HEF-C (or HEF′-C). For a positive integer m, assume that there exists* $\delta\in[0,1)$ *such that Assertion* (S) *(or* (S')*) is verified. Then, there are* $\kappa_j>0$, $j=1,2,3,\cdots,u$, $u\geq 4$, $u\in\mathbb{N}$, *with* $\sum_{j=1}^{u}\kappa_j\in(0,1]$ *so that*

$$\phi\left[Q\left(\widehat{\Im},\omega,\infty\right)\right]\leq\frac{1}{1-\delta}\left\|\omega-\widehat{\Im}\omega\right\|,\ \textit{for all } \omega\in\Omega,$$

where $\widehat{\Im}$ *is the u-fold AM linked to a HEF-C (or HEF′-C).*

Proof. Because $\Im$ is an HEF-C, there is $b_j\in(0,\infty)$, $j=1,2,3,\cdots,u$, $u\geq 4$, $u\in\mathbb{N}$, fulfilling the inequality (1.1). Consider $\kappa_1=\frac{1}{\sum_{j=1}^{u}b_j+1}>0$ and $\kappa_s=\frac{b_s}{\sum_{j=1}^{u}b_j+1}\geq 0$, $s=2,3,\cdots,u$.

Assume that $\omega\in\Omega$ is an arbitrary element. Since the sequence $\left\{\phi\left[Q(\widehat{\Im},\omega,m)\right]\right\}$ is increasing, we get

$$\phi\left[Q(\widehat{\Im},\omega,\infty)\right]=\sup\left\{\phi\left[Q(\widehat{\Im},\omega,m)\right]:m\in\mathbb{N}\right\}.$$

Then (2.7) is fulfilled if we prove that

$$\phi\left[Q\left(\widehat{\Im},\omega,m\right)\right]\leq\frac{1}{1-\delta}\left\|\omega-\widehat{\Im}\omega\right\|,\ m\in\mathbb{N}.$$

Assume that m is a positive integer. Utilizing Remark 2.5 there is $\widehat{\Im}^s\omega\in Q\left(\widehat{\Im},\omega,m\right)$, where $s\in[1,m]$ in order that

$$\left\|\omega-\widehat{\Im}^s\omega\right\|=\phi\left[Q(\widehat{\Im},\omega,m)\right].$$

It follows from the triangle inequality and Lemma 2.1 that

$$\begin{aligned}\left\|\omega-\widehat{\mathfrak{I}}^{s}\omega\right\| &\leq \left\|\omega-\widehat{\mathfrak{I}}\omega\right\|+\left\|\widehat{\mathfrak{I}}\omega-\widehat{\mathfrak{I}}^{s}\omega\right\| \\ &\leq \left\|\omega-\widehat{\mathfrak{I}}\omega\right\|+\delta\phi\left[Q(\widehat{\mathfrak{I}},\omega,m)\right] \\ &= \left\|\omega-\widehat{\mathfrak{I}}\omega\right\|+\delta\left\|\omega-\widehat{\mathfrak{I}}^{s}\omega\right\|.\end{aligned}$$

Hence,

$$\phi\left[Q(\widehat{\mathfrak{I}},\omega,m)\right]=\left\|\omega-\widehat{\mathfrak{I}}^{s}\omega\right\|\leq\frac{1}{1-\delta}\left\|\omega-\widehat{\mathfrak{I}}\omega\right\|,\ \text{for all } m\in\mathbb{N}.$$

A comparable conclusion for HEF'-C with Assertion (S') can be reached by employing reasoning akin to the ones mentioned earlier. □

Theorem 2.3. *Let $\mathfrak{I}$ be an HEF′-C (or HEF′-C) on an NS $(\Omega,\|.\|)$. For a positive integer m, assume that there exists $\delta\in[0,1)$ such that Assertion (S) (or (S')) is satisfied. Then, there are $\kappa_j>0$, $j=1,2,3,\cdots,u$, $u\geq 4$, $u\in\mathbb{N}$, with $\sum_{j=1}^{u}\kappa_j\in(0,1]$ so that the assumptions below hold, provided that Ω is a $\widehat{\mathfrak{I}}$−orbital BS:*

(i) The m-fold AM $\widehat{\mathfrak{I}}$ associated with $\mathfrak{I}$ has a unique FP;
(ii) KI defined by $\omega_m=\widehat{\mathfrak{I}}\omega_{m-1}$, for any $\omega_0\in\Omega$ converges to a unique FP of $\widehat{\mathfrak{I}}$.

