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Research article

The Hermitian solution to a matrix inequality under linear constraint

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Abstract: In this paper, the necessary and sufficient conditions under which the matrix inequality $C^*XC \geq D$ ($> D$) subject to the linear constraint $A^*XA = B$ is solvable are deduced by means of the spectral decompositions of some matrices and the generalized singular value decomposition of a matrix pair. An explicit expression of the general Hermitian solution is also provided. One numerical example demonstrates the effectiveness of the proposed method.

Keywords: matrix inequality; matrix equation; Hermitian solution; spectral decomposition; generalized singular value decomposition
Mathematics Subject Classification: 15A24, 15A57

1. Introduction

Throughout this paper, we denote the complex $m \times n$ matrix space by $\mathbb{C}^{m\times n}$, the set of all Hermitian matrices in $\mathbb{C}^{m\times m}$ by $\mathbb{C}_{\mathrm{H}}^{m\times m}$, the conjugate transpose, the range space, and the Moore-Penrose generalized inverse of a matrix A by A^*, $\mathcal{R}(A)$ and $A^\dagger$, respectively. I_n is the $n \times n$ identity matrix and $A \geq 0$ (> 0) means that $A \in \mathbb{C}^{n\times n}$ is Hermitian nonnegative definite (positive definite). $P_{\mathcal{L}}$ represents the orthogonal projector on the subspace $\mathcal{L}$. Given a matrix $A \in \mathbb{C}^{m\times n}$, $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$ are two orthogonal projectors. Besides, $\|\cdot\|_F$ stands for the Frobenius norm.

A lot of work with the Hermitian solutions of the matrix equations have been published [1–7]. However, the problem of solving the Hermitian solution of the matrix inequality under the symmetric matrix equation constraint is considered by only a few authors. To the best of our knowledge, the constrained matrix inequality:

$$C^*XC \geq D\ (> D), \text{ s.t. } A^*XA = B, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{m\times p}$ and $D \in \mathbb{C}^{p\times p}$, and $X \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ is an unknown matrix to be determined, was first investigated by Liu [8]. Observe that Liu derived the necessary and sufficient conditions for the solvability of (1.1) using the maximal and minimal ranks and inertias of the matrix polynomials. In this paper, we provide an alternative approach to find the Hermitian solution of (1.1). Compared with the approach proposed by Liu [8], the method used in this paper is from a new perspective, using the decompositions of matrices to reduce the calculations of the rank and inertia, which makes it easier to obtain the representation of the Hermitian solution of (1.1).

In what follows, we discuss the necessary and sufficient conditions for the solvability of (1.1) using the spectral decompositions (SDs) and the generalized singular value decomposition (GSVD) of some matrices, and give an explicit representation of the general Hermitian solution of (1.1) when the consistent conditions are satisfied. Further, one numerical example shows that the introduced method is correct.

2. Preliminary lemmas

In order to solve the Hermitian solution of (1.1), we need the following lemmas.

Lemma 1. ([9]) If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}_{\mathrm{H}}^{n\times n}$, then the matrix equation $A^*XA = B$ has a solution $X \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ if and only if

$$\mathcal{R}(B) \subseteq \mathcal{R}(A^*). \tag{2.1}$$

In which case, the general Hermitian solution can be expressed as

$$X = (A^*)^{\dagger}BA^{\dagger} + E_A Z + Z^* E_A, \tag{2.2}$$

where $Z \in \mathbb{C}^{m\times m}$ is an arbitrary matrix.

Lemma 2. ([10, 11]) Assume that $A_1 \in \mathbb{C}^{p\times m}$, $A_2 \in \mathbb{C}^{n\times p}$ and $A_3 \in \mathbb{C}_{\mathrm{H}}^{p\times p}$. Then the matrix equation $A_1XA_2 + (A_1XA_2)^* = A_3$ has a solution if and only if

$$E_{A_1}A_3E_{A_1} = 0, \ \ F_{A_2}A_3F_{A_2} = 0, \ \ [A_1, \ A_2{}^*][A_1, \ A_2{}^*]^{\dagger}A_3 = A_3.$$

