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Research article

Space-time decay rate of the 3D diffusive and inviscid Oldroyd-B system

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Abstract: We investigate the space-time decay rates of solutions to the 3D Cauchy problem of the compressible Oldroyd-B system with diffusive properties and without viscous dissipation. The main novelties of this paper involve two aspects: On the one hand, we prove that the weighted rate of k-th order spatial derivative (where $0 \leq k \leq 3$) of the global solution (ρ, u, η, τ) is $t^{-\frac{3}{4}+\frac{k}{2}+\gamma}$ in the weighted Lebesgue space L^2_γ. On the other hand, we show that the space-time decay rate of the m-th order spatial derivative (where $m \in [0, 2]$) of the extra stress tensor of the field in L^2_γ is $(1+t)^{-\frac{5}{4}-\frac{m}{2}+\gamma}$, which is faster than that of the velocity. The proofs are based on delicate weighted energy methods and interpolation tricks.

Keywords: Oldroyd-B system; diffusive properties; without viscous dissipation; space-time decay rates; weighted Sobolev space
Mathematics Subject Classification: 35B40, 35Q35, 74H40, 76N17

1. Introduction

We investigate the space-time decay rates of strong solutions to the diffusive Oldroyd-B system, which describes the motion of viscoelastic fluids in $\mathbb{R}^3$. The system takes the following form in the space-time cylinder $Q_T = \mathbb{R}^3 \times (0, T]$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\mu + \nu)\nabla \operatorname{div} u = \operatorname{div}\left(\mathbb{T} - \left(kL\eta + \iota\eta^2\right)\mathbb{I}\right), \\ \eta_t + \operatorname{div}(\eta u) = \alpha\Delta\eta, \\ \mathbb{T}_t + \operatorname{div}(u\mathbb{T}) - \left(\nabla u\mathbb{T} + \mathbb{T}\nabla^T u\right) = \alpha\Delta\mathbb{T} + \frac{kA_0}{2\lambda}\eta\mathbb{I} - \frac{A_0}{2\lambda}\mathbb{T}, \end{cases} \tag{1.1}$$

where $(x, t) \in \mathbb{R}^3 \times [0, +\infty]$. Let $P = P(\rho) = a\rho^\xi$, $\rho = \rho(x, t) > 0$, $u = u(x, t) \in \mathbb{R}^3$ and $\mathbb{T}(x, t) = \mathbb{T}_{i,j}(x, t) \in \mathbb{R}^3$ denote the pressure, the density, the velocity field, and the extra stress tensor of the field respectively. In these expressions, the constants $a > 0$, $\xi > 1$, and $\alpha > 0$, where α represents the center-of-mass diffusion coefficient of the system. The viscosity coefficients $\mu \geq 0$ and ν satisfy $2\mu + 3\nu \geq 0$. The polymer number density

$$\eta(x,t) = \int_{\mathbb{R}^3} \psi(x,t,q)\mathrm{d}q,$$

where η represents the integral of the probability density function ψ, a microscopic variable used in the modeling of dilute polymer chains. In addition, ι, k, L, and A_0 are known positive constants, and their meanings can be found in [3]. $\mathbb{T}$ is a positive symmetric matrix in Q_T, where $1 \leq i, j \leq 3$.

1.1. History of the problem

Let us provide essential explanations regarding the above model. The system (1.1) is a crucial model employed to characterize the motion of viscoelastic fluids. This model takes the form of the micro-macro compressible Navier-Stokes-Fokker-Planck model, delineating the motion of dilute polymer fluids under the Hookbell-Hookean setting. Barrett originally derived this formulation in [3], and additional physical background can be found in [3,7]. It is worth mentioning that the diffusive Oldroyd-B model for viscoelastic rate-type fluids has been extensively studied in [1, 20, 21]. Additionally, the diffusive Oldroyd-B model can be obtained as a macroscopic closure of the Fokker-Planck-Navier-Stokes systems, as discussed in [4, 15].

For the incompressible diffusive Oldroyd-B model, existence and uniqueness results are available in [8–10]. Regarding the long-term behavior of solutions, comprehensive discussions can be found in [12, 24, 26] and references therein. For the compressible diffusive Oldroyd-B model, Fang and Zi established the existence of local strong solutions and introduced a novel blow-up criterion in [11]. The global existence of small classical solutions is explored by Zhu in [31] for the Sobolev space H^s with $s \geq 5$, and [29,32] for the critical Besov spaces. As for the long-term behavior of global solutions, we refer to [16, 25, 29, 30].

For the Oldroyd-B system (1.1) without viscous dissipation, Liu, Wang, and Wen [17] established the global-in-time existence and obtained optimal time decay rates for the strong solution. In order to address the loss of regularity for the velocity and achieve smallness of the initial data independent of the viscosity, they introduced a new unknown $\tau_{i,j} = \mathbb{T}_{i,j} - k\eta\mathbb{I}_{i,j}$ to derive new dissipative estimates of velocity. Following the approach in [17], we reformulate the system (1.1). To see this, we introduce a change of variables by

$$(\rho, u, \eta, \tau)(x,t) \to (\rho' + \tilde{\rho}, \beta u', \eta' + \tilde{\rho}, \tau)(x,t), \tag{1.2}$$

where β is a positive constant. The initial conditions are given by

$$(\rho', u', \eta', \tau)(x,t)|_{t=0} = (\rho'_0, u'_0, \eta'_0, \tau_0) \to (0,0,0,0). \tag{1.3}$$

For simplicity, we removed all $'$ in the new system. And then the system (1.1) without viscous dissipation (i.e., $\mu = \nu = 0$) in Q_T is equivalent to the following system:

$$\begin{cases} \rho_t + r_1 \operatorname{div} u = S_1, \\ u_t + r_1 \nabla\rho + r_2 \nabla\eta - r_3 \operatorname{div} \tau = S_2, \\ \eta_t + \beta\tilde{\eta} \operatorname{div} u - \alpha\Delta\eta = S_3, \\ \tau_t + \frac{A_0}{2\lambda}\tau - \alpha\Delta\tau - \beta k\tilde{\eta}\left(\nabla u + \nabla^T u\right) = S_4, \end{cases} \tag{1.4}$$

with the initial condition

$$(\rho, u, \eta, \tau)(x, 0) = (\rho_0, u_0, \eta_0, \tau_0)(x) \to (0, 0, 0, 0), \quad \text{as } |x| \to \infty, \tag{1.5}$$

where

$$\begin{cases} S_1 = -\beta \operatorname{div}(\rho u), \\ S_2 = -\beta u \cdot \nabla u + H(\rho)\nabla\rho + G(\rho)[(k(L-1) + 2\iota\tilde{\eta})\nabla\eta - \operatorname{div}\tau] - \frac{2\iota}{\beta(\rho+\tilde{\rho})}\eta\nabla\eta, \\ S_3 = -\beta \operatorname{div}(\eta u), \\ S_4 = -\beta \operatorname{div}(u\tau) + \beta\left(\nabla u\tau + \tau\nabla^T u\right) + \beta k\eta\left(\nabla u + \nabla^T u\right), \end{cases} \tag{1.6}$$

and

$$\beta = \frac{\sqrt{P(\tilde{\rho})}}{\tilde{\rho}}, \quad r_1 = \sqrt{P(\tilde{\rho})}, \quad r_2 = \frac{k(L-1) + 2\tilde{\eta}\xi}{\sqrt{P(\tilde{\rho})}}, \quad r_3 = \frac{1}{\sqrt{P(\tilde{\rho})}}.$$

$H(\rho)$ and $G(\rho)$ are given nonlinear functions of ρ

$$H(\rho) = \frac{1}{\beta}\left(\frac{P(\tilde{\rho})}{\tilde{\rho}} - \frac{P(\tilde{\rho}+\rho)}{\tilde{\rho}+\rho}\right), \qquad G(\rho) = \frac{1}{\beta}\left(\frac{1}{\tilde{\rho}} - \frac{1}{\rho+\tilde{\rho}}\right).$$

Building upon the above conclusions, when the initial perturbation is small in Sobolev space, the global solution of the Cauchy problem (1.4)–(1.5) has been proved in the Sobolev space $H^3(\mathbb{R}^3)$ by Liu et al. in [17]. Moreover, if the initial perturbation is additionally bounded in $L^1(\mathbb{R}^3)$, the solution exhibits the following decay estimates:

$$\|\nabla^k(\rho - \tilde{\rho}, u, \eta - \tilde{\eta})(t)\|_{L^2(\mathbb{R}^3)} \le \tilde{C}_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \quad for \quad k = 0, 1, 2, 3, \tag{1.7}$$

$$\|\nabla^k\tau(t)\|_{L^2(\mathbb{R}^3)} \le \tilde{C}_1(1+t)^{-\frac{5}{4}-\frac{k}{2}} \quad for \quad k = 0, 1, 2, \tag{1.8}$$

$$\|\nabla^3\tau(t)\|_{L^2(\mathbb{R}^3)} \le \tilde{C}_2(1+t)^{-\frac{9}{4}}. \tag{1.9}$$

The space-time decay rate of strong solutions has been receiving increasing attention. Below, we will discuss the progress concerning the space-time decay in the weighted Sobolev space H^γ_ℓ. In [23], Takahashi first established the space-time decay rate of strong solutions to the Navier-Stokes equations. Furthermore, Kukavica et al. extended the weighted decay rate of the strong solution in $L^p_\gamma(2 \le p \le \infty)$ for $n(n \ge 2)$ dimensions in [13, 14]. For more results concerning the space-time decay rate in L^p_γ, we refer to [6, 18, 19, 28] and references therein.

However, to the best of our knowledge, there has been no result on the space-time decay rate of the 3D Cauchy problem of the compressible Oldroyd-B system with diffusive properties and without viscous dissipation up to now. The main motivation of this paper is to provide a definitive answer to this issue. More precisely, based on the time-decay estimates of [17], we demonstrate that the weighted rate of $k(0 \le k \le 3)$-th order spatial derivative of the global solution (ρ, u, η, τ) is $t^{-\frac{3}{4}+\frac{k}{2}+\gamma}$ in the weighted Lebesgue space L^2_γ. Moreover, we also establish that the space-time decay rate of $m\,(\in [0, 2])$-th order spatial derivative of the extra stress tensor of the field in L^2_γ is $(1+t)^{-\frac{5}{4}-\frac{m}{2}+\gamma}$, which is notably faster than that of the velocity. The proofs rely on delicate weighted energy methods and interpolation tricks.

1.2. Notations

Let's introduce the notations typically used in this paper. We use L^p to denote the usual Lebesgue space $L^p\left(\mathbb{R}^3\right)$ with the norm $\|\cdot\|_{L^p}$, and H^ℓ to denote Sobolev spaces $H^\ell\left(\mathbb{R}^3\right) = W^{\ell,2}\left(\mathbb{R}^3\right)$ with the norm $\|\cdot\|_{H^\ell}$. For any $\gamma \in \mathbb{R}$, we denote the weighted Lebesgue space by $L^p_\gamma\left(\mathbb{R}^3\right)$ (where $2 \le p < +\infty$) with respect to the spatial variables:

$$L^p_\gamma\left(\mathbb{R}^3\right) \triangleq \left\{ f(x) : \mathbb{R}^3 \to \mathbb{R}, \|f\|^p_{L^p_\gamma(\mathbb{R}^3)} \triangleq \int_{\mathbb{R}^3} |x|^{p\gamma} |f(x)|^p \, \mathrm{d}x < +\infty \right\}.$$

And then, for any $\gamma \in \mathbb{R}$, we can define the weighted Sobolev space H^k_γ as follows:

$$H^k_\gamma\left(\mathbb{R}^3\right) := \left\{ f(x) \in L^p_\gamma\left(\mathbb{R}^3\right) \mid \|f\|^2_{H^k_\gamma} := \sum_{l \le k} \left\|\nabla^l f\right\|^2_{L^2_\gamma} < \infty \right\}.$$

Denote $L^2\left(\mathbb{R}^3\right) := L^2_0\left(\mathbb{R}^3\right)$ and $H^k\left(\mathbb{R}^3\right) := H^k_0\left(\mathbb{R}^3\right)$. In addition, we denote $\|(a,b)\|_X \triangleq \|a\|_X + \|b\|_X$ for simplicity, and denote δ by a sufficiently small constant independent of time. The notation $a \lesssim b$ means that $a \le Cb$ for a generic positive constant $C > 0$ depends only on the parameters relevant to the problem. Moreover, we drop the x-dependence of differential operators; hence, $\nabla f = \nabla_x f = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$, and ∇^k to denote any partial derivative ∂^α with multi-index α, where $|\alpha| = k$.

1.3. Main results

Before presenting our main results, let's provide a brief summary of time decay estimates for the compressible Oldroyd-B system, both with and without viscous dissipation, as discussed in [17, 25]. These findings are summarized in the following two propositions:

Proposition 1.1. (see [25]) We assume that $L \ge 1$, $\iota \ge 0$ in system (1.1). There exists a positive constant δ_1, which is small enough, such that if

$$\|(\rho - \rho_\infty, u_0, \eta - \eta_\infty, \mathbb{T}_0 - \mathbb{T}_\infty)\|^2_{H^3(\mathbb{R}^3)} \le \delta_1,$$

where

$$(\rho, u, \eta, \mathbb{T})(x, 0) = (\rho_0, u_0, \eta_0, \mathbb{T}_0)(x) \to (\rho_\infty, 0, \eta_\infty, \mathbb{T}_\infty), \quad |x| \to +\infty, \tag{1.10}$$

and $\mathbb{T}_\infty$ is a matrix with $\mathbb{T}_\infty = k\eta_\infty\mathbb{I}$. Additionally, we suppose that $\|(\rho_0 - \rho_\infty, u_0, \eta_0 - \eta_\infty, \operatorname{div}\mathbb{T}_0)\|_{L^1}$ is bounded. Then the diffusive Oldroyd-B system with viscous dissipation (1.1) with the initial data (1.10) admits a unique global strong solution $(\rho, u, \eta, \mathbb{T})$, which satisfies the following time-decay estimates:

$$\|\nabla^m(\rho - \rho_\infty, u, \eta - \eta_\infty, \mathbb{T} - \mathbb{T}_\infty(t))\| \le \tilde{C}(1+t)^{-\frac{3}{4}-\frac{m}{2}}, \quad m = 0, 1,$$

$$\|\nabla^m(\rho - \rho_\infty, u, \eta - \eta_\infty, \mathbb{T} - \mathbb{T}_\infty(t))\| \le \tilde{C}(1+t)^{-\frac{7}{4}}, \quad m = 2, 3.$$

for all $0 \le t \le T$, where $\tilde{C}$ is a positive constant independent of t.

Proposition 1.2. (see [17]) We assume that $(\rho_0 - \tilde{\rho}, u_0, \eta_0 - \tilde{\eta}, \tau_0) \in H^3(\mathbb{R})$, and $(\rho_0 - \tilde{\rho}, u_0, \eta_0 - \tilde{\eta}, \operatorname{div}\tau_0) \in L^1\left(\mathbb{R}^3\right)$, the parameters $L \ge 1$ and $\iota \ge 0$. There exists a positive constant δ_2, which is small enough, such that if

$$\|(\rho_0 - \tilde{\rho}, u_0, \eta_0 - \tilde{\eta}, \tau_0)\|_{H^3(\mathbb{R}^3)} \le \delta_2,$$

the diffusive Oldroyd-B system without viscous dissipation (1.4) with the initial data (1.5) admits a unique global strong solution (ρ, u, η, τ), which satisfies the following time-decay estimates:

$$\|\nabla^k(\rho, u, \eta)(t)\|_{L^2(\mathbb{R}^3)} \le \tilde{C}_0(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \qquad k = 0, 1, 2, 3,$$

$$\|\nabla^k\tau(t)\|_{L^2(\mathbb{R}^3)} \le \tilde{C}_1(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \qquad k = 0, 1, 2,$$

$$\|\nabla^3\tau(t)\|_{L^2(\mathbb{R}^3)} \le \tilde{C}_2(1+t)^{-\frac{9}{4}}.$$

for all $0 \le t \le T$, where $\tilde{C}_0$, $\tilde{C}_1$ and $\tilde{C}_2$ are positive constants that are independent of t.

With the help of time-decay estimates in [17], our main results are concerned with the following space-time decay rates of the strong solutions in weighted Lebesgue space L^2_γ.