Proof. Utilizing reasoning akin to that in the proof of Lemma 2.1, for $\kappa_1=\frac{1}{\sum_{j=1}^{u}b_j+1}>0$ and $\kappa_s=\frac{b_s}{\sum_{j=1}^{u}b_j+1}\geq 0$, $s=2,3,\cdots,u$, one has

$$\left\|\widehat{\mathfrak{I}}\omega-\widehat{\mathfrak{I}}\theta\right\|\leq c\max\left\{\|\omega-\theta\|,\left\|\omega-\widehat{\mathfrak{I}}\omega\right\|,\left\|\theta-\widehat{\mathfrak{I}}\theta\right\|,\left\|\theta-\widehat{\mathfrak{I}}\omega\right\|,\left\|\omega-\widehat{\mathfrak{I}}\theta\right\|\right\}. \tag{2.8}$$

Consider $\omega_0\in\Omega$. Describe the KI $\{\omega_m\}$ as $\omega_m=\widehat{\mathfrak{I}}\omega_{m-1}=\widehat{\mathfrak{I}}^m\omega_0$, $m\in\mathbb{N}$.

Next, we demonstrate that the sequence of iterates $\{\omega_m\}$ forms a Cauchy sequence. Assume that m and j are positive integers with $j<m$. From Lemma 2.1, one can write

$$\begin{aligned}\left\|\omega_j-\omega_m\right\| &= \left\|\widehat{\mathfrak{I}}^j\omega_0-\widehat{\mathfrak{I}}^m\omega_0\right\| \\ &= \left\|\widehat{\mathfrak{I}}\widehat{\mathfrak{I}}^{j-1}\omega_0-\widehat{\mathfrak{I}}\widehat{\mathfrak{I}}^{m-1}\omega_0\right\| \\ &= \left\|\widehat{\mathfrak{I}}\omega_{j-1}-\widehat{\mathfrak{I}}^{m-j-1}\omega_{j-1}\right\| \\ &\leq \delta\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-1},m-j+1)\right].\end{aligned}$$

It follows from Remark 2.5 that there is an integer z, $z\in[1,m-j+1]$ in order that

$$\left\|\omega_{j-1}-\omega_{j+z-1}\right\|=\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-1},m-j+1)\right].$$

Utilizing Lemma 2.1, we get

$$\left\|\omega_{j-1}-\omega_{j+z-1}\right\| = \left\|\widehat{\mathfrak{I}}\omega_{j-2}-\widehat{\mathfrak{I}}^{z+1}\omega_{j-2}\right\|$$

$$\leq \delta\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-2},z+1)\right],$$

which yields

$$\left\|\omega_{j-1}-\omega_{j+z-1}\right\| \leq \delta\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-2},m-j+2)\right].$$

Thus, one can write

$$\left\|\omega_{j}-\omega_{m}\right\| \leq \delta\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-1},m-j+1)\right] \leq \delta^{2}\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-2},m-j+2)\right].$$

Continuing with this process, we obtain

$$\left\|\omega_{j}-\omega_{m}\right\| \leq \delta\phi\left[Q(\widehat{\mathfrak{I}},\omega_{j-1},m-j+1)\right] \leq \cdots \leq \delta^{j}\phi\left[Q(\widehat{\mathfrak{I}},\omega_{0},m)\right].$$

Applying Lemma 2.2, we have

$$\left\|\omega_{j}-\omega_{m}\right\| \leq \frac{\delta^{j}}{1-\delta}\left\|\omega_{0}-\widehat{\mathfrak{I}}\omega_{0}\right\|. \tag{2.9}$$