In this case, the general solution with respect to $X \in \mathbb{C}^{m\times n}$ gives

$$X = A_1^{\dagger}(\Gamma + F_{\widetilde{L}}S_X F_{\widetilde{L}}A_1A_1{}^{\dagger})A_2{}^{\dagger} + Y - A_1{}^{\dagger}A_1YA_2A_2{}^{\dagger},$$

where $\widetilde{L} = F_{A_2}A_1A_1{}^{\dagger}$, $\Gamma = \frac{1}{2}A_3(2I_p - A_1A_1{}^{\dagger}) + \frac{1}{2}(\Psi - \Psi^*)A_1A_1{}^{\dagger}$, and $\Psi = 2\widetilde{L}^{\dagger}F_{A_2}A_3 + (I_p - \widetilde{L}^{\dagger}F_{A_2})A_3\widetilde{L}^{\dagger}\widetilde{L}$, $Y \in \mathbb{C}^{m\times n}$ and $S_X \in \mathbb{C}^{p\times p}$ are arbitrary matrices with $S_X^* = -S_X$.

Lemma 3. ([12, 13]) Given matrices $\widetilde{A} \in \mathbb{C}^{p\times m}$ and $\widetilde{D} \in \mathbb{C}^{p\times m}$. Let the singular value decomposition (SVD) of the matrix $\widetilde{A}$ be $\widetilde{A} = \widetilde{P}\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\widetilde{Q}^*$, where $\Sigma = \mathrm{diag}(\alpha_1, \ \cdots, \ \alpha_r) > 0$, $r = \mathrm{rank}(\widetilde{A})$ and $\widetilde{P} = [\widetilde{P}_1, \ \widetilde{P}_2] \in \mathbb{C}^{p\times p}$, $\widetilde{Q} = [\widetilde{Q}_1, \ \widetilde{Q}_2] \in \mathbb{C}^{m\times m}$ are unitary matrices with $\widetilde{P}_1 \in \mathbb{C}^{p\times r}$, $\widetilde{Q}_1 \in \mathbb{C}^{m\times r}$ and $\widetilde{Q}_2 \in \mathbb{C}^{m\times(m-r)}$. Then:

(i) The matrix equation $\widetilde{A}Y = \widetilde{D}$ has a Hermitian nonnegative definite solution $Y \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ if and only if $\widetilde{D}\widetilde{A}^* \geq 0$, $\mathcal{R}(\widetilde{D}) = \mathcal{R}(\widetilde{D}\widetilde{A}^*)$, and the general nonnegative definite solution can be expressed as

$$Y = Y_0 + F_{\widetilde{A}}KF_{\widetilde{A}},$$

where $Y_0 = \widetilde{A}^{\dagger}\widetilde{D} + F_{\widetilde{A}}(\widetilde{A}^{\dagger}\widetilde{D})^* + F_{\widetilde{A}}\widetilde{D}^*(\widetilde{D}\widetilde{A}^*)^{\dagger}\widetilde{D}F_{\widetilde{A}}$, and $K \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ is an arbitrary nonnegative definite matrix.

(ii) The matrix equation $\widetilde{A}Y = \widetilde{D}$ has a Hermitian positive definite solution if and only if $\widetilde{A}\widetilde{A}^{\dagger}\widetilde{D} = \widetilde{D}$, $\widetilde{P}_1^*\widetilde{D}\widetilde{A}^*\widetilde{P}_1 > 0$, and the general positive definite solution is

$$Y = Y_0 + F_{\widetilde{A}}KF_{\widetilde{A}},$$

where $Y_0 = \widetilde{A}^{\dagger}\widetilde{D} + F_{\widetilde{A}}(\widetilde{A}^{\dagger}\widetilde{D})^* + F_{\widetilde{A}}\widetilde{D}^*(\widetilde{D}\widetilde{A}^*)^{\dagger}\widetilde{D}F_{\widetilde{A}}$, and $K \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ is an arbitrary positive definite matrix.