Theorem 1.3. Let (ρ, u, η, τ) be the strong solution to the Cauchy problem (1.4)–(1.5). In addition, we assume that $(\rho_0 - \tilde{\rho}, u_0, \eta_0 - \tilde{\eta}, \operatorname{div}\tau_0) \in L^1\left(\mathbb{R}^3\right) \bigcap H^3_\gamma\left(\mathbb{R}^3\right), \gamma \geqslant 0$. Then, if there exists a small constant $\delta_0 > 0$, such that

$$\|(\rho_0 - \tilde{\rho}, u_0, \eta_0 - \tilde{\eta}, \tau_0)\|_{H^3} \le \delta_0,$$

then there exists a large enough T such that

$$\|\nabla^k(\rho, u, \eta)(t)\|_{L^2_\gamma} \le Ct^{-\frac{3}{4}-\frac{k}{2}+\gamma}, \tag{1.11}$$

$$\|\nabla^m\tau(t)\|_{L^2_\gamma} \le Ct^{-\frac{5}{4}-\frac{m}{2}+\gamma}, \tag{1.12}$$

$$\|\nabla^3\tau(t)\|_{L^2_\gamma} \le Ct^{-\frac{9}{4}+\gamma}, \tag{1.13}$$

for all $t > T$, $0 \le k \le 3$, and $0 \le m \le 2$, where C is a positive constant independent of t.

Remark 1.4. We can derive the space-time decay rates of the smooth solution (ρ, u, η, τ) in weighted Lebesgue space L^p_γ by applying Gagliardo-Nirenberg-Sobolev inequality and the weighted interpolation inequality. For any $f(s) \in L^2\left(\mathbb{R}^3\right) \cap \dot{H}^2\left(\mathbb{R}^3\right)$, we have $\|f(s)\|_{L^\infty} \le \|f(s)\|_{L^2}^{\frac{1}{4}} \|f(s)\|_{H^2}^{\frac{3}{4}}$ in $\mathbb{R}^3$. Therefore, we obtain the estimate $\left\||x|^\gamma \nabla^k(\rho, u, \eta, \tau)(t)\right\|_{L^\infty}$ from $\left\||x|^\gamma \nabla^k(\rho, u, \eta, \tau)(t)\right\|_{L^2}$ and $\left\||x|^\gamma \nabla^k(\rho, u, \eta, \tau)(t)\right\|_{\dot{H}^2}$. More specifically, for the case of $\gamma \ge 2$ and $k = 0$, using the Gagliardo-Nirenberg-Sobolev inequality, we have

$$\begin{aligned}
&\||x|^\gamma(\rho, u, \eta, \tau)(t)\|_{L^\infty} \\
\le& C\,\||x|^\gamma(\rho, u, \eta, \tau)(t)\|_{L^2}^{\frac{1}{4}} \left\|\nabla^2(|x|^\gamma(\rho, u, \eta, \tau)(t))\right\|_{L^2}^{\frac{3}{4}} \\
\le& C\,\|(\rho, u, \eta, \tau)(t)\|_{L^2_\gamma}^{\frac{1}{4}} (\left\|\nabla^2(\rho, u, \eta, \tau)(t)\right\|_{L^2_\gamma} + \|\nabla(\rho, u, \eta, \tau)(t)\|_{L^2_{\gamma-1}} + \|(\rho, u, \eta, \tau)(t)\|_{L^2_{\gamma-2}})^{\frac{3}{4}} \\
\le& Ct^{-\frac{3}{2}+\gamma}.
\end{aligned}$$

On the other hand, one has the following facts by applying the weighted interpolation inequality, and (1.10)–(1.12)

$$\begin{aligned}
\|(\rho, u, \eta, \tau)(t)\|_{L^p_\gamma} &\le C\,\|(\rho, u, \eta, \tau)(t)\|_{L^2_\gamma}^{\frac{2}{p}} \||x|^\gamma(\rho, u, \eta, \tau)(t)\|_{L^\infty}^{1-\frac{2}{p}} \\
&\le Ct^{-\frac{3}{2}(1-\frac{1}{p})+\gamma}.
\end{aligned}$$

In a similar way, for the case of $\gamma \geq 2$ and $k = 1$, one has

$$\begin{aligned}\|\nabla(\rho,u,\eta,\tau)(t)\|_{L^p_\gamma} \leq & C\,\|\nabla(\rho,u,\eta,\tau)(t)\|_{L^2_\gamma}^{\frac{2}{p}}\,\||x|^\gamma\nabla(\rho,u,\eta,\tau)(t)\|_{L^\infty}^{1-\frac{2}{p}}\\ \leq & Ct^{(-\frac{5}{4}+\gamma)\frac{2}{p}}t^{(-\frac{11}{4}+\gamma)(1-\frac{2}{p})}\\ \leq & Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}+\gamma}.\end{aligned}$$

For the case $\gamma \in [0,2)$, the corresponding results can be derived from the weighted interpolation inequality. Consequently, by employing the interpolation inequality, we can show that there exists a large enough T such that

$$\left\|\nabla^k(\rho,u,\eta,\tau)(t)\right\|_{L^p_\gamma} \leq Ct^{-\frac{3}{2}(1-\frac{1}{p})-\frac{k}{2}+\gamma}, \tag{1.14}$$

for $t > T$, $2 \leq p \leq \infty$, $k = 0, 1$, and $\gamma \geq 0$, where C is a positive constant independent of t.

Remark 1.5. In this paper, we analyze the space-time decay rates of $\|(\rho,u,\eta,\tau)(t)\|_{L^2_\gamma}$ for $k(\in[0,3])$th-order derivative. Additionally, we determine the sharp space-time decay rate for $k(\in[0,2])$th-order derivative of the variable τ, which revealed the difference between the polymer number density η and the extra stress tensor $\mathbb{T}$.

Now, let us outline the strategies employed in proving Theorem 1.3 and explain the primary challenges encountered in the process. Initially, we introduced

$$\mathrm{E}(t) := \|(\rho,u,\eta,\tau)\|^2_{L^2_\gamma},$$

and then use the delicately weighted energy estimates and interpolation tricks to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \leq C_0t^{-\frac{5}{4}}\mathrm{E}(t) + C_1t^{-\frac{3}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2t^{-\frac{3}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}}. \tag{1.15}$$

In the process of deriving the aforementioned energy inequality, we encounter four trouble terms arising from the dissipative structure of the original system. If these terms didn't exist, we could achieve better space-time decay rates for the solutions. To illustrate this issue, let us consider the zero-th order space-time decay rates of the solutions. In the derivation of zero-th order weighted-energy estimate, we come across these troublesome terms $r_1\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\,\mathrm{div}\,u\,\mathrm{d}x$, $r_1\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla\rho\,\mathrm{d}x$, $r_2\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla\eta\,\mathrm{d}x$ and $r_2\int_{\mathbb{R}^3}|x|^{2\gamma}\eta\,\mathrm{div}\,u\,\mathrm{d}x$. By using integration by parts, we obtain

$$\begin{aligned}&\left|r_1\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\rho u\,\mathrm{d}x\right| + \left|r_2\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\eta u\,\mathrm{d}x\right|\\ \lesssim & \|\rho\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} + \|\eta\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}\\ \lesssim & \|(\rho,\eta)\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}},\end{aligned}$$

and then we get the weighted energy inequality (3.11)

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho\|^2_{L^2_\gamma} + \|u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\tau\|^2_{L^2_\gamma})\\ &+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla\tau\|^2_{L^2_\gamma}\\ \lesssim & t^{-\frac{5}{4}}\|(\rho,u,\eta,\tau)\|^2_{L^2_\gamma} + \|(\rho,\eta)\|_{L^2_\gamma}\|u\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|u\|_{L^2}^{\frac{1}{\gamma}} + \|(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}\|(u,\eta,\tau)\|_{L^2}^{\frac{2}{\gamma}},\end{aligned}$$

which is exactly (1.15). Therefore, one has

$$\mathrm{E}(t) \leq Ct^{-\frac{3}{2}+2\gamma},$$

which implies

$$\|(\rho, u, \eta, \tau)(t)\|_{L^2_{\gamma_0}} \leq C\|(\rho, u, \eta, \tau)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}} \|(\rho, u, \eta, \tau)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}} \leq Ct^{-\frac{3}{4}+\gamma_0},$$

for all $t > T$, $\gamma_0 \in [0, \gamma]$, and $[0, \frac{3}{2}] \subset [0, \gamma](\gamma > \frac{3}{2})$. This implies that the decay rate of the zero–th order of the solution in L^2_γ is $t^{-\frac{3}{4}+\gamma}$.

However, in the absence of the troublesome terms, we can derive a new weighted energy inequality as follows:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho\|^2_{L^2_\gamma} + \|u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\tau\|^2_{L^2_\gamma}) \\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla\tau\|^2_{L^2_\gamma} \\
\lesssim & t^{-\frac{5}{4}}\|(\rho, u, \eta, \tau)\|^2_{L^2_\gamma} + \|(u, \eta, \tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}\|(u, \eta, \tau)\|_{L^2}^{\frac{2}{\gamma}},
\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \leq C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{3}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}},$$

where C_0 and C_1 are positive constants independent of t. If $\gamma > \frac{3}{2}$, then we can apply Lemma 2.4 with $\alpha_0 = \frac{5}{4} > 1, \alpha_1 = \frac{3}{2\gamma} < 1, \beta_1 = \frac{\gamma-1}{\gamma} < 1, \gamma_1 = \frac{1-\alpha_1}{1-\beta_1} = -\frac{3}{2} + \gamma > 0$ to get

$$\mathrm{E}(t) \leq Ct^{-\frac{3}{2}+\gamma},$$

which yields

$$\|(\rho, u, \eta, \tau)(t)\|_{L^2_{\gamma_0}} \leq C\|(\rho, u, \eta, \tau)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}} \|(\rho, u, \eta, \tau)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}} \leq Ct^{-\frac{3}{4}+\frac{\gamma_0}{2}},$$

for all $t > T$, $\gamma_0 \in [0, \gamma]$, and $[0, \frac{3}{2}] \subset [0, \gamma](\gamma > \frac{3}{2})$. This implies that the decay rate of the zero-th order of the solution in L^2_γ is $t^{-\frac{3}{4}+\frac{\gamma}{2}}$.

By employing a similar method, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) &\leq C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{5}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{5}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}} + C_3 t^{-\frac{7}{2}+2\gamma} \quad for \quad k = 1, \\
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) &\leq C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{7}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{7}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}} + C_3 t^{-\frac{9}{2}+2\gamma} \quad for \quad k = 2, \\
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) &\leq C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{9}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{9}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}} + C_3 t^{-\frac{11}{2}+2\gamma} \quad for \quad k = 3.
\end{aligned}$$

The main difficulties in deducing the above estimates arise from the absence of dissipation terms of density and velocity when making the delicate weighted energy estimates. Due to the lack of the dissipation in terms $\int_{\mathbb{R}^3} |x|^\gamma \nabla^4 \rho \, \mathrm{d}x$ or $\int_{\mathbb{R}^3} |x|^\gamma \nabla^4 u \, \mathrm{d}x$ on the left-hand (3.39), it seems impossible for us to handle a new trouble term $\beta \int_{\mathbb{R}^3} |x|^{2\gamma} \rho \nabla^3 \rho \nabla^4 u \, \mathrm{d}x$. To overcome this difficulty, we fully utilize the equations. Specifically, we employ the fact that $\operatorname{div} u = \frac{\rho_t + \beta u \nabla \rho}{r_1 + \beta \rho}$ and $\rho_t = -r_1 \operatorname{div} u - \beta \operatorname{div}(\rho u)$ from $(1.4)_1$, which allows us to reduce the order of the spatial derivative of velocity. More specifically, through integration by parts, we transfer the derivative of u to another term: $\beta \int_{\mathbb{R}^3} \nabla(|x|^{2\gamma} \rho \nabla^3 \rho) \nabla^3 u \, \mathrm{d}x$. Next, one has

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^3} \nabla(|x|^{2\gamma} \rho \nabla^3 \rho) \nabla^3 u \, \mathrm{d}x \right| \\
\lesssim &\left| \beta \int_{\mathbb{R}^3} |x|^{2\gamma} \rho (\rho_t + \beta u \nabla \rho) \nabla^3 \rho \nabla^3 \left(\frac{1}{r_1 + \beta \rho} \right) \mathrm{d}x \right| + \|\nabla u\|_{L^\infty} \|\nabla^3 \rho\|_{L^2_\gamma}^2 \\
&+ \|u\|_{L^\infty} \|\nabla^3 \rho\|_{L^2_\gamma} \|\nabla^3 \rho\|_{L^2_{\gamma-1}} + \|\nabla \rho\|_{L^\infty} \|\nabla^3 \rho\|_{L^2_\gamma} \|\nabla^3 u\|_{L^2_\gamma}.
\end{aligned}
$$

Subsequently, we derive the weighted energy inequality by applying Cauchy's inequality, Gagliardo-Nirenberg-Sobolev's inequality, intricate weighted energy estimates, and interpolation tricks. For more details, we refer to the proofs from (3.41) to (3.50).

Next, utilizing the Gronwall-type lemma, we prove Lemmas 3.1–3.4. Finally, we establish the Lyapunov-type energy inequality to prove Lemma 3.5:

$$
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^m \tau\|_{L^2_\gamma}^2 + C' \|\nabla^m \tau\|_{L^2_\gamma}^2 + C'' \|\nabla^{m+1} \tau\|_{L^2_\gamma}^2 \lesssim t^{-\frac{5}{2} - m + 2\gamma}.
$$

By combining the previously obtained lemmas, we can complete the proof Theorem 1.3.

2. Reformulation

Before proving Theorem 1.3, let us list several tools that will be frequently used in the article. Firstly, we recall the well-known Sobolev interpolation inequalities.

Lemma 2.1. *(see [22]) (Gagliardo-Nirenberg-Sobolev inequality) Let* $2 \le p \le \infty$, $0 \le s, l \le k$ *and* $0 \le \theta \le 1$ *, then*

$$
\|\nabla^s f\|_{L^p} \lesssim \left\|\nabla^k f\right\|_{L^r}^\theta \left\|\nabla^l f\right\|_{L^q}^{1-\theta},
$$

where θ *is given by*

$$
\frac{s}{3} - \frac{1}{p} = \left(\frac{k}{3} - \frac{1}{r} \right) \theta + \left(\frac{l}{3} - \frac{1}{q} \right) (1 - \theta).
$$

Particularly, when $p = 3$*,* $q = r = 2$*,* $s = l = 0$*, and* $k = 1$*, one has*

$$
\|f\|_{L^3} \lesssim \|f\|_{H^1}, \tag{2.1}
$$

when $p = \infty$*,* $q = r = 2$*,* $s = 0$*,* $l = 1$*, and* $k = 2$*, we get*

$$
\|f\|_{L^\infty} \lesssim \|\nabla f\|_{H^1}, \tag{2.2}
$$

while $s = l = 0$*,* $k = 1$*,* $\theta = 1$*,* $p = q = r = 2$*, and* $\gamma > \frac{3}{2}$*, we obtain*

$$
\|f\|_{L^6_\gamma} \lesssim \|\nabla f\|_{L^2_\gamma} + \|f\|_{L^2_{\gamma-1}}. \tag{2.3}
$$

Lemma 2.2. *(see [5]) Assume that there exists a function $f(s)$ that satisfies*

$$f(s) \sim s,$$

and

$$\|f^{(k)}(s)\| \leq C(k),$$

for any integer $k \geq 1$, one has

$$\|f^{(k)}(s)\|_{L^p} \leq C(k)\|f^{(k)}(s)\|_{L^p},$$

for any integer $k \geq 0$ and $p \geq 2$, where $C(k)$ is a constant independent of t. Especially, it holds that $G(\rho) \sim \bar{O}(1)(\rho)$ in this paper, and then $\||x|^{2\gamma}\nabla^k G(\rho)\|_{L^p} \leq C\||x|^{2\gamma}\nabla^k \rho\|_{L^p}$, i.e.,

$$\|\nabla^k G(\rho)\|_{L^2_\gamma} \leq C\|\nabla^k \rho\|_{L^2_\gamma}.$$

And by the same token, it holds that $\|\nabla^k H(\rho)\|_{L^2_\gamma} \leq C\|\nabla^k \rho\|_{L^2_\gamma}$.