Passing $m \to \infty$ in (2.9), we conclude that $\{\omega_m\}$ forms a Cauchy sequence. As Ω is a $\widehat{\mathfrak{I}}$−orbital BS, there is $\omega^* \in \Omega$ such that $\omega_m \to \omega^*$ as $m \to \infty$. Clearly,

$$\begin{aligned}
\left\|\omega^*-\widehat{\mathfrak{I}}\omega^*\right\| &\leq \|\omega^*-\omega_{m+1}\| + \left\|\omega_{m+1}-\widehat{\mathfrak{I}}\omega^*\right\| \\
&= \|\omega^*-\omega_{m+1}\| + \left\|\widehat{\mathfrak{I}}\omega_m-\widehat{\mathfrak{I}}\omega^*\right\| \\
&\leq \|\omega^*-\omega_{m+1}\| + \delta\max\left\{\begin{array}{c} \|\omega_m-\omega^*\|, \|\omega_m-\omega_{m+1}\|, \left\|\omega^*-\widehat{\mathfrak{I}}\omega^*\right\|, \\ \|\omega^*-\omega_{m+1}\|, \left\|\omega_m-\widehat{\mathfrak{I}}\omega^*\right\| \end{array}\right\} \\
&\leq \|\omega^*-\omega_{m+1}\| + \delta\left\{\begin{array}{c} \|\omega_m-\omega^*\| + \|\omega_m-\omega_{m+1}\| + \left\|\omega^*-\widehat{\mathfrak{I}}\omega^*\right\| \\ +\|\omega^*-\omega_{m+1}\| + \left\|\omega_m-\widehat{\mathfrak{I}}\omega^*\right\| \end{array}\right\}.
\end{aligned}$$

Hence,

$$\left\|\omega^*-\widehat{\mathfrak{I}}\omega^*\right\| \leq \frac{1}{1-2\delta}\left\{(1+\delta)\|\omega^*-\omega_{m+1}\| + \delta\|\omega_m-\omega^*\| + \delta\|\omega_m-\omega_{m+1}\|\right\}$$

Since $\omega_m \to \omega^*$ as $m \to \infty$, we have $\left\|\omega^*-\widehat{\mathfrak{I}}\omega^*\right\| = 0$. Thus, $\omega^* = \widehat{\mathfrak{I}}\omega^*$, that is, ω^* is a FP of $\widehat{\mathfrak{I}}$. The uniqueness follows immediately from (2.8).

A comparable conclusion for HEF'-C with Assertion (S') can be reached by employing reasoning akin to the ones mentioned earlier. □

Subsequently, we will examine the well-posedness and limit shadowing property for each category of hybrid enriched contractions defined in this context.

Theorem 2.4. *Let Ω be a BS. Then, $Fix\left(\widehat{\mathfrak{I}}\right)$ is well posed, provided that $\mathfrak{I}$ is an HEF-C mapping.*

Proof. Thanks to Theorem 2.1, $\widehat{\mathfrak{I}}$ has a unique FP ω^* in Ω. Assume that $\lim_{m\to\infty}\left\|\widehat{\mathfrak{I}}\omega_m-\omega_m\right\| = 0$. By (2.3), we get

$$\|\omega_m-\omega^*\| \leq \left\|\omega_m-\widehat{\mathfrak{I}}\omega_m\right\| + \left\|\widehat{\mathfrak{I}}\omega_m-\omega^*\right\|$$

$$
\begin{aligned}
&= \left\|\omega_m - \widehat{\mathfrak{I}}\omega_m\right\| + \left\|\widehat{\mathfrak{I}}\omega_m - \widehat{\mathfrak{I}}\omega^*\right\| \\
&\leq \left\|\omega_m - \widehat{\mathfrak{I}}\omega_m\right\| \\
&\quad + \hbar\left(\|\omega_m - \omega^*\|, \left\|\omega_m - \widehat{\mathfrak{I}}\omega_m\right\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega^*\right\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega_m\right\|\right).
\end{aligned}
$$

Letting $m \to \infty$ in the above inequality, we have

$$
\lim_{m\to\infty} \|\omega_m - \omega^*\| \leq \hbar\left(\lim_{m\to\infty} \|\omega_m - \omega^*\|, 0, 0, 0\right).
$$

Using ($\hbar_2$), there is $\zeta \in [0, 1)$ such that $\lim_{m\to\infty} \|\omega_m - \omega^*\| \leq \zeta.0$, which leads to $\lim_{m\to\infty} \|\omega_m - \omega^*\| = 0$, thereby establishing the result. □

