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Research article

On strong geodeticity in the lexicographic product of graphs

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Abstract: The strong geodetic number of a graph and its edge counterpart are recent variations of the pioneering geodetic number problem. Covering every vertex and edge of G, respectively, using a minimum number of vertices and the geodesics connecting them, while ensuring that one geodesic is fixed between each pair of these vertices, is the objective of the strong geodetic number problem and its edge version. This paper investigates the strong geodetic number of the lexicographic product involving graph classes that include complete graph K_m, path P_m, cycle C_m and star $K_{1,m}$ paired with P_n and with C_n. Furthermore, the parameter is studied in the lexicographic product of, arbitrary trees with diameter-2 graphs whose geodetic number is equal to 2, $K_n - e$ with K_2 and their converses. Upper and lower bounds for the parameter are established for the lexicographic product of general graphs and in addition, the edge variant of the aforementioned problem is studied in certain lexicographic products. The strong geodetic parameters considered in this paper have pivotal applications in social network problems, thereby making them indispensable in the realm of graph theoretical research. This work contributes to the expansion of the current state of research pertaining to strong geodetic parameters in product graphs.

Keywords: strong geodetic number; lexicographic product; strong edge geodetic number; shortest path
Mathematics Subject Classification: 05C12

1. Introduction

Geodesics are crucial in graph theory, and they serve as the basis for numerous research problems [1–8]. The geodetic number of a graph, as introduced by Harary et al., is a significant invariant in graph theory, associated with the notion of geodesic convexity [9, 10]. It entails the determination of a minimum cardinality set of vertices capable of covering all the vertices of a given graph, along with the associated isometric paths that link these selected vertices. Applications of this seminal graph theoretic parameter are due in location theory, convexity theory and game theory [9]. The problem was proved to be NP-hard for general graphs [11] and NP-complete for chordal and chordal bipartite graphs [12]. Recent research on this concept includes, geodetic convexity in Kneser graphs [13], Mycielskian of graphs [14], the study of geodetic number in complimentary prisms [15], tree derived architectures [16] and partial grids [17].

Various forms of path convexities discussed in literature include geodesic convexity, detour convexity, monophonic convexity and triangle-path convexity. Geodetic number of a graph is an invariant of geodesic convexity otherwise called as metric convexity [10]. Several variants of this parameter were introduced and investigated [18–22] and one such variant namely the strong geodetic number was proposed in light of its application in simulating a problem on social networks [21]. The same concept could be used to model the position of bus terminals inside a city, aiding with facility location. Though the underlying motive of the problem is similar to the geodetic number, it gets more complex, as here the geodesics connecting the vertices are fixed between each of its pairs. The problem was investigated in grid like structures and the complicacy in determining the exact value of the parameter for an arbitrary grid, which is the simplest Cartesian product graph, is indicated in [23]. NP-completeness of the problem has been established [21] and various graph classes have been explored [23–26]. Interestingly, the edge variant was introduced first and its NP-completeness was proved [22]. The edge variant has received significant attention in the recent years, and it has been studied in the Cartesian [27] and corona product of graphs [28], complete multipartite graphs [29]. It would be relevant to emphasise here that, determining the strong edge geodetic number of the Cartesian product $P_m \,\square\, P_n$, is an open problem [27]. We have obtained the strong edge geodetic number of the lexicographic product of paths.

Almost all branches of mathematics utilise the concept of products to combine or break down their fundamental structures. There are four standard products in graph theory, each with a unique range of applications and theoretical explanations [30]. The Cartesian and lexicographic products are two significant instances of graph operations, which facilitate the construction of larger graphs from smaller graphs. The larger graphs are intricately connected to those of the related smaller graphs with respect to many characteristics [31]. Lexicographic product initially termed as composition was introduced by Harary [32] and there have been many research investigations on the lexicographic product of graphs concerning various graph theoretic parameters [33–38]. Furthermore, lexicographic product of graphs has been studied in the context of generalised distance spectrum [39] and Gromov hyperbolicity [40].

With respect to the study of geodetic parameters in product graphs, the lexicographic product of graphs was explored in the context of the geodetic number problem [41]. The strong geodetic problem has been studied in the Cartesian product of graphs, resulting in an upper bound for the parameter. However, the general lower bound was posed as a conjecture [26]. Gledel et al. introduced a new parameter namely, the strong geodetic core number, and using this concept an improved upper bound on the strong geodetic number of Cartesian products was obtained [42]. The corona and join of graphs have been investigated for the strong geodetic number [43]. This work intends to bridge the gap by investigating the lexicographic product of graphs in relation to the strong geodetic parameters, which has not yet been done. As a result, in our study, we have established exact values for general classes of graphs and also lower and upper bounds for the parameter.

The subsequent sections of the paper are organised as follows. Definitions and basic concepts are compiled in Section 2. In Section 3, we provide bounds for $Sg(G[H])$ and proceed with the study of the parameter on certain lexicographic products such as $G[K_n]$, where G is a general graph, $G[P_n]$

and $G[C_n]$ where G is isomorphic to any of the graphs P_m, C_m, $K_{1,m}$ and K_m. Furthermore, we have determined the strong geodetic number of the lexicographic product of, arbitrary trees with graphs of diameter and geodetic number both equal to two, $K_n - e$ with K_2 and their converses. In Section 4, the lexicographic product of certain graphs including $K_m[P_n]$, $P_m[K_n]$ and $P_m[P_n]$ are studied with respect to the edge version of the parameter and the paper is concluded in Section 5.

2. Basic concepts

Let $G = (V(G), E(G))$ be a connected graph with $|V(G)| \geq 2$, where the vertex and edge sets of G are denoted as $V(G)$ and $E(G)$, respectively. The *order* of G is the number of vertices in G. An arrangement of non-repetitive vertices connected through edges is a *path*. The length of the shortest path connecting two vertices s_1 and s_2 determines the *distance* between the 2 vertices [44]. A *geodesic* or an *isometric path* refers to a shortest path. The length of any longest geodesic from a vertex $s \in V$ is the *eccentricity* $e(s)$ of s and the maximum of the eccentricities of all of the vertices in G is the diameter of G denoted as $diam(G)$ [44]. If $e(s) = diam(G)$, then s is referred to as a *peripheral vertex* [44]. *Neighbours* of a vertex s, denoted as $N[s]$ are the set of vertices adjacent to s. *Extreme vertices* also known as *simplicial vertices* are those vertices whose neighbours induce a subgraph which is complete [44]. A graph G is *geodetic* if there is one unique isometric path connecting every pair of vertices in G [44]. If a vertex s in a graph G lies on a $t - t'$ geodesic in G for any two vertices t and t', then the pair of vertices (t, t') is said to geodominate s [9]. *Antipodal vertices* are the 2 vertices which are farthest from each other [44]. A graph G is termed as an *extreme geodesic graph* if every vertex in G lies on an $s - t$ geodesic, where s and t are extreme vertices in G [45]. If every vertex of G is covered by the geodesics joining any 2 vertices in η, then η is said to be a geodetic cover of G. The geodetic cover of least cardinality and the least cardinality of its geodetic covers are referred to as the geodetic basis of G and the *geodetic number* $g(G)$, respectively [44]. A set $\eta_{Sg} \subseteq V$ is called a strong geodetic set of G, if all the vertices in $V(G) \backslash \eta_{Sg}$ are covered using geodesics that are fixed between the elements of η_{Sg}, in a manner that every pair of vertices in η_{Sg} is assigned a unique geodesic. If we denote $\widetilde{I}[(s, t)]$ as the geodesic that is fixed between 2 vertices s and t of η_{Sg} and $\widetilde{I}[\eta_{Sg}] = \{\widetilde{I}[(s, t)] : s, t \in \eta_{Sg}\}$, then η_{Sg} is called a strong geodetic set if $V(\widetilde{I}[\eta_{Sg}]) = V(G)$. The least order of such a set is referred to as the *strong geodetic number* $Sg(G)$ and any such set of least order is called the strong geodetic basis [21]. A strong edge geodetic cover of G is a set $\eta_{Sg_e} \subseteq V(G)$ such that for each pair $(s_1, s_2) \in \eta_{Sg_e}$, there corresponds a fixed shortest $s_1 - s_2$ path $P_{s_1 s_2}$ and the union of the edges in all such paths is equal to $E(G)$. The *strong edge geodetic number* of G, denoted as $Sg_e(G)$, is the minimum cardinality of such covers in G [22].

K_n, P_n, C_n and $K_{1,n}$ denote the complete graph, path, cycle and star, respectively on n vertices.

Definition 2.1. *An edge $e = st$ in G is called a unique edge, if it belongs to a unique geodesic joining any 2 vertices s and t in G.*

Theorem 2.2. [43] *For a graph G with $|V(G)| \geq 2$, $Sg(G) = 2$ if and only if G is a path.*

Remark 2.3. [9] *$g(P_n) = 2$, $g(K_n) = n$, and $g(K_{1,n}) = n$.*

Remark 2.4. [9]

$$g(C_n) = \begin{cases} 3, & \text{if } n \text{ is odd;} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Definition 2.5. [30] *The lexicographic product of G and H is the graph* $G[H]$*, where*

$$V(G[H]) = \{(s,t) \mid s \in V(G),\ t \in V(H)\},$$
$$E(G[H]) = \{(s,t)(s', t') \mid ss' \in E(G) \text{ or } s = s' \text{ and } tt' \in E(H)\}$$

The graph $G[H]$ consists of 2 layers: The H-layer (a horizontal layer) and G-layer (a vertical layer). For a vertex $s \in V(G)$, we define the H-layer ${}^sH = \{(s,t) \in V(G[H]) \mid t \in V(H)\}$ and similarly for $t \in V(H)$, the G-layer $G^t = \{(s,t) \in V(G[H]) \mid s \in V(G)\}$.

The following remark is evident from Definition 2.5.

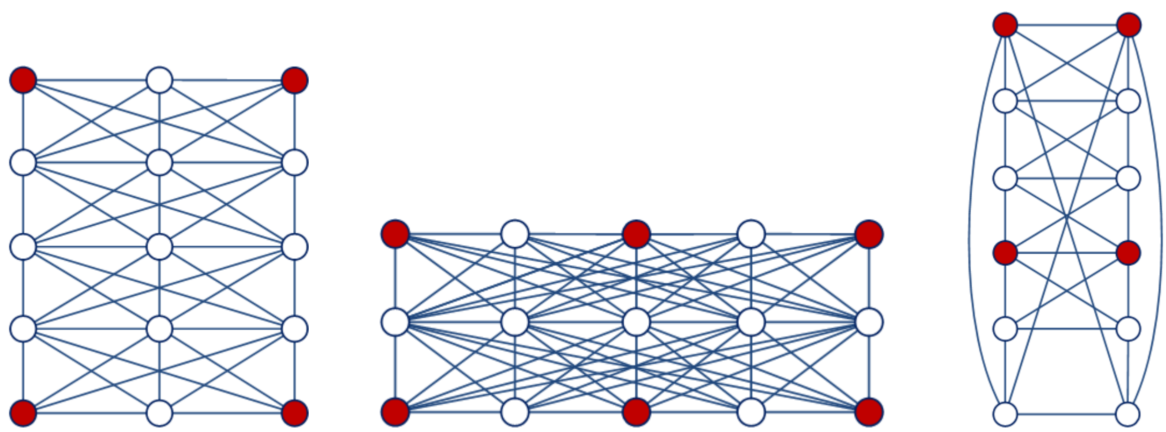
Remark 2.6. *Consider 2 vertices in* $G[H]$ *with their first coordinates to be same. The distance between those vertices depends on the second coordinate of the vertices and is* 1 *if they are adjacent in H, otherwise it is* 2*.*

Remark 2.7. *Consider 2 vertices* (s_1, t_1) *and* (s_2, t_2) *in* $G[H]$*. The distance between the 2 vertices is the distance between* s_1 *and* s_2 *in G, i.e.,* $d_{G[H]}[(s_1,t_1), (s_2,t_2)] = d_G[(s_1,s_2)]$*.*

Remark 2.8. [30] *The lexicographic product is not commutative.*

It could be seen from Figures 1(a) and 1(b) that $P_5[P_3]$ and $P_3[P_5]$ are not isomorphic with $Sg(P_5[P_3]) = 4$ and $Sg(P_3[P_5]) = 6$.