Lemma 2.3. *For the vector function $f \in C_0^\infty\left(\mathbb{R}^3\right)$ and bounded scalar function g, it holds that*

$$\left|\int_{\mathbb{R}^3}\left(\nabla |x|^{2\gamma}\right)\cdot fg \,\mathrm{d}x\right| \lesssim \|g\|_{L^2_\gamma}\|f\|_{L^2_{\gamma-1}}.$$

Proof. We compute

$$\left|\int_{\mathbb{R}^3}\left(\nabla |x|^{2\gamma}\right)\cdot fg \,\mathrm{d}x\right| = \left|2\gamma\int_{\mathbb{R}^3}|x|^{2\gamma-2}x_j\partial_i x_j g f_i \,\mathrm{d}x\right| \lesssim \|g\|_{L^2_\gamma}\|f\|_{L^2_{\gamma-1}}.$$

Thus, we complete the proof of Lemma 2.3. □

Lemma 2.4. *(see [2]) (Interpolation inequality with weights) If $p \geqslant 1$, $r \geqslant 1$, $s + \frac{n}{r} > 0$, $a + \frac{n}{p} > 0$, $b + \frac{n}{q} > 0$, and $0 \leqslant \theta \leqslant 1$, then*

$$\|f\|_{L^r_s} \leq \|f\|^{\theta}_{L^p_a}\|f\|^{1-\theta}_{L^q_b},$$

for $f \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q},$$

and

$$s = a\theta + b(1-\theta).$$

More specifically, while $s = p = q = 2$, $\theta = \frac{\gamma-1}{\gamma}$, $s = \gamma - 1$, $a = \gamma$, $b = 0$, one has

$$\|f\|_{L^2_{\gamma-1}} \leq \|f\|^{\frac{\gamma-1}{\gamma}}_{L^2_\gamma}\|f\|^{\frac{1}{\gamma}}_{L^2}. \tag{2.4}$$

Lemma 2.5. *(see [27]) (Gronwall-type Lemma) Let $\alpha_0 > 1$, $\alpha_1 < 1$, $\alpha_2 < 1$, and $\beta_1 < 1$, $\beta_2 < 2$. Assume that a continuously differential function $F : [1,\infty) \to [0,\infty)$ satisfies*

$$\frac{d}{dt}F(t) \leq C_0 t^{-\alpha_0}F(t) + C_1 t^{-\alpha_1}F(t)^{\beta_1} + C_2 t^{-\alpha_2}F(t)^{\beta_2} + C_3 t^{\gamma_1 - 1}, t \geq 1$$

and

$$F(1) \le K_0,$$

where C_0, C_1, C_2, C_3, $K_0 \ge 0$ *and* $\gamma_i = \frac{1-\alpha_i}{1-\beta_i} > 0$ *for* $i = 1, 2$. *Assume that* $\gamma_1 \ge \gamma_2$, *then there exists a constant* $\tilde{C}$ *depending on* α_0, α_1, β_1, α_2, β_2, K_0, C_i, *where* $i = 1, 2, 3$, *such that*

$$F(t) \le \tilde{C} t^{\gamma_1},$$

for all $t \ge 1$.

3. The proof of Theorem 1.3

We can make use of the precise linear approximations for (ρ, u, η, τ) found in [17] to prove Theorem 1.3.

Lemma 3.1. *Under the assumption of Theorem* 1.3, *there exists a sufficiently large* T *such that the solution* (ρ, u, η, τ) *of the system* (1.4) *with the initial data* (1.5) *has the following estimate:*

$$\|(\rho, u, \eta, \tau)(t)\|_{L^2_\gamma} \le C t^{-\frac{3}{4}+\gamma}, \tag{3.1}$$

for all $t > T$ *and* $\gamma \ge 0$, *where* C *is a positive constant independent of* t.

Proof. Multiplying $|x|^{2\gamma}\rho$, $|x|^{2\gamma}u$, $\frac{r_2}{\beta\tilde{\eta}}|x|^{2\gamma}\eta$, $\frac{r_3}{2\beta k\tilde{\eta}}|x|^{2\gamma}\tau$ by $(1.4)_1$–$(1.4)_4$, and then adding them up and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |x|^{2\gamma}\rho\rho_t \,\mathrm{d}x + r_1 \int_{\mathbb{R}^3} |x|^{2\gamma}\rho \operatorname{div} u \,\mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma} u u_t \,\mathrm{d}x + r_1 \int_{\mathbb{R}^3} |x|^{2\gamma} u \nabla\rho \,\mathrm{d}x \\
&+ r_2 \int_{\mathbb{R}^3} |x|^{2\gamma} u \nabla\eta \,\mathrm{d}x - r_3 \int_{\mathbb{R}^3} |x|^{2\gamma} u \operatorname{div}\tau \,\mathrm{d}x + \frac{r_2}{\beta\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma}\eta\eta_t \,\mathrm{d}x \\
&+ r_2 \int_{\mathbb{R}^3} |x|^{2\gamma}\eta \operatorname{div} u \,\mathrm{d}x - \frac{r_2\alpha}{\beta\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma}\eta\triangle\eta \,\mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma}\tau\tau_t \,\mathrm{d}x \\
&+ \frac{r_3}{2\beta k\tilde{\eta}} \frac{A_0}{2\lambda} \int_{\mathbb{R}^3} |x|^{2\gamma}\tau^2 \,\mathrm{d}x + \frac{r_3\alpha}{2\beta k\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma}\tau\triangle\tau \,\mathrm{d}x - \frac{r_3}{2} \int_{\mathbb{R}^3} |x|^{2\gamma}\tau(\nabla u + \nabla^T u) \,\mathrm{d}x \\
=& \int_{\mathbb{R}^3} |x|^{2\gamma}\rho S_1 \mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma} u S_2 \mathrm{d}x + \frac{r_2}{\beta\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma}\eta S_3 \mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma}\tau S_4 \mathrm{d}x.
\end{aligned} \tag{3.2}$$

Then, using integration by parts to simplify, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho\|_{L^2_\gamma}^2+\|u\|_{L^2_\gamma}^2+\frac{r_2}{\beta\tilde{\eta}}\|\eta\|_{L^2_\gamma}^2+\frac{r_3}{2\beta k\tilde{\eta}}\|\tau\|_{L^2_\gamma}^2)\\
&+\frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla\eta\|_{L^2_\gamma}^2+\frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\tau\|_{L^2_\gamma}^2+\frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla\tau\|_{L^2_\gamma}^2\\
=&-r_1\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\,\mathrm{div}\,u\,\mathrm{d}x-r_1\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla\rho\,\mathrm{d}x-r_2\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla\eta\,\mathrm{d}x\\
&-r_2\int_{\mathbb{R}^3}|x|^{2\gamma}\eta\,\mathrm{div}\,u\,\mathrm{d}x+r_3\int_{\mathbb{R}^3}|x|^{2\gamma}u\,\mathrm{div}\,\tau\,\mathrm{d}x+\frac{r_3}{2}\int_{\mathbb{R}^3}|x|^{2\gamma}\tau(\nabla u+\nabla^Tu)\,\mathrm{d}x\\
&-\frac{r_2\alpha}{\beta\tilde{\eta}}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\eta\nabla\eta\,\mathrm{d}x-\frac{r_3\alpha}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\tau\nabla\tau\,\mathrm{d}x\\
&+\int_{\mathbb{R}^3}|x|^{2\gamma}\rho S_1\mathrm{d}x+\int_{\mathbb{R}^3}|x|^{2\gamma}uS_2\mathrm{d}x+\frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\eta S_3\mathrm{d}x+\frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\tau S_4\mathrm{d}x\\
\triangleq&\sum_{i=1}^{12}J_{1,i}.
\end{aligned}\tag{3.3}$$

Initially, by using integration by parts and applying Lemma 2.3 and Young' inequality, we can obtain

$$\begin{aligned}
\sum_{i=1}^{6}J_{1,i}\lesssim&\left|r_1\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\rho u\,\mathrm{d}x\right|+\left|r_2\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\eta u\,\mathrm{d}x\right|+\left|r_3\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})u\tau\,\mathrm{d}x\right|\\
\lesssim&\|\rho\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}+\|\eta\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}+\|\tau\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}\\
\lesssim&\|(\rho,\eta)\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}}+\varepsilon r_3\|\tau\|_{L^2_\gamma}^2+Cr_3(\varepsilon)\|u\|_{L^2_{\gamma-1}}^2.
\end{aligned}\tag{3.4}$$

Applying Lemma 2.3 and Young' inequality, we can get

$$\begin{aligned}
&|J_{1,7}|+|J_{1,8}|\\
\lesssim&\frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla\eta\|_{L^2_\gamma}\|\eta\|_{L^2_{\gamma-1}}+\frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla\tau\|_{L^2_\gamma}\|\tau\|_{L^2_{\gamma-1}}\\
\lesssim&\varepsilon(\frac{r_2\alpha}{\beta\tilde{\eta}})\|\nabla\eta\|_{L^2_\gamma}^2+C\frac{r_2\alpha}{\beta\tilde{\eta}}(\varepsilon)\|\eta\|_{L^2_{\gamma-1}}^2+\varepsilon(\frac{r_3\alpha}{2\beta k\tilde{\eta}})\|\nabla\tau\|_{L^2_\gamma}^2+C\frac{r_3\alpha}{2\beta k\tilde{\eta}}(\varepsilon)\|\tau\|_{L^2_{\gamma-1}}^2.
\end{aligned}\tag{3.5}$$

Using the definitions of S_1, S_2, S_3, and S_4, Lemma 2.1, Cauchy's inequality, and Lemma 2.3, we have

$$\begin{aligned}
|J_{1,9}|=&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho u\nabla\rho\,\mathrm{d}x\right|+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho^2\nabla u\,\mathrm{d}x\right|\\
\lesssim&\|\nabla\rho\|_{L^\infty}\|u\|_{L^2_\gamma}\|\rho\|_{L^2_\gamma}+\|\nabla u\|_{L^\infty}\|\rho\|_{L^2_\gamma}^2\\
\lesssim&\|\nabla^2\rho\|_{H^1}\|u\|_{L^2_\gamma}^2+\|\nabla^2\rho\|_{H^1}\|\rho\|_{L^2_\gamma}^2+\|\nabla^2u\|_{H^1}\|\rho\|_{L^2_\gamma}^2\\
\lesssim&t^{-\frac{7}{4}}\|\rho\|_{L^2_\gamma}^2+t^{-\frac{7}{4}}\|u\|_{L^2_\gamma}^2.
\end{aligned}\tag{3.6}$$

In a similar way, we have

$$
\begin{aligned}
|J_{1,10}| =& \left|\beta \int_{\mathbb{R}^3} |x|^{2\gamma} u^2 \nabla u \, \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} u H(\rho) \nabla \rho \, \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} u G(\rho) \nabla \eta \, \mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma} u G(\rho) \operatorname{div} \tau \, \mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma} u \left(\frac{\eta}{\rho+\tilde{\rho}}\right) \nabla \eta \, \mathrm{d}x\right| \\
\lesssim& \beta \|\nabla u\|_{L^\infty} \|u\|_{L^2_\gamma}^2 + \|\nabla \rho\|_{L^\infty} \|H(\rho)\|_{L^2_\gamma} \|u\|_{L^2_\gamma} + \|G(\rho)\|_{L^\infty} \|\nabla \eta\|_{L^2_\gamma} \|u\|_{L^2_\gamma} \\
&+ \|G(\rho)\|_{L^\infty} \|\nabla \tau\|_{L^2_\gamma} \|u\|_{L^2_\gamma} + \|\frac{\eta}{\rho+\tilde{\rho}}\|_{L^\infty} \|u\|_{L^2_\gamma} \|\nabla \eta\|_{L^2_\gamma} \\
\lesssim& \|\nabla^2 u\|_{H^1} \|u\|_{L^2_\gamma}^2 + \|\nabla^2 \rho\|_{H^1} \|H(\rho)\|_{L^2_\gamma}^2 + \|\nabla^2 \rho\|_{H^1} \|u\|_{L^2_\gamma}^2 + \|\nabla G(\rho)\|_{H^1} \|\nabla \eta\|_{L^2_\gamma}^2 \\
&+ \|\nabla G(\rho)\|_{H^1} \|u\|_{L^2_\gamma}^2 + \|\nabla G(\rho)\|_{H^1} \|\nabla \tau\|_{L^2_\gamma}^2 + \|\nabla \left(\frac{\eta}{\rho+\tilde{\rho}}\right)\|_{H^1} \|u\|_{L^2_\gamma} \|\nabla \eta\|_{L^2_\gamma}^2 \\
\lesssim& t^{-\frac{5}{4}} \|u\|_{L^2_\gamma}^2 + t^{-\frac{7}{4}} \|\rho\|_{L^2_\gamma}^2 + t^{-\frac{5}{4}} \|\nabla(\eta, \tau)\|_{L^2_\gamma}^2,
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
|J_{1,11}| =& \left|\frac{r_2}{\tilde{\eta}} \int_{\mathbb{R}^3} t|x|^{2\gamma} \eta u \nabla \eta \, \mathrm{d}x\right| + \left|\frac{r_2}{\tilde{\eta}} \int_{\mathbb{R}^3} |x|^{2\gamma} \eta^2 \nabla u \, \mathrm{d}x\right| \\
\lesssim& \|u\|_{L^\infty} \|\eta\|_{L^2_\gamma} \|\nabla \eta\|_{L^2_\gamma} + \|\nabla u\|_{L^\infty} \|\eta\|_{L^2_\gamma}^2 \\
\lesssim& \|\nabla u\|_{H^1} \|\eta\|_{L^2_\gamma}^2 + \|\nabla u\|_{H^1} \|\nabla \eta\|_{L^2_\gamma}^2 + \|\nabla^2 u\|_{H^1} \|\eta\|_{L^2_\gamma}^2 \\
\lesssim& t^{-\frac{5}{4}} \|\eta\|_{L^2_\gamma}^2 + t^{-\frac{5}{4}} \|\nabla \eta\|_{L^2_\gamma}^2,
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
|J_{1,12}| \lesssim& \|\nabla \tau\|_{L^\infty} \|u\|_{L^2_\gamma} \|\tau\|_{L^2_\gamma} + \|\nabla u\|_{L^\infty} \|\tau\|_{L^2_\gamma}^2 + \|\nabla u\|_{L^\infty} \|\eta\|_{L^2_\gamma} \|\tau\|_{L^2_\gamma} \\
\lesssim& \|\nabla^2 \tau\|_{H^1} \|u\|_{L^2_\gamma} \|\tau\|_{L^2_\gamma} + \|\nabla^2 u\|_{H^1} \|\tau\|_{L^2_\gamma}^2 + \|\nabla^2 u\|_{H^1} \|\eta\|_{L^2_\gamma} \|\tau\|_{L^2_\gamma} \\
\lesssim& t^{-\frac{7}{4}} \|(u, \eta, \tau)\|_{L^2_\gamma}^2.
\end{aligned} \tag{3.9}
$$

Substituting (3.4) to (3.9) into (3.3), we conclude that there exists a sufficiently large T_1 and a sufficiently small ε, such that

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\|\rho\|_{L^2_\gamma}^2 + \|u\|_{L^2_\gamma}^2 + \frac{r_1}{\beta \tilde{\eta}} \|\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k \tilde{\eta}} \|\tau\|_{L^2_\gamma}^2) \\
&+ \frac{r_2 \alpha}{\beta \tilde{\eta}} \|\nabla \eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k \tilde{\eta}} \frac{A_0}{2\lambda} \|\tau\|_{L^2_\gamma}^2 + \frac{r_3 \alpha}{2\beta k \tilde{\eta}} \|\nabla \tau\|_{L^2_\gamma}^2 \\
\lesssim& t^{-\frac{5}{4}} \|(\rho, u, \eta)\|_{L^2_\gamma}^2 + t^{-\frac{7}{4}} \|\tau\|_{L^2_\gamma}^2 + \|(\rho, \eta)\|_{L^2_\gamma} \|u\|_{L^2_{\gamma-1}} + \|(u, \eta, \tau)\|_{L^2_{\gamma-1}}^2,
\end{aligned} \tag{3.10}
$$

for all $t > T_1$. Using the interpolation inequality with weights $\|f\|_{L^2_{\gamma-1}} \lesssim \|f\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}} \|f\|_{L^2}^{\frac{1}{\gamma}}$, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\rho\|_{L^2_\gamma}^2 + \|u\|_{L^2_\gamma}^2 + \frac{r_2}{\beta\tilde{\eta}}\|\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\|\tau\|_{L^2_\gamma}^2) \\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\tau\|_{L^2_\gamma}^2 + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla\tau\|_{L^2_\gamma}^2 \\
\lesssim & t^{-\frac{5}{4}}\|(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2 + \|(\rho,\eta)\|_{L^2_\gamma}\|u\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|u\|_{L^2}^{\frac{1}{\gamma}} + \|(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}\|(u,\eta,\tau)\|_{L^2}^{\frac{2}{\gamma}} \\
\lesssim & t^{-\frac{5}{4}}\|(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2 + t^{-\frac{3}{4\gamma}}\|(\rho,\eta)\|_{L^2_\gamma}^{\frac{2\gamma-1}{\gamma}} + t^{-\frac{3}{2\gamma}}\|(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}.
\end{aligned} \tag{3.11}$$

Denoting $\mathrm{E}(t) := \|(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2$, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \le C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{3}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{3}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}},$$

where C_0, C_1, and C_2 are positive constants independent of t. If $\gamma > \frac{3}{2}$, then we can apply Lemma 2.5 with $\alpha_0 = \frac{5}{4} > 1$, $\alpha_1 = \frac{3}{4\gamma} < 1$, $\beta_1 = \frac{2\gamma-1}{2\gamma} < 1$, $\alpha_2 = \frac{3}{2\gamma} < 1$, $\beta_2 = \frac{\gamma-1}{\gamma} < 1$, $\gamma_1 = \frac{1-\alpha_1}{1-\beta_1} = -\frac{3}{2} + 2\gamma > 0$, $\gamma_2 = \frac{1-\alpha_2}{1-\beta_2} = -\frac{3}{2} + \gamma > 0$, $\gamma_1 > \gamma_2$. Thus, for all $t > T$, one has

$$\mathrm{E}(t) \le Ct^{-\frac{3}{2}+2\gamma}, \tag{3.12}$$

which implies

$$\|(\rho,u,\eta,\tau)(t)\|_{L^2_{\gamma_0}} \le C\|(\rho,u,\eta,\tau)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}}\|(\rho,u,\eta,\tau)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}} \le Ct^{-\frac{3}{4}+\gamma_0},$$

for all $t > T$, $\gamma_0 \in [0,\gamma]$, and $[0,\frac{3}{2}] \subset [0,\gamma](\gamma > \frac{3}{2})$. Thus, the proof of Lemma 3.1 has been completed. □

Lemma 3.2. *Under the assumption of Theorem* 1.3*, there exists a sufficiently large T such that the solution* (ρ,u,η,τ) *of the system* (1.4) *with the initial data* (1.5) *has the following estimate:*

$$\|\nabla(\rho,u,\eta,\tau)(t)\|_{L^2_\gamma} \le Ct^{-\frac{5}{4}+\gamma}, \tag{3.13}$$

for all $t > T$ and $\gamma \ge 0$, where C is a positive constant independent of t.