Theorem 2.5. *Let Ω be a BS. Then, $Fix\left(\widehat{\mathfrak{I}}\right)$ is well posed, provided that $\mathfrak{I}$ is an HEF′-C mapping.*

Proof. The conclusion can be derived by employing reasoning analogous to that in the proof of Theorem 2.4. □

Theorem 2.6. *Let Ω be a BS and $\mathfrak{I}$ be a HEF-C (resp., HEF′-C). Then, $Fix\left(\widehat{\mathfrak{I}}\right)$ exhibits the limit shadowing property in Ω.*

Proof. From Theorem 2.1 (resp., Theorem 2.2), we conclude that $\widehat{\mathfrak{I}}$ owns a unique FP ω^* in Ω. Hence, $\widehat{\mathfrak{I}}^m\omega^* = \omega^*$ for any $m \in \mathbb{N}$, assume that $\lim_{m\to\infty} \left\|\widehat{\mathfrak{I}}\omega_m - \omega_m\right\| = 0$. It is clear that

$$
\begin{aligned}
\left\|\omega_m - \widehat{\mathfrak{I}}^m\omega^*\right\| &= \|\omega_m - \omega^*\| \\
&\leq \left\|\omega_m - \widehat{\mathfrak{I}}\omega_m\right\| + \left\|\widehat{\mathfrak{I}}\omega_m - \widehat{\mathfrak{I}}\omega^*\right\| \\
&\leq \left\|\omega_m - \widehat{\mathfrak{I}}\omega_m\right\| \\
&\quad + \hbar\left(\|\omega_m - \omega^*\|, \left\|\omega_m - \widehat{\mathfrak{I}}\omega_m\right\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega^*\right\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega_m\right\|\right) \\
&\quad \left(\text{resp., } \hbar\left(\|\omega_m - \omega^*\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega_m\right\|, \left\|\omega_m - \widehat{\mathfrak{I}}\omega^*\right\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega_m\right\|\right)\right).
\end{aligned}
$$

Setting $m \to \infty$ in the above inequality, and we have

$$
\begin{aligned}
\lim_{m\to\infty} \left\|\omega_m - \widehat{\mathfrak{I}}^m\omega^*\right\| &\leq \hbar\left(\lim_{m\to\infty} \|\omega_m - \omega^*\|, 0, 0, 0\right) \\
&\quad \left(\text{resp., } \hbar\left(\begin{array}{c} \lim_{m\to\infty} \|\omega_m - \omega^*\|, \lim_{m\to\infty} \|\omega^* - \omega_{m+1}\|, \\ \lim_{m\to\infty} \|\omega_m - \omega^*\|, \lim_{m\to\infty} \|\omega^* - \omega_{m+1}\| \end{array}\right)\right).
\end{aligned}
$$

By ($\hbar_2$) $\left(\text{resp., } \left(\hbar_5'\right)\right)$, we have $\lim_{m\to\infty} \left\|\omega_m - \widehat{\mathfrak{I}}^m\omega^*\right\| = 0$, and this completes the proof. □

To study UH stablility, we introduce the following theorems:

Theorem 2.7. *Let Ω be a BS and $\mathfrak{I}$ be an HEF-C that fulfills the condition below:*

($\hbar_5$) *there is $\zeta \in (0, 1)$ so that $\hbar(\varkappa, \varrho, \xi, \upsilon) \leq \zeta\varkappa + \xi$ for all $\varkappa, \varrho, \xi, \upsilon \in \mathbb{R}_+$.*

Then, the FP equation $\widehat{\mathfrak{I}}\omega = \omega$ is UH stable.

Proof. Thanks to Theorem 2.1, $\widehat{\mathfrak{I}}$ has a unique FP ω^* in Ω. Let $\epsilon > 0$ and $\upsilon^* \in \Omega$ be an $\epsilon-$solution, i.e.,

$$\left\|\upsilon^* - \widehat{\mathfrak{I}}\upsilon^*\right\| \leq \epsilon.$$

As $\omega^* \in \Omega$ and $\left\|\omega^* - \widehat{\mathfrak{I}}\omega^*\right\| = 0 \leq \epsilon$, then $\omega^* \in \Omega$ is an $\epsilon-$solution too. Using $(\hbar_5)$, we have

