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***Research article***

# Repdigits base $\eta$ as sum or product of Perrin and Padovan numbers

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**Abstract:** Let $\{E_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ be sequences of Perrin and Padovan numbers, respectively. We have found all repdigits that can be written as the sum or product of $E_n$ and $P_m$ in the base $\eta$, where $2 \leq \eta \leq 10$ and $m \leq n$. In addition, we have applied the theory of linear forms in logarithms of algebraic numbers and Baker-Davenport reduction method in Diophantine approximation approaches.

**Keywords:** exponential Diophantine equation; repdigit; linear forms in logarithms; Padovan numbers; Perrin numbers
**Mathematics Subject Classification:** 11B39, 11D45, 11D61, 11J70, 11J86

## 1. Introduction

The sequence of Padovan numbers $\{P_n\}_{n\geq 0}$ is defined by the following recurrence sequence:

$$P_n = \begin{cases} 1 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ P_{n-2} + P_{n-3} & \text{if } n \geq 3. \end{cases}$$

The first few terms are as follows:

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, ...$$

The Perrin sequence $\{E_n\}_{n\geq 0}$ that is derived from the recurrence relation is as follows:

$$E_n = \begin{cases} 3 & \text{if } n = 0, \\ 0 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ E_{n-2} + E_{n-3} & \text{if } n \geq 3. \end{cases}$$

The first few terms are as follows:

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, ...$$

The Padovan and Perrin numbers are the sequences $A000931$ and $A001608$ respectively, in the online encyclopedia of integer sequences (OEIS).

A natural number $N$ is called a base $\eta$ repdigit if it is of the form

$$N = d\left(\frac{\eta^b - 1}{\eta - 1}\right), \quad \text{where } 1 \leq d \leq \eta - 1 \text{ and for some } b \geq 1.$$

When $\eta = 10$, the number $N$ is a repdigit. Recently, many mathematicians have investigated the solutions to the Diophantine equations that involve repdigits and linear recurrence sequences. Lomelí and Hernández [1] showed that the only repdigits that can be written as sums of two Padovan numbers are $11, 22, 33, 44, 66, 77, 88, 3333$.

Trojovský [2] found all repdigits that can be written as sums of Fibonacci and Tribonacci numbers. Bednařík and Trojovská [3] studied the repdigits that can be expressed as products of Fibonacci and Tribonacci numbers. Erduvan et al. [4–6] expressed all repdigits in base $b$ as products of two Fibonacci, two Lucas, two Pell, and two Pell-Lucas numbers. In 2022, the same authors [7] examined all repdigits in base $b$ that are represented as the difference between two Fibonacci numbers.

Moreover, Bhoi and Ray [8] demonstrated that only Perrin numbers that are expressible as the sum of two repdigits are $P_{11}$ and $P_{20}$. Rihane and Togbé [9] found all repdigits that can be expressed as products of consecutive Padovan or Perrin numbers. One year later, the same authors [10] investigated Padovan and Perrin numbers as a product of two repdigits. Adédji et al. [11] found all Padovan and Perrin numbers, which are products of two repdigits in base $b$, and showed that $P_{25}$ and $T_{22}$ are the largest Padovan and Perrin numbers in that form, respectively.

Subsequently, Adédji et al. [12] showed that Padovan or Perrin numbers are concatenations of two distinct base $b$ repdigits with $2 \leq b \leq 9$. They also found that the largest Padovan and Perrin numbers are concatenations of two distinct base $b$ repdigits, $P_{26}$ and $E_{24}$, respectively. Adédji et al. [13] considered the Padovan and Perrin numbers to be expressible as products of two generalized Lucas numbers. Duman et al. [14] showed that $2, 3, 4, 5, 7, 9, 12, 16, 28, 37, 49, 86$ and $114$ are the only Padovan numbers that can be expressed as the sum of two repdigits. The findings are as follows

**Theorem 1.1.** *The only solutions of the Diophantine equation*

$$E_n + P_m = d\left(\frac{\eta^b - 1}{\eta - 1}\right), \tag{1.1}$$

*as non-negative integers for* $m \leq n$, $1 \leq d \leq \eta - 1$, $2 \leq \eta \leq 10$, *and* $b \geq 2$ *are given in Table A.1.*

**Theorem 1.2.** *The only solutions of the Diophantine equation*

$$E_n \cdot P_m = d\left(\frac{\eta^b - 1}{\eta - 1}\right), \tag{1.2}$$

*as non-negative integers for* $m \leq n$, $1 \leq d \leq \eta - 1$, $2 \leq \eta \leq 10$, *and* $b \geq 2$ *are given in Table A.2.*

**Remark.** We know that $P_0 = P_1 = P_2 = 1$ and $P_3 = P_4 = 2$. In both theorems above, we assume that $m \leq n$ and $m \neq 0, 1, 3$.

## 2. Auxiliary results

### *2.1. Padovan and Perrin sequences*

This section includes various Padovan and Perrin sequence features that are relevant to our theorems. The characteristic polynomial of the Padovan and Perrin sequences is given by

$$x^3 - x - 1 = 0,$$

with roots $\theta_1, \theta_2$, and $\theta_3$, where

$$\theta_1 = \frac{s_1 + s_2}{6}, \quad \theta_2 = \frac{-s_1 - s_2 + i\sqrt{3}\,(s_1 - s_2)}{12}, \quad \theta_3 = \bar{\theta}_2,$$

and

$$s_1 = \sqrt[3]{108 + 12\sqrt{69}}, \quad s_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$c_1 = \frac{(1-\theta_2)(1-\theta_3)}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{1+\theta_1}{-\theta_1^2 + 3\theta_1 + 1},$$
$$c_2 = \frac{(1-\theta_1)(1-\theta_3)}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{1+\theta_2}{-\theta_2^2 + 3\theta_2 + 1},$$
$$c_3 = \frac{(1-\theta_1)(1-\theta_2)}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{1+\theta_3}{-\theta_3^2 + 3\theta_3 + 1}.$$

Binet's formulas for Padovan and Perrin sequences are respectively defined by

$$P_n = c_1\theta_1^n + c_2\theta_2^n + c_3\theta_3^n, \quad \text{for all } n \geq 0, \tag{2.1}$$

and

$$E_n = \theta_1^n + \theta_2^n + \theta_3^n, \quad \text{for all } n \geq 0. \tag{2.2}$$

Numerically, we have

$$1.32 < \theta_1 < 1.33,$$
$$0.86 < |\theta_2| = |\theta_3| = \theta_1^{-1/2} < 0.87,$$
$$0.72 < c_1 < 0.73,$$
$$0.24 < |c_2| = |c_3| < 0.25.$$

Given that $\theta_2 = \theta_1^{-1/2}e^{iR}$ and $\theta_3 = \theta_1^{-1/2}e^{-iR}$ for some $R \in (0, 2\pi)$, it follows that

$$P_n = c_1\theta_1^n + e_n, \quad \text{with } |e_n| < \frac{1}{\theta_1^{n/2}}, \quad \text{for all } n \geq 1, \tag{2.3}$$

and

$$E_n = \theta_1^n + l_n, \quad \text{with } |l_n| < \frac{2}{\theta_1^{n/2}}, \quad \text{for all } n \geq 1. \tag{2.4}$$