Throughout this paper, we abbreviate a geodetic set as g-set, a strong geodetic set as Sg-set and its corresponding edge variant as Sg_e-set, a geodetic basis as g-basis, a strong geodetic basis as Sg-basis and its corresponding edge variant as Sg_e-basis.



(a) **(b)** **(c)**

Figure 1. Coloured vertices denote: (a) the Sg-basis of $P_5[P_3]$; (b) the Sg-basis of $P_3[P_5]$; (c) the Sg-basis of $C_6[K_2]$.

3. The strong geodetic number of certain lexicographic products

Bounds for $Sg(G[H])$ are established in this section. Moreover, $Sg(G[K_n])$ and the strong geodetic number of the lexicographic product of some general graphs including K_m, P_m, C_m, $K_{1,m}$ with P_n and with C_n are determined. It could be seen that, the strong geodetic number for the graphs considered is dependent on the geodetic number of one of the elements in the product. The strong geodetic number of the cartesian product of $K_n - e$ with K_2 was explored in [26] and it was proved that $Sg((K_n - e) \square K_2) = Sg(K_n - e) = n - 1$. We have computed $Sg((K_n - e)[K_2]) = 2n - 4$, and it is evident that it is not equal to $Sg(K_n - e)$. The geodetic number of the lexicographic product of arbitrary trees with diameter-2 graphs whose geodetic number is equal to 2 was determined in [41]. We have computed $Sg(G[H])$ when G and H are isomorphic to these graphs and we have also studied the parameter in the converse of both the aforementioned products.

The following result is obtained directly from Remark 2.7.

Proposition 3.1. *Two vertices (s_1, t_1) and (s_2, t_2) of $G[H]$ are antipodal if and only if s_1 and s_2 are antipodal in G.*

Proposition 3.2. *Any geodesic in $G[H]$ connecting the vertices lying in different H-layers traverses either vertically or diagonally.*

Proof. Consider two vertices (s_1, t_1) and (s_2, t_2) in $G[H]$ that lie in different H-layers say, ${}^{s_1}H$ and ${}^{s_2}H$. Let P be the geodesic which connects the 2 vertices. Suppose if P traverses through a horizontal edge, then $d_{G[H]}[(s_1, t_1), (s_2, t_2)] = d_G[(s_1, s_2)] + 1$, a contradiction, by Remark 2.7. □

Theorem 3.3. $Sg(G[H]) \geq 4$.

Proof. Let $\eta_{Sg}(G[H])$ and $\widetilde{I}(\eta_{Sg}(G[H]))$ denote an Sg-basis of $G[H]$ and the geodesics that are fixed between the vertices in $\eta_{Sg}(G[H])$, respectively. Assume that $Sg(G[H]) < 4$. Then, it should be either 2 or 3. As G and H are non-trivial connected graphs, $G[H]$ cannot be a path and hence by Theorem 2.2, $Sg(G[H])$ must be more than 2. Let $(s_1, t_1), (s_2, t_2), (s_3, t_3)$ be the three elements of $\eta_{Sg}(G[H])$. Now, three cases arise.

Case 1: $s_1 = s_2 = s_3 = s$ (say).

Clearly, the vertices (s, t_1), (s, t_2) and (s, t_3) must lie in the same H-layer and denote the layer as ${}^{s}H$.

By Remark 2.6, $d_{G[H]}[(s, t_i), (s, t_j)]$ is either 1 or 2 for every $1 \leq i, j \leq 3$, and $i \neq j$.

Figure 2. Schematic representation for Subcase 1.1 in Theorem 3.3.

Subcase 1.1: $d_{G[H]}[(s, t_1), (s, t_2)] = 1$.

Then, $d_{G[H]}[(s, t_2), (s, t_3)] = 2$ (see Figure 2(a)) and $d_{G[H]}[(s, t_1), (s, t_3)] = 2$. For if, $d_{G[H]}[(s, t_2), (s, t_3)] = 1$, then (s, t_2) becomes an internal vertex of the $(s, t_1) \sim (s, t_3)$ geodesic by our assumption and hence, (s, t_2) need not be included in $\eta_{Sg}(G[H])$. Also, if $d_{G[H]}[(s, t_1), (s, t_3)] = 1$, then (s, t_3) is another horizontal neighbour of (s, t_1) and since (s, t_2) is already a neighbour of of (s, t_1) which is included in $\eta_{Sg}(G[H])$, (s, t_1) becomes an internal vertex of the $(s, t_2) \sim (s, t_3)$ geodesic by our assumption and hence it need not be included in $\eta_{Sg}(G[H])$.

Now, we proceed to prove that $Sg(G[H]) \geq 4$. Choose another vertex $s_k \in V(G)$ in such a way that its corresponding vertices (s_k, t_1), (s_k, t_2) and (s_k, t_3) in $G[H]$, lie in ${}^{s_k}H$. Let (s_k, t_1) be an internal vertex in $\widetilde{I}[(s, t_2), (s, t_3)]$ using the geodesic $(s, t_2) \sim (s_k, t_1) \sim (s, t_3)$ and let (s_k, t_2) be the vertex covered in the ${}^{s_k}H$-layer by using the geodesic $(s, t_1) \sim (s_k, t_2) \sim (s, t_3)$. Then, the vertex (s_k, t_3) would be left uncovered in the ${}^{s_k}H$-layer, a contradiction. See Figure 2(b). Similarly, when we consider all the pairs in $d_{G[H]}[(s, t_i), (s, t_j)]$, $1 \leq i, j \leq 3$, and $i \neq j$, we get a contradiction.

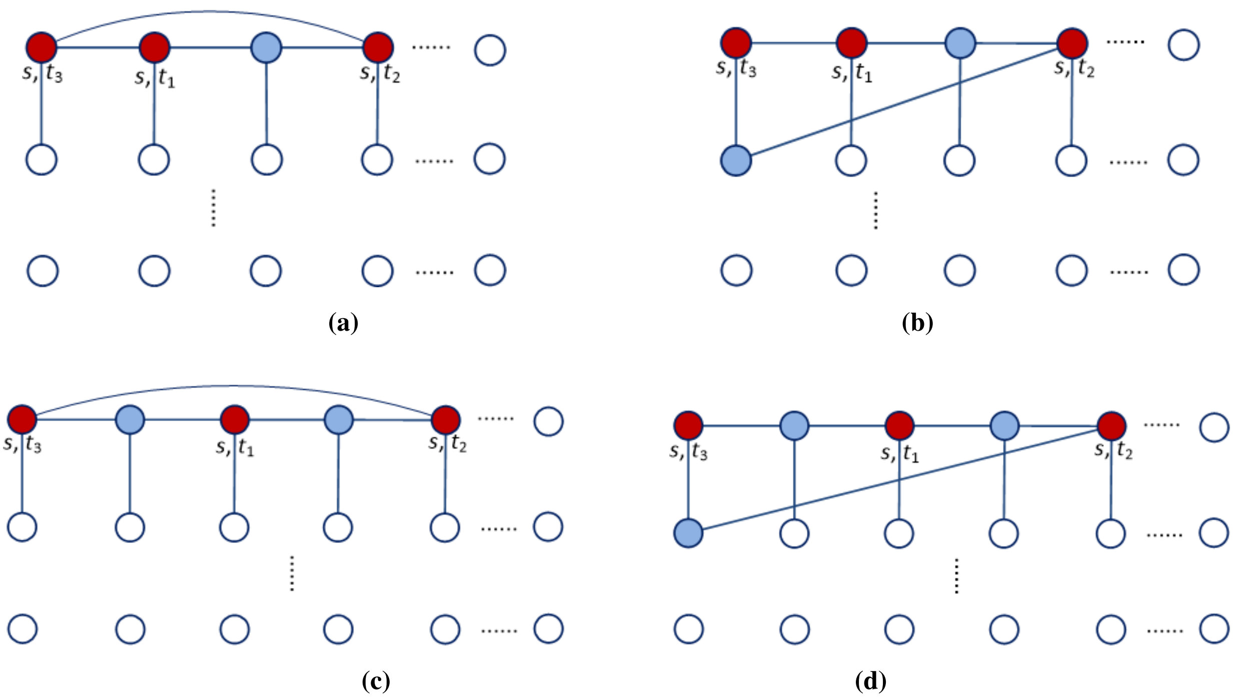


Figure 3. Schematic representation for Subcase 1.2 in Theorem 3.3.

Subcase 1.2: $d_{G[H]}[(s, t_1), (s, t_2)] = 2$.

Then, $d_{G[H]}[(s, t_1), (s, t_3)]$ and $d_{G[H]}[(s, t_2), (s, t_3)]$ are either 1 or 2. We consider all the possible cases. If $d_{G[H]}[(s, t_1), (s, t_3)] = 1$, and $d_{G[H]}[(s, t_2), (s, t_3)] = 1$, and then only one additional vertex from $G[H]$ could be covered by the $(s, t_1) \sim (s, t_2)$ geodesic. Refer Figure 3(a). If $d_{G[H]}[(s, t_1), (s, t_3)] = 1$, and $d_{G[H]}[(s, t_2), (s, t_3)] = 2$, then 2 vertices each are covered by the geodesics $(s, t_1) \sim (s, t_2)$ and $(s, t_2) \sim (s, t_3)$. See Figure 3(b). If $d_{G[H]}[(s, t_1), (s, t_3)] = 2$, and $d_{G[H]}[(s, t_2), (s, t_3)] = 1$, then 2 vertices each are covered by the geodesics $(s, t_1) \sim (s, t_2)$ and $(s, t_1) \sim (s, t_3)$. See Figure 3(c). If $d_{G[H]}[(s, t_1), (s, t_3)] = 2$, and $d_{G[H]}[(s, t_2), (s, t_3)] = 2$, then three vertices are covered by the geodesics $(s, t_1) \sim (s, t_2)$, $(s, t_1) \sim (s, t_3)$, and $(s, t_2) \sim (s, t_3)$. See Figure 3(d). In Figures 3(a)–3(d), the red-coloured vertices are the vertices that are chosen in $\eta_{Sg}(G[H])$, the vertices coloured in blue depicts the vertices that are geodominated and the vertices with no colour are the ones that are left uncovered. In all the cases, if the covered vertices belong to ${}^{s}H$, then the vertical neighbours of all the vertices considered in $\eta_{Sg}(G[H])$ would be left uncovered, a contradiction. If the vertices that are covered belong to different H-layers other than ${}^{s}H$, then the horizontal neighbours of the vertices (s_1, t_1), (s_2, t_2), (s_3, t_3) in ${}^{s}H$ would be left uncovered, a contradiction.

Case 2: $s_1 = s_2 \neq s_3$.

Let $s_1 = s_2 = s$ (say). Then, (s, t_1) and (s, t_2) lie in the same H-layer and (s_3, t_3) lies in another

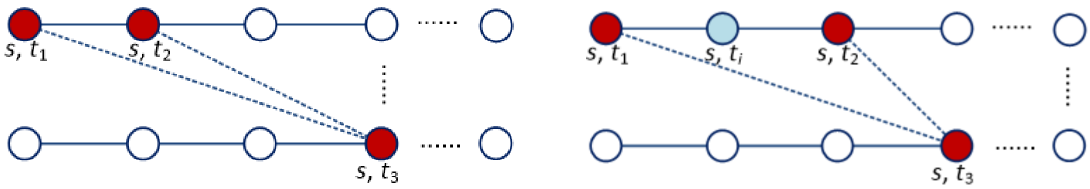
H-layer. By Remark 2.6, $d_{G[H]}[(s, t_1), (s, t_2)] = 1$ or 2. Now, two subcases arise.