Proof. Multiplying $|x|^{2\gamma}\nabla\rho$, $|x|^{2\gamma}\nabla u$, $\frac{r_2}{\beta\tilde{\eta}}|x|^{2\gamma}\nabla\eta$, $\frac{r_3}{2\beta k\tilde{\eta}}|x|^{2\gamma}\nabla\tau$ by $\nabla(1.4)_1 - \nabla(1.4)_4$, and then adding them up and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\rho\nabla\rho_t \,\mathrm{d}x + r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\rho\nabla \operatorname{div} u \,\mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla u_t \,\mathrm{d}x\\
&+ r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla^2\rho \,\mathrm{d}x + r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla^2\eta \,\mathrm{d}x - r_3\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla \operatorname{div}\tau \,\mathrm{d}x\\
&+ \frac{r_2}{\beta\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\eta\nabla\eta_t \,\mathrm{d}x + r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\eta\nabla \operatorname{div} u \,\mathrm{d}x - \frac{r_2\alpha}{\beta\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\eta\nabla\triangle\eta \,\mathrm{d}x\\
&+ \frac{r_3}{2\beta k\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\tau\nabla\tau_t \,\mathrm{d}x + \frac{r_3}{2\beta k\tilde\eta}\frac{A_0}{2\lambda}\int_{\mathbb{R}^3} |x|^{2\gamma}(\nabla\tau)^2 \,\mathrm{d}x\\
&- \frac{r_3\alpha}{2\beta k\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\tau\nabla\triangle\tau \,\mathrm{d}x - \frac{r_3}{2}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\tau\nabla(\nabla u + \nabla^T u) \,\mathrm{d}x\\
=&\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\rho\nabla S_1 \mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla S_2 \mathrm{d}x\\
&+ \frac{r_2}{\beta\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\eta\nabla S_3 \mathrm{d}x + \frac{r_3}{2\beta k\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\tau\nabla S_4 \mathrm{d}x.
\end{aligned} \tag{3.14}$$

Then, using integration by parts to simplify, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\rho\|^2_{L^2_\gamma} + \|\nabla u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde\eta}\|\nabla\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde\eta}\|\nabla\tau\|^2_{L^2_\gamma})\\
&+ \frac{r_2\alpha}{\beta\tilde\eta}\|\nabla^2\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde\eta}\frac{A_0}{2\lambda}\|\nabla\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde\eta}\|\nabla^2\tau\|^2_{L^2_\gamma}\\
=&-r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\rho\nabla \operatorname{div} u \,\mathrm{d}x - r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla^2\rho \,\mathrm{d}x - r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\eta\nabla \operatorname{div} u \,\mathrm{d}x\\
&- r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla^2\eta \,\mathrm{d}x + r_3\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla \operatorname{div}\tau \,\mathrm{d}x + \frac{r_3}{2}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\tau\nabla(\nabla u + \nabla^T u) \,\mathrm{d}x\\
&- \frac{r_2\alpha}{\beta\tilde\eta}\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla\eta\nabla^2\eta \,\mathrm{d}x - \frac{r_3\alpha}{2\beta k\tilde\eta}\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla\tau\nabla^2\tau \,\mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\rho\nabla S_1 \mathrm{d}x\\
&+ \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla S_2 \mathrm{d}x + \frac{r_2}{\beta\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\eta\nabla S_3 \mathrm{d}x + \frac{r_3}{2\beta k\tilde\eta}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla\tau\nabla S_4 \mathrm{d}x\\
\triangleq&\sum_{i=1}^{12} J_{2,i}.
\end{aligned} \tag{3.15}$$

Initially, by using integration by parts and applying Cauchy' inequality and Lemma 2.3, we can obtain

$$\begin{aligned}
\sum_{i=1}^{6} J_{2,i} &\lesssim \left|r_1\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla\rho\nabla u \,\mathrm{d}x\right| + \left|r_2\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla\eta\nabla u \,\mathrm{d}x\right| + \left|r_3\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla\tau\nabla u \,\mathrm{d}x\right|\\
&\lesssim \|\nabla\rho\|_{L^2_\gamma}\|\nabla u\|_{L^2_{\gamma-1}} + \|\nabla\eta\|_{L^2_\gamma}\|\nabla u\|_{L^2_{\gamma-1}} + \|\nabla\tau\|_{L^2_\gamma}\|\nabla u\|_{L^2_{\gamma-1}}\\
&\lesssim \|\nabla(\rho,\eta)\|_{L^2_\gamma}\|\nabla u\|_{L^2_{\gamma-1}} + \varepsilon r_3\|\nabla\tau\|^2_{L^2_\gamma} + Cr_3(\varepsilon)\|\nabla u\|^2_{L^2_{\gamma-1}}.
\end{aligned} \tag{3.16}$$

Applying Lemma 2.3 and Young' inequality, we can get

$$
\begin{aligned}
&|J_{2,7}| + |J_{2,8}| \\
\lesssim&\frac{r_2\alpha}{\beta\tilde{\eta}} \left\|\nabla^2\eta\right\|_{L^2_\gamma} \|\nabla\eta\|_{L^2_{\gamma-1}} + \frac{r_3\alpha}{2\beta k\tilde{\eta}} \left\|\nabla^2\tau\right\|_{L^2_\gamma} \|\nabla\tau\|_{L^2_{\gamma-1}} \\
\lesssim&\varepsilon(\frac{r_2\alpha}{\beta\tilde{\eta}}) \left\|\nabla^2\eta\right\|^2_{L^2_\gamma} + C\frac{r_2\alpha}{\beta\tilde{\eta}}(\varepsilon) \|\nabla\eta\|^2_{L^2_{\gamma-1}} + \varepsilon(\frac{r_3\alpha}{2\beta k\tilde{\eta}}) \left\|\nabla^2\tau\right\|^2_{L^2_\gamma} + C\frac{r_3\alpha}{2\beta k\tilde{\eta}}(\varepsilon) \|\nabla\tau\|^2_{L^2_{\gamma-1}}.
\end{aligned}
\tag{3.17}
$$

By the definitions of S_1, S_2, S_3, and S_4, and Lemma 2.1, Cauchy's inequality, and Lemma 2.3, we have

$$
\begin{aligned}
|J_{2,9}| =&\left|\beta\int_{\mathbb{R}^3} |x|^{2\gamma} u\nabla\rho\nabla^2\rho \,\mathrm{d}x\right| + \left|\beta\int_{\mathbb{R}^3} |x|^{2\gamma} \rho\nabla\rho\nabla^2 u \,\mathrm{d}x\right| + \left|\beta\int_{\mathbb{R}^3} |x|^{2\gamma} \nabla u\nabla\rho\nabla\rho \,\mathrm{d}x\right| \\
\lesssim&\|\nabla^2\rho\|_{L^3}\|\nabla\rho\|_{L^2_\gamma}\|u\|_{L^6_\gamma} + \|\nabla^2 u\|_{L^3}\|\nabla\rho\|_{L^2_\gamma}\|\rho\|_{L^6_\gamma} + \|\nabla u\|_{L^\infty}\|\nabla\rho\|^2_{L^2_\gamma} \\
\lesssim&\|\nabla^2\rho\|_{H^1}\|\nabla\rho\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} + \|\nabla^2\rho\|_{H^1}\|\nabla\rho\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} \\
&+ \|\nabla^2 u\|_{H^1}\|\nabla\rho\|_{L^2_\gamma}\|\rho\|_{L^2_{\gamma-1}} + \|\nabla^2 u\|_{H^1}\|\nabla\rho\|^2_{L^2_\gamma} \\
\lesssim&t^{-\frac{5}{4}-\frac{2}{4}}t^{-\frac{3}{4}+\gamma-1}\|\nabla\rho\|_{L^2_\gamma} + t^{-\frac{7}{4}}\|\nabla\rho\|^2_{L^2_\gamma} + t^{-\frac{7}{4}}\|\nabla u\|^2_{L^2_\gamma} \\
\lesssim&t^{-\frac{5}{4}}\|\nabla(\rho,u)\|^2_{L^2_\gamma} + t^{-\frac{7}{2}+2\gamma}.
\end{aligned}
\tag{3.18}
$$

In a similar way, one has

$$
\begin{aligned}
|J_{2,10}| =&\left|\beta\int_{\mathbb{R}^3} |x|^{2\gamma}(\nabla u)^3 \,\mathrm{d}x\right| + \left|\beta\int_{\mathbb{R}^3} |x|^{2\gamma} u\nabla u\nabla^2 u \,\mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla H(\rho)\nabla\rho \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u H(\rho)\nabla^2\rho \,\mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla G(\rho)\nabla\eta \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u G(\rho)\nabla^2\eta \,\mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla G(\rho)\,\mathrm{div}\,\tau \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u G(\rho)\nabla\,\mathrm{div}\,\tau \,\mathrm{d}x\right| \\
&+ \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla u\nabla\left(\frac{\eta}{\rho+\tilde{\rho}}\right)\nabla\eta \,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^3} |x|^{2\gamma}\left(\frac{\eta}{\rho+\tilde{\rho}}\right)\nabla u\nabla^2\eta \,\mathrm{d}x\right|,
\end{aligned}
$$

therefore

$$
\begin{aligned}
|J_{2,10}| \lesssim&\beta\|\nabla u\|_{L^\infty}\|\nabla u\|^2_{L^2_\gamma} + \beta\|\nabla^2 u\|_{L^3}\|\nabla u\|_{L^2_\gamma}\|u\|_{L^6_\gamma} + \|\nabla H(\rho)\|_{L^\infty}\|\nabla\rho\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} \\
&+ \|\nabla^2\rho\|_{L^3}\|\nabla u\|_{L^2_\gamma}\|H(\rho)\|_{L^6_\gamma} + |k(L-1) + 2\tilde{\eta}\iota|\|\nabla G(\rho)\|_{L^\infty}\|\nabla\eta\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} \\
&+ |k(L-1) + 2\tilde{\eta}\iota|\|G(\rho)\|_{L^\infty}\|\nabla^2\eta\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} \\
&+ \|G(\rho)\|_{L^\infty}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} + \|\nabla G(\rho)\|_{L^\infty}\|\nabla\tau\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} \\
&+ \|\nabla(\frac{\eta}{\rho+\tilde{\rho}})\|_{L^\infty}\|\nabla u\|_{L^2_\gamma}\|\nabla\eta\|_{L^2_\gamma} + \|\frac{\eta}{\rho+\tilde{\rho}}\|_{L^\infty}\|\nabla^2\eta\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma} \\
\lesssim&\|\nabla^2 u\|_{H^1}\|\nabla u\|^2_{L^2_\gamma} + \|\nabla^2 u\|_{H^1}\|\nabla u\|_{L^2_\gamma}\|u\|_{L^2_{\gamma-1}} + \|\nabla^2 H(\rho)\|_{H^1}\|\nabla\rho\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_\gamma} \\
&+ \|\nabla^2\rho\|_{H^1}\|\nabla u\|^2_{L^2_\gamma} + \|\nabla^2\rho\|_{H^1}\|\nabla u\|_{L^2_\gamma}\|\rho\|_{L^2_{\gamma-1}} + \|\nabla^2 G(\rho)\|_{H^1}\|\nabla\eta\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma}
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
&+\|\nabla G(\rho)\|_{H^1}\|\nabla^2\eta\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma}+\|\nabla G(\rho)\|_{H^1}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma}\\
&+\|\nabla^2 G(\rho)\|_{H^1}\|\nabla\tau\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma}+\|\nabla^2(\frac{\eta}{\rho+\tilde\rho})\|_{H^1}\|\nabla\eta\|^2_{L^2_\gamma}\\
&+\|\nabla^2(\frac{\eta}{\rho+\tilde\rho})\|_{H^1}\|\nabla u\|^2_{L^2_\gamma}+\|\nabla(\frac{\eta}{\rho+\tilde\rho})\|_{H^1}\|\nabla^2\eta\|^2_{L^2_\gamma}+\|\nabla(\frac{\eta}{\rho+\tilde\rho})\|_{H^1}\|\nabla u\|^2_{L^2_\gamma}\\
\lesssim& t^{-\frac54}\|\nabla(\rho,u,\eta)\|^2_{L^2_\gamma}+t^{-\frac54}\|\nabla^2(\eta,\tau)\|^2_{L^2_\gamma}+t^{-\frac72+2\gamma},
\end{aligned}
$$

and

$$
\begin{aligned}
|J_{2,11}|=&\left|\frac{r_2}{\tilde\eta}\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla\eta\nabla^2\eta\,\mathrm{d}x\right|+\left|\frac{r_2}{\tilde\eta}\int_{\mathbb{R}^3}|x|^{2\gamma}\eta\nabla\eta\nabla^2 u\,\mathrm{d}x\right|+\left|\frac{r_2}{\tilde\eta}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\eta\nabla\eta\nabla u\,\mathrm{d}x\right|\\
\lesssim&\|u\|_{L^\infty}\|\nabla\eta\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma}+\|\nabla^2u\|_{L^3}\|\nabla\eta\|_{L^2_\gamma}\|\eta\|_{L^6_\gamma}+\|\nabla u\|_{L^\infty}\|\nabla\eta\|^2_{L^2_\gamma}\\
\lesssim&\|\nabla u\|_{H^1}\|\nabla\eta\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma}+\|\nabla^2u\|_{H^1}\|\nabla\eta\|^2_{L^2_\gamma}+\|\nabla^2u\|_{H^1}\|\nabla\eta\|_{L^2_\gamma}\|\eta\|_{L^2_{\gamma-1}}\\
\lesssim& t^{-\frac54}\|\nabla\eta\|^2_{L^2_\gamma}+t^{-\frac54}\|\nabla^2\eta\|^2_{L^2_\gamma}+t^{-\frac72+2\gamma},
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
|J_{2,12}|\lesssim&\|\nabla^2u\|_{L^3}\|\nabla\tau\|_{L^2_\gamma}\|\tau\|_{L^6_\gamma}+\|u\|_{L^\infty}\|\nabla\tau\|_{L^2_\gamma}\|\nabla^2\tau\|_{L^2_\gamma}+\|\nabla u\|_{L^\infty}\|\nabla\tau\|^2_{L^2_\gamma}\\
&+\|\nabla u\|_{L^\infty}\|\nabla\eta\|_{L^2_\gamma}\|\nabla\tau\|_{L^2_\gamma}+\|\nabla^2u\|_{L^3}\|\nabla\tau\|_{L^2_\gamma}\|\eta\|_{L^6_\gamma}\\
\lesssim&\|\nabla^2u\|_{H^1}\|\nabla\tau\|^2_{L^2_\gamma}+\|\nabla^2u\|_{H^1}\|\nabla\tau\|_{L^2_\gamma}\|\tau\|_{L^2_{\gamma-1}}+\|\nabla u\|_{H^1}\|\nabla\tau\|_{L^2_\gamma}\|\nabla^2\tau\|_{L^2_\gamma}\\
&+\|\nabla^2u\|_{H^1}\|\nabla\eta\|_{L^2_\gamma}\|\nabla\tau\|_{L^2_\gamma}+\|\nabla^2u\|_{H^1}\|\nabla\tau\|_{L^2_\gamma}\|\nabla\eta\|_{L^2_\gamma}\\
&+\|\nabla^2u\|_{H^1}\|\nabla\tau\|_{L^2_\gamma}\|\eta\|_{L^2_{\gamma-1}}+\|\nabla^2u\|_{H^1}\|\nabla\tau\|^2_{L^2_\gamma}\\
\lesssim& t^{-\frac54}\|\nabla\tau\|^2_{L^2_\gamma}+t^{-\frac74}\|\nabla\eta\|^2_{L^2_\gamma}+t^{-\frac54}\|\nabla^2\tau\|^2_{L^2_\gamma}+t^{-\frac72+2\gamma}.
\end{aligned}
\tag{3.21}
$$