$$\begin{aligned} \|\omega^* - \upsilon^*\| &= \left\|\widehat{\mathfrak{I}}\omega^* - \upsilon^*\right\| \\ &\leq \left\|\widehat{\mathfrak{I}}\omega^* - \widehat{\mathfrak{I}}\upsilon^*\right\| + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq \hbar\left(\|\omega^* - \upsilon^*\|, \left\|\omega^* - \widehat{\mathfrak{I}}\omega^*\right\|, \left\|\upsilon^* - \widehat{\mathfrak{I}}\upsilon^*\right\|, \left\|\upsilon^* - \widehat{\mathfrak{I}}\omega^*\right\|\right) + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &= \hbar\left(\|\omega^* - \upsilon^*\|, 0, \left\|\upsilon^* - \widehat{\mathfrak{I}}\upsilon^*\right\|, \|\upsilon^* - \omega^*\|\right) + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq \zeta\|\omega^* - \upsilon^*\| + 2\left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq \zeta\|\omega^* - \upsilon^*\| + 2\epsilon, \end{aligned}$$

which yields

$$\|\omega^* - \upsilon^*\| \leq U\epsilon,$$

where $U = \frac{1}{1-\zeta}$. Hence, the result is proved. □

Theorem 2.8. *Let Ω be a BS and $\mathfrak{I}$ be an HEF′-C that fulfills the condition below:*

$(\hbar_6')$ *there is $\zeta \in (0, \frac{1}{3})$ so that $\hbar(\varkappa, \varrho, \xi, \upsilon) \leq \zeta(2\varkappa + \xi)$ for all $\varkappa, \varrho, \xi, \upsilon \in \mathbb{R}_+$.*

Then, the FP equation $\widehat{\mathfrak{I}}\omega = \omega$ is UH stable.

Proof. Thanks to Theorem 2.2, $\widehat{\mathfrak{I}}$ has a unique FP ω^* in Ω. Let $\epsilon > 0$ and $\upsilon^* \in \Omega$ be an $\epsilon-$solution, i.e.,

$$\left\|\upsilon^* - \widehat{\mathfrak{I}}\upsilon^*\right\| \leq \epsilon.$$

As $\omega^* \in \Omega$ and $\left\|\omega^* - \widehat{\mathfrak{I}}\omega^*\right\| = 0 \leq \epsilon$, then $\omega^* \in \Omega$ is an $\epsilon-$solution too. Using $(\hbar_6')$, we can write

$$\begin{aligned} \|\omega^* - \upsilon^*\| &= \left\|\widehat{\mathfrak{I}}\omega^* - \upsilon^*\right\| \\ &\leq \left\|\widehat{\mathfrak{I}}\omega^* - \widehat{\mathfrak{I}}\upsilon^*\right\| + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq \hbar\left(\|\omega^* - \upsilon^*\|, \left\|\upsilon^* - \widehat{\mathfrak{I}}\omega^*\right\|, \left\|\omega^* - \widehat{\mathfrak{I}}\upsilon^*\right\|, \left\|\upsilon^* - \widehat{\mathfrak{I}}\omega^*\right\|\right) + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq \zeta\left(2\|\omega^* - \upsilon^*\| + \left\|\omega^* - \widehat{\mathfrak{I}}\upsilon^*\right\|\right) + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq \zeta\left(2\|\omega^* - \upsilon^*\| + \left(\|\omega^* - \upsilon^*\| + \left\|\upsilon^* - \widehat{\mathfrak{I}}\upsilon^*\right\|\right)\right) + \left\|\widehat{\mathfrak{I}}\upsilon^* - \upsilon^*\right\| \\ &\leq 3\zeta\|\omega^* - \upsilon^*\| + (1 + \zeta)\epsilon, \end{aligned}$$

which yields

$$\|\omega^* - \upsilon^*\| \leq U\epsilon,$$

where $U = \frac{1+\zeta}{1-3\zeta}$. Hence, the result is proved. □

3. The relation between $Fix(\mathfrak{I})$ and $Fix\left(\widehat{\mathfrak{I}}\right)$

Assuming the existence of an FP of a u-fold AM linked to an HEF-C mapping $\mathfrak{I}$ (or HEF'-C), we aim to investigate essential conditions for the equivalence of FP sets between the u-fold AM and the related ECM.