Subcase 2.1: $d_{G[H]}[(s, t_1), (s, t_2)] = 1$.

By Proposition 3.2, the geodesic which connects the vertex (s_3, t_3) with the vertices (s, t_1) and (s, t_2) in the ${}^{s}H$ layer traverses either diagonally or vertically. Hence, the neighbouring vertices of (s_3, t_3) in the layer ${}^{s_3}H$ could not be covered, a contradiction. In Figure 4(a), the vertices coloured in red are the elements of $\eta_{Sg}(G[H])$, and the vertices that are not coloured are the uncovered vertices.

Subcase 2.2: $d_{G[H]}[(s, t_1), (s, t_2)] = 2$.

The geodesic which connects the vertex (s_3, t_3) with the vertices (s, t_1) and (s, t_2) in the ${}^{s}H$ layer could cover only an extra vertex (s, t_i) in the ${}^{s}H$ layer along with the vertices covered in the Subcase 2.1. Hence, the neighbouring vertices of (s_3, t_3) in the layer ${}^{s_3}H$ could not be covered, a contradiction. In Figure 4(b), the vertices coloured in red are the elements of $\eta_{Sg}(G[H])$, the blue-coloured ones are the geodominated vertices, and the vertices that are not coloured are the uncovered vertices.



(a) (b)

Figure 4. Schematic representation for (a) Subcase 2.1 in Theorem 3.3 (b) Subcase 2.2 in Theorem 3.3.

The case $s_1 \neq s_2 = s_3$ is similar to Case 2, and hence, we omit the proof.

Case 3: $s_1 \neq s_2 \neq s_3$.

By Proposition 3.2, the geodesic connecting the vertices $(s_1, t_1), (s_2, t_2), (s_3, t_3)$ traverses either vertically or diagonally and hence the neighbours of these vertices in their respective H-layers would be left uncovered, a contradiction. See Figure 5, where the members of $\eta_{Sg}(G[H])$ are depicted in red colour, the blue-coloured vertices are geodominated and the uncoloured vertices are the ones that are left uncovered.

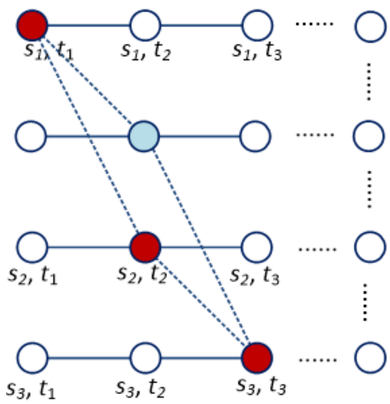


Figure 5. Schematic representation for Case 3 in Theorem 3.3.

□

Two of the lexicographic product graphs with strong geodetic number equal to 4 are depicted in Figures 1(a) and 1(c).

Theorem 3.4. *If G is a graph of order $m \geq 4$, and H is neither a complete graph nor a complete bipartite graph of order $n \geq 4$, then, $Sg(G[H]) \leq Sg(G)\{\lfloor \frac{n}{2} \rfloor + 1\}$.*

Proof. Let $\eta_{Sg}(G) = \{s_1, s_2, s_3, \ldots, s_k\}$, $k \leq m$ represent an Sg-basis of G. Define $T = (\eta_{Sg}(G) \times V(H)) - \{(s_i, t_j)|(s_i, t_j) \in \text{even } G\text{-layer}\}$. Evidently, $|T| = Sg(G)\{\lfloor \frac{n}{2} \rfloor + 1\}$. The geodesics that are fixed between the vertices in T cover $V(G[H])$. $Sg(G[H]) \leq Sg(G)\{\lfloor \frac{n}{2} \rfloor + 1\}$. □

Lemma 3.5. *For graphs G and H of orders m and n, respectively, $m, n \geq 5$, any Sg-set of $G[H]$ contains at least $\lceil \frac{n}{2} \rceil$ vertices that lie in alternate G-layers, from the first H-layer.*

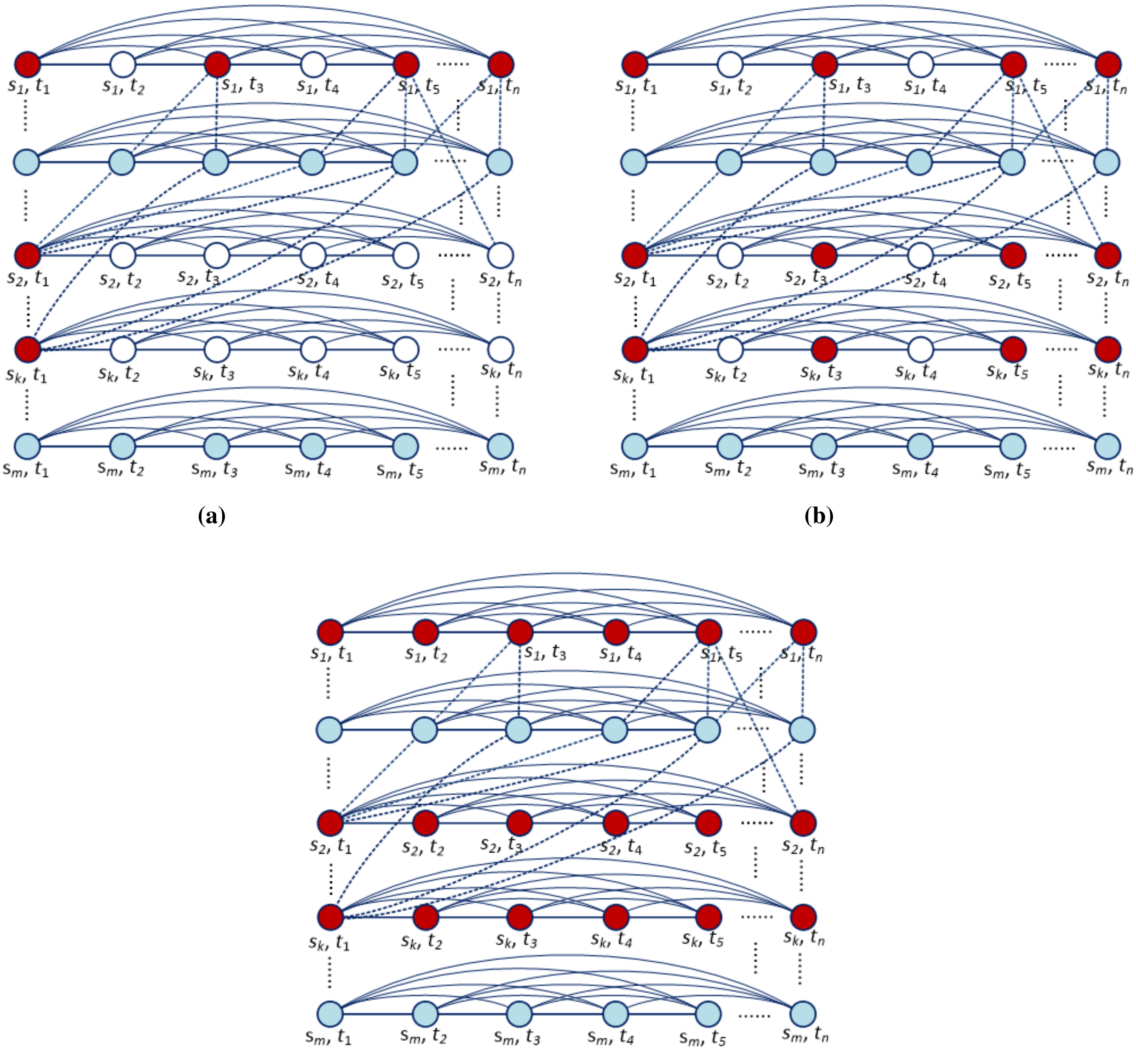
Proof. Let $V(G) = \{s_1, s_2, s_3, \ldots, s_m\}$, and $V(H) = \{t_1, t_2, t_3, \ldots, t_n\}$. Let $\eta_{Sg}(G[H])$ represent an Sg-set of $G[H]$. To choose vertices in $\eta_{Sg}(G[H])$, we start with a pair of vertices that are antipodal. By Remark 2.6, any 2 vertices in the same H-layer in $G[H]$ are connected by geodesics of length either 1 or 2. Let (s_1, t_1) and (s_k, t_k) be antipodal vertices in $G[H]$ lying in ${}^{s_1}H$ and ${}^{s_k}H$ respectively. By Proposition 3.2, the geodesic connecting these 2 vertices covers only the vertices in $G[H]$ lying in the H-layers that are positioned in between ${}^{s_1}H$ and ${}^{s_k}H$. Also this implies that the vertices lying in the ${}^{s_1}H$-layer could be covered only by a geodesic that connects vertices in the ${}^{s_1}H$-layer. Since the length of a geodesic that connects any two vertices in the ${}^{s_1}H$-layer is atmost 2, every alternate vertex in the ${}^{s_1}H$-layer must be chosen in $\eta_{Sg}(G[H])$. Hence, $\eta_{Sg}(G[H]) \geq \lceil \frac{n}{2} \rceil$. □

Lemma 3.6. *Let G and H be graphs of orders m, n respectively, $m, n \geq 5$. Any Sg-set of $G[H]$ contains at least $|\eta_g(G)|$ vertices from the first G-layer.*

Proof. Let $V(G) = \{s_1, s_2, s_3, \ldots, s_m\}$ and $V(H) = \{t_1, t_2, t_3, \ldots, t_n\}$. Let $\eta_g(G) = \{s_1, s_2, s_3, \ldots, s_l\}$, $l \leq m$ represent a g-basis of G and $\eta_{Sg}(G[H]) = \{(s_1, t_1), (s_2, t_2), (s_3, t_3), \ldots, (s_k, t_k)\}$, $k \leq m$ represent an Sg-set of $G[H]$. Assume that $\eta_{Sg}(G[H])$ contains less than $|\eta_g(G)|$ vertices from the first G-layer. This implies that a vertex say (s_j, t_1), where s_j is a member of $\eta_g(G)$ does not belong to $\eta_{Sg}(G[H])$. Since $(s_1, t_1) \in \eta_{Sg}(G[H])$, all of the alternate vertices in ${}^{s_1}H$ should be chosen in $\eta_{Sg}(G[H])$, by Lemma 3.5. The geodesics that connect (s_1, t_1) and the remaining vertices in $\eta_{Sg}(G[H])$ does not cover (s_j, t_1). Also (s_j, t_1) is not geodominated by any of the pair of vertices in $\eta_{Sg}(G[H])$. Thus, if a vertex (s_j, t_1) from $G[H]$ where $s_j \in \eta_g(G)$, is not included in $\eta_{Sg}(G[H])$, then (s_j, t_1) and the vertices lying in ${}^{s_j}H$ are left uncovered by the geodesics connecting the remaining vertices in $\eta_{Sg}(G[H])$, a contradiction. □

Lemma 3.7. *If G and H are graphs of orders m and n, respectively, $m, n \geq 5$, and if G contains k pendant vertices, then, any Sg-set of $G[H]$ contains at least k vertices.*

Proof. A pendant vertex in G is a member of any Sg-basis of G. Hence, by Lemma 3.6, the proof follows directly. □



(a) **(b)** **(c)**

Figure 6. The red coloured vertices denote the vertices that are chosen in $\eta_{Sg}(G[K_n])$, blue coloured vertices denote the geodominated vertices and the uncoloured vertices are the uncovered vertices.