Using the method of induction, we can prove that

$$\theta_1^{n-2} \leq P_n \leq \theta_1^{n-1}, \quad \text{for all } n \geq 4, \tag{2.5}$$

and

$$\theta_1^{n-2} \leq E_n \leq \theta_1^{n+1}, \quad \text{for all } n \geq 2. \tag{2.6}$$

*2.2. Linear forms in logarithms*

**Definition 2.1.** *(Absolute logarithmic height) Let $\gamma$ be an algebraic number of degree $d$ with the following minimal polynomial:*

$$c_0x^d + c_1x^{d-1} + \ldots + c_d = c_0 \prod_{i=1}^{d} \left(x - \gamma^{(i)}\right) \in \mathbb{Z}[x],$$

*where $\gamma^{(i)}$ denotes conjugates of $\gamma$, and $c_i$ values are relatively prime to each other with $c_0 > 0$. Then the logarithmic height of $\gamma$ is given by*

$$h(\gamma) = \frac{1}{d}\left(\log c_0 + \sum_{i=1}^{d} \log\left(\max\left\{\left|\gamma^{(i)}\right|, 1\right\}\right)\right). \tag{2.7}$$

*If $\gamma = \frac{a}{b}$ is a rational number with $gcd(a, b) = 1$ and $b > 0$, then $h(\gamma) = \log(\max\{|a|, b\})$.*

The following are the properties of the logarithmic height function, which will be utilized in the subsequent sections of this paper:

$$h(\eta \pm \gamma) \le h(\eta) + h(\gamma) + \log 2, \tag{2.8}$$

$$h\left(\eta\gamma^{\pm 1}\right) \le h(\eta) + h(\gamma), \tag{2.9}$$

$$h\left(\eta^k\right) = |k|h(\eta). \tag{2.10}$$

To prove the validity of Theorems 1.1 and 1.2, we use the modified version of the Matveev result [15], as stated by Bugeaud et al. [16, Theorem 9.4].

**Theorem 2.2.** *Let $\mathbb{L}$ be the real algebraic number field of degree $D$ over $\mathbb{Q}$. Let $\gamma_1, \ldots, \gamma_t \in \mathbb{L}$ be positive real algebraic numbers, and let $b_1, b_2, \ldots, b_t$ be nonzero integers such that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

*is not zero. Then*

$$\log|\Lambda| > (-1.4)\left(30^{t+3}\right)\left(t^{4.5}\right)\left(D^2\right)(A_1 \ldots A_t)(1 + \log D)(1 + \log B),$$

*where*

$$B \ge \max\{|b_1|, \ldots, |b_t|\},$$

*and*

$$A_i \ge \max\left\{Dh(\gamma_i), \left|\log(\gamma_i)\right|, 0.16\right\}, 1 \le i \le t.$$

*2.3. De Weger reduction method*

We need the variation of the Baker-Davenport reduction method developed by de Weger [17] to reduce the upper bound. Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given and $x_1, x_2 \in \mathbb{Z}$ be unknowns.

Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{2.11}$$

Let $c, \delta$ be positive constants. We set $X = \max\{|x_1|, |x_2|\}$ and let $X_0, Y$ be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{2.12}$$

$$Y \le X \le X_0. \tag{2.13}$$

**Case 1:** If $\beta = 0$ in Eq (2.11), then

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Set $\vartheta = -\vartheta_1/\vartheta_2$. We assume that $x_1$ and $x_2$ are relatively prime. The continued fraction expansion of $\vartheta$ is represented by $[a_0, a_1, a_2, \ldots]$. The $k$-th convergent of $\vartheta$ is denoted by $p_k/q_k$, where $k = 0, 1, 2, \ldots$. Without a loss of generality, we may assume that $|\vartheta_1| < |\vartheta_2|$ and $x_1 > 0$. We obtain the following lemma.

**Lemma 2.3.** *( [17, Lemma 3.2]) Let*

$$A = \max_{0 \le k \le Y_0} a_{k+1},$$

*where*

$$Y_0 = -1 + \frac{\log\left(\sqrt{5}X_0 + 1\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}.$$

*If* (2.12) *and* (2.13) *hold for* $x_1, x_2$ *and* $\beta = 0$*, then*

$$Y < \frac{1}{\delta}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{2.14}$$

**Case 2:** If $\beta \neq 0$ in Eq (2.11), then we get

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2,$$

where $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Let $p/q$ be a convergent of $\vartheta$ with $q > X_0$. The distance between a real number $m$ and the nearest integer is represented by $||m|| = \min\{|m - n| : n \in \mathbb{Z}\}$. We have the following lemma.

**Lemma 2.4.** *( [17, Lemma 3.3]) Suppose that*

$$||q\psi|| > \frac{2X_0}{q}.$$

*Then the solutions of* (2.12) *and* (2.13) *satisfy*

$$Y < \frac{1}{\delta}\log\left(\frac{q^2 c}{|\vartheta_2|\, X_0}\right). \tag{2.15}$$

To prove our theorems, we present the following findings.

**Lemma 2.5.** *( [18, Lemma 7]) If* $r \ge 1$, $S \ge (4r^2)^r$, *and* $\frac{L}{(\log L)^r} < S$*, then*

$$L < 2^r S (\log S)^r.$$

**Lemma 2.6.** *( [17, Lemma 2.2]) Let* $v, x \in \mathbb{R}$ *and* $0 < v < 1$*. If* $|x| < v$*, then*

$$|\log(1 + x)| < \frac{-\log(1 - v)}{v}|x|.$$

## 3. Main results

### 3.1. *The proof of Theorem 1.1*

In this section, we present the proof of Theorem 1.1. The main process is detailed below.

#### 3.1.1. Relation between $n$ and $b$

The set of solutions for the Diophantine equation given by Eq (1.1) in the range of $2 \leq m \leq n < 350$ and for $m \neq 3$ can be obtained by using the Maple program; the solutions are presented in Table A.1. Considering the remaining case, we assume that $n \geq 350$ and $m \geq 2$, where $m \neq 3$. By combining inequalities (2.5) and (2.6) in Eq (1.1), we obtain

$$2^{b-1} \leq \eta^{b-1} < d\left(\frac{\eta^b - 1}{\eta - 1}\right) = E_n + P_m \leq \theta_1^{n+1} + \theta_1^{m-1} < 2\theta_1^{n+1},$$

and it is clear that $\theta_1 < 1.33$; then,

$$2^{b-1} < 2\theta_1^{n+1} < 2 \cdot 2^{n+1} < 2^{n+2}.$$

We conclude that

$$b \leq n + 2 < n + 3. \tag{3.1}$$

#### 3.1.2. Finding an upper bound for $n$

Putting Eq (2.4) in Eq (1.1), we have

$$(\theta_1^n + l_n) + P_m = d\left(\frac{\eta^b - 1}{\eta - 1}\right),$$

$$\theta_1^n - \frac{d\eta^b}{\eta - 1} = -P_m - l_n - \frac{d}{\eta - 1}.$$

Taking the absolute value for both sides of the above equation, we get

$$\left|\theta_1^n - \frac{d\eta^b}{\eta - 1}\right| = \left|-P_m - l_n - \frac{d}{\eta - 1}\right| < P_m + |l_n| + \frac{d}{\eta - 1} < \theta_1^{m-1} + 1.1.$$