3. Main results

According to Lemma 1, we know that the matrix equation $A^*XA = B$ has a Hermitian solution if and only if the condition (2.1) holds, then the general solution is given by (2.2), where $Z \in \mathbb{C}^{m\times m}$ is an arbitrary matrix. Substituting (2.2) into the first matrix inequality of (1.1) yields

$$C^*E_AZC + C^*Z^*E_AC \geq D - C^*(A^*)^{\dagger}BA^{\dagger}C \ (> D - C^*(A^*)^{\dagger}BA^{\dagger}C). \tag{3.1}$$

Clearly, the inequality (3.1) can be equivalently written as

$$GZC + C^*Z^*G^* = K - \widetilde{D}, \tag{3.2}$$

where $\widetilde{D} = -D + C^*(A^*)^{\dagger}BA^{\dagger}C$, $G = C^*E_A$ and $K \geq 0$ (> 0) is an unknown matrix to be determined. By utilizing Lemma 2, we see that Eq (3.2) with respect to Z is solvable if and only if the following three matrix equations hold simultaneously:

$$E_GKE_G = E_G\widetilde{D}E_G, \tag{3.3}$$

$$F_CKF_C = F_C\widetilde{D}F_C, \tag{3.4}$$

$$[G,\ C^*][G,\ C^*]^{\dagger}(K - \widetilde{D}) = K - \widetilde{D}. \tag{3.5}$$

Now, we will seek the solvability conditions with respect to $K \geq 0$ (> 0) such that (3.3)–(3.5) are consistent. If let $P_{\mathcal{L}} = [G,\ C^*][G,\ C^*]^{\dagger}$, Eq (3.5) can be equivalently written as

$$(I_p - P_{\mathcal{L}})K = (I_p - P_{\mathcal{L}})\widetilde{D}. \tag{3.6}$$

From Lemma 3, Eq (3.6) has a Hermitian nonnegative definite solution $K \in \mathbb{C}^{p\times p}$ if and only if

$$(I_p - P_{\mathcal{L}})\widetilde{D}(I_p - P_{\mathcal{L}}) \geq 0,\ \mathcal{R}\left((I_p - P_{\mathcal{L}})\widetilde{D}\right) = \mathcal{R}\left((I_p - P_{\mathcal{L}})\widetilde{D}(I_p - P_{\mathcal{L}})\right), \tag{3.7}$$

in this case, the general Hermitian nonnegative definite solution can be expressed as

$$K = K_0 + P_{\mathcal{L}}SP_{\mathcal{L}}, \tag{3.8}$$

where

$$K_0 = \widetilde{D} - P_{\mathcal{L}}\widetilde{D}P_{\mathcal{L}} + P_{\mathcal{L}}\widetilde{D}(I_p - P_{\mathcal{L}})[(I_p - P_{\mathcal{L}})\widetilde{D}(I_p - P_{\mathcal{L}})]^{\dagger}(I_p - P_{\mathcal{L}})\widetilde{D}P_{\mathcal{L}}, \tag{3.9}$$

and $S \in \mathbb{C}_{\mathrm{H}}^{p\times p}$ is an arbitrary nonnegative matrix.

Assume that the SD of $I_p - P_{\mathcal{L}}$ is

$$I_p - P_{\mathcal{L}} = Q\begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}Q^* = Q_1Q_1^*, \tag{3.10}$$

where $s = \text{rank}(I_p - P_{\mathcal{L}})$ and $Q = [Q_1,\ Q_2]$ is a unitary matrix with $Q_1 \in \mathbb{C}^{p\times s}$ and $Q_2 \in \mathbb{C}^{p\times(p-s)}$. By Lemma 3, Eq (3.6) has a Hermitian positive definite solution if and only if

$$Q_1^* \widetilde{D} Q_1 > 0, \tag{3.11}$$

then the general Hermitian positive definite solution can be formulated as

$$K = K_0 + P_{\mathcal{L}} S P_{\mathcal{L}}, \tag{3.12}$$

where K_0 is given by (3.9), and S is an arbitrary Hermitian positive definite matrix. Due to $\mathcal{L} = \mathcal{R}(G) + \mathcal{R}(C^*)$, then

$$P_{\mathcal{L}} GG^\dagger = GG^\dagger = GG^\dagger P_{\mathcal{L}},\ P_{\mathcal{L}} C^\dagger C = C^\dagger C = C^\dagger C P_{\mathcal{L}}.$$