Substituting (3.16) to (3.21) into (3.15), we conclude that there exists a sufficiently large T_1 and a sufficiently small ε, such that

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\rho\|^2_{L^2_\gamma}+\|\nabla u\|^2_{L^2_\gamma}+\frac{r_2}{\beta\tilde\eta}\|\nabla\eta\|^2_{L^2_\gamma}+\frac{r_3}{2\beta k\tilde\eta}\|\nabla\tau\|^2_{L^2_\gamma})\\
&+\frac{r_2\alpha}{\beta\tilde\eta}\|\nabla^2\eta\|^2_{L^2_\gamma}+\frac{r_3}{2\beta k\tilde\eta}\frac{A_0}{2\lambda}\|\nabla\tau\|^2_{L^2_\gamma}+\frac{r_3\alpha}{2\beta k\tilde\eta}\|\nabla^2\tau\|^2_{L^2_\gamma}\\
\lesssim& t^{-\frac54}\|\nabla(\rho,u,\eta,\tau)\|^2_{L^2_\gamma}+\|\nabla(\rho,\eta)\|_{L^2_\gamma}\|\nabla u\|_{L^2_{\gamma-1}}+\|\nabla(u,\eta,\tau)\|^2_{L^2_{\gamma-1}}+t^{-\frac72+2\gamma},
\end{aligned}
\tag{3.22}
$$

for all $t>T_1$. Using the interpolation inequality with weights $\|\nabla f\|_{L^2_{\gamma-1}}\lesssim\|\nabla f\|^{\frac{\gamma-1}{\gamma}}_{L^2_\gamma}\|\nabla f\|^{\frac1\gamma}_{L^2}$, we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla\rho\|_{L^2_\gamma}^2+\|\nabla u\|_{L^2_\gamma}^2+\frac{r_2}{\beta\tilde{\eta}}\|\nabla\eta\|_{L^2_\gamma}^2+\frac{r_3}{2\beta k\tilde{\eta}}\|\nabla\tau\|_{L^2_\gamma}^2)\\
&+\frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^2\eta\|_{L^2_\gamma}^2+\frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla\tau\|_{L^2_\gamma}^2+\frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^2\tau\|_{L^2_\gamma}^2\\
\lesssim& t^{-\frac{5}{4}}\|\nabla(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2+\|\nabla(\rho,\eta)\|_{L^2_\gamma}\|\nabla u\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|\nabla u\|_{L^2}^{\frac{1}{\gamma}}\\
&+\|\nabla(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}\|\nabla(u,\eta,\tau)\|_{L^2}^{\frac{2}{\gamma}}+t^{-\frac{7}{2}+2\gamma}\\
\lesssim& t^{-\frac{5}{4}}\|\nabla(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2+t^{-\frac{5}{4\gamma}}\|\nabla(\rho,\eta)\|_{L^2_\gamma}^{\frac{2\gamma-1}{\gamma}}+t^{-\frac{5}{2\gamma}}\|\nabla(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}+t^{-\frac{7}{2}+2\gamma}.
\end{aligned} \tag{3.23}$$

Denoting $\mathrm{E}(t) := \|\nabla(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t)\le C_0t^{-\frac{5}{4}}\mathrm{E}(t)+C_1t^{-\frac{5}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}}+C_2t^{-\frac{5}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}}+C_3t^{-\frac{7}{2}+2\gamma},$$

where C_0, C_1, C_2, and C_3 are positive constants independent of t. If $\gamma>\frac{5}{2}$, then we can apply Lemma 2.5 with $\alpha_0=\frac{5}{4}>1$, $\alpha_1=\frac{5}{4\gamma}<1$, $\beta_1=\frac{2\gamma-1}{2\gamma}<1$, $\alpha_2=\frac{5}{2\gamma}<1$, $\beta_2=\frac{\gamma-1}{\gamma}<1$, $\gamma_1=\frac{1-\alpha_1}{1-\beta_1}=-\frac{5}{2}+2\gamma>0$, $\gamma_2=\frac{1-\alpha_2}{1-\beta_2}=-\frac{5}{2}+\gamma>0$, $\gamma_1>\gamma_2$, $\gamma_1-1=-\frac{7}{2}+2\gamma$. Thus, for all $t>T$, one has

$$\mathrm{E}(t)\le Ct^{-\frac{5}{2}+2\gamma}, \tag{3.24}$$

we get the fact

$$\|\nabla(\rho,u,\eta,\tau)(t)\|_{L^2_{\gamma_0}}\le C\|\nabla(\rho,u,\eta,\tau)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}}\|\nabla(\rho,u,\eta,\tau)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}}\le Ct^{-\frac{5}{4}+\gamma_0},$$

for all $t>T$, $\gamma_0\in[0,\gamma]$, and $[0,\frac{5}{2}]\subset[0,\gamma](\gamma>\frac{5}{2})$. Thus, the proof of Lemma 3.2 has been completed. □

Lemma 3.3. *Under the assumption of Theorem* 1.3*, there exists a sufficiently large T such that the solution (ρ,u,η,τ) of the system* (1.4) *with the initial data* (1.5) *has the following estimate:*

$$\left\|\nabla^2(\rho,u,\eta,\tau)(t)\right\|_{L^2_\gamma}\le Ct^{-\frac{7}{4}+\gamma}, \tag{3.25}$$

for all $t>T$ and $\gamma\ge 0$, where C is a positive constant independent of t.

Proof. Multiplying $|x|^{2\gamma}\nabla^2\rho$, $|x|^{2\gamma}\nabla^2u$, $\frac{r_2}{\beta\tilde{\eta}}|x|^{2\gamma}\nabla^2\eta$, $\frac{r_3}{2\beta k\tilde{\eta}}|x|^{2\gamma}\nabla^2\tau$ by $\nabla^2(1.4)_1-\nabla^2(1.4)_4$, and then adding them up and integrating on $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2\rho_t \,\mathrm{d}x + r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2 \operatorname{div} u \,\mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^2 u_t \,\mathrm{d}x \\
&+ r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^3\rho \,\mathrm{d}x + r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^3\eta \,\mathrm{d}x - r_3\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^2 \operatorname{div}\tau \,\mathrm{d}x \\
&+ \frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\eta\nabla^2\eta_t \,\mathrm{d}x + r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\eta\nabla^2 \operatorname{div} u \,\mathrm{d}x - \frac{r_2\alpha}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\eta\nabla^2\triangle\eta \,\mathrm{d}x \\
&+ \frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\tau\nabla^2\tau_t \,\mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\int_{\mathbb{R}^3} |x|^{2\gamma}(\nabla^2\tau)^2 \,\mathrm{d}x \qquad (3.26)\\
&- \frac{r_3\alpha}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\tau\nabla^2\triangle\tau \,\mathrm{d}x - \frac{r_3}{2}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\tau\nabla^2(\nabla u + \nabla^T u) \,\mathrm{d}x \\
=&\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2 S_1 \mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^2 S_2 \mathrm{d}x \\
&+ \frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\eta\nabla^2 S_3 \mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\tau\nabla^2 S_4 \mathrm{d}x.
\end{aligned}
$$

Then, using integration by parts to simplify, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^2\rho\|_{L^2_\gamma}^2 + \|\nabla^2 u\|_{L^2_\gamma}^2 + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^2\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^2\tau\|_{L^2_\gamma}^2) \\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^3\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^2\tau\|_{L^2_\gamma}^2 + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|_{L^2_\gamma}^2 \\
=& - r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2 \operatorname{div} u \,\mathrm{d}x - r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^3\rho \,\mathrm{d}x - r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\eta\nabla^2 \operatorname{div} u \,\mathrm{d}x \\
&- r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^3\eta \,\mathrm{d}x + r_3\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^2 \operatorname{div}\tau \,\mathrm{d}x + \frac{r_3}{2}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\tau\nabla^2(\nabla u + \nabla^T u) \,\mathrm{d}x \qquad (3.27)\\
&- \frac{r_2\alpha}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla^2\eta\nabla^3\eta \,\mathrm{d}x - \frac{r_3\alpha}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla^2\tau\nabla^3\tau \,\mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2 S_1 \mathrm{d}x \\
&+ \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2 u\nabla^2 S_2 \mathrm{d}x + \frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\eta\nabla^2 S_3 \mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\tau\nabla^2 S_4 \mathrm{d}x \\
\triangleq& \sum_{i=1}^{12} J_{3,i}.
\end{aligned}
$$

Initially, by using integration by parts and applying Lemma 2.3, we can obtain

$$
\begin{aligned}
&\sum_{i=1}^{6} J_{3,i} \\
\lesssim& \left| r_1\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla^2\rho\nabla^2 u \,\mathrm{d}x \right| + \left| r_2\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla^2\eta\nabla^2 u \,\mathrm{d}x \right| + \left| r_3\int_{\mathbb{R}^3} \nabla(|x|^{2\gamma})\nabla^2\tau\nabla^2 u \,\mathrm{d}x \right| \qquad (3.28)\\
\lesssim& \left\|\nabla^2\rho\right\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_{\gamma-1}} + \left\|\nabla^2\eta\right\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_{\gamma-1}} + \left\|\nabla^2\tau\right\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_{\gamma-1}} \\
\lesssim& \left\|\nabla^2(\rho,\eta)\right\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_{\gamma-1}} + \varepsilon r_3\left\|\nabla^2\tau\right\|_{L^2_\gamma}^2 + Cr_3(\varepsilon)\|\nabla^2 u\|_{L^2_{\gamma-1}}^2.
\end{aligned}
$$

By using Lemma 2.3 and Cauchy's inequality, we obtain

$$\begin{aligned}
&|J_{3,7}| + |J_{3,8}| \\
\lesssim& \frac{r_2\alpha}{\beta\tilde{\eta}} \left\|\nabla^3\tau\right\|_{L^2_\gamma} \left\|\nabla^2\tau\right\|_{L^2_{\gamma-1}} + \frac{r_3\alpha}{2\beta k\tilde{\eta}} \left\|\nabla^3\eta\right\|_{L^2_\gamma} \left\|\nabla^2\eta\right\|_{L^2_{\gamma-1}} \\
\lesssim& \varepsilon(\frac{r_2\alpha}{\beta\tilde{\eta}}) \left\|\nabla^3\eta\right\|^2_{L^2_\gamma} + C\frac{r_2\alpha}{\beta\tilde{\eta}}(\varepsilon) \left\|\nabla^2\eta\right\|^2_{L^2_{\gamma-1}} + \varepsilon(\frac{r_3\alpha}{2\beta k\tilde{\eta}}) \left\|\nabla^3\tau\right\|^2_{L^2_\gamma} + C\frac{r_3\alpha}{2\beta k\tilde{\eta}}(\varepsilon) \left\|\nabla^2\tau\right\|^2_{L^2_{\gamma-1}}.
\end{aligned} \tag{3.29}$$

By the definitions of S_1, S_2, S_3, and S_4, Lemma 2.1, Cauchy's inequality, and Lemma 2.3, we get

$$\begin{aligned}
|J_{3,9}| =& \left|\beta \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2(\rho\nabla u)\,\mathrm{d}x\right| + \left|\beta \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^2\rho\nabla^2(u\nabla\rho)\,\mathrm{d}x\right| \\
\lesssim& ||\nabla^3\rho||_{L^3}||\nabla^2\rho||_{L^2_\gamma}||u||_{L^6_\gamma} + ||\nabla^3 u||_{L^3}||\nabla^2\rho||_{L^2_\gamma}||\rho||_{L^6_\gamma} \\
&+ ||\nabla\rho||_{L^\infty}||\nabla^2\rho||_{L^2_\gamma}||\nabla^2 u||_{L^2_\gamma} + ||\nabla u||_{L^\infty}||\nabla^2\rho||^2_{L^2_\gamma} \\
\lesssim& ||\nabla^3\rho||_{H^1}||\nabla^2\rho||_{L^2_\gamma}||u||_{L^2_{\gamma-1}} + ||\nabla^3\rho||_{H^1}||\nabla^2\rho||_{L^2_\gamma}||\nabla u||_{L^2_\gamma} \\
&+ ||\nabla^3 u||_{H^1}||\nabla^2\rho||_{L^2_\gamma}||\rho||_{L^2_{\gamma-1}} + ||\nabla^3 u||_{H^1}||\nabla^2\rho||_{L^2_\gamma}||\nabla\rho||_{L^2_\gamma} \\
&+ ||\nabla^2\rho||_{H^1}||\nabla^2\rho||_{L^2_\gamma}||\nabla^2 u||_{L^2_\gamma} + ||\nabla^2 u||_{H^1}||\nabla^2\rho||^2_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}-\frac{4}{4}}t^{-\frac{3}{4}+\gamma-1}||\nabla^2\rho||_{L^2_\gamma} + t^{-\frac{5}{4}-\frac{4}{4}}t^{-\frac{5}{4}+\gamma}||\nabla^2\rho||^2_{L^2_\gamma} + t^{-\frac{5}{4}}||\nabla^2\rho||^2_{L^2_\gamma} + t^{-\frac{7}{4}}||\nabla^2 u||^2_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}}||\nabla^2\rho||^2_{L^2_\gamma} + t^{-\frac{7}{4}}||\nabla^2 u||^2_{L^2_\gamma} + t^{-\frac{9}{2}+2\gamma} + t^{-\frac{11}{2}+2\gamma}.
\end{aligned} \tag{3.30}$$

In a similar way, one has

$$\begin{aligned}
|J_{3,10}| \lesssim& \beta||\nabla u||_{L^\infty}||\nabla^2 u||^2_{L^2_\gamma} + \beta||\nabla^3 u||_{L^3}||\nabla^2 u||_{L^2_\gamma}||u||_{L^6_\gamma} + ||\nabla\rho||_{L^\infty}||\nabla^2 u||_{L^2_\gamma}||\nabla^2 H(\rho)||_{L^2_\gamma} \\
&+ \beta||\nabla^3\rho||_{L^3}||\nabla^2 u||_{L^2_\gamma}||H(\rho)||_{L^6_\gamma} + ||G(\rho)||_{L^\infty}||\nabla^3\eta||_{L^2_\gamma}||\nabla^2 u||_{L^2_\gamma} \\
&+ (||\nabla\eta||_{L^\infty} + ||\nabla\tau||_{L^\infty})||\nabla^2 G(\rho)||_{L^2_\gamma}||\nabla^2 u||_{L^2_\gamma} + ||G(\rho)||_{L^\infty}||\nabla^2 u||_{L^2_\gamma}||\nabla^3\tau||_{L^2_\gamma} \\
&+ ||\nabla^2(\frac{\eta}{\rho+\tilde{\rho}})||_{L^3}||\nabla^2 u||_{L^2_\gamma}||\nabla\eta||_{L^6_\gamma} + ||\frac{\eta}{\rho+\tilde{\rho}}||_{L^\infty}||\nabla^2 u||_{L^2_\gamma}||\nabla^3\eta||_{L^2_\gamma} \\
\lesssim& ||\nabla^2 u||_{H^1}||\nabla^2 u||^2_{L^2_\gamma} + ||\nabla^3 u||_{H^1}||\nabla^2 u||_{L^2_\gamma}||\nabla u||_{L^2_\gamma} + ||\nabla^3 u||_{H^1}||\nabla^2 u||_{L^2_\gamma}||u||_{L^2_{\gamma-1}} \\
&+ ||\nabla^2\rho||_{H^1}||\nabla^2 H(\rho)||_{L^2_\gamma}||\nabla^2 u||_{L^2_\gamma} + ||\nabla^3\rho||_{H^1}||\nabla^2 u||_{L^2_\gamma}||\nabla H(\rho)||_{L^2_\gamma} \\
&+ ||\nabla^3\rho||_{H^1}||\nabla^2 u||_{L^2_\gamma}||H(\rho)||_{L^2_{\gamma-1}} + ||\nabla^2\eta||_{H^1}||\nabla^2 u||^2_{L^2_\gamma}||\nabla^2 G(\rho)||^2_{L^2_\gamma} \\
&+ ||\nabla G(\rho)||_{H^1}||\nabla^3\eta||^2_{L^2_\gamma}||\nabla^2 u||^2_{L^2_\gamma} + ||\nabla G(\rho)||_{H^1}||\nabla^2 u||^2_{L^2_\gamma}||\nabla^3\tau||^2_{L^2_\gamma} \\
&+ (||\nabla^2\eta||_{H^1} + ||\nabla^2\tau||_{H^1})||\nabla^2 G(\rho)||_{L^2_\gamma}||\nabla^2 u||_{L^2_\gamma} \\
&+ ||\nabla^2(\frac{\eta}{\rho+\tilde{\rho}})||_{H^1}||\nabla^2 u||_{L^2_\gamma}||\nabla^2\eta||_{L^2_\gamma} + ||\nabla^2(\frac{\eta}{\rho+\tilde{\rho}})||_{H^1}||\nabla^2 u||_{L^2_\gamma}||\nabla\eta||_{L^2_{\gamma-1}} \\
&+ ||\nabla(\frac{\eta}{\rho+\tilde{\rho}})||_{H^1}||\nabla^2 u||^2_{L^2_\gamma} + ||\nabla(\frac{\eta}{\rho+\tilde{\rho}})||_{H^1}||\nabla^3\eta||^2_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}}||\nabla^2(\rho,u,\eta,\tau)||^2_{L^2_\gamma} + t^{-\frac{5}{4}}||\nabla^3(\eta,\tau)||^2_{L^2_\gamma} + t^{-\frac{9}{2}+2\gamma} + t^{-\frac{11}{2}+2\gamma},
\end{aligned} \tag{3.31}$$