Dividing both sides by $\theta_1^n$, we deduce that

$$\left|1 - \frac{d\eta^b\theta_1^{-n}}{\eta - 1}\right| < \theta_1^{m-n}(\theta_1^{-1} + 1.1\theta_1^{-m}) < \frac{1.5}{\theta_1^{n-m}}. \tag{3.2}$$

Let

$$\Lambda_1 := \frac{d\eta^b\theta_1^{-n}}{\eta - 1} - 1.$$

We claim that $\Lambda_1$ is nonzero. Suppose that $\Lambda_1 = 0$, which implies that

$$\frac{d\eta^b}{\eta - 1} = \theta_1^n.$$

However, this is contradictory since $\theta_1^n \notin \mathbb{Q}$ for any $n > 0$. Hence $\Lambda_1 \neq 0$. We apply Theorem 2.2 to the left hand side of inequality (3.2) in consideration of the parameter $t := 3$, we set

$$(\gamma_1, b_1) := (\theta_1, -n), \quad (\gamma_2, b_2) := (\eta, b), \quad (\gamma_3, b_3) := \left(\frac{d}{\eta - 1}, 1\right).$$

Since $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$, it follows that $D := [\mathbb{L}, \mathbb{Q}] = 3$. The heights of $\gamma_1, \gamma_2, \gamma_3$ can be calculated as follows:

$$h(\gamma_1) = \frac{\log(\theta_1)}{3}, \quad h(\gamma_2) = \log(\eta) \leq \log(10) < 2.31,$$

and

$$h(\gamma_3) \leq h(d) + h(\eta - 1) \leq 2\log(9) < 4.4.$$

Thus, we can take $A_1 := 0.29$, $A_2 := 6.93$ and $A_3 := 13.2$. Since $b \leq n + 2$, taking $B := n + 2 \geq \max\{|-n|, |b|, |1|\}$, by Theorem 2.2, we get

$$\frac{1.5}{\theta_1^{n-m}} > |\Lambda_1| > \exp\{-G(1 + \log(n + 2))(0.29)(6.93)(13.2)\},$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(9)(1 + \log(3))$. By simplifying the computation, we obtain

$$(n - m)\log(\theta_1) < 7.2 \cdot 10^{13}(1 + \log(n + 2)). \tag{3.3}$$

Rewrite Eq (1.1) as follows

$$(\theta_1^n + l_n) + (c_1\theta_1^m + e_m) = d\left(\frac{\eta^b - 1}{\eta - 1}\right),$$

$$\theta_1^n + c_1\theta_1^m - \frac{d\eta^b}{\eta - 1} = -l_n - e_m - \frac{d}{\eta - 1}.$$

Taking the absolute value of the above equation, we get

$$\left|\theta_1^n(1 + c_1\theta_1^{m-n}) - \frac{d\eta^b}{\eta - 1}\right| = \left|-l_n - e_m - \frac{d}{\eta - 1}\right| < 1.9,$$

for $n \geq 350$ and $m \geq 2$. Dividing both sides by $\theta_1^n\left(1 + c_1\theta_1^{m-n}\right)$, we obtain

$$\left|1 - \frac{d\eta^b\theta_1^{-n}\left(1 + c_1\theta_1^{m-n}\right)^{-1}}{\eta - 1}\right| < 1.9 \cdot \theta_1^{-n}(1 + c_1\theta_1^{m-n})^{-1} < \frac{2.1}{\theta_1^n}. \tag{3.4}$$

We apply that, for $n \geq 350$ and $m \geq 2$, the inequality

$$(1 + c_1\theta_1^{m-n})^{-1} < 1.1,$$

holds.

Let

$$\Lambda_2 := \frac{d\eta^b\theta_1^{-n}(1 + c_1\theta_1^{m-n})^{-1}}{\eta - 1} - 1.$$

Note that $\Lambda_2$ is nonzero. Suppose that $\Lambda_2 = 0$, which implies that

$$\frac{d\eta^b}{\eta - 1} = \theta_1^n(1 + c_1\theta_1^{m-n}).$$

But, this is a contradiction since $\theta_1^n(1 + c_1\theta_1^{m-n}) \notin \mathbb{Q}$ for any $n \geq m \geq 2$. Hence $\Lambda_2 \neq 0$. By applying Theorem 2.2 to the left-hand side of inequality (3.4) and considering the parameter $t := 3$, we have

$$(\gamma_1, b_1) := (\theta_1, -n), \quad (\gamma_2, b_2) := (\eta, b), \quad (\gamma_3, b_3) := \left(\frac{d(1 + c_1\theta_1^{m-n})^{-1}}{\eta - 1}, 1\right).$$

We have that $D := [\mathbb{L}, \mathbb{Q}] = 3$, where $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$. By using the definition of logarithmic height, we deduce that

$$h(\gamma_1) = \frac{\log(\theta_1)}{3}, \quad h(\gamma_2) = \log(\eta) \leq \log(10) < 2.31,$$

and

$$\begin{aligned} h(\gamma_3) &= h\left(\frac{d(1 + c_1\theta_1^{m-n})^{-1}}{\eta - 1}\right) \\ &\leq h(d) + h(c_1) + (n - m)h(\theta_1) + h(\eta - 1) + \log(2) \\ &< 2\log(9) + \frac{\log(23)}{3} + (n - m)\frac{\log(\theta_1)}{3} + \log(2) < 6.13 + (n - m)\frac{\log(\theta_1)}{3}. \end{aligned}$$

So we can take $A_1 := 0.29$, $A_2 := 6.93$ and $A_3 := 18.39 + (n - m)\log(\theta_1)$. Also, by Eq (3.1), we can choose $B := n + 2 \geq \max\{| - n|, |b|, |1|\}$. According to Theorem 2.2, we get

$$\frac{2.1}{\theta_1^n} > |\Lambda_2| > \exp\{-G(1 + \log(n + 2))(0.29)(6.93)(18.39 + (n - m)\log(\theta_1))\}, \tag{3.5}$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(9)(1 + \log(3))$. Putting inequality (3.3) in the above inequality, by simple calculation, we get

$$\frac{n}{(\log(n))^2} < 5.57 \cdot 10^{27}.$$

We consider the fact that $1 + \log(n + 2) < 2\log(n)$ for all $n > 5$. Now we apply Lemma 2.5, taking $S := 5.57 \cdot 10^{27}$, $L := n$, and $r := 2$. With the help of Maple, we can obtain

$$n < 9.1 \cdot 10^{31}.$$

3.1.3. Reducing the upper bound of $n$

In this section, we attempt to reduce the upper bound of $n$ by using Lemmas 2.3 and 2.4.
Let

$$y_1 := \log(\Lambda_1 + 1) = b\log(\eta) - n\log(\theta_1) + \log\left(\frac{d}{\eta - 1}\right).$$

In inequality (3.2), assume that $n - m \geq 2$; then, we get

$$|\Lambda_1| = |e^{y_1} - 1| < \frac{1.5}{\theta_1^{n-m}} < 0.86.$$

Choosing $v = 0.86$ in Lemma 2.6, we obtain

$$|y_1| = |\log(\Lambda_1 + 1)| = -\frac{\log(1-0.86)}{0.86} \cdot \frac{1.5}{\theta_1^{n-m}} < \frac{3.5}{\theta_1^{n-m}}.$$

It follows that

$$0 < \left| n\left(-\log(\theta_1)\right) + b\log(\eta) + \log\left(\frac{d}{\eta-1}\right) \right| < 3.5 \cdot \exp\left(-(n-m)\cdot\log(\theta_1)\right).$$

We consider the following two cases of the above inequality.