Substituting (3.8) ((3.12)) into (3.3) and (3.4), we can obtain that

$$(P_{\mathcal{L}} - GG^\dagger) S (P_{\mathcal{L}} - GG^\dagger) = (I_p - GG^\dagger) W (I_p - GG^\dagger), \tag{3.13}$$

$$(P_{\mathcal{L}} - C^\dagger C) S (P_{\mathcal{L}} - C^\dagger C) = (I_p - C^\dagger C) W (I_p - C^\dagger C), \tag{3.14}$$

where $W = \widetilde{D} - K_0$. According to (3.9), it follows that

$$(P_{\mathcal{L}} - GG^\dagger) W (P_{\mathcal{L}} - GG^\dagger) = (I_p - GG^\dagger) W (I_p - GG^\dagger), \tag{3.15}$$

$$(P_{\mathcal{L}} - C^\dagger C) W (P_{\mathcal{L}} - C^\dagger C) = (I_p - C^\dagger C) W (I_p - C^\dagger C). \tag{3.16}$$

By Eqs (3.15) and (3.16), Eqs (3.13) and (3.14) are equivalent to

$$(P_{\mathcal{L}} - GG^\dagger) S (P_{\mathcal{L}} - GG^\dagger) = (P_{\mathcal{L}} - GG^\dagger) W (P_{\mathcal{L}} - GG^\dagger), \tag{3.17}$$

$$(P_{\mathcal{L}} - C^\dagger C) S (P_{\mathcal{L}} - C^\dagger C) = (P_{\mathcal{L}} - C^\dagger C) W (P_{\mathcal{L}} - C^\dagger C). \tag{3.18}$$

It is easily verified that $P_{\mathcal{L}} - GG^\dagger$ and $P_{\mathcal{L}} - C^\dagger C$ are orthogonal projection operators, then there exists unitary matrices U and V such that

$$P_{\mathcal{L}} - GG^\dagger = U \begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix} U^* = U_1 U_1^*,\ \ P_{\mathcal{L}} - C^\dagger C = V \begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix} V^* = V_1 V_1^*, \tag{3.19}$$

where $a = \text{rank}(P_{\mathcal{L}} - GG^\dagger)$, $b = \text{rank}(P_{\mathcal{L}} - C^\dagger C)$, $U_1 \in \mathbb{C}^{p\times a}$ and $V_1 \in \mathbb{C}^{p\times b}$ are full column rank unitary matrices. Substituting (3.19) into (3.17) and (3.18), we can get that

$$U_1^* S U_1 = U_1^* W U_1,\ \ V_1^* S V_1 = V_1^* W V_1. \tag{3.20}$$

Let the GSVD [14] of the matrix pair $[U_1,\ V_1]$ be:

$$U_1 = M \Delta_1 N_1^*,\ \ V_1 = M \Delta_2 N_2^*, \tag{3.21}$$

where $M \in \mathbb{C}^{p\times p}$ is a nonsingular matrix, and $N_1 \in \mathbb{C}^{a\times a}$, $N_2 \in \mathbb{C}^{b\times b}$ are unitary matrices, and

$$\Delta_1 = \begin{array}{cc} \begin{bmatrix} I & 0 \\ 0 & \Upsilon \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{array}{c} a-e \\ e \\ f-a \\ p-f \end{array} \\ \begin{array}{cc} a-e & \ \ e \end{array} & \end{array},\quad \Delta_2 = \begin{array}{cc} \begin{bmatrix} 0 & 0 \\ \Theta & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} & \begin{array}{c} a-e \\ e \\ f-a \\ p-f \end{array} \\ \begin{array}{cc} e & \ \ b-e \end{array} & \end{array},$$

$f = \operatorname{rank}([U_1,\ V_1]) = a + b - e$, $\Upsilon = \operatorname{diag}(\delta_1,\ \delta_2,\ \cdots,\ \delta_e)$ and $\Theta = \operatorname{diag}(\vartheta_1,\ \vartheta_2,\ \cdots,\ \vartheta_e)$ with $1 > \delta_1 \geq \delta_2 \geq \cdots \geq \delta_e > 0$, $0 < \vartheta_1 \leq \vartheta_2 \leq \cdots \leq \vartheta_e < 1$, $\delta_i^2 + \vartheta_i^2 = 1$, $i = 1,\ 2,\ \cdots,\ e$. Substituting (3.21) into (3.20) and partitioning the matrix M^*WM as:

$$M^*WM = \begin{array}{cc} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12}^* & M_{22} & M_{23} & M_{24} \\ M_{13}^* & M_{23}^* & M_{33} & M_{34} \\ M_{14}^* & M_{24}^* & M_{34}^* & M_{44} \end{bmatrix} & \begin{array}{c} a-e \\ e \\ f-a \\ p-f \end{array} \\ \begin{array}{cccc} a-e & e & f-a & p-f \end{array} & \end{array}. \tag{3.22}$$