and

$$
\begin{aligned}
|J_{3,11}| =& \left|\frac{r_2}{\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^2\eta\nabla^2(\eta\nabla u)\,\mathrm{d}x\right| + \left|\frac{r_2}{\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^2\eta\nabla^2(u\nabla\eta)\,\mathrm{d}x\right| \\
\lesssim& \|u\|_{L^\infty}\|\nabla^2\eta\|_{L^2_\gamma}\|\nabla^3\eta\|_{L^2_\gamma} + \|\nabla^3 u\|_{L^3}\|\nabla^2\eta\|_{L^2_\gamma}\|\eta\|_{L^6_\gamma} \\
&+ \|\nabla u\|_{L^\infty}\|\nabla^2\eta\|^2_{L^2_\gamma} + \|\nabla\eta\|_{L^\infty}\|\nabla^2 u\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma} \\
\lesssim& \|\nabla u\|_{H^1}\|\nabla^2\eta\|_{L^2_\gamma}\|\nabla^3\eta\|^2_{L^2_\gamma} + \|\nabla^3 u\|_{H^1}\|\nabla^2\eta\|_{L^2_\gamma}(\|\nabla\eta\|_{L^2_\gamma} + \|\eta\|_{L^2_{\gamma-1}}) \\
&+ \|\nabla^2 u\|_{H^1}\|\nabla^2\eta\|^2_{L^2_\gamma} + \|\nabla^2\eta\|_{H^1}\|\nabla^2 u\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}}\|\nabla^2\eta\|^2_{L^2_\gamma} + t^{-\frac{5}{4}}\|\nabla^3\eta\|^2_{L^2_\gamma} + t^{-\frac{7}{4}}\|\nabla^2 u\|^2_{L^2_\gamma} + t^{-\frac{9}{2}+2\gamma} + t^{-\frac{11}{2}+2\gamma},
\end{aligned} \tag{3.32}
$$

and

$$
\begin{aligned}
|J_{3,12}| \lesssim& \|\nabla^3 u\|_{L^3}\|\nabla^2\tau\|_{L^2_\gamma}\|\tau\|_{L^6_\gamma} + \|u\|_{L^\infty}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla^3\tau\|_{L^2_\gamma} \\
&+ \|\nabla u\|_{L^\infty}\|\nabla^2\tau\|^2_{L^2_\gamma} + \|\nabla\tau\|_{L^\infty}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_\gamma} \\
&+ \|\nabla^3 u\|_{L^3}\|\nabla^2\tau\|_{L^2_\gamma}\|\eta\|_{L^6_\gamma} + \|\nabla u\|_{L^\infty}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma} \\
\lesssim& \|\nabla^3 u\|_{H^1}\|\nabla^2\tau\|_{L^2_\gamma}(\|\nabla\tau\|_{L^2_\gamma} + \|\tau\|_{L^2_{\gamma-1}}) + \|\nabla u\|_{H^1}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla^3\tau\|_{L^2_\gamma} \\
&+ \|\nabla^2 u\|_{H^1}\|\nabla^2\tau\|^2_{L^2_\gamma} + \|\nabla^2\tau\|_{H^1}\|\nabla^2\tau\|^2_{L^2_\gamma} + \|\nabla^2\tau\|_{H^1}\|\nabla^2 u\|^2_{L^2_\gamma} \\
&+ \|\nabla^3 u\|_{H^1}\|\nabla^2\tau\|_{L^2_\gamma}(\|\nabla\eta\|_{L^2_\gamma} + \|\eta\|_{L^2_{\gamma-1}}) + \|\nabla^2 u\|_{H^1}\|\nabla^2\tau\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}}\|\nabla^2(u,\eta,\tau)\|^2_{L^2_\gamma} + t^{-\frac{5}{4}}\|\nabla^3\tau\|^2_{L^2_\gamma} + t^{-\frac{9}{2}+2\gamma} + t^{-\frac{11}{2}+2\gamma}.
\end{aligned} \tag{3.33}
$$

Substituting (3.28) into (3.33) into (3.27), we conclude that there exists a sufficiently large T_1 and a sufficiently small ε, such that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^2\rho\|^2_{L^2_\gamma} + \|\nabla^2 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^2\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^2\tau\|^2_{L^2_\gamma}) \\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^2\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}}\|\nabla^2(\rho,u,\eta,\tau)\|^2_{L^2_\gamma} + \|\nabla^2(\rho,\eta)\|_{L^2_\gamma}\|\nabla^2 u\|_{L^2_{\gamma-1}} + \|\nabla^2(u,\eta,\tau)\|^2_{L^2_{\gamma-1}} + t^{-\frac{9}{2}+2\gamma},
\end{aligned} \tag{3.34}
$$

for all $t > T_1$. Using the interpolation inequality with weights $\left\|\nabla^2 f\right\|_{L^2_{\gamma-1}} \lesssim \left\|\nabla^2 f\right\|^{\frac{\gamma-1}{\gamma}}_{L^2_\gamma}\left\|\nabla^2 f\right\|^{\frac{1}{\gamma}}_{L^2}$, we can obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^2\rho\|^2_{L^2_\gamma} + \|\nabla^2 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^2\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^2\tau\|^2_{L^2_\gamma}) \\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^2\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma} \\
\lesssim& t^{-\frac{5}{4}}\|\nabla^2(\rho,u,\eta,\tau)\|^2_{L^2_\gamma} + \|\nabla^2(\rho,\eta)\|_{L^2_\gamma}\|\nabla^2 u\|^{\frac{\gamma-1}{\gamma}}_{L^2_\gamma}\|\nabla^2 u\|^{\frac{1}{\gamma}}_{L^2} \\
&+ \|\nabla^2(u,\eta,\tau)\|^{\frac{2\gamma-2}{\gamma}}_{L^2_\gamma}\|\nabla^2(u,\eta,\tau)\|^{\frac{2}{\gamma}}_{L^2} + t^{-\frac{9}{2}+2\gamma} \\
\lesssim& t^{-\frac{5}{4}}\|\nabla^2(\rho,u,\eta,\tau)\|^2_{L^2_\gamma} + t^{-\frac{7}{4\gamma}}\|\nabla^2(\rho,\eta)\|^{\frac{2\gamma-1}{\gamma}}_{L^2_\gamma} + t^{-\frac{7}{2\gamma}}\|\nabla^2(u,\eta,\tau)\|^{\frac{2\gamma-2}{\gamma}}_{L^2_\gamma} + t^{-\frac{9}{2}+2\gamma}.
\end{aligned} \tag{3.35}
$$

Denoting $\mathrm{E}(t) := \left\|\nabla^2(\rho, u, \eta, \tau)\right\|_{L^2_\gamma}^2$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \le C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{7}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{7}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}} + C_3 t^{-\frac{9}{2}+2\gamma},$$

where C_0, C_1, C_2, and C_3 are positive constants independent of t. If $\gamma > \frac{7}{2}$, then we can apply Lemma 2.5 with $\alpha_0 = \frac{5}{4} > 1$, $\alpha_1 = \frac{7}{4\gamma} < 1$, $\beta_1 = \frac{2\gamma-1}{2\gamma} < 1$, $\alpha_2 = \frac{7}{2\gamma} < 1$, $\beta_2 = \frac{\gamma-1}{\gamma} < 1$, $\gamma_1 = \frac{1-\alpha_1}{1-\beta_1} = -\frac{7}{2} + 2\gamma > 0$, $\gamma_2 = \frac{1-\alpha_2}{1-\beta_2} = -\frac{7}{2} + \gamma > 0$, $\gamma_1 > \gamma_2$, $\gamma_1 - 1 = -\frac{9}{2} + 2\gamma$. Thus, for all $t > T$,

$$\mathrm{E}(t) \le Ct^{-\frac{7}{2}+2\gamma}, \tag{3.36}$$

we get the fact

$$\|\nabla^2(\rho, u, \eta, \tau)(t)\|_{L^2_{\gamma_0}} \le C\|\nabla^2(\rho, u, \eta, \tau)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}} \|\nabla^2(\rho, u, \eta, \tau)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}} \le Ct^{-\frac{7}{4}+\gamma_0},$$

for all $t > T$, $\gamma_0 \in [0, \gamma]$, and $[0, \frac{7}{2}] \subset [0, \gamma](\gamma > \frac{7}{2})$. Thus, the proof of Lemma 3.3 has been completed. □

Lemma 3.4. *Under the assumption of Theorem* 1.3*, there exists a sufficiently large T such that the solution (ρ, u, η, τ) of the system* (1.4) *with the initial data* (1.5) *has the following estimate:*

$$\left\|\nabla^3(\rho, u, \eta, \tau)(t)\right\|_{L^2_\gamma} \le Ct^{-\frac{9}{4}+\gamma}, \tag{3.37}$$

for all $t > T$ and $\gamma \ge 0$, where C is a positive constant independent of t.

Proof. Multiplying $|x|^{2\gamma}\nabla^3\rho$, $|x|^{2\gamma}\nabla^3 u$, $\frac{r_2}{\beta\tilde{\eta}}|x|^{2\gamma}\nabla^3\eta$, $\frac{r_3}{2\beta k\tilde{\eta}}|x|^{2\gamma}\nabla^3\tau$ by $\nabla^3(1.4)_1 - \nabla^3(1.4)_4$, and then adding them up and integrating on $\mathbb{R}^3$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\rho\nabla^3\rho_t \,\mathrm{d}x + r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\rho\nabla^3 \operatorname{div} u \,\mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\nabla^3 u_t \,\mathrm{d}x \\
&+ r_1\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\nabla^4\rho \,\mathrm{d}x + r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\nabla^4\eta \,\mathrm{d}x - r_3\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\nabla^3 \operatorname{div}\tau \,\mathrm{d}x \\
&+ \frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\eta\nabla^3\eta_t \,\mathrm{d}x + r_2\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\eta\nabla^3 \operatorname{div} u \,\mathrm{d}x - \frac{r_2\alpha}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\eta\nabla^3\triangle\eta \,\mathrm{d}x \\
&+ \frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\tau\nabla^3\tau_t \,\mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\int_{\mathbb{R}^3} |x|^{2\gamma}(\nabla^3\tau)^2 \,\mathrm{d}x \\
&- \frac{r_3\alpha}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\tau\nabla^3\triangle\tau \,\mathrm{d}x - \frac{r_3}{2}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\tau\nabla^3(\nabla u + \nabla^T u) \,\mathrm{d}x \\
=&\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\rho\nabla^3 S_1 \mathrm{d}x + \int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3 u\nabla^3 S_2 \mathrm{d}x \\
&+ \frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\eta\nabla^3 S_3 \mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3} |x|^{2\gamma}\nabla^3\tau\nabla^3 S_4 \mathrm{d}x.
\end{aligned} \tag{3.38}$$

Then, using integration by parts to simplify, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^3\rho\|_{L^2_\gamma}^2 + \|\nabla^3 u\|_{L^2_\gamma}^2 + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|_{L^2_\gamma}^2)\\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^4\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^3\tau\|_{L^2_\gamma}^2 + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^4\tau\|_{L^2_\gamma}^2\\
=& - r_1\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\rho\nabla^3 \operatorname{div} u\,\mathrm{d}x - r_1\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^4\rho\,\mathrm{d}x - r_2\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\eta\nabla^3\operatorname{div} u\,\mathrm{d}x\\
&- r_2\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^4\eta\,\mathrm{d}x + r_3\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^3\operatorname{div}\tau\,\mathrm{d}x + \frac{r_3}{2}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\tau\nabla^3(\nabla u + \nabla^T u)\,\mathrm{d}x\\
&- \frac{r_2\alpha}{\beta\tilde{\eta}}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^3\eta\nabla^4\eta\,\mathrm{d}x - \frac{r_3\alpha}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^3\tau\nabla^4\tau\,\mathrm{d}x + \int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\rho\nabla^3 S_1\mathrm{d}x\\
&+ \int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^3 S_2\mathrm{d}x + \frac{r_2}{\beta\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\eta\nabla^3 S_3\mathrm{d}x + \frac{r_3}{2\beta k\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\tau\nabla^3 S_4\mathrm{d}x\\
\triangleq& \sum_{i=1}^{12} J_{4,i}.
\end{aligned} \tag{3.39}$$

Initially, by applying integration by parts and using Lemma 2.3, we can obtain

$$\begin{aligned}
&\sum_{i=1}^{6} J_{4,i}\\
\lesssim& \left|r_1\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^3\rho\nabla^3 u\,\mathrm{d}x\right| + \left|r_2\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^3\eta\nabla^3 u\,\mathrm{d}x\right|\\
&+ \left|r_3\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^3\tau\nabla^3 u\,\mathrm{d}x\right|\\
\lesssim& \left\|\nabla^3\rho\right\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} + \left\|\nabla^3\eta\right\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} + \left\|\nabla^3\tau\right\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}}\\
\lesssim& \left\|\nabla^3(\rho,\eta)\right\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} + \varepsilon r_3\left\|\nabla^3\tau\right\|_{L^2_\gamma}^2 + Cr_3(\varepsilon)\|\nabla^3 u\|_{L^2_{\gamma-1}}^2.
\end{aligned} \tag{3.40}$$

By using integration by parts, Lemma 2.3, and Cauchy's inequality

$$\begin{aligned}
&|J_{4,7}| + |J_{4,8}|\\
\lesssim&\frac{r_2\alpha}{\beta\tilde{\eta}}\left\|\nabla^4\tau\right\|_{L^2_\gamma}\left\|\nabla^3\tau\right\|_{L^2_{\gamma-1}} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\left\|\nabla^4\eta\right\|_{L^2_\gamma}\left\|\nabla^3\eta\right\|_{L^2_{\gamma-1}}\\
\lesssim&\varepsilon(\frac{r_2\alpha}{\beta\tilde{\eta}})\left\|\nabla^4\eta\right\|_{L^2_\gamma}^2 + C\frac{r_2\alpha}{\beta\tilde{\eta}}(\varepsilon)\left\|\nabla^3\eta\right\|_{L^2_{\gamma-1}}^2\\
&+ \varepsilon(\frac{r_3\alpha}{2\beta k\tilde{\eta}})\left\|\nabla^4\tau\right\|_{L^2_\gamma}^2 + C\frac{r_3\alpha}{2\beta k\tilde{\eta}}(\varepsilon)\left\|\nabla^3\tau\right\|_{L^2_{\gamma-1}}^2.
\end{aligned} \tag{3.41}$$

By the definitions of S_1, S_2, S_3, and S_4, Lemma 2.1, Cauchy's inequality and Lemma 2.3, we get

$$\begin{aligned}
|J_{4,9}| =& \left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\rho\nabla^3(u\nabla\rho)\,\mathrm{d}x\right| + \left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\rho\nabla^3(\rho\nabla u)\,\mathrm{d}x\right|\\
\lesssim& \left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla^3\rho\nabla^4\rho\,\mathrm{d}x\right| + \left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^4 u\,\mathrm{d}x\right|
\end{aligned} \tag{3.42}$$

$$
\begin{aligned}
&+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla u(\nabla^3\rho)^2\,\mathrm{d}x\right|+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\rho\nabla^3\rho\nabla^3u\,\mathrm{d}x\right|\\
\lesssim&\left|\beta\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma}u)(\nabla^3\rho)^2\,\mathrm{d}x\right|+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^4u\,\mathrm{d}x\right|\\
&+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla u(\nabla^3\rho)^2\,\mathrm{d}x\right|+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla\rho\nabla^3\rho\nabla^3u\,\mathrm{d}x\right|\\
\lesssim&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^4u\,\mathrm{d}x\right|+\|\nabla u\|_{L^\infty}\|\nabla^3\rho\|^2_{L^2_\gamma}\\
&+\|u\|_{L^\infty}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^3\rho\|_{L^2_{\gamma-1}}+\|\nabla\rho\|_{L^\infty}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^3u\|_{L^2_\gamma}\\
\lesssim&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^4u\,\mathrm{d}x\right|+t^{-\frac{5}{4}}\|\nabla^3\rho\|^2_{L^2_\gamma}+t^{-\frac{7}{4}}\|\nabla^3u\|^2_{L^2_\gamma}+\|\nabla^3\rho\|^2_{L^2_{\gamma-1}}.
\end{aligned}
$$