**Case 1:** If $1 \le d < \eta - 1$ and $3 \le \eta \le 10$, we see that $\beta \neq 0$; then, applying Lemma 2.4, we have

$$c := 3.5, \quad \delta := \log(\theta_1), \quad \psi := \frac{\log\left(\frac{d}{\eta-1}\right)}{\log(\eta)},$$

$$\vartheta := \frac{\log(\theta_1)}{\log(\eta)}, \quad \vartheta_1 := -\log(\theta_1), \quad \vartheta_2 := \log(\eta), \quad \beta := \log\left(\frac{d}{\eta-1}\right).$$

We know that $\vartheta$ is irrational. We take $X_0 := 9.2 \cdot 10^{31}$, which is an upper bound for both $b$ and $n$. Using Maple programming, we obtain Table 1, so we get

$$n - m < \frac{1}{\log(\theta_1)} \cdot \log\left(\frac{q_{61}^2 \cdot 3.5}{|\log(3)| \cdot 9.2 \cdot 10^{31}}\right) < 300.41.$$

Therefore, we obtain that $n - m \le 300$.

**Table 1.** Results of reducing the upper bound of $n - m$ for $\beta \neq 0$.

| $\eta$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $q_k$ | $q_{61}$ | $q_{71}$ | $q_{63}$ | $q_{72}$ | $q_{56}$ | $q_{62}$ | $q_{66}$ | $q_{74}$ |
| $n-m$ | 300 | 280 | 285 | 295 | 295 | 294 | 288 | 297 |

**Case 2:** If $d = \eta - 1$ where $2 \le \eta \le 10$, then $\beta = 0$. Now applying Lemma 2.3, we have that $X_0 := 9.2 \cdot 10^{31}$.

$$Y_0 = -1 + \frac{\log(\sqrt{5}\cdot 9.2\cdot 10^{31} + 1)}{\log(\frac{1+\sqrt{5}}{2})} = 153.618,$$

and

$$A = \max_{0 \le k \le 153} a_{k+1}.$$

Upon inspection by using Maple programming, we can find Table 2 for all possibilities of $\eta$, we get

$$n - m < \frac{1}{\log(\theta_1)} \cdot \log\left(\frac{3.5(303+2)9.2\cdot 10^{31}}{|\log(2)|}\right) < 287.$$

**Table 2.** Results of reducing the upper bound of $n - m$ for $\beta = 0$.

| $\eta$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $A$ | 303 | 151 | 160 | 433 | 195 | 627 | 241 | 278 | 564 |
| $n-m$ | 287 | 283 | 283 | 286 | 282 | 286 | 283 | 283 | 285 |

Hence, in both cases, $n-m \leq 300$. Substituting this into inequality (3.5), we obtain that $n < 2.9 \cdot 10^{17}$. Let

$$y_2 := \log(\Lambda_2 + 1) = b\log(\eta) - n\log(\theta_1) + \log\left(\frac{d}{(\eta-1)(1+c_1\theta_1^{m-n})}\right).$$

Then, by inequality (3.4), we have

$$|\Lambda_2| = |e^{y_2} - 1| < \frac{2.1}{\theta_1^n} < 0.25,$$

Based on our assumption, $n \geq 350$. Given Lemma 2.6, we choose $v = 0.25$. Thus

$$|y_2| = |\log(\Lambda_2 + 1)| = -\frac{\log(1-0.25)}{0.25} \cdot \frac{2.1}{\theta_1^n} < \frac{2.5}{\theta_1^n}.$$

It follows that

$$0 < \left| n\left(-\log(\theta_1)\right) + b\log(\eta) + \log\left(\frac{d}{(\eta-1)(1+c_1\theta_1^{m-n})}\right)\right| < 2.5 \cdot \exp\left(-(n) \cdot \log(\theta_1)\right).$$

In the above inequality, according to the de Weger reduction method, we obtain that $\beta \neq 0$; then, applying Lemma 2.4, we have

$$c := 2.5, \quad \delta := \log(\theta_1), \quad \psi := \frac{\log\left(\frac{d}{(\eta-1)(1+c_1\theta_1^{m-n})}\right)}{\log(\eta)},$$

$$\vartheta := \frac{\log(\theta_1)}{\log(\eta)}, \quad \vartheta_1 := -\log(\theta_1), \quad \vartheta_2 := \log(\eta), \quad \beta := \log\left(\frac{d}{(\eta-1)(1+c_1\theta_1^{m-n})}\right),$$

where $\vartheta$ denotes an irrational number. For $0 \leq (n-m) \leq 300$ and $X_0 := 3 \cdot 10^{17}$, our calculations with the help of Maple programming find Table 3, so we get

$$n < \frac{1}{\log(\theta_1)} \cdot \log\left(\frac{q_{48}^2 \cdot 2.5}{|\log(3)| \cdot 3 \cdot 10^{17}}\right) < 321.6.$$

Therefore, we can obtain a contradiction based on our assumption that $n \geq 350$. Theorem 1.1 is proved.

**Table 3.** Results of reducing the upper bound of $n$.

| $\eta$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $q_k$ | $q_{63}$ | $q_{48}$ | $q_{57}$ | $q_{53}$ | $q_{62}$ | $q_{47}$ | $q_{51}$ | $q_{50}$ | $q_{56}$ |
| $n$ | 309 | 321 | 306 | 308 | 313 | 318 | 308 | 319 | 308 |

**Corollary 3.1.** *The largest repdigits in base $\eta$ of the Diophantine equation given by Eq* (1.1) *are as follows:*

$$E_{16} + P_{14} = 90 + 37 = 127 = [1111111]_2, \quad E_{17} + P_4 = 119 + 2 = 121 = [11111]_3,$$
$$E_{18} + P_{10} = 158 + 12 = 170 = [2222]_4, \quad E_{26} + P_{16} = 1497 + 65 = 1562 = [22222]_5,$$
$$E_{21} + P_{19} = 367 + 151 = 518 = [2222]_6, \quad E_{20} + P_{16} = 277 + 65 = 342 = [666]_7,$$
$$E_{15} + P_7 = 68 + 5 = 73 = [111]_8, \quad E_{22} + P_{19} = 486 + 151 = 637 = [777]_9,$$

$$E_{16} + P_{12} = 90 + 21 = 111 = [111]_{10}.$$

*3.2. The proof of Theorem 1.2*

In this section, we present the following subsections to prove Theorem 1.2.