We will commence with the subsequent observation, established for AMs $\mathfrak{I}_\vartheta$ and double AMs $\mathfrak{I}_{\kappa_1,\kappa_2}$.

Remark 3.1. Assume that $\mathfrak{I}$ is a self-mapping on an NS Ω. For $\kappa_j > 0$, $j = 1,2,3,\cdots,u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \in (0,1]$, the u-fold AM $\widehat{\mathfrak{I}} : \Omega \to \Omega$ linked to $\mathfrak{I}$ is described as

$$\widehat{\mathfrak{I}} = (1-\kappa_1-\kappa_2-\kappa_3-\cdots-\kappa_u)\,I + \kappa_1\mathfrak{I} + \kappa_2\mathfrak{I}^2 + \kappa_3\mathfrak{I}^3 + \cdots + \kappa_u\mathfrak{I}^u,$$

and has the property $Fix(\mathfrak{I}) \subseteq Fix(\widehat{\mathfrak{I}})$.

Next, we analyze the conditions ensuring the equivalence of $Fix(\mathfrak{I})$ and $Fix(\widehat{\mathfrak{I}})$.

Theorem 3.1. *Let Ω be a BS and $\mathfrak{I}$ be an HEF-C (resp., HEF′-C). Suppose that $\kappa_j > 0$, $j = 1,2,3,\cdots,u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u}\kappa_j \in (0,1]$ fulfilling the following hypothesis:*

(H_1) for all $h_j \in (0,1)$, $j = 1,2,3,\cdots,u$ with $\sum_{j=1}^{u} h_j \in [0,1)$ and $\varkappa \in Fix(\widehat{\mathfrak{I}})$,

$$\left\|\varkappa - \mathfrak{I}\varkappa\right\| \leq \left\|\varkappa - \left(1 - \sum_{j=2}^{u} h_j\right)\mathfrak{I}\varkappa - h_2\mathfrak{I}^2\varkappa - h_3\mathfrak{I}^3\varkappa - \cdots - h_u\mathfrak{I}^u\varkappa\right\|. \tag{3.1}$$

Then, $Fix(\mathfrak{I}) = Fix(\widehat{\mathfrak{I}})$.

Proof. We know from Remark 3.1 that $Fix(\mathfrak{I}) \subseteq Fix(\widehat{\mathfrak{I}})$. To demonstrate the reverse, suppose $Fix(\widehat{\mathfrak{I}})$ is not empty. Otherwise, the conclusion is self-evident. According to Theorem 2.1 (resp., Theorem 2.2), we obtain $Fix(\widehat{\mathfrak{I}}) \neq \emptyset$. If $\varkappa \in Fix(\widehat{\mathfrak{I}})$, then there is $\kappa_j > 0$, $j = 1,2,3,\cdots,u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u}\kappa_j \in (0,1]$ such that

$$\varkappa = (1-\kappa_1-\kappa_2-\kappa_3-\cdots-\kappa_u)\,\varkappa + \kappa_1\mathfrak{I}\varkappa + \kappa_2\mathfrak{I}^2\varkappa + \kappa_3\mathfrak{I}^3\varkappa + \cdots + \kappa_u\mathfrak{I}^u\varkappa.$$

Put $h_j = \frac{\kappa_j}{\sum_{j=1}^{u}\kappa_j}$, $j = 1,2,3,\cdots,u$, in (3.1), and we have

$$\begin{aligned}
&\left\|\varkappa - \mathfrak{I}\varkappa\right\| \\
\leq\;& \left\|\varkappa - \frac{\kappa_1}{\sum_{j=1}^{u}\kappa_j}\mathfrak{I}\varkappa - \frac{\kappa_2}{\sum_{j=1}^{u}\kappa_j}\mathfrak{I}^2\varkappa - \frac{\kappa_3}{\sum_{j=1}^{u}\kappa_j}\mathfrak{I}^3\varkappa - \cdots - \frac{\kappa_u}{\sum_{j=1}^{u}\kappa_j}\mathfrak{I}^u\varkappa\right\| \\
=\;& \frac{1}{\sum_{j=1}^{u}\kappa_j}\left\|\varkappa - (1-\kappa_1-\kappa_2-\kappa_3-\cdots-\kappa_u)\,z - \kappa_1\mathfrak{I}\varkappa - \kappa_2\mathfrak{I}^2\varkappa - \kappa_3\mathfrak{I}^3\varkappa - \cdots - \kappa_u\mathfrak{I}^u\varkappa\right\| \\
=\;& \left\|\varkappa - \widehat{\mathfrak{I}}\varkappa\right\| = 0.
\end{aligned}$$