Theorem 3.8. *If G is a graph and H is isomorphic to K_n where $|V(G)|$, $|V(H)| \geq 4$, then $Sg(G[H])$ $= g(G)|V(H)|$.*

Proof. Let $V(G) = \{s_1, s_2, s_3, \ldots, s_m\}$, $V(H) = \{t_1, t_2, t_3, \ldots, t_n\}$. Let $\eta_g(G) = \{s_1, s_2, s_3, \ldots, s_k\}$, $k \leq m$ be a g-basis of G and $\eta_{Sg}(G[H])$ represent an Sg-basis of $G[H]$. We now claim that $Sg(G[K_n]) \geq g(G)|V(H)|$. By Lemma 3.6, $Sg(G[K_n]) \geq g(G)$ and by Lemma 3.5, every alternate vertex from ${}^{s_1}H$ must be chosen in $\eta_{Sg}(G[H])$. The geodesics connecting these vertices could cover only vertices in the H-layers, ${}^{s_x}H$, $s_x \notin \{s_1, s_2, s_3 \ldots, s_k\}$ as illustrated in Figure 6(a). Hence, the vertices of $G[K_n]$ whose first coordinate is an element of the geodetic basis of G should be chosen from the alternate G-layers also. Refer Figure 6(b). Therefore, $\eta_{Sg}(G[H]) = \{(s_1, t_1), (s_1, t_3), (s_1, t_5),$

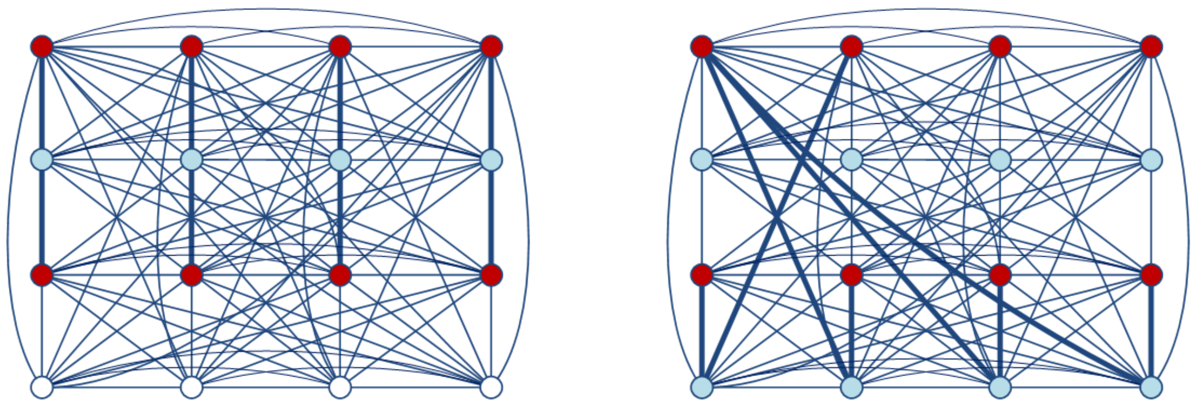
$\ldots$, (s_1, t_l), (s_2, t_1), (s_2, t_3), (s_2, t_5), $\ldots$, (s_2, t_l), (s_3, t_1), (s_3, t_3), (s_3, t_5), $\ldots$, (s_3, t_l), $\ldots$, (s_k, t_1), (s_k, t_3), (s_k, t_5), $\ldots$, $(s_k, t_l)\}$, where $l \in \{1, 3, 5, \ldots, n-2, n\}$, if n is odd and $l \in \{1, 3, 5, \ldots, n-1, n\}$, if n is even.

When geodesics are fixed between these vertices, the following conditions arise:

- The vertical or the slanting geodesics could cover all the vertices that lie in the H-layers ${}^{s_x}H$, $s_x \notin \{s_1, s_2, s_3 \ldots, s_k\}$. See Figure 6 (b).
- As the vertices in ${}^{s_l}H$, $s_l \in \{s_1, s_2, s_3 \ldots, s_k\}$ are adjacent with each other, the vertices $\{(s_i, t_2), (s_i, t_4), (s_i, t_6), \ldots, (s_i, t_{n-1})\}$, $1 \leq i \leq k$, when n is odd and the vertices $\{(s_i, t_2), (s_i, t_4), (s_i, t_6), \ldots, (s_i, t_{n-2})\}$, $1 \leq i \leq k$, when n is even are not covered by any of the geodesics. See Figure 6(b).

To cover the remaining uncovered vertices, every vertex in ${}^{s_1}H$, ${}^{s_2}H$, $\ldots$, ${}^{s_k}H$ must be chosen in $\eta_{Sg}(G[H])$ as shown in Figure 6 (c). Hence, $\eta_{Sg}(G[H]) \geq g(G)|V(H)|$.

For the upper bound, set $\eta_{Sg}(G[H]) = (\eta_g(G) \times V(H))$. Clearly $|\eta_{Sg}(G[H])| = g(G)|V(H)|$. We claim that $\eta_{Sg}(G[H])$ is an Sg-set of $G[H]$. This is proved by initially fixing vertical geodesics between vertices of $\eta_{Sg}(G[H])$ in the same G-layers in a similar manner as they could be fixed in G between the first coordinate of these vertices, which are members of $\eta_g(G)$. Refer Figure 7(a), where the darkened edges constitute the geodesic connecting the chosen vertices. Subsequently, the geodesics are fixed between the vertices of $\eta_{Sg}(G[H])$ that lie in different G-layers as shown in Figure 7(b). This approach ensures that the uncovered vertices in each G-layer are covered and hence, $V(G[H])$ is covered. Therefore, $Sg(G[H]) \leq g(G)|V(H)|$. □



(a) **(b)**

Figure 7. The red-coloured vertices denote the vertices that are chosen in $\eta_{Sg}(C_4[K_4])$, blue coloured vertices denote the geodominated vertices, and the uncoloured vertices are the uncovered vertices.

Corollary 3.9. *For an extreme geodesic graph G, and H is isomorphic to K_n, $Sg(G[K_n]) = Sg(G)|V(H)|$.*

Proof. Since G is an extreme geodesic graph, the set of all extreme vertices of G forms a unique g-basis and an Sg-basis of G. Hence, by Theorem 3.8, the result follows. □

Theorem 3.10. *For $m, n \geq 5$, if G is isomorphic to one of P_m, C_m, K_m, $K_{1,m}$ and if H is isomorphic to P_n, then* $Sg(G[H])= \begin{cases} g(G)\lceil\frac{n}{2}\rceil, & n \text{ is odd;} \\ g(G)(\frac{n}{2}+1), & n \text{ is even.} \end{cases}$

Proof. Assume that G and H are two graphs with $V(G) = \{s_1, s_2, s_3, \ldots, s_m\}$, and $V(H) = \{t_1, t_2, t_3, \ldots, t_n\}$, respectively. Let $\eta_{Sg}(G[H])$ be a set which represents an Sg-set of $G[H]$, $\eta_g(G)$ denote the g-basis of G, and $\widetilde{I}(\eta_g(G))$ denote the geodesics that are fixed between the vertices in $\eta_g(G)$.

Case 1: $G \cong P_m$, $H \cong P_n$, n is odd.

By Remark 2.3, we know that $g(P_n) = 2$. Let $\eta_g(G) = \{s_1, s_2\}$ be a set which represents a g-basis of G. By Proposition 3.1, (s_1, t_1), and (s_2, t_1) are antipodal vertices in $G[H]$. Also since there are only 2 antipodal vertices in G, the antipodal vertices in $G[H]$ lie in ${}^{s_1}H$ and ${}^{s_2}H$. By Lemma 3.5, every alternate vertex from ${}^{s_1}H$ should be chosen in $\eta_{Sg}(G[H])$, and since vertices in ${}^{s_2}H$ could not be covered by any vertical or slanting geodesic, every alternate vertex from ${}^{s_2}H$ should be chosen in $\eta_{Sg}(G[H])$. Hence, by Lemma 3.5 and Lemma 3.6, $Sg(P_m[P_n]) \geq 2\lceil\frac{n}{2}\rceil$.

Now, choose $\eta_{Sg}(G[H]) = (\eta_g(G) \times V(H)) - \{(s_1, t_j), (s_2, t_j) \mid j = 2, 4, 6, \ldots n-1\}$. Evidently, $|\eta_{Sg}(G[H])| = 2\lceil\frac{n}{2}\rceil$. We claim that $\eta_{Sg}(G[H])$ is an Sg-set of $G[H]$. This could be proved by fixing one geodesic between each pair of vertices of $\eta_{Sg}(G[H])$ in a G-layer in a similar manner as they could be fixed between the first coordinate of these vertices, which are members of $\eta_g(G)$ and clearly these geodesics are vertical geodesics. The vertices that are left uncovered in that G-layer are covered by the isometric paths that are fixed between vertices that lie in different G-layers. However, those vertices lying in the even G-layers, $G^{t_{2i}}$ are possibly left uncovered. Now, the slanting geodesics of the form $(s_i, t_j) \sim (s_{i+a}, t_{j+b}) \sim (s_{i'}, t_{j'})$, for some a, b and s_i, $s_{i'} \in \eta_g(G)$ cover the remaining vertices. By using this approach $V(P_m[P_n])$ is covered and hence $Sg(P_m[P_n]) \leq 2\lceil\frac{n}{2}\rceil$.

When n is even, the proof for the lower bound is analogous to the case when n is odd and for finding the upper bound, choose $\eta_{Sg}(G[H]) = (\eta_g(G) \times V(H)) - \{(s_1, t_j), (s_2, t_j) \mid j = 2, 4, 6, \ldots n-2\}$. This approach ensures that $V(P_m[P_n])$ is covered and $Sg(P_m[P_n]) \leq 2(\frac{n}{2}+1)$.

Case 2: $G \cong C_m$, $H \cong P_n$, both m and n are odd.

By Remark 2.4, we know that $g(C_m) = 3$. Let $\eta_g(G) = \{s_1, s_2, s_3\}$ be a g-basis of G. We claim that $\eta_{Sg}(G[H]) \geq 3\lceil\frac{n}{2}\rceil$. By Lemma 3.5, $\eta_{Sg}(G[H]) \geq \lceil\frac{n}{2}\rceil$. Suppose if $\lceil\frac{n}{2}\rceil \leq \eta_{Sg}(G[H]) < 3\lceil\frac{n}{2}\rceil$. Without loss of generality, we assume $\eta_{Sg}(G[H]) = 2\lceil\frac{n}{2}\rceil$. This implies that only 2 vertices say (s_1, t_1) and (s_2, t_1) that are antipodal in $G[H]$ are chosen from the two H-layers say ${}^{s_1}H$ and ${}^{s_2}H$ and atleast $\lceil\frac{n}{2}\rceil$ alternative vertices are chosen from ${}^{s_1}H$ and ${}^{s_2}H$ in $\eta_{Sg}(G[H])$. The geodesics are fixed among these vertices in a similar manner as they could be fixed in G between the first coordinate of these vertices, which are members of $\eta_g(G)$. The uncovered vertices in the upper half portion of $G[H]$ are covered by the horizontal and vertical isometric paths between the chosen vertices lying in the different G and H-layers. However, the vertices lying in ${}^{s_3}H$ are left uncovered. Hence, by Lemma 3.5 and Lemma 3.6, $\eta_{Sg}(G[H]) \geq 3\lceil\frac{n}{2}\rceil$.