3.2.1. Relation between $n$ and $b$

All solutions to the Diophantine equation given by Eq (1.2) in the range of $2 \leq m \leq n < 350$, and for $m \neq 3$, with the help of the Maple program, are presented in Table A.2. Considering the remaining possibility, we assume that $n \geq 350$ and $m \geq 2$ where $m \neq 3$. By combining inequalities (2.5) and (2.6) together in Eq (1.2), we obtain

$$2^{b-1} \leq \eta^{b-1} < d\left(\frac{\eta^b - 1}{\eta - 1}\right) = E_n \cdot P_m \leq \theta_1^{n+1} \cdot \theta_1^{m-1} < \theta_1^{n+1} \cdot \theta_1^{n} < \theta_1^{2n+1},$$

and it is clear that $\theta_1 < 1.33$; then,

$$2^{b-1} < \theta_1^{2n+1} < 2^{2n+1}.$$

We conclude that

$$b \leq 2n + 1 < 2n + 2. \tag{3.6}$$

3.2.2. Finding an upper bound for $n$

Putting Eqs (2.3) and (2.4) in Eq (1.2), we have

$$(\theta_1^n + l_n) \cdot (c_1\theta_1^m + e_m) = d\left(\frac{\eta^b - 1}{\eta - 1}\right),$$

$$c_1\theta_1^{n+m} - \frac{d\eta^b}{\eta - 1} = -\theta_1^n e_m - c_1\theta_1^m l_n - e_m l_n - \frac{d}{\eta - 1}.$$

Taking the absolute value of both sides, we get

$$\left|c_1\theta_1^{n+m} - \frac{d\eta^b}{\eta - 1}\right| = \left|-\theta_1^n e_m - c_1\theta_1^m l_n - e_m l_n - \frac{d}{\eta - 1}\right| \leq \theta_1^n|e_m| + c_1\theta_1^m|l_n| + |e_m||l_n| + \frac{d}{\eta - 1}.$$

Dividing both sides by $c_1\theta_1^{n+m}$, we deduce that

$$\begin{aligned}
\left|1 - \frac{d\eta^b\theta_1^{-(n+m)}}{c_1(\eta - 1)}\right| &\leq \frac{|e_m|}{c_1\theta_1^m} + \frac{|l_n|}{\theta_1^n} + \frac{|e_m||l_n|}{c_1\theta_1^{n+m}} + \frac{d}{c_1\theta_1^{n+m}(\eta - 1)} \\
&< \frac{1}{c_1\theta_1^{3m/2}} + \frac{2}{\theta_1^{3n/2}} + \frac{2}{c_1\theta_1^{3(n+m)/2}} + \frac{1}{c_1\theta_1^{n+m}} \\
&< \frac{1}{c_1\theta_1^m} + \frac{2}{c_1\theta_1^m} + \frac{2}{c_1\theta_1^{n+m}} + \frac{1}{c_1\theta_1^{n+m}} < \frac{1}{c_1\theta^m}\left(3 + \frac{3}{\theta_1^n}\right) < \frac{3.1}{c_1\theta^m}.
\end{aligned}$$

It can be seen that

$$\left|1 - \frac{d\eta^b\theta_1^{-(n+m)}}{c_1(\eta - 1)}\right| < \frac{4.31}{\theta^m}. \tag{3.7}$$

Let

$$\Lambda_3 := \frac{d\eta^b \theta_1^{-(n+m)}}{c_1(\eta - 1)} - 1.$$

We want to show that $\Lambda_3$ is nonzero. We assume that $\Lambda_3 = 0$, which implies that

$$\frac{d\eta^b}{\eta - 1} = c_1 \theta_1^{n+m}.$$

However, this is contradictory because $c_1\theta_1^{n+m} \notin \mathbb{Q}$ is an irrational number. Hence $\Lambda_3 \neq 0$.

By applying Theorem 2.2 and $t := 3$ on the left hand side of inequality (3.7). We can take

$$(\gamma_1, b_1) := (\theta_1, -(n+m)), \quad (\gamma_2, b_2) := (\eta, b), \quad (\gamma_3, b_3) := \left(\frac{d}{c_1(\eta - 1)}, 1\right).$$

Since $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$, then $D := [\mathbb{L}, \mathbb{Q}] = 3$. Furthermore, we can obtain the heights of $\gamma_1$, $\gamma_2$ and $\gamma_3$; thus,

$$h(\gamma_1) = \frac{\log(\theta_1)}{3}, \quad h(\gamma_2) = \log(\eta) \leq \log(10) < 2.31,$$

and

$$h(\gamma_3) \leq h(d) + h(c_1) + h(\eta - 1) \leq 2\log(9) + \frac{\log(23)}{3} < 5.43.$$

Thus, we can take $A_1 := 0.29$, $A_2 := 6.93$ and $A_3 := 16.3$. Also, since $b \leq 2n + 1$ and $m \leq n$, we can take $B := 2n + 1 \geq \max\{|-(n+m)|, |b|, |1|\}$. By Theorem 2.2, we get

$$\frac{4.31}{\theta_1^m} > |\Lambda_3| > \exp\{-G(1 + \log(2n+1))(0.29)(6.93)(16.3)\},$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(9)(1 + \log(3))$. By simplifying the computation, we obtain

$$m\log(\theta_1) < 8.9 \cdot 10^{13}(1 + \log(2n+1)). \tag{3.8}$$

Again, rewrite Eq (1.2) as

$$(\theta_1^n + l_n) \cdot P_m = d\left(\frac{\eta^b - 1}{\eta - 1}\right),$$

$$\theta_1^n - \frac{d\eta^b}{P_m(\eta - 1)} = -l_n - \frac{d}{P_m(\eta - 1)}.$$

Taking the absolute values for both sides of the above equation, we get

$$\left|\theta_1^n - \frac{d\eta^b}{P_m(\eta - 1)}\right| = \left|-l_n - \frac{d}{P_m(\eta - 1)}\right| \leq |l_n| + \frac{d}{P_m(\eta - 1)} < 2.5,$$

for $n \geq 350$ and $m \geq 2$. Dividing both sides by $\theta_1^n$, we obtain

$$\left|1 - \frac{d\eta^b \theta_1^{-n}}{P_m(\eta - 1)}\right| < \frac{2.5}{\theta_1^n}. \tag{3.9}$$

Let

$$\Lambda_4 := \frac{d\eta^b \theta_1^{-n}}{P_m(\eta - 1)} - 1.$$

We claim that $\Lambda_4$ is nonzero. Suppose that $\Lambda_4 = 0$, which implies that

$$\frac{d\eta^b}{P_m(\eta - 1)} = \theta_1^n.$$

But this is a contradiction since $\theta_1^n \notin \mathbb{Q}$ for any $n \geq 350$. Hence $\Lambda_4 \neq 0$.