Hence, $\varkappa \in Fix(\mathfrak{I})$. Therefore $Fix(\mathfrak{I}) = Fix(\widehat{\mathfrak{I}})$. □

We can also obtain equality between $Fix(\mathfrak{I})$ and $Fix(\widehat{\mathfrak{I}})$ in another way, as follows:

Theorem 3.2. *Let Ω be a BS and $\Im$ be an HEF-C (resp., HEF′-C). Suppose that there exist $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \kappa_j \in (0, 1]$ and $\vartheta \in [0, 1)$ such that*

(H_2) for all $\omega \in \Omega$, we get

$$\left\|\widehat{\Im}\omega - \Im\omega\right\| \leq \vartheta \left\|\omega - \Im\omega\right\|.$$

Then, $Fix(\Im) = Fix(\widehat{\Im})$.

Proof. From Remark 3.1, we have $Fix(\Im) \subseteq Fix(\widehat{\Im})$. Based on Theorem 2.1 (resp., Theorem 2.2), we conclude that $Fix(\widehat{\Im}) \neq \emptyset$. If $\varkappa \in Fix(\widehat{\Im})$, one has

$$\left\|\varkappa - \Im\varkappa\right\| = \left\|\widehat{\Im}\varkappa - \Im\varkappa\right\| \leq \vartheta \left\|\varkappa - \Im\varkappa\right\|,$$

which implies that $\left\|\varkappa - \Im\varkappa\right\| = 0$. Thus, $\varkappa \in Fix(\Im)$. Hence, $Fix(\widehat{\Im}) \subseteq Fix(\Im)$. Hence $Fix(\Im) = Fix(\widehat{\Im})$. □

Subsequently, we derive an approximation of an FP for an HEF-C (resp., HEF'-C) by employing the KI method for $\widehat{\Im}$.

Theorem 3.3. *Let Ω be a BS and $\Im$ be an HEF-C (resp., HEF′-C). Suppose that (H_1) or (H_2) are satisfied. Then,*

(i) $\Im$ possesses a unique FP in Ω;
(ii) KI defined by $\omega_m = \widehat{\Im}\omega_{m-1}$, for any $\omega_0 \in \Omega$ converges to a unique FP of $\Im$.

Proof. According to Theorem 2.1 (resp., Theorem 2.2), there are $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \kappa_j \in (0, 1]$ such that $\widehat{\Im}$ is described as

$$\widehat{\Im} = (1 - \kappa_1 - \kappa_2 - \kappa_3 - \cdots - \kappa_u) I + \kappa_1 \Im + \kappa_2 \Im^2 + \kappa_3 \Im + \cdots + \kappa_u \Im^u$$

and has a unique FP $\omega^* \in \Omega$, which can be achieved through KI (2.1) for $\omega_0 \in \Omega$. Since κ_j ($j = 1, 2, 3, \cdots, u$) fulfills hypothesis (H_1) or (H_2), the result follows immediately by Theorem 3.1 or Theorem 3.2. □

We finish this manuscript with revisiting the concept of the periodic point property (PPP) for a self-mapping $\Im$ described on Ω.

Definition 3.1. Assume that Ω is a nonempty set. We say that a mapping $\Im : \Omega \to \Omega$ has the PPP Ξ if for every $m \in \mathbb{N}$, $Fix(\Im) = Fix(\Im^m)$.

Remark 3.2. (i) For all $m \in \mathbb{N}$, $Fix(\Im) \subset Fix(\Im^m)$. Nevertheless, the reverse is not necessarily valid in all cases.

(ii) The mapping $\Im$ owns the PPP Ξ if $\Im_\vartheta$ owns the PPP Ξ; indeed, $Fix(\Im) = Fix(\Im_\vartheta)$.

Now, we investigate the conditions that ensure a self-mapping $\Im$, which meets the hybrid ECM, and possesses the PPP Ξ.