Next, we make full use of the dissipative structure of the system (1.4) to deal with the trouble term

$$
N_1\triangleq\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^4u\,\mathrm{d}x\right|.
$$

More precisely, based on the fact that $\operatorname{div}u=\frac{\rho_t+\beta u\nabla\rho}{r_1+\beta\rho}$, we can reduce the order of the spatial derivative of velocity. First, we rewrite the trouble N_1 as follows:

$$
\begin{aligned}
N_1=&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^3\left(\frac{\rho_t+\beta u\nabla\rho}{r_1+\beta\rho}\right)\mathrm{d}x\right|\\
=&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho(\rho_t+\beta u\nabla\rho)\nabla^3\rho\nabla^3\left(\frac{1}{r_1+\beta\rho}\right)\mathrm{d}x\right|\\
&+3\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla(\rho_t+\beta u\nabla\rho)\nabla^2\left(\frac{1}{r_1+\beta\rho}\right)\mathrm{d}x\right|\\
&+3\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^2(\rho_t+\beta u\nabla\rho)\nabla\left(\frac{1}{r_1+\beta\rho}\right)\mathrm{d}x\right|\\
&+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{1}{r_1+\beta\rho}\rho\nabla^3\rho_t\nabla^3\rho\,\mathrm{d}x\right|+\beta^2\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{1}{r_1+\beta\rho}\rho u\nabla^4\rho\nabla^3\rho\,\mathrm{d}x\right|\\
&+3\beta^2\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{1}{r_1+\beta\rho}\rho\nabla u\nabla^3\rho\nabla^3\rho\,\mathrm{d}x\right|+3\beta^2\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{1}{r_1+\beta\rho}\rho\nabla^2u\nabla^2\rho\nabla^3\rho\,\mathrm{d}x\right|\\
&+\beta^2\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{1}{r_1+\beta\rho}\rho\nabla^3u\nabla\rho\nabla^3\rho\,\mathrm{d}x\right|\\
\triangleq&\sum_{i=1}^{8}N_{1,i}.
\end{aligned}
\tag{3.43}
$$

Noticing that $\rho_t=-r_1\operatorname{div}u-\beta\operatorname{div}(\rho u)$ from $(1.4)_1$, we have

$$
\begin{aligned}
|N_{1,1}|=&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}\rho(\rho_t+\beta u\nabla\rho)\nabla^3\rho\nabla^3\left(\frac{1}{r_1+\beta\rho}\right)\mathrm{d}x\right|\\
\le&\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho(\beta u\operatorname{div}\rho+\beta\rho\operatorname{div}u+\beta u\nabla\rho+r_1\operatorname{div}u)\nabla^3\left(\frac{1}{r_1+\beta\rho}\right)\mathrm{d}x\right|
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|\nabla^3(\frac{1}{r_1+\beta\rho})\|_{L^2}\|\nabla^3\rho\|_{L^2_\gamma}\||x|^\gamma\rho\|_{L^\infty}(\|u\|_{L^\infty}\|\operatorname{div}\rho\|_{L^\infty}+\|\operatorname{div}u\|_{L^\infty}\|\rho\|_{L^\infty})\\
&\quad+\|\nabla^3(\frac{1}{r_1+\beta\rho})\|_{L^2}\|\nabla^3\rho\|_{L^2_\gamma}\||x|^\gamma\rho\|_{L^\infty}(\|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty}+r_1\|\operatorname{div}u\|_{L^\infty})\\
&\lesssim \|\nabla^3\rho\|_{H^1}\|\nabla^3\rho\|_{L^2_\gamma}\||x|^\gamma\rho\|_{L^\infty}(\|\nabla u\|_{H^1}\|\nabla\operatorname{div}\rho\|_{H^1}+\|\nabla\operatorname{div}u\|_{H^1}\|\nabla\rho\|_{H^1})\\
&\quad+\|\nabla^3\rho\|_{H^1}\|\nabla^3\rho\|_{L^2_\gamma}\||x|^\gamma\rho\|_{L^\infty}(\|\nabla u\|_{H^1}\|\nabla^2\rho\|_{H^1}+r_1\|\nabla\operatorname{div}u\|_{H^1})\\
&\lesssim t^{-\frac{5}{4}}\|\nabla^3\rho\|^2_{L^2_\gamma}+t^{-\frac{11}{2}+2\gamma},
\end{aligned}
\tag{3.44}
$$

where we have used the fact that $(\frac{1}{r_1+\beta\rho})\sim O(1)(\rho)$. By using the Gagliardo-Nirenberg-Sobolev inequality and Cauchy's inequality, we have

$$
\begin{aligned}
\||x|^\gamma\rho\|_{L^\infty}&\lesssim \|\nabla(|x|^\gamma\rho)\|^{\frac{1}{2}}_{L^2}\|\nabla^2(|x|^\gamma\rho)\|^{\frac{1}{2}}_{L^2}\\
&\lesssim (\|\nabla^2(|x|^\gamma\rho)\|_{L^2}+\|\nabla(|x|^\gamma\rho)\|_{L^2})\\
&\lesssim \|\rho\|_{L^2_{\gamma-2}}+\|\rho\|_{L^2_{\gamma-1}}+\|\nabla\rho\|_{L^2_{\gamma-1}}+\|\nabla\rho\|_{L^2_\gamma}+\|\nabla^2\rho\|_{L^2_\gamma}\\
&\lesssim t^{-\frac{5}{4}+\gamma}.
\end{aligned}
\tag{3.45}
$$

In a similar way, one has

$$
\begin{aligned}
|N_{1,2}|\le&\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla(r_1\operatorname{div}u-\beta\operatorname{div}(\rho u)+\beta u\nabla\rho)\nabla^2\left(\frac{1}{r_1+\beta\rho}\right)\,\mathrm{d}x\right|\\
&\lesssim \|\nabla^2(\frac{1}{r_1+\beta\rho})\|_{L^2}\|\nabla^3\rho\|_{L^2_\gamma}\||x|^\gamma\rho\|_{L^\infty}(\|\nabla^2u\|_{L^\infty}\|\rho\|_{L^\infty}+\|u\|_{L^\infty}\|\nabla^2\rho\|_{L^\infty})\\
&\quad+\|\nabla^2(\frac{1}{r_1+\beta\rho})\|_{L^2}\|\nabla^3\rho\|_{L^2_\gamma}\||x|^\gamma\rho\|_{L^\infty}(\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^\infty}+r_1\|\nabla\operatorname{div}u\|_{L^\infty})\\
&\lesssim t^{-\frac{5}{4}}\|\nabla^3\rho\|^2_{L^2_\gamma}+t^{-\frac{11}{2}+2\gamma},
\end{aligned}
\tag{3.46}
$$

and

$$
\begin{aligned}
|N_{1,3}|\le&\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\rho\nabla^3\rho\nabla^2(r_1\operatorname{div}u-\beta\operatorname{div}(\rho u)+\beta u\nabla\rho)\nabla(\frac{1}{r_1+\beta\rho})\,\mathrm{d}x\right|\\
&\lesssim \|\nabla(\frac{1}{r_1+\beta\rho})\|_{L^3}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^2\rho\|_{L^6_\gamma}\|\rho\|_{L^\infty}\|\operatorname{div}u\|_{L^\infty}\\
&\quad+\|\nabla(\frac{1}{r_1+\beta\rho})\|_{L^\infty}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^3u\|_{L^2_\gamma}\|\rho\|_{L^\infty}\|\rho\|_{L^\infty}\\
&\quad+\|\nabla(\frac{1}{r_1+\beta\rho})\|_{L^3}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^2u\|_{L^6_\gamma}\|\rho\|_{L^\infty}\|\operatorname{div}\rho\|_{L^\infty}\\
&\quad+\|\nabla(\frac{1}{r_1+\beta\rho})\|_{L^\infty}\|\nabla^3\rho\|^2_{L^2_\gamma}\|\rho\|_{L^\infty}\|u\|_{L^\infty}\\
&\quad+r_1\|\rho\|_{L^\infty}\|\nabla(\frac{1}{r_1+\beta\rho})\|_{L^\infty}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^2\operatorname{div}u\|_{L^2_\gamma}\\
&\lesssim t^{-\frac{5}{4}}\|\nabla^3(u,\rho)\|^2_{L^2_\gamma}+t^{-\frac{11}{2}+2\gamma}.
\end{aligned}
\tag{3.47}
$$

By applying integration by parts, we arrive at

$$
\begin{aligned}
|N_{1,4}|+|N_{1,5}| \lesssim & \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{1}{r_1+\beta\rho}\rho\nabla^3\rho\nabla^3\rho\,\mathrm{d}x-\frac{\beta}{2}\int_{\mathbb{R}^3}(|x|^{2\gamma}\frac{\rho}{r_1+\beta\rho})_t|\nabla^3\rho|^2\,\mathrm{d}x\\
&-\frac{\beta^2}{2}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma}\frac{\rho u}{r_1+\beta\rho})|\nabla^3\rho|^2\,\mathrm{d}x\\
\lesssim & \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{\rho}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x+\frac{\beta}{2}\left|\int_{\mathbb{R}^3}|x|^{2\gamma}(\frac{\rho}{\beta\rho})_t|\nabla^3\rho|^2\,\mathrm{d}x\right| \qquad (3.48)\\
&+\frac{\beta}{2}\left|\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\frac{\rho}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x\right|+\left|\frac{\beta^2}{2}\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma}\frac{\rho u}{r_1+\beta\rho})|\nabla^3\rho|^2\,\mathrm{d}x\right|\\
\lesssim & \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{\rho}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x+\varepsilon\|\nabla^3\rho\|_{L^2_\gamma}^2+\|\nabla^3\rho\|_{L^2_{\gamma-1}}^2.
\end{aligned}
$$

Similar to (3.44), we have

$$
\begin{aligned}
&|N_{1,6}|+|N_{1,7}|+|N_{1,8}|\\
\lesssim & 3\beta^2\|\frac{1}{r_1+\beta\rho}\|_{L^\infty}\|\rho\|_{L^\infty}\|\nabla u\|_{L^\infty}\|\nabla^3\rho\|_{L^2_\gamma}^2\\
&+3\beta^2\|\frac{1}{r_1+\beta\rho}\|_{L^3}\|\rho\|_{L^\infty}\|\nabla^2 u\|_{L^\infty}\|\nabla^2\rho\|_{L^6_\gamma}\|\nabla^3\rho\|_{L^2_\gamma} \qquad (3.49)\\
&+\beta^2\|\frac{1}{r_1+\beta\rho}\|_{L^\infty}\|\rho\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\|\nabla^3\rho\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_\gamma}\\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3(u,\rho)\|_{L^2_\gamma}^2+t^{-\frac{11}{2}+2\gamma}.
\end{aligned}
$$

Substituting (3.43)–(3.49) into (3.42) gives

$$
|J_{4,9}|\lesssim\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{\rho}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x+t^{-\frac{5}{4}}\|\nabla^3(u,\rho)\|_{L^2_\gamma}^2+\|\nabla^3\rho\|_{L^2_{\gamma-1}}^2+t^{-\frac{11}{2}+2\gamma}. \qquad (3.50)
$$

By applying integration by parts, Lemma 2.1, Cauchy's inequality, Minkowski's inequality, and Lemma 2.3, one has

$$
\begin{aligned}
|J_{4,10}|=&\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}(\nabla^3 u)^2\nabla u\,\mathrm{d}x\right|+\left|\beta\int_{\mathbb{R}^3}|x|^{2\gamma}u\nabla^3 u\nabla^4 u\,\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 uH(\rho)\nabla^4\rho\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^3 H(\rho)\nabla\rho\,\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^3 G(\rho)\nabla\eta\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^3 G(\rho)\,\mathrm{div}\,\tau\,\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 uG(\rho)\nabla^4\eta\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 uG(\rho)\nabla^3\,\mathrm{div}\,\tau\,\mathrm{d}x\right| \qquad (3.51)\\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3 u\nabla^3(\frac{\eta}{\rho+\tilde{\rho}})\nabla\eta\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{\eta}{\rho+\tilde{\rho}}\nabla^3 u\nabla^4\eta\,\mathrm{d}x\right|\\
\triangleq&\sum_{i=1}^{10}H_{4,i},
\end{aligned}
$$

where

$$
\begin{aligned}
|H_{4,1}|+|H_{4,2}| \lesssim & \|\nabla u\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}^2+\|u\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3 u\|_{L^2_\gamma}^2+\|\nabla^3 u\|_{L^2_{\gamma-1}}^2 .
\end{aligned}
\tag{3.52}
$$

Similar to the delicate weighted energy estimates for the trouble term $|J_{4,9}|$, we get

$$
|H_{4,3}| \lesssim \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{H(\rho)}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x+t^{-\frac{5}{4}}\|\nabla^3(u,\rho)\|_{L^2_\gamma}^2+\|\nabla^3\rho\|_{L^2_{\gamma-1}}^2+t^{-\frac{11}{2}+2\gamma}. \tag{3.53}
$$

By applying integration by parts, Lemma 2.3, and Cauchy's inequality, we have

$$
\begin{aligned}
& |H_{4,4}|+|H_{4,5}|+|H_{4,6}| \\
\lesssim & \|\nabla^3 H(\rho)\|_{L^3}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla\rho\|_{L^6_\gamma}+\|\nabla^3 G(\rho)\|_{L^3}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla\eta\|_{L^6_\gamma} \\
& +\|\nabla^3 G(\rho)\|_{L^3}\|\nabla^3 u\|_{L^2_\gamma}\|\operatorname{div}\tau\|_{L^6_\gamma} \\
\lesssim & \|\nabla^3 H(\rho)\|_{H^1}\|\nabla^3 u\|_{L^2_\gamma}(\|\nabla^2\rho\|_{L^2_\gamma}+\|\nabla\rho\|_{L^2_{\gamma-1}})+\|\nabla^3 G(\rho)\|_{H^1}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^2\eta\|_{L^2_\gamma} \\
& +\|\nabla^3 G(\rho)\|_{H^1}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla\eta\|_{L^2_{\gamma-1}}+\|\nabla^3 G(\rho)\|_{H^1}\|\nabla^3 u\|_{L^2_\gamma}(\|\nabla\operatorname{div}\tau\|_{L^2_\gamma}+\|\operatorname{div}\tau\|_{L^2_{\gamma-1}}) \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3 u\|_{L^2_\gamma}^2+t^{-\frac{11}{2}+2\gamma}+t^{-\frac{13}{2}+2\gamma},
\end{aligned}
\tag{3.54}
$$

and

$$
\begin{aligned}
& |H_{3,7}|+|H_{3,8}|+|H_{3,9}|+|H_{3,10}| \\
\lesssim & \|G(\rho)\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^4\eta\|_{L^2_\gamma}+\|G(\rho)\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^3\operatorname{div}\tau\|_{L^2_\gamma} \\
& +\|\nabla^3(\frac{\eta}{\rho+\tilde{\rho}})\|_{L^3}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla\eta\|_{L^6_\gamma}+\|\frac{\eta}{\rho+\tilde{\rho}}\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^4\eta\|_{L^2_\gamma} \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3 u\|_{L^2_\gamma}^2+t^{-\frac{5}{4}}\|\nabla^4(\eta,\tau)\|_{L^2_\gamma}^2+t^{-\frac{11}{2}+2\gamma}+t^{-\frac{13}{2}+2\gamma}.
\end{aligned}
\tag{3.55}
$$

Therefore

$$
\begin{aligned}
|J_{4,10}| \lesssim & \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{H(\rho)}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x+t^{-\frac{5}{4}}\|\nabla^3(\rho,u)\|_{L^2_\gamma}^2 \\
& +t^{-\frac{5}{4}}\|\nabla^4(\eta,\tau)\|_{L^2_\gamma}^2+\|\nabla^3(u,\rho)\|_{L^2_{\gamma-1}}^2+t^{-\frac{11}{2}+2\gamma}.
\end{aligned}
\tag{3.56}
$$