By applying Theorem 2.2 and $t := 3$ to the left hand side of inequality (3.9), we can take

$$(\gamma_1, b_1) := (\theta_1, -n), \quad (\gamma_2, b_2) := (\eta, b), \quad (\gamma_3, b_3) := \left(\frac{d}{P_m(\eta - 1)}, 1\right).$$

Since $\mathbb{L} := \mathbb{Q}(\gamma_1, \gamma_2, \gamma_3)$, then $D := [\mathbb{L}, \mathbb{Q}] = 3$. Moreover, the heights of $\gamma_1$,$\gamma_2$ and $\gamma_3$ were found to be as follows:

$$h(\gamma_1) = \frac{\log(\theta_1)}{3}, \quad h(\gamma_2) = \log(\eta) \leq \log(10) < 2.31,$$

and

$$\begin{aligned} h(\gamma_3) = h\left(\frac{d}{P_m(\eta - 1)}\right) &\leq h(d) + h(P_m) + h(\eta - 1) \\ &< 2\log(9) + m\left(\frac{\log(\theta_1)}{3}\right) < 4.4 + m\left(\frac{\log(\theta_1)}{3}\right). \end{aligned}$$

We can take $A_1 := 0.29$, $A_2 := 6.93$ and $A_3 := 13.2 + m\log(\theta_1)$. Also, by Eq (3.6), can choose $B := 2n + 1 \geq \max\{|-n|, |b|, |1|\}$. According to Theorem 2.2, we get

$$\frac{2.5}{\theta_1^n} > |\Lambda_4| > \exp\{-G(1 + \log(2n + 1))(0.29)(6.93)(13.2 + m\log(\theta_1))\}, \tag{3.10}$$

where $G = (1.4)\left(30^6\right)\left(3^{4.5}\right)(9)(1 + \log(3))$. By substituting inequality (3.8) into the above inequality and simplifying the computation, we obtain

$$\frac{n}{(\log(n))^2} < 6.9 \cdot 10^{27}.$$

We use the fact that $1 + \log(2n + 1) < 2\log(n)$ for all $n > 6$. Now, we apply Lemma 2.5 by taking $S := 6.9 \cdot 10^{27}$, $L := n$, and $r := 2$. With the help of Maple, we can obtain

$$n < 1.13 \cdot 10^{32}.$$

### 3.2.3. Reducing the upper bound of $n$

In this section, we apply Lemmas 2.3 and 2.4 to reduce the upper bound of $n$.

Let

$$y_3 := \log(\Lambda_3 + 1) = b\log(\eta) - (n + m)\log(\theta_1) + \log\left(\frac{d}{c_1(\eta - 1)}\right).$$

In inequality (3.7), assume that $m \geq 6$; then, we get

$$|\Lambda_3| = |e^{y_3} - 1| < \frac{4.31}{\theta_1^m} < 0.8.$$

By applying Lemma 2.6 and choosing $v = 0.8$, we obtain

$$|y_3| = |\log(\Lambda_3 + 1)| = -\frac{\log(1-0.8)}{0.8} \cdot \frac{4.31}{\theta_1^m} < \frac{8.7}{\theta_1^m}.$$

It can be seen that

$$0 < \left|(n+m)\left(-\log(\theta_1)\right) + b\log(\eta) + \log\left(\frac{d}{c_1(\eta-1)}\right)\right| < 8.7 \cdot \exp\left(-(m)\cdot\log(\theta_1)\right).$$

According to the de Weger reduction method, we see that $\beta \neq 0$; then, applying Lemma 2.4, we have

$$c := 8.7, \quad \delta := \log(\theta_1), \quad \psi := \frac{\log\left(\frac{d}{c_1(\eta-1)}\right)}{\log(\eta)},$$

$$\vartheta := \frac{\log(\theta_1)}{\log(\eta)}, \quad \vartheta_1 := -\log(\theta_1), \quad \vartheta_2 := \log(\eta), \quad \beta := \log\left(\frac{d}{c_1(\eta-1)}\right).$$

Clearly, $\vartheta$ is an irrational number. We can take $X_0 := 2.26 \cdot 10^{32}$, which is the upper bound for both $b$ and $n$ since $b < 2n + 1 < 2.26 \cdot 10^{32}$. Using Maple programming, we computed Table 4, we get

$$m < \frac{1}{\log(\theta_1)} \cdot \log\left(\frac{q_{76}^2 \cdot 8.7}{|\log(10)| \cdot 2.26 \cdot 10^{32}}\right) < 306.33.$$

Therefore, we consider that $m \leq 306$. Substituting this into inequality (3.10), we obtain that $n < 2.75 \cdot 10^{17}$.

**Table 4.** Results of reducing the upper bound of $m$.

| $\eta$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $q_k$ | $q_{76}$ | $q_{61}$ | $q_{73}$ | $q_{66}$ | $q_{73}$ | $q_{57}$ | $q_{62}$ | $q_{66}$ | $q_{76}$ |
| $m$ | 290 | 300 | 288 | 305 | 295 | 297 | 294 | 288 | 306 |

Let

$$y_4 := \log(\Lambda_4 + 1) = b\log(\eta) - n\log(\theta_1) + \log\left(\frac{d}{P_m(\eta-1)}\right).$$

It is obvious that

$$|\Lambda_4| = |e^{y_4} - 1| < \frac{2.5}{\theta_1^n} < 0.25,$$

by our assumption that $n \geq 350$. Given Lemma 2.6, we can choose $v = 0.25$. Thus

$$|y_4| = |\log(\Lambda_4 + 1)| = -\frac{\log(1-0.25)}{0.25} \cdot \frac{2.5}{\theta_1^n} < \frac{2.9}{\theta_1^n}.$$

It follows that

$$0 < \left| n\left(-\log(\theta_1)\right) + b\log(\eta) + \log\left(\frac{d}{P_m(\eta-1)}\right) \right| < 2.9 \cdot \exp\left(-(n)\cdot\log(\theta_1)\right).$$

For the above inequality, we have the following two cases.
**Case 1:** If $4 \le m \le 306$ and $1 \le d \le \eta - 1$, then $\beta \neq 0$. By applying Lemma 2.4, we have

$$c := 2.9, \quad \delta := \log(\theta_1), \quad \psi := \frac{\log\left(\frac{d}{P_m(\eta-1)}\right)}{\log(\eta)},$$
$$\vartheta := \frac{\log(\theta_1)}{\log(\eta)}, \quad \vartheta_1 := -\log(\theta_1), \quad \vartheta_2 := \log(\eta), \quad \beta := \log\left(\frac{d}{P_m(\eta-1)}\right),$$

where $\vartheta$ denotes an irrational number. We can take $X_0 := 5.5 \cdot 10^{17}$, which is the upper bound for both $b$ and $n$ since $b < 2n + 1 < 5.5 \cdot 10^{17}$. We obtained the results in the Table 5 with the help of Maple programming, except for the following cases:

$$(\eta, d, m) \in \left\{ \begin{array}{c} (2,1,4),(2,1,6),(2,1,11),(3,2,5),(3,2,9),(4,3,6), \\ (4,3,11),(5,4,7),(7,6,8),(7,6,15),(9,8,9) \end{array} \right\}.$$

$$n < \frac{1}{\log(\theta_1)} \cdot \log\left(\frac{q_{41}^2 \cdot 2.9}{|\log(7)| \cdot 5.5 \cdot 10^{17}}\right) < 231.06.$$

For these exceptional cases, we obtained the following equations

$$\left\{ \begin{array}{c} 2E_n = 2^b - 1, 4E_n = 2^b - 1, 16E_n = 2^b - 1, 3E_n = 3^b - 1, 9E_n = 3^b - 1, 4E_n = 4^b - 1, \\ 16E_n = 4^b - 1, 5E_n = 5^b - 1, 7E_n = 7^b - 1, 49E_n = 7^b - 1, 9E_n = 9^b - 1 \end{array} \right\},$$

respectively, which are impossible in the region defined above.