Then, it follows from Lemma 2.6 of [15] that:

(i) Equation (3.20) has a solution $S \geq 0$ if and only if

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} \geq 0, \quad \begin{bmatrix} M_{22} & M_{23} \\ M_{23}^* & M_{33} \end{bmatrix} \geq 0, \tag{3.23}$$

and the general Hermitian nonnegative definite solution of Eq (3.20) is

$$S = (M^*)^{-1} \begin{bmatrix} \Omega(S_{13}) & \Omega(S_{13})J_1 \\ J_1^*\Omega(S_{13}) & J_2 + J_1^*\Omega(S_{13})J_1 \end{bmatrix} M^{-1}, \tag{3.24}$$

where

$$\Omega(S_{13}) \triangleq \begin{bmatrix} M_{11} & M_{12} & S_{13} \\ M_{12}^* & M_{22} & M_{23} \\ S_{13}^* & M_{23}^* & M_{33} \end{bmatrix}, \tag{3.25}$$

with

$$S_{13} = M_{12}M_{22}^{\dagger}M_{23} + (M_{11} - M_{12}M_{22}^{\dagger}M_{12}^*)^{\frac{1}{2}} J_3 (M_{33} - M_{23}^*M_{22}^{\dagger}M_{23})^{\frac{1}{2}}, \tag{3.26}$$

and $J_1 \in \mathbb{C}^{f\times(p-f)}$, $J_2 \in \mathbb{C}^{(p-f)\times(p-f)}$ are arbitrary matrices with $J_2 \geq 0$, and $J_3 \in \mathbb{C}^{(a-e)\times(f-a)}$ is an arbitrary contraction matrix (that is, the maximum singular value with respect to J_3 cannot exceed 1).

(ii) Equation (3.20) has a Hermitian positive solution of S if and only if

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} > 0, \quad \begin{bmatrix} M_{22} & M_{23} \\ M_{23}^* & M_{33} \end{bmatrix} > 0, \tag{3.27}$$

and the representation of the general solution for Eq (3.20) can be expressed as

$$S = (M^*)^{-1} \begin{bmatrix} \Omega(S_{13}) & J_4 \\ J_4^* & J_5 + J_4^*[\Omega(S_{13})]^{-1}J_4 \end{bmatrix} M^{-1}, \tag{3.28}$$

where $\Omega(S_{13})$ is given by (3.25) with

$$S_{13} = M_{12}M_{22}^{-1}M_{23} + (M_{11} - M_{12}M_{22}^{-1}M_{12}^*)^{\frac{1}{2}} J_6 (M_{33} - M_{23}^*M_{22}^{-1}M_{23})^{\frac{1}{2}}, \tag{3.29}$$

and $J_4 \in \mathbb{C}^{f\times(p-f)}$, $J_5 \in \mathbb{C}^{(p-f)\times(p-f)}$ are arbitrary matrices with $J_5 > 0$, and $J_6 \in \mathbb{C}^{(a-e)\times(f-a)}$ is an arbitrary strict contraction matrix (that is, the maximum singular value with respect to J_6 is less than 1).

Furthermore, when the conditions (3.3)–(3.5) hold simultaneously, the general expression with respect to Z in Eq (3.2) is

$$Z = G^{\dagger}(\Gamma + F_{\widetilde{L}} N_X F_{\widetilde{L}} GG^{\dagger})C^{\dagger} + Y - G^{\dagger}GYCC^{\dagger}, \tag{3.30}$$

where

$$\Gamma = \frac{1}{2}(P_{\mathcal{L}} S P_{\mathcal{L}} - W)(2I_p - GG^{\dagger}) + \frac{1}{2}(\Psi - \Psi^*)GG^{\dagger}, \tag{3.31}$$

with

$$\Psi = 2\widetilde{L}^{\dagger} F_C (P_{\mathcal{L}} S P_{\mathcal{L}} - W) + (I_p - \widetilde{L}^{\dagger} F_C)(P_{\mathcal{L}} S P_{\mathcal{L}} - W)\widetilde{L}^{\dagger}\widetilde{L}, \tag{3.32}$$

and S is given by Eq (3.24) ((3.28)), $\widetilde{L} = F_C GG^{\dagger}$, and $Y \in \mathbb{C}^{m\times m}$, $N_X \in \mathbb{C}^{p\times p}$ are arbitrary matrices with $N_X^* = -N_X$. By substituting (3.30) into (2.2), we can get the Hermitian solution of (1.1).