Similarly, one has

$$
\begin{aligned}
|J_{4,11}| = & \left|\frac{r_2}{\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\eta\nabla^3(u\nabla\eta)\,\mathrm{d}x\right|+\left|\frac{r_2}{\tilde{\eta}}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^3\eta\nabla^3(\eta\nabla u)\,\mathrm{d}x\right| \\
\lesssim & \|\nabla\eta\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^3\eta\|_{L^2_\gamma}+\|u\|_{L^\infty}\|\nabla^3\eta\|_{L^2_\gamma}\|\nabla^4\eta\|_{L^2_\gamma}+\|\nabla u\|_{L^\infty}\|\nabla^3\eta\|_{L^2_\gamma}^2 \\
& +\|\eta\|_{L^\infty}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^4\eta\|_{L^2_\gamma}+\|\eta\|_{L^\infty}\|\nabla^3\eta\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} \\
\lesssim & \|\nabla^2\eta\|_{H^1}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^3\eta\|_{L^2_\gamma}+\|\nabla u\|_{H^1}\|\nabla^3\eta\|_{L^2_\gamma}\|\nabla^4\eta\|_{L^2_\gamma}+\|\nabla^2 u\|_{H^1}\|\nabla^3\eta\|_{L^2_\gamma}^2 \\
& +\|\nabla\eta\|_{H^1}\|\nabla^3 u\|_{L^2_\gamma}\|\nabla^4\eta\|_{L^2_\gamma}+\|\nabla\eta\|_{H^1}\|\nabla^3\eta\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3(u,\eta)\|_{L^2_\gamma}^2+t^{-\frac{5}{4}}\|\nabla^4\eta\|_{L^2_\gamma}^2+\|\nabla^3 u\|_{L^2_{\gamma-1}}^2,
\end{aligned}
\tag{3.57}
$$

and

$$
\begin{aligned}
|J_{4,12}| \lesssim & \|\tau\|_{L^\infty}\|\nabla^3\tau\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} + \|\tau\|_{L^\infty}\|\nabla^4\tau\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_\gamma} + \|\nabla u\|_{L^\infty}\|\nabla^3\tau\|^2_{L^2_\gamma} \\
& + \|\nabla\tau\|_{L^\infty}\|\nabla^3\tau\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_\gamma} + \|u\|_{L^\infty}\|\nabla^3\tau\|_{L^2_\gamma}\|\nabla^4\tau\|_{L^2_\gamma} \\
& + \|\nabla u\|_{L^\infty}\|\nabla^3\tau\|_{L^2_\gamma}\|\nabla^3\eta\|_{L^2_\gamma} + \|\eta\|_{L^\infty}\|\nabla^3\eta\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} \\
& + \|\nabla\tau\|_{L^\infty}\|\nabla^3\tau\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_\gamma} + \|\eta\|_{L^\infty}\|\nabla^4\tau\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_\gamma} \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3(u,\eta,\tau)\|^2_{L^2_\gamma} + t^{-\frac{5}{4}}\|\nabla^4\tau\|^2_{L^2_\gamma} + \|\nabla^3 u\|^2_{L^2_{\gamma-1}}.
\end{aligned} \tag{3.58}
$$

Substituting (3.40) to (3.58) into (3.39), we conclude that there exists a sufficiently large T_1 and a sufficiently small ε, such that

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^3\rho\|^2_{L^2_\gamma} + \|\nabla^3 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma}) \\
& - \frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{\rho+H(\rho)}{r_1+\beta\rho}|\nabla^3\rho|^2\,\mathrm{d}x + \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^4\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^3\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^4\tau\|^2_{L^2_\gamma} \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3(\rho,u,\eta,\tau)\|^2_{L^2_\gamma} + \|\nabla^3(\rho,\eta)\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} + \|\nabla^3(u,\eta,\tau)\|^2_{L^2_{\gamma-1}} + t^{-\frac{11}{2}+2\gamma},
\end{aligned} \tag{3.59}
$$

for all $t > T_1$. We define

$$
K(t) = \|\nabla^3\rho\|^2_{L^2_\gamma} + \|\nabla^3 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma} - \beta\int_{\mathbb{R}^3}|x|^{2\gamma}\frac{\rho+H(\rho)}{r_1+\beta\rho}|\nabla^3\rho|^2\mathrm{d}x.
$$

Then, it is clear that

$$
\begin{aligned}
& \underline{C}(\|\nabla^3\rho\|^2_{L^2_\gamma} + \|\nabla^3 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma}) \\
\leq & K(t) \leq \overline{C}(\|\nabla^3\rho\|^2_{L^2_\gamma} + \|\nabla^3 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma}),
\end{aligned} \tag{3.60}
$$

where $\underline{C}$ and $\overline{C}$ are two positive constants. Therefore, one has the following fact:

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^3\rho\|^2_{L^2_\gamma} + \|\nabla^3 u\|^2_{L^2_\gamma} + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|^2_{L^2_\gamma}) \\
& + \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^4\eta\|^2_{L^2_\gamma} + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^3\tau\|^2_{L^2_\gamma} + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^4\tau\|^2_{L^2_\gamma} \\
\lesssim & t^{-\frac{5}{4}}\|\nabla^3(\rho,u,\eta,\tau)\|^2_{L^2_\gamma} + \|\nabla^3(\rho,\eta)\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_{\gamma-1}} + \|\nabla^3(u,\eta,\tau)\|^2_{L^2_{\gamma-1}} + t^{-\frac{11}{2}+2\gamma}.
\end{aligned} \tag{3.61}
$$

Using the interpolation inequality with weights $\left\|\nabla^3 f\right\|_{L^2_{\gamma-1}} \lesssim \left\|\nabla^3 f\right\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}} \left\|\nabla^3 f\right\|_{L^2}^{\frac{1}{\gamma}}$, we can obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^3\rho\|_{L^2_\gamma}^2 + \|\nabla^3 u\|_{L^2_\gamma}^2 + \frac{r_2}{\beta\tilde{\eta}}\|\nabla^3\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\|\nabla^3\tau\|_{L^2_\gamma}^2)\\
&+ \frac{r_2\alpha}{\beta\tilde{\eta}}\|\nabla^4\eta\|_{L^2_\gamma}^2 + \frac{r_3}{2\beta k\tilde{\eta}}\frac{A_0}{2\lambda}\|\nabla^3\tau\|_{L^2_\gamma}^2 + \frac{r_3\alpha}{2\beta k\tilde{\eta}}\|\nabla^4\tau\|_{L^2_\gamma}^2\\
\lesssim& t^{-\frac{5}{4}}\|\nabla^3(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2 + \|\nabla^3(\rho,\eta)\|_{L^2_\gamma}\|\nabla^3 u\|_{L^2_\gamma}^{\frac{\gamma-1}{\gamma}}\|\nabla^3 u\|_{L^2}^{\frac{1}{\gamma}} \\
&+ \|\nabla^3(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}}\|\nabla^3(u,\eta,\tau)\|_{L^2}^{\frac{2}{\gamma}} + t^{-\frac{11}{2}+2\gamma}\\
\lesssim& t^{-\frac{5}{4}}\|\nabla^3(\rho,u,\eta,\tau)\|_{L^2_\gamma}^2 + t^{-\frac{9}{4\gamma}}\|\nabla^3(\rho,\eta)\|_{L^2_\gamma}^{\frac{2\gamma-1}{\gamma}} + t^{-\frac{9}{2\gamma}}\|\nabla^3(u,\eta,\tau)\|_{L^2_\gamma}^{\frac{2\gamma-2}{\gamma}} + t^{-\frac{11}{2}+2\gamma}.
\end{aligned}
\tag{3.62}
$$

Denoting $\mathrm{E}(t) := \left\|\nabla^3(\rho,u,\eta,\tau)\right\|_{L^2_\gamma}^2$, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(t) \le C_0 t^{-\frac{5}{4}}\mathrm{E}(t) + C_1 t^{-\frac{9}{4\gamma}}\mathrm{E}(t)^{\frac{2\gamma-1}{2\gamma}} + C_2 t^{-\frac{9}{2\gamma}}\mathrm{E}(t)^{\frac{\gamma-1}{\gamma}} + t^{-\frac{11}{2}+2\gamma},
$$

where C_0, C_1, C_2 are positive constants independent of t. If $\gamma > \frac{7}{2}$, then we can apply Lemma 2.5 with $\alpha_0 = \frac{5}{4} > 1$, $\alpha_1 = \frac{9}{4\gamma} < 1$, $\beta_1 = \frac{2\gamma-1}{2\gamma} < 1$, $\alpha_2 = \frac{9}{2\gamma} < 1$, $\beta_2 = \frac{\gamma-1}{\gamma} < 1$, $\gamma_1 = \frac{1-\alpha_1}{1-\beta_1} = -\frac{9}{2} + 2\gamma > 0$, $\gamma_2 = \frac{1-\alpha_2}{1-\beta_2} = -\frac{9}{2} + \gamma > 0$, $\gamma_1 > \gamma_2$. Thus, for all $t > T$, one has

$$
\mathrm{E}(t) \le C t^{-\frac{9}{2}+2\gamma}, \tag{3.63}
$$

we get the fact

$$
\|\nabla^3(\rho,u,\eta,\tau)(t)\|_{L^2_{\gamma_0}} \le C\|\nabla^3(\rho,u,\eta,\tau)(t)\|_{L^2}^{1-\frac{\gamma_0}{\gamma}}\|\nabla^3(\rho,u,\eta,\tau)(t)\|_{L^2_\gamma}^{\frac{\gamma_0}{\gamma}} \le C t^{-\frac{9}{4}+\gamma_0},
$$

for all $t > T$, $\gamma_0 \in [0,\gamma]$, and $[0,\frac{9}{2}] \subset [0,\gamma](\gamma > \frac{9}{2})$. Thus, the proof of Lemma 3.4 has been completed. □

Lemma 3.5. *Under the assumption of Theorem* 1.3*, there exists a sufficiently large T such that the solution* (ρ,u,η,τ) *of the system* (1.4) *with the initial data* (1.5) *has the following estimate:*

$$
\|\nabla^m\tau(t)\|_{L^2_\gamma} \le C t^{-\frac{5}{4}-\frac{m}{2}+\gamma}, \tag{3.64}
$$

for all $t > T$, $\gamma \ge 0$, and $0 \le m \le 2$, where C is a positive constant independent of t.

Proof. Multiplying $|x|^{2\gamma}\nabla^m\tau$ by $\nabla^m(1.4)_4$, and then adding them up, and integrating on $\mathbb{R}^3$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m\tau_t\,\mathrm{d}x + \frac{A_0}{2\lambda}\int_{\mathbb{R}^3}|x|^{2\gamma}(\nabla^m\tau)^2\,\mathrm{d}x\\
&-\alpha\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m\triangle\tau\,\mathrm{d}x - \beta k\tilde{\eta}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m(\nabla u + \nabla^T u)\,\mathrm{d}x\\
=&\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m S_4\mathrm{d}x.
\end{aligned}
\tag{3.65}
$$

Then, using integration by parts to simplify, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m\tau\|_{L^2_\gamma}^2+\frac{A_0}{2\lambda}\|\nabla^m\tau\|_{L^2_\gamma}^2+\alpha\|\nabla^{m+1}\tau\|_{L^2_\gamma}^2\\
=&-\alpha\int_{\mathbb{R}^3}\nabla(|x|^{2\gamma})\nabla^{m+1}\tau\nabla^m\tau\,\mathrm{d}x+\beta k\tilde{\eta}\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m(\nabla u+\nabla^T u)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m S_4\mathrm{d}x\\
\triangleq&\sum_{i=1}^{3}J_{5,i}.
\end{aligned} \tag{3.66}$$

Initially, using Lemma 2.3 and Cauchy's inequality, we can obtain

$$\begin{aligned}
\left|J_{5,1}\right|&\lesssim\alpha\left\|\nabla^{m+1}\tau\right\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_{\gamma-1}}\\
&\lesssim\varepsilon\alpha\left\|\nabla^{m+1}\tau\right\|_{L^2_\gamma}^2+C\alpha(\varepsilon)\|\nabla^m\tau\|_{L^2_{\gamma-1}}^2.
\end{aligned} \tag{3.67}$$

Applying Lemma 2.3, Lemmas 3.2–3.4 and Hölder's inequality, one has

$$\begin{aligned}
|J_{5,2}|&\lesssim\left\|\nabla^{m+1}u\right\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_\gamma}\\
&\lesssim t^{-\frac{5}{4}-\frac{m}{2}+\gamma}\|\nabla^m\tau\|_{L^2_\gamma}\\
&\lesssim t^{-\frac{5}{2}-m+2\gamma}+\frac{1}{2}\|\nabla^m\tau\|_{L^2_\gamma}^2.
\end{aligned} \tag{3.68}$$

By using Lemma 2.1, Hölder's inequality, Cauchy's inequality, we can get the following weighted estimate

$$\begin{aligned}
|J_{5,3}|\lesssim&\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m(\mathrm{div}(u\tau))\,\mathrm{d}x\right|+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m(\nabla u\tau+\tau\nabla^T u)\,\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^m(\eta(\nabla u+\nabla^T u))\,\mathrm{d}x\right|\\
\lesssim&\left|\sum_{j=0}^{m}C_m^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^j u\nabla^{m-j+1}\tau\,\mathrm{d}x\right|+\left|\sum_{j=0}^{m}C_m^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^j\tau\nabla^{m-j+1}u\,\mathrm{d}x\right|\\
&+\left|\sum_{j=0}^{m}C_m^j\int_{\mathbb{R}^3}|x|^{2\gamma}\nabla^m\tau\nabla^j\eta\nabla^{m-j+1}u\,\mathrm{d}x\right|\\
\lesssim&\sum_{j=0}^{m}C_m^j\|\nabla^j u\|_{L^\infty}\|\nabla^{m-j+1}\tau\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_\gamma}+\sum_{j=0}^{m}C_m^j\|\nabla^j\tau\|_{L^\infty}\|\nabla^{m-j+1}u\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_\gamma}\\
&+\sum_{j=0}^{m}C_m^j\|\nabla^j\eta\|_{L^\infty}\|\nabla^{m-j+1}u\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_\gamma}\\
\lesssim&\sum_{j=0}^{m}C_m^j\|\nabla^{j+1}u\|_{H^1}\|\nabla^{m-j+1}\tau\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_\gamma}+\sum_{j=0}^{m}C_m^j\|\nabla^{j+1}\tau\|_{H^1}\|\nabla^{m-j+1}u\|_{L^2_\gamma}\|\nabla^m\tau\|_{L^2_\gamma}
\end{aligned}$$

$$+\sum_{j=0}^{m} C_m^j \|\nabla^{j+1}\eta\|_{H^1} \|\nabla^{m-j+1}u\|_{L^2_\gamma} \|\nabla^m \tau\|_{L^2_\gamma}$$

$$\lesssim \sum_{j=0}^{m} t^{-\frac{5}{4}-\frac{j}{2}} \times t^{-\frac{5}{4}-\frac{m-j}{2}+\gamma} \|\nabla^m \tau\|_{L^2_\gamma}$$

$$\lesssim t^{-\frac{5}{2}-m+2\gamma} + t^{-\frac{5}{4}} \|\nabla^m \tau\|^2_{L^2_\gamma}. \tag{3.69}$$

Substituting (3.67) to (3.69) into (3.66), we conclude that there exists a sufficiently large T_1 and a sufficiently small ε, such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m \tau\|^2_{L^2_\gamma} + C'\|\nabla^m \tau\|^2_{L^2_\gamma} + C''\|\nabla^{m+1} \tau\|^2_{L^2_\gamma} \lesssim t^{-\frac{5}{2}-m+2\gamma}. \tag{3.70}$$

for all $t > T_1$, where C' and C'' are positive constant independent of t. By the Gronwall's argument, one has

$$\|\nabla^m \tau(t)\|_{L^2_\gamma} \lesssim t^{-\frac{5}{4}-\frac{m}{2}+\gamma}, \tag{3.71}$$

for all $0 \leq m \leq 2$ and $0 \leq \gamma$. Thus, the proof of Theorem 1.3 has been completed. □

4. Conclusions

In this paper, we studied the space-time decay rates of solutions to the Cauchy problem of the compressible Oldroyd-B system with diffusive properties and without viscous dissipation in three dimensions. More precisely, we demonstrated that the weighted rate of $k(0 \leq k \leq 3)$-th order spatial derivative of the global solution (ρ, u, η, τ) is $t^{-\frac{3}{4}+\frac{k}{2}+\gamma}$ in the weighted Lebesgue space L^2_γ. And we further explained the reason why the decay estimates cannot achieve better results. Moreover, we also establish that the space-time decay rate of $m\,(\in [0,2])$-th order spatial derivative of the extra stress tensor of the field in L^2_γ is $(1+t)^{-\frac{5}{4}-\frac{m}{2}+\gamma}$, which is faster than that of the velocity.

Author contributions

All authors have contributed significantly to the development of this article. Yangyang Chen: Conceptualization, Methodology, Validation, Writing-original draft and editing; Yixuan Song: Formal analysis, Validation, Writing-original draft and editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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