Now, choose $\eta_{Sg}(G[H]) = (\eta_g(G) \times V(H)) - \{(s_i, t_j)$, where (s_i, t_j) lies in the alternate (even) G-layers$\}$. Clearly, $|\eta_{Sg}(G[H])| = 3\lceil\frac{n}{2}\rceil$. We claim that $\eta_{Sg}(G[H])$ is an Sg-set of $G[H]$. To establish this, the geodesics in the G-layers are initially fixed among the vertices of $\eta_{Sg}(G[H])$ in a similar manner as the geodesics could be fixed between the first coordinate of these vertices, which are members of $\eta_g(G)$. The vertices that are left uncovered in that G-layer are covered by the isometric paths that are fixed between vertices that lie in different G-layers. Also, the vertices lying in the even

G-layers, $G^{t_{2i}}$ would not be covered. These vertices are of two types viz., those lying in the H-layers, ${}^{s_i}H$, $s_i \in \eta_g(G)$, and those lying in the H-layers, ${}^{s_k}H, s_k \notin \eta_g(G)$. To cover the vertices lying in the H-layers, ${}^{s_i}H, s_i \in \eta_g(G)$, fix the geodesics $(s_1, t_1) \sim (s_1, t_3)$, $(s_1, t_3) \sim (s_1, t_5)$, $\dots$, $(s_1, t_{n-2}) \sim (s_1, t_n)$. The vertices lying in the H-layers, ${}^{s_k}H, s_k \notin \eta_g(G)$, are such that the alternate vertices (s_k, t_1), (s_k, t_3), $\dots$, (s_k, t_{n-2}), (s_k, t_n) are already covered by the geodesics that were fixed as they were fixed in G between the first coordinate of these vertices, which are members of $\eta_g(G)$. Now, the slanting geodesics of the form $(s_i, t_j) \sim (s_{i+a}, t_{j+b}) \sim (s_{i'}, t_{j'})$, for some a and b and s_i, $s_{i'} \in \eta_g(G)$ cover the remaining vertices. By using this approach every vertex of $C_m[P_n]$ is covered and therefore $Sg(C_m[P_n]) \leq 3\lceil \frac{n}{2} \rceil$.

Similarly, it can be proved that $Sg(C_m[P_n]) = 2\lceil \frac{n}{2} \rceil$, when m and n are even and odd, respectively. The proof for the cases when m and n are odd and even respectively and both m and n are even are analogous to the previous case.

Case 3: $G \cong K_m$, $H \cong P_n$, n is odd.

By Remark 2.3, we know that $g(K_m) = m$. By Lemma 3.5, $Sg(K_m[P_n]) \geq \lceil n/2 \rceil$, and by Lemma 3.6, all the m vertices from G^{t_1} need to be chosen in $\eta_{Sg}(G[H])$. Hence, $\eta_{Sg}(G[H]) \geq m$. The geodesics connecting the m vertices (from G^{t_1}) to the $\lceil n/2 \rceil$ vertices (from ${}^{s_1}H$) traverse either diagonally or vertically. Hence, by Proposition 3.2, except for the $\lceil n/2 \rceil$ vertices lying in ${}^{s_1}H$, the other vertices positioned in the remaining H-layers would be left uncovered. Every alternative vertex from the H layers ${}^{s_2}H$, ${}^{s_3}H$, $\dots$, ${}^{s_m}H$ must be chosen in any Sg-set of $G[H]$ in order to cover these vertices. Therefore, $|\eta_{Sg}(G[H])| \geq m\lceil n/2 \rceil$.

For the upper bound, we choose the set $\eta_{Sg}(G[H]) = \{(s_1, t_1), (s_1, t_3), (s_1, t_5), \dots, (s_1, t_l), (s_2, t_1), (s_2, t_3), (s_2, t_5), \dots, (s_2, t_l), (s_3, t_1), (s_3, t_3), (s_3, t_5), \dots, (s_3, t_l), \dots, (s_k, t_1), (s_k, t_3), (s_k, t_5), \dots, (s_k, t_l)\}$, where $l \in \{1, 3, 5, \dots, n-2, n\}$, when n is odd.

When n is even, the lower bound could be obtained similarly. Choose the set $\eta_{Sg}(G[H]) = \{(s_1, t_1), (s_1, t_3), (s_1, t_5), \dots, (s_1, t_l), (s_2, t_1), (s_2, t_3), (s_2, t_5), \dots, (s_2, t_l), (s_3, t_1), (s_3, t_3), (s_3, t_5), \dots, (s_3, t_l), \dots, (s_k, t_1), (s_k, t_3), (s_k, t_5), \dots, (s_k, t_l)\}$, $l \in \{1, 3, 5, \dots, n-1, n\}$. Figure 8 depicts $K_4[P_4]$ along with its Sg-basis. $V(G[H])$ could be covered by the geodesics connecting these vertices and $|\eta_{Sg}(G[H])| = m(n/2 + 1)$. Hence, $Sg(K_m[P_n]) \leq m(n/2 + 1)$. □

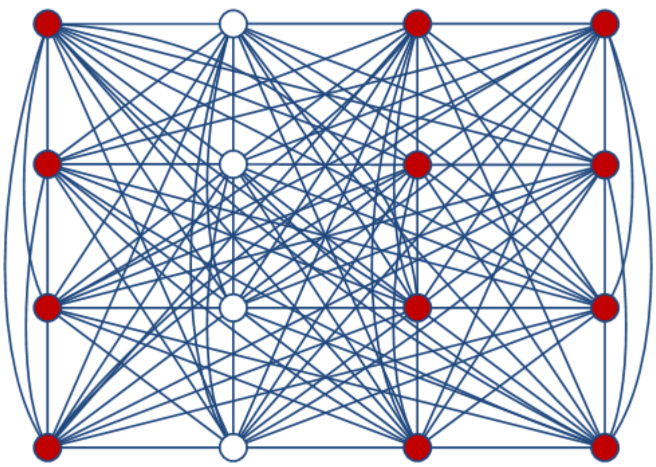


Figure 8. Coloured vertices denote the Sg-basis of $K_4[P_4]$.

Theorem 3.11. *For $m, n \geq 5$, if G is isomorphic to one of P_m, C_m, K_m, $K_{1,m}$ and if H is isomorphic to C_n, then $Sg(G[H])= g(G)\lceil \frac{n}{2} \rceil$.*

Proof. Assume that $V(G) = \{s_1, s_2, s_3, \dots, s_m\}$, $H \cong C_n$ and $V(H) =\{t_1, t_2, t_3, \dots, t_n\}$. Let $\eta_g(G) = \{s_1, s_2, s_3, \dots, s_k\}$ be a g-basis of G. Using Lemma 3.5, $Sg(G[H]) \geq \lceil \frac{n}{2} \rceil$, and by Lemma 3.6, $Sg(G[H]) \geq |\eta_g(G)|$. The geodesics connecting the vertices from G^{t_1} to the $\lceil n/2 \rceil$ vertices in ${}^{s_1}H$ traverse either diagonally or vertically. Hence, by Proposition 3.2, except for the $\lceil n/2 \rceil$ vertices lying in ${}^{s_1}H$, the remaining vertices from the other H-layers would be left uncovered. To cover these vertices, every alternate vertex from ${}^{s_2}H$, ${}^{s_3}H$, ..., ${}^{s_k}H$ must be chosen in any Sg-set of $G[H]$, and hence, $Sg(G[H]) \geq g(G)\lceil n/2 \rceil$. Now, we proceed to derive the upper bound.
Case 1: $G \cong P_m$, m is odd.
Subcase 1.1: n is odd.

Set $\eta_{Sg}(G[H]) = (\eta_g(G) \times V(H)) - \{(s_i, t_j)$, where (s_i, t_j) lies in the alternate (even) G-layers$\}$. Clearly, $|\eta_{Sg}(G[H])| = 3\lceil n/2 \rceil$. Our claim is that $\eta_{Sg}(G[H])$ is an Sg-set of $G[H]$. The geodesics in G-layers are fixed between vertices from $\eta_{Sg}(G[H])$ in the similar manner as they could be fixed in G between the first coordinate of these vertices, which are members of $\eta_g(G)$. The possible vertices that are uncovered are those lying in the even G-layers, $G^{t_{2i}}$. These vertices are of 2 types: those lying in the H-layers, ${}^{s_i}H$, $s_i \in \eta_g(G)$, and those lying in the H-layers, ${}^{s_k}H$, $s_k \notin \eta_g(G)$. To cover the vertices lying in the H-layers, ${}^{s_i}H$, $s_i \in \eta_g(G)$, fix the geodesics $(s_1, t_1) \sim (s_1, t_3)$, $(s_1, t_3) \sim (s_1, t_5)$, ..., $(s_1, t_{n-2}) \sim (s_1, t_n)$. The vertices lying in the H-layers of the form ${}^{s_k}H$, $s_k \notin \eta_g(G)$, are such that the alternate vertices $(s_k, t_1), (s_k, t_3), \dots, (s_k, t_{n-2}), (s_k, t_n)$ are already covered by geodesics that were fixed as it could be done in G between the first coordinate of these vertices, which are members of $\eta_g(G)$. Now, the geodesics of the form $(s_i, t_j) \sim (s_{i+p}, t_{j+q}) \sim (s_{i'}, t_{j'})$, for some p and q cover the remaining vertices in ${}^{s_k}H$. By this approach, $V(G[H])$ is covered and therefore $Sg(G[H]) \leq 3\lceil \frac{n}{2} \rceil$.

The proof is analogous for the remaining cases and hence it is omitted. □

Lemma 3.12. *Let $G \cong (K_n - e)$, $n \geq 4$ and $H \cong K_2$. If s is a simplicial vertex in G, then the vertices of the form (s, t) in $G[H]$ where $t \in V(H)$ are simplicial vertices in $G[H]$.*

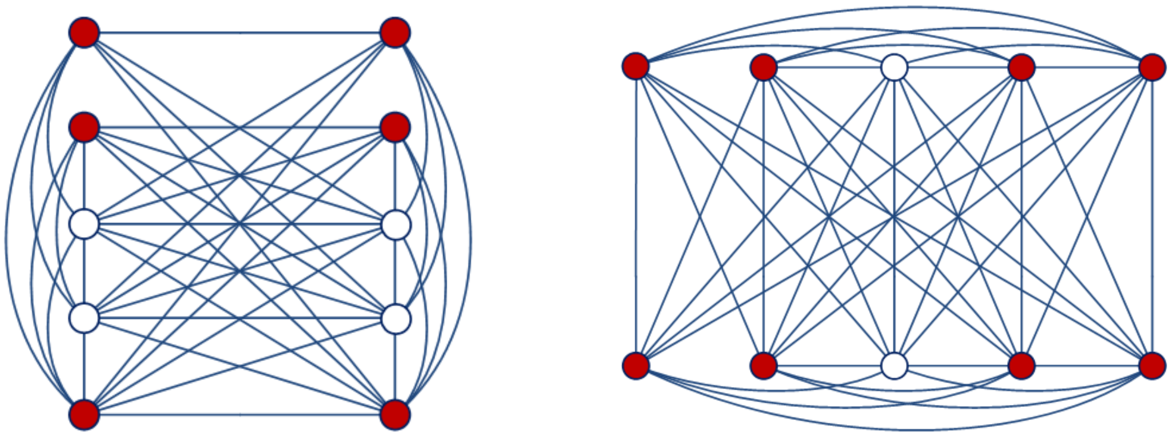
Proof. Let $V(K_n - e) = \{u, v, s_1, s_2, s_3, \dots, s_{n-2}\}$, where $e = uv$, and let $V(K_2) = \{t_1, t_2\}$. $V((K_n - e)[K_2]) = \{(u, t_1), (u, t_2), (v, t_1), (v, t_2), (s_1, t_1), (s_1, t_2), (s_2, t_1), (s_2, t_2), \dots, (s_{n-2}, t_1), (s_{n-2}, t_2)\}$. It could be verified that, u and v are simplicial vertices in $K_n - e$. Since, for all $1 \leq i \leq n-2$, $N(s_i, t_1)$ and $N(s_i, t_2)$ are equal, (s_i, t_1) and (s_i, t_2) are called as twins. By definition of lexicographic product of graphs, every vertex in G^{t_1} except the vertices (u, t_1) and (v, t_1) are adjacent with each other and also with their twins in G^{t_2}. Similarly every vertex except the vertices (u, t_2) and (v, t_2) in G^{t_2} are adjacent with each other and with their twins in G^{t_1}. Hence $\langle N(u, t_1) \rangle$, $\langle N(u, t_2) \rangle$, $\langle N(v, t_1) \rangle$ and $\langle N(v, t_2) \rangle$ are cliques that are isomorphic to K_{2n-3}, which implies that the vertices (u, t_1), (u, t_2), (v, t_1) and (v, t_2) are simplicial in $G[H]$. □

Theorem 3.13. *If $G \cong (K_n - e)$, $n \geq 4$, and $H \cong K_2$, then $Sg(G[H]) = 2n - 4$.*

Proof. Let $e = (u, v) \in E(G)$, and $V(G) = \{u, v, s_1, s_2, s_3, \dots, s_{n-2}\}$. Let $\eta_{Sg}(G[H]) = \{{}^uH, {}^vH, {}^{s_3}H, {}^{s_4}H, \dots, {}^{s_{n-2}}H\}$ be an Sg-set of $G[H]$. The geodesics that are fixed between these vertices cover the vertices in $G[H]$. Evidently, $|\eta_{Sg}(G[H])| = 4 + 2(n-4) = 2n-4$. Hence, $Sg(G[H]) \leq 2n-4$.