With the above discussion, we can obtain the following theorem.

Theorem 1. Given matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{m\times p}$ and $D \in \mathbb{C}^{p\times p}$. Let $\widetilde{D} = -D + C^*(A^*)^{\dagger} BA^{\dagger} C$, $G = C^* E_A$, $\mathcal{L} = \mathcal{R}(G) + \mathcal{R}(C^*)$, and let the SDs of the matrices $I_p - P_{\mathcal{L}}$, $P_{\mathcal{L}} - GG^{\dagger}$ and $P_{\mathcal{L}} - C^{\dagger}C$ be respectively given by (3.10) and (3.19). Suppose that the GSVD of the matrix pair $[U_1,\ V_1]$ is given by (3.21), and the partition of the matrix M^*WM is given by (3.22). Then,

(i) The matrix inequality $C^*XC \geq D$ s.t. $A^*XA = B$ has a Hermitian solution if and only if the conditions (2.1), (3.7) and (3.23) hold, in which case, the general Hermitian solution $X \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ can be expressed as

$$\begin{aligned} X = {} & (A^*)^{\dagger} BA^{\dagger} + E_A G^{\dagger}(\Gamma + F_{\widetilde{L}} N_X F_{\widetilde{L}} GG^{\dagger})C^{\dagger} + E_A Y - E_A G^{\dagger} GYCC^{\dagger} \\ & + (C^{\dagger})^*(\Gamma^* - GG^{\dagger} F_{\widetilde{L}} N_X F_{\widetilde{L}})(G^{\dagger})^* E_A + Y^* E_A - CC^{\dagger} Y^* G^{\dagger} G E_A, \end{aligned} \tag{3.33}$$

where $W = \widetilde{D} - K_0$, $\widetilde{L} = F_C GG^{\dagger}$, and K_0, S, $\Omega(S_{13})$, S_{13}, Γ and Ψ are respectively given by (3.9), (3.24)–(3.26), (3.31) and (3.32), and $N_X \in \mathbb{C}^{p\times p}$, $Y \in \mathbb{C}^{m\times m}$, $J_1 \in \mathbb{C}^{f\times(p-f)}$ and $J_2 \in \mathbb{C}^{(p-f)\times(p-f)}$ are arbitrary matrices with $N_X^* = -N_X$ and $J_2 \geq 0$, and $J_3 \in \mathbb{C}^{(a-e)\times(f-a)}$ is an arbitrary contraction matrix.

(ii) The matrix inequality $C^*XC > D$ s.t. $A^*XA = B$ has a Hermitian solution $X \in \mathbb{C}_{\mathrm{H}}^{m\times m}$ if and only if the conditions (2.1), (3.11) and (3.27) are satisfied, in this case, the Hermitian solution can be expressed as

$$\begin{aligned} X = {} & (A^*)^{\dagger} BA^{\dagger} + E_A G^{\dagger}(\Gamma + F_{\widetilde{L}} N_X F_{\widetilde{L}} GG^{\dagger})C^{\dagger} + E_A Y - E_A G^{\dagger} GYCC^{\dagger} \\ & + (C^{\dagger})^*(\Gamma^* - GG^{\dagger} F_{\widetilde{L}} N_X F_{\widetilde{L}})(G^{\dagger})^* E_A + Y^* E_A - CC^{\dagger} Y^* G^{\dagger} G E_A, \end{aligned} \tag{3.34}$$

where $W = \widetilde{D} - K_0$, $\widetilde{L} = F_C GG^{\dagger}$, and K_0, S, $\Omega(S_{13})$, S_{13}, Γ and Ψ are respectively given by (3.9), (3.28), (3.25), (3.29), (3.31) and (3.32), and $N_X \in \mathbb{C}^{p\times p}$, $Y \in \mathbb{C}^{m\times m}$, $J_4 \in \mathbb{C}^{f\times(p-f)}$ and $J_5 \in \mathbb{C}^{(p-f)\times(p-f)}$ are arbitrary matrices with $N_X^* = -N_X$ and $J_5 > 0$, and $J_6 \in \mathbb{C}^{(a-e)\times(f-a)}$ is an arbitrary strict contraction matrix.