**Table 5.** Results of reducing the upper bound of $n$ for $\beta \neq 0$.

| $\eta$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $q_k$ | $q_{47}$ | $q_{38}$ | $q_{43}$ | $q_{46}$ | $q_{49}$ | $q_{41}$ | $q_{38}$ | $q_{38}$ | $q_{45}$ |
| $n$ | 201 | 215 | 198 | 202 | 208 | 231 | 223 | 212 | 221 |

**Case 2:** If $m = 2$ and $d = \eta - 1$, then $\beta = 0$. Now, applying Lemma 2.3, we have that $X_0 := 5.5 \cdot 10^{17}$.

$$Y_0 = -1 + \frac{\log(\sqrt{5} \cdot 5.5 \cdot 10^{17} + 1)}{\log(\frac{1+\sqrt{5}}{2})} = 85.55,$$

and

$$A = \max_{0 \le k \le 85} a_{k+1}.$$

Upon inspection by Maple programming, we can obtain Table 6 in all possible cases of $\eta$, we get

$$n < \frac{1}{\log(\theta_1)} \cdot \log\left(\frac{2.9(303+2)5.5 \cdot 10^{17}}{|\log(2)|}\right) < 170.$$

In both cases, we obtained a contradiction based on the assumption that $n \geq 350$. Theorem 1.2 is proved.

**Table 6.** Results of reducing the upper bound of $n$ for $\beta = 0$.

| $\eta$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $A$ | 303 | 151 | 160 | 433 | 65 | 627 | 241 | 107 | 49 |
| $n$ | 170 | 166 | 165 | 168 | 161 | 169 | 165 | 162 | 160 |

**Corollary 3.2.** *The largest repdigits in base $\eta$ of the Diophantine equation given by Eq* (1.2) *are as follows:*

$$\begin{aligned} &E_{14} \cdot P_7 = 51 \cdot 5 = 255 = [11111111]_2, && E_8 \cdot P_6 = 10 \cdot 4 = 40 = [1111]_3, \\ &E_{14} \cdot P_7 = 51 \cdot 5 = 255 = [3333]_4, && E_{13} \cdot P_{11} = 39 \cdot 16 = 624 = [4444]_5, \\ &E_7 \cdot P_7 = 7 \cdot 5 = 35 = [55]_6, && E_9 \cdot P_6 = 12 \cdot 4 = 48 = [66]_7, \\ &E_9 \cdot P_5 = 12 \cdot 3 = 36 = [44]_8, && E_{13} \cdot P_8 = 39 \cdot 7 = 273 = [333]_9, \end{aligned}$$

$$E_{11} \cdot P_6 = 22 \cdot 4 = 88 = [88]_{10}.$$

## Author contributions

Hunar Sherzad Taher: Writing–original draft, Methodology; Saroj Kumar Dash: Supervision, Data curation, Visualization. We used Maple software programming to calculate the results. All authors have reviewed the results and approved the final version of the manuscript.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in creating this article.

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## Conflict of interest

The authors declare that there are no competing interests.

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## Appendix: Tables that include solutions for Theorems 1.1 and 1.2

**Table A.1.** The only solutions of the Diophantine equation given by Eq (1.1) for non-negative integers.

| $n$ | $m$ | $E_n + P_m$ | $\eta$ | $d$ | $b$ | $\eta$-repdigits |
|---|---|---|---|---|---|---|
| 2 | 2 | 3 | 2 | 1 | 2 | 11 |
| 3 | 2 | 4 | 3 | 1 | 2 | 11 |
| 4 | 2 | 3 | 2 | 1 | 2 | 11 |
| 4 | 4 | 4 | 3 | 1 | 2 | 11 |
| 5 | 2 | 6 | 5 | 1 | 2 | 11 |
| 5 | 4 | 7 | 2 | 1 | 3 | 111 |
| 5 | 4 | 7 | 6 | 1 | 2 | 11 |
| 5 | 5 | 8 | 3 | 2 | 2 | 22 |
| 5 | 5 | 8 | 7 | 1 | 2 | 11 |
| 6 | 2 | 6 | 5 | 1 | 2 | 11 |
| 6 | 4 | 7 | 2 | 1 | 3 | 111 |
| 6 | 4 | 7 | 6 | 1 | 2 | 11 |
| 6 | 5 | 8 | 3 | 2 | 2 | 22 |
| 6 | 5 | 8 | 7 | 1 | 2 | 11 |
| 6 | 6 | 9 | 8 | 1 | 2 | 11 |
| 7 | 2 | 8 | 3 | 2 | 2 | 22 |
| 7 | 2 | 8 | 7 | 1 | 2 | 11 |
| 7 | 4 | 9 | 8 | 1 | 2 | 11 |
| 7 | 5 | 10 | 4 | 2 | 2 | 22 |
| 7 | 5 | 10 | 9 | 1 | 2 | 11 |
| 7 | 6 | 11 | 10 | 1 | 2 | 11 |
| 7 | 7 | 12 | 5 | 2 | 2 | 22 |
| 8 | 2 | 11 | 10 | 1 | 2 | 11 |
| 8 | 4 | 12 | 5 | 2 | 2 | 22 |
| 8 | 5 | 13 | 3 | 1 | 3 | 111 |
| 8 | 6 | 14 | 6 | 2 | 2 | 22 |
| 8 | 7 | 15 | 2 | 1 | 4 | 1111 |
| 8 | 7 | 15 | 4 | 3 | 2 | 33 |
| 9 | 2 | 13 | 3 | 1 | 3 | 111 |
| 9 | 4 | 14 | 6 | 2 | 2 | 22 |
| 9 | 5 | 15 | 2 | 1 | 4 | 1111 |
| 9 | 5 | 15 | 4 | 3 | 2 | 33 |
| 9 | 6 | 16 | 7 | 2 | 2 | 22 |
| 9 | 9 | 21 | 4 | 1 | 3 | 111 |
| 9 | 9 | 21 | 6 | 3 | 2 | 33 |
| 10 | 2 | 18 | 5 | 3 | 2 | 33 |
| 10 | 2 | 18 | 8 | 2 | 2 | 22 |
| 10 | 5 | 20 | 9 | 2 | 2 | 22 |