As u and v are simplicial vertices in G, by Lemma 3.12, ${}^uH$ and ${}^vH$ are simplicial vertices in $G[H]$, and hence they should be included in $\eta_{Sg}(G[H])$. $\eta_{Sg}(G[H]) = \{(u, 1), (u, 2), (v, 1), (v, 2)\}$.

The diameter of G is 2, and hence, by Proposition 3.1, the diameter of $G[H]$ is two. Therefore, the geodesics connecting the vertices in $\eta_{Sg}(G[H])$ could cover only four other vertices from 2 H-layers say, ${}^{s_1}H$ and ${}^{s_2}H$. It could be seen that, $G[H] - \{{}^{u}H,\ {}^{v}H,\ {}^{s_1}H,\ {}^{s_2}H\} \cong K_{2(n-4)}$, and hence along with the previously chosen vertices, every vertex from $G[H] - \{{}^{u}H,\ {}^{v}H,\ {}^{s_1}H,\ {}^{s_2}H\}$ should be chosen in $\eta_{Sg}(G[H])$. Figure 9(a) depicts $(K_5 - e)[K_2]$ along with its Sg-basis. Evidently, $|\eta_{Sg}(G[H])| = 4 + 2(n-4) = 2n-4$. Therefore, $Sg(G[H]) \geq 2n-4$. $\square$



(a) **(b)**

Figure 9. Coloured vertices denote: (a) the Sg-basis of $(K_5 - e)[K_2]$; (b) the Sg-basis of $K_2[K_5 - e]$.

Theorem 3.14. *If G is an arbitrary tree with $V(G) \geq 3$, and H is a graph with $g(H) = diam(H) = 2$, then $Sg(G[H]) = Sg(G)Sg(H)$.*

Proof. Assume that G is a tree with l leaves. Let $\eta_{Sg}(G[H])$ be an Sg-basis of $G[H]$. We know that the set of pendant vertices of G forms an Sg-basis of G. Let $\eta_{Sg}(G) = \{s_1,\ s_2,\ s_3,\ \ldots,\ s_l\}$ be an Sg-basis of G and $\eta_{Sg}(H) = \{t_1,\ t_2,\ t_3,\ \ldots,\ t_k\}$ be an Sg-basis of H for which $g(H) = diam(H) = 2$. Set $\eta_{Sg}(G[H]) = \{(s_1,t_1),\ (s_1,t_2),\ (s_1,t_3),\ \ldots,\ (s_1,t_k),\ (s_2,t_1),\ (s_2,t_2),\ (s_2,t_3),\ \ldots,\ (s_2,t_k),\ \ldots,\ (s_l,t_1),\ (s_l,t_2),\ (s_l,t_3),\ \ldots,\ (s_l,t_k)\}$. i.e., set $\eta_{Sg}(G[H]) = \eta_{Sg}(G) \times \eta_{Sg}(H)$. To show that $\eta_{Sg}(G) \times \eta_{Sg}(H)$ is an Sg-set of $G[H]$, initially the geodesics are fixed in the H-layers connecting the vertices of $\eta_{Sg}(G[H])$ similar to the manner as they were fixed in $\widetilde{I}[\eta_{Sg}(H)]$. The uncovered vertices are those that are located in between the H-layers, ${}^{s_i}H$ and ${}^{s_j}H$, where $1 \leq i, j \leq l$. To cover those vertices, the geodesics are fixed in G-layers between the vertices in $\eta_{Sg}(G[H])$ similar to the manner as it is done in $\widetilde{I}[\eta_{Sg}(G)]$. The further uncovered vertices are covered by the geodesics connecting the vertices that lie in 2 different G-layers. This procedure ensures that $V(G[H])$ is covered. Hence, $\eta_{Sg}(G[H]) \leq Sg(G)Sg(H)$.

To derive the lower bound, we prove two claims. First, we claim that, $\eta_{Sg}(G[H]) \bigcap {}^{s_z}H \neq \phi$, where $s_z \in \eta_{Sg}(G)$. Suppose if $\eta_{Sg}(G[H]) \bigcap {}^{s_z}H = \phi$, then $\eta_{Sg}(G[H])$ does not contain vertices from ${}^{s_z}H$, where $s_z \in \eta_{Sg}(G)$. By Remark 2.7, we know that if s_1 and s_2 are antipodal vertices in G, then (s_1, t_j) and (s_2, t_j) are antipodal vertices in $G[H]$. But by our assumption, if a pair of vertices (s_x, t_j) and (s_y, t_j) are included in $\eta_{Sg}(G[H])$, then s_x and s_y are non-pendant vertices in G for some x, y and hence the geodesics connecting vertices from $\eta_{Sg}(G[H])$ are not the longest ones in $G[H]$.

This implies that some of the vertices in the G-layers would not be covered by any of the geodesics connecting the vertices from $\eta_{Sg}(G[H])$, a contradiction to $\eta_{Sg}(G[H])$. Hence, $\eta_{Sg}(G[H]) \bigcap {}^{s_z}H \neq \phi$, where $s_z \in \eta_{Sg}(G)$.

Our next claim is that $\eta_{Sg}(G[H]) \bigcap G^{t_z} \neq \phi$, where $t_z \in \eta_{Sg}(H)$. Suppose if $\eta_{Sg}(G[H]) \bigcap G^{t_z} = \phi$, for every $t_z \in \eta_{Sg}(H)$. Then, $\eta_{Sg}(G[H])$ consists of vertices of the form (s_p, t_q) where $s_p \in \eta_{Sg}(G)$ and $t_q \notin \eta_{Sg}(H)$. By Remark 2.6, the distance between any 2 vertices in ${}^{s_p}H$ is at most 2. Since $diam(H) = 2$, by our assumption, the vertices in ${}^{s_p}H$ that are not included in $\eta_{Sg}(G[H])$ could not be covered by a horizontal geodesic. Further it could be observed that the vertices in ${}^{s_p}H$ could not be internal vertices of any vertical or slanting geodesic as $s_p \in \eta_{Sg}(G)$. This implies that the neighbours of (s_p, t_q) in ${}^{s_p}H$ could not be covered by any geodesic connecting the vertices of $\eta_{Sg}(G[H])$, a contradiction to $\eta_{Sg}(G[H])$. Hence, $\eta_{Sg}(G[H]) \geq Sg(G)Sg(H)$ and $Sg(G[H]) = Sg(G)Sg(H)$. □

Now, we obtain the results by swapping G and H in Theorems 3.13 and 3.14. Since the lexicographic product of graphs is not commutative, these results add importance.

Theorem 3.15. *If $G \cong K_2$ and $H \cong K_n - e$, $n \geq 4$, then $Sg(G[H]) = 2(n-1)$.*

Proof. Let $V(G) = \{s_1, s_2\}$, $V(H) = \{u, v, t_1, t_2, \ldots, t_{n-2}\}$, where $e = (u, v) \in E(H)$. Let $\eta_{Sg}(G[H])$ denote an Sg-set of $G[H]$. It is evident that $G[H] - \{(s_1, u), (s_1, v), (s_2, u), (s_2, v)\} \cong K_{2(n-2)}$. The length of the longest geodesic in $G[H]$ is 2 and those geodesics in $G[H]$ are the ones connecting the vertices (s_1, u), (s_1, v) and (s_2, u), (s_2, v). Two vertices from $K_{2(n-2)}$ are covered by these geodesics. Hence, the aforementioned four vertices should be included in $\eta_{Sg}(G[H])$ along with the $2(n-2) - 2$ vertices from $G[H] - \{(s_1, u), (s_1, v), (s_2, u), (s_2, v)\}$. Figure 9(b) depicts $K_2[K_5 - e]$ along with its Sg-basis. $|\eta_{Sg}(G[H])| = 4 + 2(n-2) - 2 = 2(n-1)$, and $Sg(G[H]) \geq 2(n-1)$.

Choose $\eta_{Sg}(G[H]) = \{(s_1, u), (s_1, v), (s_2, u), (s_2, v)\} \bigcup \{(s_1, t_i) : 2 \leq i \leq n-2\} \bigcup \{(s_2, t_j) : 2 \leq j \leq n-2\}$. Clearly, $|\eta_{Sg}(G[H])| = 2(n-1)$, and when geodesics are fixed between each pair of vertices in $\eta_{Sg}(G[H])$, $V(G[H])$ is covered. Hence, $Sg(G[H]) \leq 2(n-1)$. □

The term 2-geodesic used in Theorem 3.16 refers to a geodesic of length 2.