4. Numerical algorithm and numerical example

According to Theorem 1, we can describe the numerical algorithm to solve the Hermitian solution of (1.1) as follows.

Algorithm 1.

(1) Input matrices $A,\ B,\ C$ and D.

(2) If the condition (2.1) holds, then continue; or else, (1.1) has no solution $X \in \mathbb{C}_{\mathrm{H}}^{m\times m}$.

(3) Compute matrices $\widetilde{D},\ G$ and $I_p - P_{\mathcal{L}}$.

(4) If the condition (3.7) holds, then continue; or else, the matrix inequality $C^*XC \geq D$ s.t. $A^*XA = B$ has no solution X.

(5) Compute the SD of the matrix $I_p - P_{\mathcal{L}}$ by (3.10).

(6) If the condition (3.11) holds, then continue; or else, the matrix inequality $C^*XC > D$ s.t. $A^*XA = B$ has no solution X.

(7) Compute the matrices K_0 and W in the light of (3.9), (3.13) and (3.14), respectively.

(8) Compute the SDs of the matrices $P_{\mathcal{L}} - GG^\dagger$ and $P_{\mathcal{L}} - C^\dagger C$ by (3.19).

(9) Compute the GSVD of the matrix pair $[U_1,\ V_1]$ by (3.21).

(10) Compute the matrix M^*WM by (3.22).

(11) (i) If the condition (3.23) holds, then continue; or else, the matrix inequality $C^*XC \geq D$ s.t. $A^*XA = B$ has no solution X.
(ii) If the condition (3.27) holds, then continue; or else, the matrix inequality $C^*XC > D$ s.t. $A^*XA = B$ has no solution X.

(12) (i) Select matrices $J_1,\ J_2 \geq 0$ and a contraction matrix J_3, compute matrices S, $\Omega(S_{13})$ and S_{13} based on (3.24)–(3.26), respectively.
(ii) Select matrices $J_4,\ J_5 > 0$ and a strict contraction matrix J_6, compute matrices S, $\Omega(S_{13})$ and S_{13} by (3.28), (3.25) and (3.29), respectively.

(13) Calculate matrices $Z,\ \Gamma$ and Ψ on the basis of (3.30)–(3.32), respectively.

(14) (i) Select matrices N_X and Y, compute the Hermitian solution of $C^*XC \geq D$ s.t. $A^*XA = B$ in the light of (3.33).
(ii) Select matrices N_X and Y, compute the Hermitian solution of $C^*XC > D$ s.t. $A^*XA = B$ based on (3.34).

Remark 1. Through careful statistics, the amount of computations required by Algorithm 1 is about $146p^3 + 84m^3 + 46m^2p + 22mp^2 + 2mnp + n^2p + 9n^3 + 9mn^2 + 5m^2n$ flops, where the generalized inverse matrices $C^\dagger$, $G^\dagger$ and $\widetilde{L}^\dagger$ are calculated by the SVDs of the matrices C, G and $\widetilde{L}$. Further, to compute the time complexity of Algorithm 1, the readers can see a survey [16].

Example 1. Let $m = 6,\ n = 5$ and $p = 7$. The matrices $A,\ B,\ C$ and D are presented by

$$A = \begin{bmatrix} 0.6937 & 0.4421 & 0.2510 & 0.5662 & 0.8973 \\ 0.3921 & 0.5509 & 0.2658 & 0.4114 & 0.4982 \\ 0.8097 & 1.0134 & 0.7290 & 0.6978 & 0.8002 \\ 0.3821 & 0.7560 & 0.2989 & 0.8028 & 0.7199 \\ 0.9898 & 1.2502 & 1.0575 & 1.2697 & 1.1179 \\ 0.9704 & 1.1141 & 0.5695 & 1.1198 & 1.3391 \end{bmatrix},$$