Lemma 3.1. *Assume that Ω is a BS and $\Im$ is an HEF-C (resp., HEF′-C). Assume also there are $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \kappa_j \in (0, 1]$ and*

(H) for all $\epsilon > 0$, *there are* $\omega, \theta \in \Omega$ *so that*

$$\left\|\omega - \widehat{\mathfrak{I}}\theta\right\| < \epsilon \Rightarrow \left\|\omega - \widehat{\mathfrak{I}}^j\theta\right\| < \frac{\epsilon}{j},\ j = 1, 2, \cdots, u.$$

Then, the FP of $\mathfrak{I}$ *aligns with that of* $\widehat{\mathfrak{I}}^j$ $(j = 1, 2, 3, \cdots, u, u \geq 4, u \in \mathbb{N})$.

Proof. Thanks to Theorem 2.1 (resp., Theorem 2.2), there are $\kappa_j > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, with $\sum_{j=1}^{u} \kappa_j \in (0, 1]$ such that $\widehat{\mathfrak{I}}$ owns a unique FP $\omega^* \in \Omega$ and the KI defined by $\omega_m = \widehat{\mathfrak{I}}\omega_{m-1}$, $m \in \mathbb{N}$ converges to a unique FP of $\mathfrak{I}$. Therefore, for every $\frac{\epsilon}{j} > 0$, $j = 1, 2, 3, \cdots, u$, $u \geq 4$, $u \in \mathbb{N}$, there is $M(j) \in \mathbb{N}$ with $m(j) \geq M(j)$ such that

$$0 < \left\|\omega^* - \widehat{\mathfrak{I}}\omega_{m(j)}\right\| \leq \frac{\epsilon}{j},\ j = 1, 2, 3, \cdots, u, u \geq 4, u \in \mathbb{N}.$$

Using Hypothesis (H), for $m(j) \geq M(j)$, one has

$$\left\|\omega^* - \mathfrak{I}^j\omega_{m(j)}\right\| \leq \frac{\epsilon}{j},\ j = 1, 2, 3, \cdots, u, u \geq 4, u \in \mathbb{N}.$$

Put $W = \max\{M(1), M(2), \cdots, M(u)\}$. For $m > W$, we can write

$$\begin{aligned}
\left\|\omega^* - \widehat{\mathfrak{I}}\omega_m\right\| &= \left\|\sum_{j=1}^{u} \kappa_j \left(\omega^* - \mathfrak{I}^j\omega_m\right)\right\| \\
&\leq \sum_{j=1}^{u} \left\|\kappa_j \left(\omega^* - \mathfrak{I}^j\omega_m\right)\right\| \\
&\leq \sum_{j=1}^{u} \kappa_j \frac{\epsilon}{j} \leq \sum_{j=1}^{u} \kappa_j \epsilon = \epsilon.
\end{aligned}$$

Hence, $\left\|\omega^* - \mathfrak{I}^j\omega_m\right\| \to 0$, $j = 1, 2, 3, \cdots, u, u \geq 4, u \in \mathbb{N}$ as $m \to \infty$ and for an arbitrary ϵ. Therefore, ω^* is an FP of $\mathfrak{I}^j$, $j = 1, 2, 3, \cdots, u, u \geq 4, u \in \mathbb{N}$, and this aligns with the FP of $\widehat{\mathfrak{I}}$. □

Theorem 3.4. *Let* Ω *be a BS and* $\mathfrak{I}$ *be an HEF-C (resp., HEF′-C). If the hypotheses* (H_1) *or* (H_1) *and* (H) *are satisfied, then* $\mathfrak{I}$ *admits the PPP* Ξ.

Proof. The proof follows immediately from Theorem 3.3 and Lemma 3.1. □

4. Conclusions

In this paper, we examine the necessary conditions for the u-fold AM and weakly enriched contractions to have equal sets of FPs. Additionally, we illustrate that an appropriate KI algorithm can effectively approximate an FP of a u-fold AM as well as the two enriched contractions. Also, we delve into the well-posedness, limit shadowing property, and UH stability of the u-fold AM. Furthermore, we establish necessary conditions that guarantee the PPP for each of the illustrated, strengthened contractions.

Author Contributions

All authors contributed equally to the writing of this article. All authors have accepted responsibility for entire content of the manuscript and approved its submission.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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