Theorem 3.16. *If G is a diameter-2 graph with $g(G) = 2$, $V(G) \geq 4$, and H is any arbitrary tree with $V(H) \geq 8$, then $Sg(G[H]) = 2(|L| + \lceil A/2 \rceil)$, where L denotes the set of pendant vertices of H, A denotes the set of consecutive non-pendant vertices of H that are left uncovered after fixing 2-geodesics between every pair of vertices in L, where at least one vertex in A is the neighbour of a pendant vertex, $|L| \geq 3$ and $|A| > 3$.*

Proof. Let L denote the set of all pendant vertices of H and A denote the set of consecutive non-pendant vertices of H that are left uncovered after fixing 2-geodesics between every pair of vertices in L. Let $\eta_{Sg}(G[H])$ denote an Sg-set of $G[H]$. Since G is a diameter-2 graph, by Remark 2.7, the distance between any two antipodal vertices in $G[H]$ is 2. Hence, throughout the proof, we consider that the vertices to be chosen in $\eta_{Sg}(G[H])$ are from the H-layers ${}^{s_1}H$ and ${}^{s_2}H$ where s_1 and s_2 are antipodal in G. We first claim that the set of vertices in $G[H]$ of the form (s_1, t_j) and (s_2, t_j), where t_j is a pendant vertex in H should be included in $\eta_{Sg}(G[H])$. Suppose if the vertices of the above form are not included in $\eta_{Sg}(G[H])$. Then, the vertices (s_i, t_j), where t_j is a non-pendant vertex in H should be included in $\eta_{Sg}(G[H])$ from ${}^{s_1}H$ and ${}^{s_2}H$. We know that in a tree, a pendant vertex could not be an internal vertex of a geodesic connecting any 2 non-pendant vertices of the tree. Also, each vertex in ${}^{s_1}H$ and ${}^{s_2}H$ either belongs to $\eta_{Sg}(G[H])$ or is an internal vertex of a geodesic connecting any 2 vertices in ${}^{s_1}H$ and ${}^{s_2}H$, respectively, i.e., Vertices in ${}^{s_1}H$ and ${}^{s_2}H$ could not be internal vertices of a geodesic connecting 2 vertices from ${}^{s_1}H$ and ${}^{s_2}H$, respectively. Hence, if the vertices of the form (s_1, t_j) and (s_2, t_j), where t_j is a pendant vertex in H is not included in $\eta_{Sg}(G[H])$, then those vertices could not be covered by any geodesic connecting the vertices of ${}^{s_1}H$ and ${}^{s_2}H$. Next, we claim that along with the $2|L|$ vertices, at least $2\lceil A/2 \rceil$ vertices should be included in $\eta_{Sg}(G[H])$. Since the distance between any 2 vertices in ${}^{s_1}H$ is at most 2, the set of vertices $\{(s_1, t_j)$: t_j is not an internal vertex of a 2-geodesic connecting vertices in $L\}$ could not be covered by any geodesic connecting the vertices of ${}^{s_1}H$ and, similarly, the set of vertices $\{(s_2, t_j)$: t_j is not an internal vertex of a 2-geodesic connecting vertices in $L\}$ could not be covered by any geodesic connecting the vertices of ${}^{s_2}H$ in $G[H]$. Hence, the vertices (s_1, t_u), (s_2, t_u), where t_u is an alternate non-pendant vertex in H, that is left uncovered by the 2-geodesics connecting vertices in L, should be chosen from ${}^{s_1}H$ and ${}^{s_2}H$. Therefore, $Sg(G[H]) \geq 2(|L| + \lceil A/2 \rceil)$.

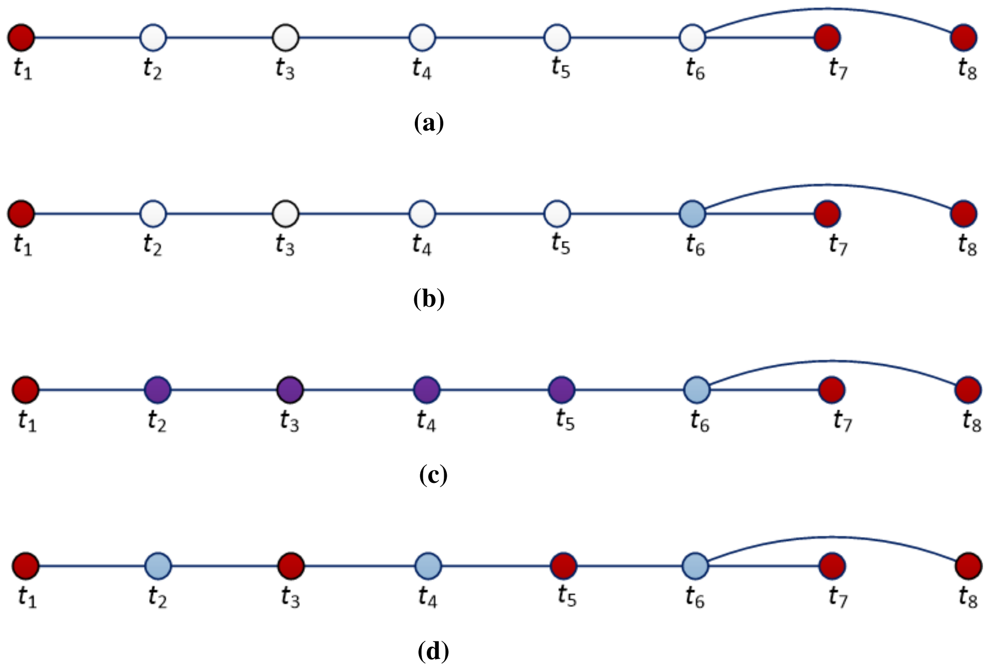


Figure 10. Illustration for choosing the vertices in H as mentioned in Theorem 3.16.

We now proceed to derive the upper bound. For convenience, we consider H and choose the suitable vertices in H, and then the corresponding vertices in $G[H]$ are chosen in $\eta_{Sg}(G[H])$. Initially, choose the pendant vertices of H. See Figure 10(a). Now, set $\eta_{Sg}(G[H]) = \{(s_1, t_u), (s_2, t_u) \mid t_u \in L\}$. Fix 2-geodesics between each pair of vertices of L. In Figure 10(b), the vertices covered by the 2-geodesics connecting the pendant vertices are shown in blue colour. If the distance between a pair of vertices in L is greater than 2, then we consider the non-pendant vertices between that pair of vertices as uncovered and those vertices are collected in the set A. In Figure 10(c), the vertices t_2, t_3, t_4 and t_5 are collected in A. Now, set $\eta_{Sg}(G[H]) = \{(s_1, t_u), (s_2, t_u) \mid t_u \in L\} \bigcup \{(s_1, t_x), (s_1, t_y), (s_2, t_x), (s_2, t_y) \mid t_x, t_y \in A$ and $d_H(t_x, t_y) = 2$ in $H\}$. Among the vertices of A, vertices, such as t_x and t_y where $d_H(t_x, t_y) = 2$, are considered and their corresponding vertices in $G[H]$ are chosen in $\eta_{Sg}(G[H])$. In Figure 10(d), the finally chosen vertices are coloured in red and the covered vertices are the blue-coloured ones. When the horizontal geodesics are fixed between the vertices chosen in ${}^{s_1}H$ and ${}^{s_2}H$ all the vertices in ${}^{s_1}H$ and ${}^{s_2}H$, respectively, are geodominated and the vertical and slanting geodesics that are fixed between the vertices of ${}^{s_1}H$ and ${}^{s_2}H$ geodominate all of the vertices in the remaining H-layers. In this way, every vertex of $G[H]$ is covered. Hence, $Sg(G[H]) \leq 2(|L| + \lceil A/2 \rceil)$.

Figure 11 depicts the lexicographic product of a graph G, whose diameter and geodetic number are both equal to 2 with an arbitrary tree along with its Sg-basis.

□

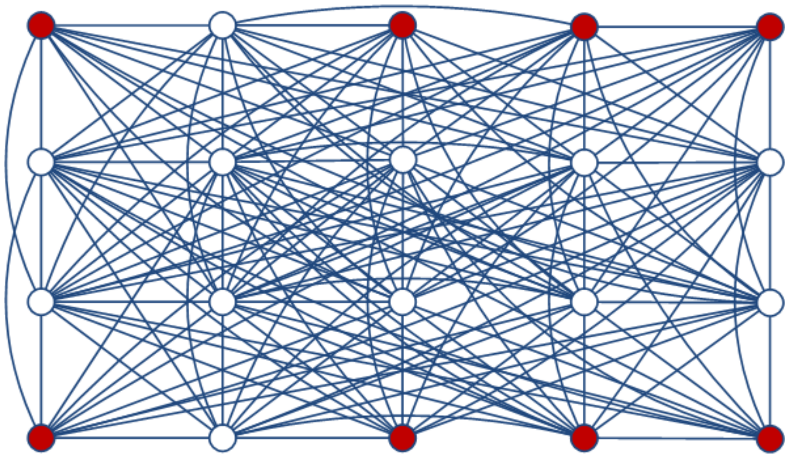


Figure 11. Coloured vertices denote the strong geodetic basis of $G[H]$, where $G \cong K_4 - e$ and H is an arbitrary tree

4. The strong edge geodetic number of certain lexicographic products

The strong edge geodetic number of the lexicographic product of certain general graphs that include $K_m[P_n]$, $P_m[K_n]$ and $P_m[P_n]$ are determined in this section.

As $Sg_e(G) \geq Sg(G)$ for a graph G, the following result is obtained directly from Theorem 3.3.

Theorem 4.1. $Sg_e(G[H]) \geq 4$.

Theorem 4.2. *For graphs G and H of orders $m \geq 5$ and $n \geq 5$, respectively, $Sg_e(G[H])= mn$ if either*

- G is isomorphic to K_m and H is isomorphic to P_n, or
- G is isomorphic to P_m and H is isomorphic to K_n.

Proof. Let $V(G) = \{s_1, s_2, s_3, \ldots, s_m\}$, $V(H) = \{t_1, t_2, t_3, \ldots, t_n\}$, and $\eta_{Sg_e}(G[H])$ denote an Sg_e-set of $G[H]$.

Case 1: $G \cong K_m$, $H \cong P_n$. Every vertical edge $e = ((s_1, t_1), (s_2, t_1))$ in $G[H]$ is covered only by a geodesic connecting its end vertices, i.e., both (s_1, t_1) and (s_2, t_1) have to be chosen in $\eta_{Sg_e}(G[H])$. This implies that all the mn vertices have to be chosen in $\eta_{Sg_e}(G[H])$. Also, if a vertex say (s_j, t_j) is not included in $\eta_{Sg_e}(G[H])$, then the edges incident with that vertex would be left uncovered. Hence, $\eta_{Sg_e}(G[H])$ is an Sg_e-set of minimum cardinality and $Sg_e(G[H]) = mn$.

Case 2: $G \cong P_m$ and $H \cong K_n$. The horizontal edge $e = ((s_1, t_1), (s_1, t_2))$ in $G[H]$ is covered by a geodesic connecting its end vertices, i.e., both (s_1, t_1) and (s_1, t_2) have to be chosen in $\eta_{Sg_e}(G[H])$, and hence, all the mn vertices have to be chosen in $\eta_{Sg_e}(G[H])$. Suppose a vertex say (s_j, t_j) is not included in $\eta_{Sg_e}(G[H])$ then the edges incident with that vertex remains uncovered. Hence, $\eta_{Sg_e}(G[H])$ is an Sg_e-set of minimum cardinality and $Sg_e(G[H]) = mn$.