| $n$ | $m$ | $E_n + P_m$ | $\eta$ | $d$ | $b$ | $\eta$-repdigits |
|---|---|---|---|---|---|---|
| 10 | 6 | 21 | 4 | 1 | 3 | 111 |
| 10 | 6 | 21 | 6 | 3 | 2 | 33 |
| 10 | 7 | 22 | 10 | 2 | 2 | 22 |
| 10 | 8 | 24 | 5 | 4 | 2 | 44 |
| 10 | 8 | 24 | 7 | 3 | 2 | 33 |
| 10 | 9 | 26 | 3 | 2 | 3 | 222 |
| 11 | 4 | 24 | 5 | 4 | 2 | 44 |
| 11 | 4 | 24 | 7 | 3 | 2 | 33 |
| 11 | 6 | 26 | 3 | 2 | 3 | 222 |
| 11 | 7 | 27 | 8 | 3 | 2 | 33 |
| 11 | 9 | 31 | 2 | 1 | 5 | 11111 |
| 11 | 9 | 31 | 5 | 1 | 3 | 111 |
| 12 | 2 | 30 | 9 | 3 | 2 | 33 |
| 12 | 4 | 31 | 2 | 1 | 5 | 11111 |
| 12 | 4 | 31 | 5 | 1 | 3 | 111 |
| 12 | 5 | 32 | 7 | 4 | 2 | 44 |
| 12 | 6 | 33 | 10 | 3 | 2 | 33 |
| 12 | 8 | 36 | 8 | 4 | 2 | 44 |
| 12 | 11 | 45 | 8 | 5 | 2 | 55 |
| 12 | 12 | 50 | 9 | 5 | 2 | 55 |
| 13 | 2 | 40 | 3 | 1 | 4 | 1111 |
| 13 | 2 | 40 | 7 | 5 | 2 | 55 |
| 13 | 2 | 40 | 9 | 4 | 2 | 44 |
| 13 | 5 | 42 | 4 | 2 | 3 | 222 |
| 13 | 6 | 43 | 6 | 1 | 3 | 111 |
| 13 | 7 | 44 | 10 | 4 | 2 | 44 |
| 13 | 9 | 48 | 7 | 6 | 2 | 66 |
| 13 | 11 | 55 | 10 | 5 | 2 | 55 |
| 13 | 12 | 60 | 9 | 6 | 2 | 66 |
| 14 | 5 | 54 | 8 | 6 | 2 | 66 |
| 14 | 6 | 55 | 10 | 5 | 2 | 55 |
| 14 | 9 | 60 | 9 | 6 | 2 | 66 |
| 14 | 10 | 63 | 2 | 1 | 6 | 111111 |
| 14 | 10 | 63 | 4 | 3 | 3 | 333 |
| 14 | 10 | 63 | 8 | 7 | 2 | 77 |
| 14 | 14 | 88 | 10 | 8 | 2 | 88 |
| 15 | 4 | 70 | 9 | 7 | 2 | 77 |
| 15 | 7 | 73 | 8 | 1 | 3 | 111 |

| $n$ | $m$ | $E_n + P_m$ | $\eta$ | $d$ | $b$ | $\eta$-repdigits |
|---|---|---|---|---|---|---|
| 15 | 9 | 77 | 10 | 7 | 2 | 77 |
| 15 | 10 | 80 | 3 | 2 | 4 | 2222 |
| 15 | 10 | 80 | 9 | 8 | 2 | 88 |
| 16 | 2 | 91 | 9 | 1 | 3 | 111 |
| 16 | 5 | 93 | 5 | 3 | 3 | 333 |
| 16 | 9 | 99 | 10 | 9 | 2 | 99 |
| 16 | 12 | 111 | 10 | 1 | 3 | 111 |
| 16 | 14 | 127 | 2 | 1 | 7 | 1111111 |
| 17 | 4 | 121 | 3 | 1 | 5 | 11111 |
| 17 | 7 | 124 | 5 | 4 | 3 | 444 |
| 17 | 14 | 156 | 5 | 1 | 4 | 1111 |
| 18 | 10 | 170 | 4 | 2 | 4 | 2222 |
| 20 | 16 | 342 | 7 | 6 | 3 | 666 |
| 21 | 19 | 518 | 6 | 2 | 4 | 2222 |
| 22 | 19 | 637 | 9 | 7 | 3 | 777 |
| 26 | 16 | 1562 | 5 | 2 | 5 | 22222 |

**Table A.2.** The only solutions of the Diophantine equation given by Eq (1.2) for non-negative integers.

| $n$ | $m$ | $E_n \cdot P_m$ | $\eta$ | $d$ | $b$ | $\eta$-repdigits | $n$ | $m$ | $E_n \cdot P_m$ | $\eta$ | $d$ | $b$ | $\eta$-repdigits |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 3 | 2 | 1 | 2 | 11 | 8 | 5 | 30 | 9 | 3 | 2 | 33 |
| 4 | 4 | 4 | 3 | 1 | 2 | 11 | 8 | 6 | 40 | 3 | 1 | 4 | 1111 |
| 5 | 2 | 5 | 4 | 1 | 2 | 11 | 8 | 6 | 40 | 7 | 5 | 2 | 55 |
| 5 | 4 | 10 | 4 | 2 | 2 | 22 | 8 | 6 | 40 | 9 | 4 | 2 | 44 |
| 5 | 4 | 10 | 9 | 1 | 2 | 11 | 8 | 7 | 50 | 9 | 5 | 2 | 55 |
| 5 | 5 | 15 | 2 | 1 | 4 | 1111 | 8 | 8 | 70 | 9 | 7 | 2 | 77 |
| 5 | 5 | 15 | 4 | 3 | 2 | 33 | 9 | 2 | 12 | 5 | 2 | 2 | 22 |
| 6 | 2 | 5 | 4 | 1 | 2 | 11 | 9 | 4 | 24 | 5 | 4 | 2 | 44 |
| 6 | 4 | 10 | 4 | 2 | 2 | 22 | 9 | 4 | 24 | 7 | 3 | 2 | 33 |
| 6 | 4 | 10 | 9 | 1 | 2 | 11 | 9 | 5 | 36 | 8 | 4 | 2 | 44 |
| 6 | 5 | 15 | 2 | 1 | 4 | 1111 | 9 | 6 | 48 | 7 | 6 | 2 | 66 |
| 6 | 5 | 15 | 4 | 3 | 2 | 33 | 9 | 7 | 60 | 9 | 6 | 2 | 66 |
| 6 | 6 | 20 | 9 | 2 | 2 | 22 | 10 | 7 | 85 | 4 | 1 | 4 | 1111 |
| 7 | 2 | 7 | 2 | 1 | 3 | 111 | 11 | 2 | 22 | 10 | 2 | 2 | 22 |
| 7 | 2 | 7 | 6 | 1 | 2 | 11 | 11 | 4 | 44 | 10 | 4 | 2 | 44 |
| 7 | 4 | 14 | 6 | 2 | 2 | 22 | 11 | 5 | 66 | 10 | 6 | 2 | 66 |
| 7 | 5 | 21 | 4 | 1 | 3 | 111 | 11 | 6 | 88 | 10 | 8 | 2 | 88 |
| 7 | 5 | 21 | 6 | 3 | 2 | 33 | 13 | 6 | 156 | 5 | 1 | 4 | 1111 |
| 7 | 6 | 28 | 6 | 4 | 2 | 44 | 13 | 8 | 273 | 9 | 3 | 3 | 333 |
| 7 | 7 | 35 | 6 | 5 | 2 | 55 | 13 | 10 | 468 | 5 | 3 | 4 | 3333 |
| 8 | 2 | 10 | 4 | 2 | 2 | 22 | 13 | 11 | 624 | 5 | 4 | 4 | 4444 |
| 8 | 2 | 10 | 9 | 1 | 2 | 11 | 14 | 7 | 255 | 2 | 1 | 8 | 11111111 |
| 8 | 4 | 20 | 9 | 2 | 2 | 22 | 14 | 7 | 255 | 4 | 3 | 4 | 3333 |

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