□

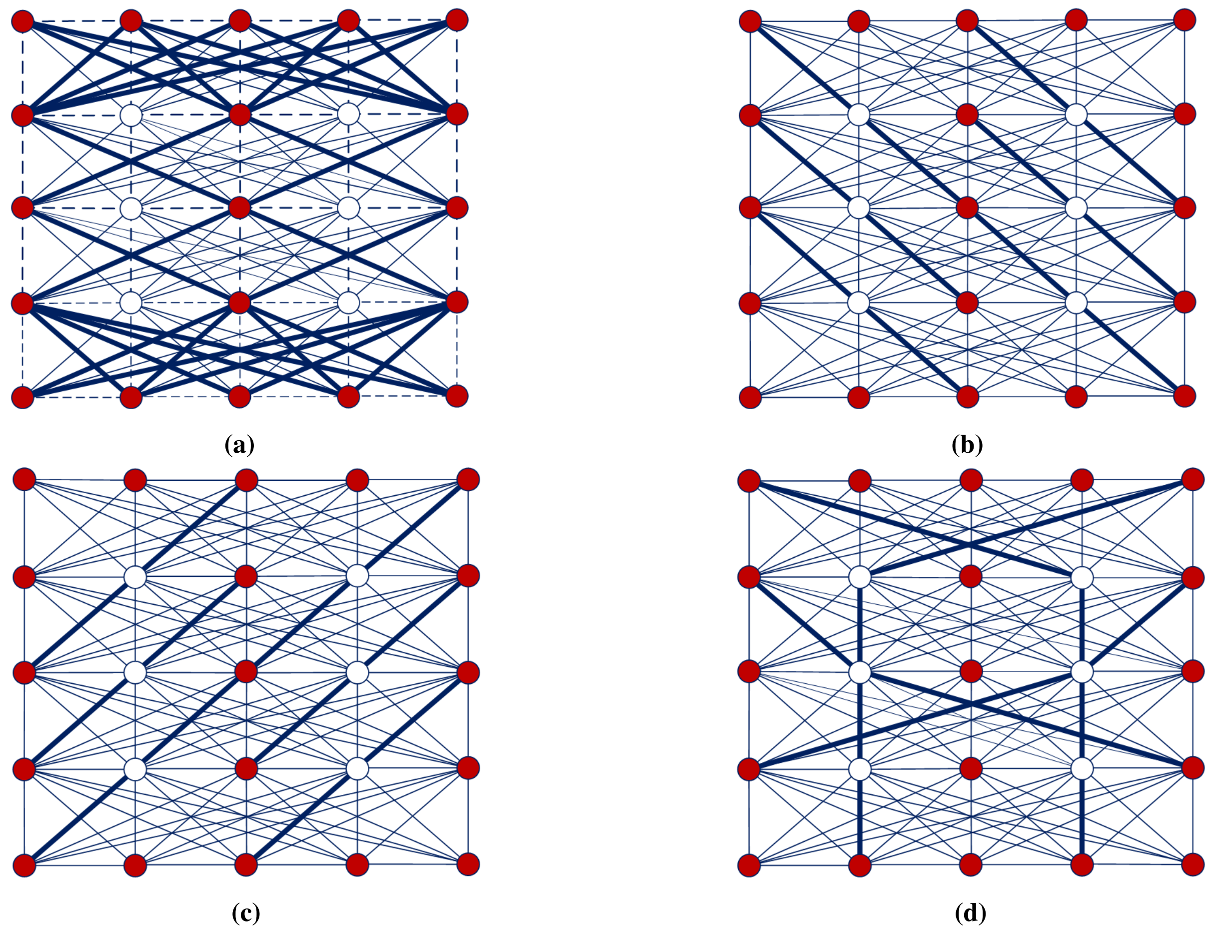


Figure 12. Coloured vertices denote the elements of $\eta_{Sg_e}(P_5[P_5])$ and the darkened and dotted lines denote the various geodesics that are fixed between the vertices in $\eta_{Sg_e}(P_5[P_5])$.

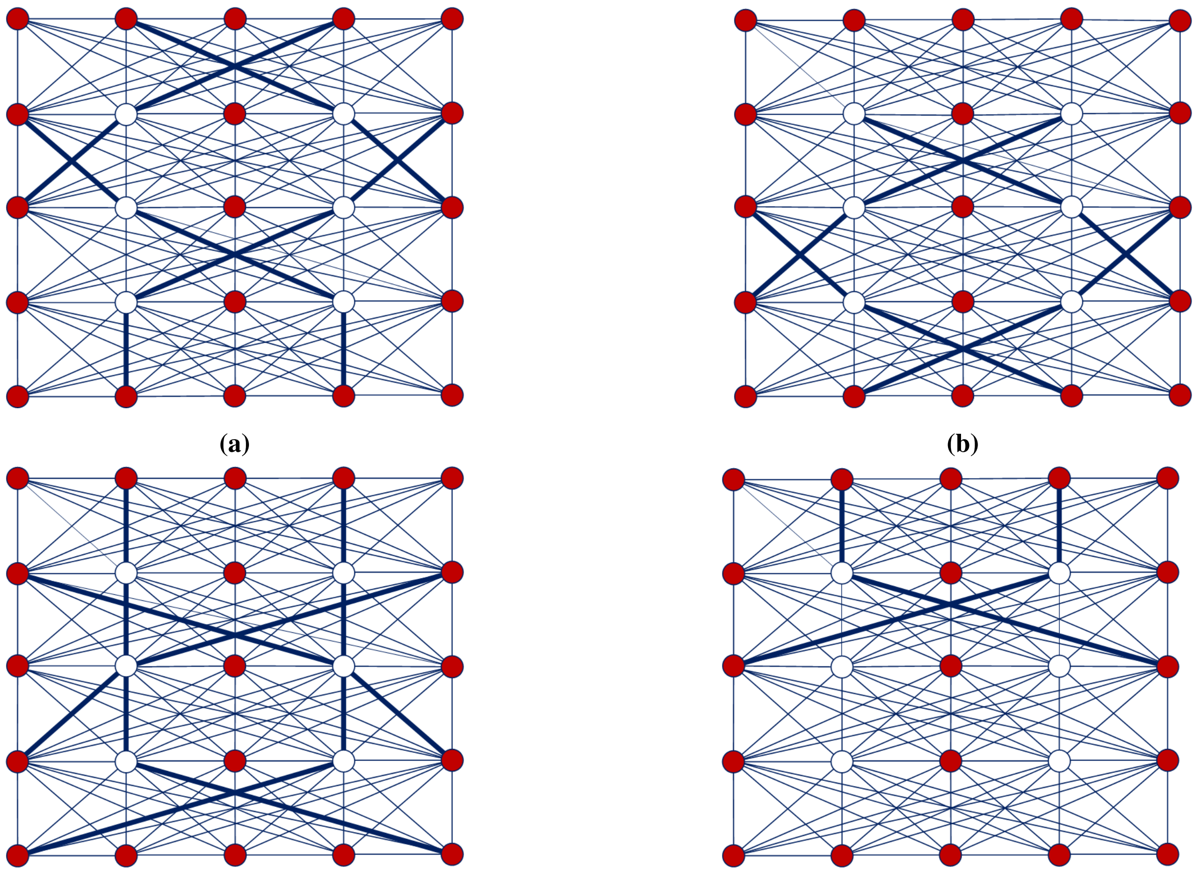
Theorem 4.3. *Let G and H be isomorphic to the paths of orders $m \geq 5$ and $n \geq 5$, respectively.*

$$Sg_e(G[H]) = \begin{cases} 2n + (m-2)\lceil \frac{n}{2} \rceil, & \text{if } n \text{ is odd;} \\ 2n + (m-2)(\frac{n}{2} + 1), & \text{if } n \text{ is even.} \end{cases}$$

Proof. Assume that $V(G) = \{s_1, s_2, s_3, \ldots, s_m\}$, $V(H) = \{t_1, t_2, t_3, \ldots, t_n\}$, and $\eta_{Sg_e}(G[H])$ is an Sg_e-set of $G[H]$. Evidently, there are m H-layers and n G-layers in $P_m[P_n]$. We begin by proving the lower bound. The vertices lying in the first and the last H-layers, i.e., ${}^{s_1}H$ and ${}^{s_m}H$ are peripheral vertices and the edges that are incident with the vertices in these 2 layers are unique edges. These edges lie in the beginning and end of the geodesics. Hence, to cover those edges, all of the vertices lying in ${}^{s_1}H$ and ${}^{s_m}H$ need to be chosen in any Sg_e-set of G. Hence, $Sg_e(P_m[P_n]) \geq 2n$.

Case 1: n is odd.

We consider the remaining $m-2$ H-layers, since vertices lying in ${}^{s_1}H$ and ${}^{s_m}H$ are already selected in the Sg_e-set. Every horizontal edge in $G[H]$ is a unique edge, and hence, every alternate vertex in all the $m-2$ H-layers must be chosen to cover the horizontal edges. The vertical and slanting edges are covered by the slanting geodesics that connects the vertices $(s_i, t_j), (s_{i+q}, t_{j+r})$, where $1 \le i, i+q \le m$, and $1 \le j,\ j+r \le n$. Since there are $\lceil n/2 \rceil$ alternate G-layers starting from G^{t_1}, and $m-2$ vertices in each G-layer, $Sg_e(P_m[P_n]) \ge 2n + (m-2)\lceil n/2 \rceil$. The different geodesics covering the edges of $P_5[P_5]$ are shown in Figures 12 and 13.



(a) **(b)** **(c)** **(d)**

Figure 13. Coloured vertices denote the elements of $\eta_{Sg_e}(P_5[P_5])$ and the darkened lines denote the various geodesics that are fixed between the vertices in $\eta_{Sg_e}(P_5[P_5])$.

Case 2: n is even.

We consider the $m-2$ H-layers in $G[H]$. As mentioned in Case 1, alternate vertices from all the $m-2$ H-layers should be chosen in $\eta_{Sg_e}(G[H])$. Since n is even, there are $n/2$ alternate G-layers. The last vertex in each H-layer should be chosen in $\eta_{Sg_e}(G[H])$, as the horizontal edge incident with it is a unique edge. Hence, $Sg_e(P_m[P_n]) \ge 2n + (m-2)((n/2)+1)$.

To derive the upper bound, consider the set $\eta_{Sg_e}(G[H]) = (V(G) \times V(H)) - \{(s_i, t_j)\}$, where

- $(s_i, t_j) \in \{(G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-2}}, G^{t_n}) \bigcap {}^{s_2}H, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-2}}, G^{t_n}) \bigcap {}^{s_3}H, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-2}}, G^{t_n}) \bigcap {}^{s_4}H, \ldots, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-2}}, G^{t_n}) \bigcap {}^{s_{m-2}}H, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-2}}, G^{t_n}) \bigcap {}^{s_{m-1}}H\}$, when n is even.

- $(s_i, t_j) \in \{(G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-3}}, G^{t_{n-1}}) \bigcap {}^{s_2}H, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-3}}, G^{t_{n-1}}) \bigcap {}^{s_3}H, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-3}}, G^{t_{n-1}}) \bigcap {}^{s_4}H, \ldots, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-3}}, G^{t_{n-1}}) \bigcap {}^{s_{m-2}}H, (G^{t_2}, G^{t_4}, G^{t_6}, \ldots, G^{t_{n-3}}, G^{t_{n-1}}) \bigcap {}^{s_{m-1}}H\}$, when n is odd.

Clearly, $\eta_{Sg_e}(G[H]) = \begin{cases} 2n + (m-2)\lceil \frac{n}{2} \rceil & n \text{ is odd,} \\ 2n + (m-2)(\frac{n}{2} + 1) & n \text{ is even,} \end{cases}$
and all the edges of $G[H]$ are covered by fixing one geodesic between each pair of vertices in $\eta_{Sg_e}(G[H])$. Hence, $Sg_e(P_m[P_n]) \leq \begin{cases} 2n + (m-2)\lceil \frac{n}{2} \rceil & n \text{ is odd,} \\ 2n + (m-2)(\frac{n}{2} + 1) & n \text{ is even.} \end{cases}$ $\square$

5. Conclusions

We have studied the strong geodetic number and its edge counterpart in the lexicographic product of graphs G and H. Lower bound for $Sg(G[H])$ has been established and since $Sg_e(G[H]) \geq Sg(G[H])$, the lower bound holds for the edge variant as well. An upper bound for $Sg(G[H])$ is derived, where G is general graph and H is neither a complete graph nor a complete bipartite graph. We have determined $Sg(G[H])$ for certain graph classes including P_m, C_m, K_m, $K_{1,m}$, diameter-2 graphs whose geodetic number is 2, arbitrary trees, and $K_n - e$. We have observed that the strong geodetic number is dependent on the geodetic number of one of the factor graphs involved in the product. Further, $Sg_e(K_m[P_n])$, $Sg_e(P_m[K_n])$ and $Sg_e(P_m[P_n])$ are determined.

Although determining $Sg(G[H])$ for general graphs G and H is the final objective, finding an upper bound for general graphs is itself a quite challenging problem. The strong edge geodetic problem is yet to be solved for $G[H]$ where G and H are general graphs. Though the parameter is explored in the Cartesian product $G \,\square\, H$, the strong edge geodetic number of $P_m \,\square\, P_n$ is still open. Other product graphs including strong product and direct product of graphs are also unexplored with respect to the strong geodetic parameters.

Author contributions

S. Gajavalli: Conceptualization, formal analysis, methodology, writing-original draft; A. Berin Greeni: Conceptualization, investigation, methodology, supervision, validation, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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