$$B = \begin{bmatrix} 5.0119 & 6.0743 & 3.9509 & 5.8627 & 6.2797 \\ 6.0743 & 7.3551 & 4.7827 & 7.1049 & 7.6140 \\ 3.9509 & 4.7827 & 3.1096 & 4.6212 & 4.9531 \\ 5.8627 & 7.1049 & 4.6212 & 6.8579 & 7.3460 \\ 6.2797 & 7.6140 & 4.9531 & 7.3460 & 7.8667 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.7991 & 0.6564 & 0.4484 & 0.5550 & 0.4410 & 0.2278 & 0.4740 \\ 1.0514 & 0.7819 & 1.6430 & 0.5919 & 1.7605 & 1.1620 & 1.2337 \\ 0.8563 & 0.7862 & 0.8861 & 0.5520 & 0.7799 & 0.4598 & 0.8968 \\ 0.6053 & 0.4850 & 0.8488 & 0.3291 & 0.7518 & 0.5847 & 0.7347 \\ 0.7185 & 0.7218 & 0.9206 & 0.4474 & 0.7571 & 0.4532 & 0.9563 \\ 1.1549 & 0.8773 & 1.3560 & 0.6328 & 1.1260 & 0.9894 & 1.1950 \end{bmatrix},$$

$$D = \begin{bmatrix} -5.2636 & -7.9819 & -2.6335 & -2.2506 & -1.2401 & -10.0230 & -0.1972 \\ -7.9819 & -11.1637 & -5.6897 & -3.8471 & -3.8013 & -12.9962 & -2.3591 \\ -2.6335 & -5.6897 & -0.2158 & -0.8759 & 0.9846 & -7.5319 & 1.9655 \\ -2.2506 & -3.8471 & -0.8759 & -1.0109 & -0.1707 & -5.1224 & 0.3619 \\ -1.2401 & -3.8013 & 0.9846 & -0.1707 & 1.6595 & -5.5533 & 2.6673 \\ -10.0230 & -12.9962 & -7.5319 & -5.1224 & -5.5533 & -15.5151 & -4.0797 \\ -0.1972 & -2.3591 & 1.9655 & 0.3619 & 2.6673 & -4.0797 & 3.0450 \end{bmatrix}.$$

It is easy to validate that the stated conditions (2.1), (3.7) and (3.23) are satisfied. In fact, $\|F_A B\|_F = 1.6310 \times 10^{-15}$, the eigenvalues of the matrix $(I_p - P_{\mathcal{L}})\widetilde{D}(I_p - P_{\mathcal{L}})$ are 0.0000, 0.0000, 0.0000, 0.0000, 0.1016, 0.2585, 4.3461, $\|E_{(I_p-P_{\mathcal{L}})\widetilde{D}}(I_p - P_{\mathcal{L}})\widetilde{D}(I_p - P_{\mathcal{L}})\|_F = 1.3311 \times 10^{-15}$, and $\|E_{(I_p-P_{\mathcal{L}})\widetilde{D}(I_p-P_{\mathcal{L}})}(I_p - P_{\mathcal{L}})\widetilde{D}\|_F = 1.8544 \times 10^{-15}$, the eigenvalues of the matrix $\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix} = \begin{bmatrix} 0.4079 & -0.2813 \\ -0.2813 & 0.1940 \end{bmatrix}$ are 0.6018, 2.1184×10^{-5}, and the matrix $\begin{bmatrix} M_{22} & M_{23} \\ M_{23}^* & M_{33} \end{bmatrix}$ is 0. According to Algorithm 1 and choosing the matrices $J_1 = 0$, $J_2 = I_5$, $N_X = 0$ and $Y = I_6$, we can obtain that

$$X = \begin{bmatrix} 3.7060 & 6.0203 & 15.4870 & 9.2670 & -8.1450 & -12.5926 \\ 6.0203 & -12.7748 & 4.5979 & 49.2869 & -10.9899 & -16.3238 \\ 15.4870 & 4.5979 & 16.3503 & -0.5303 & -1.4077 & -21.2418 \\ 9.2670 & 49.2869 & -0.5303 & -7.4863 & 2.6776 & -23.5101 \\ -8.1450 & -10.9899 & -1.4077 & 2.6776 & -1.7205 & 12.6190 \\ -12.5926 & -16.3238 & -21.2418 & -23.5101 & 12.6190 & 32.0736 \end{bmatrix}.$$

The absolute error is estimated by

$$\|A^*XA - B\|_F = 2.6065 \times 10^{-14},$$

and the eigenvalues of $(C^*XC - D)$ are 0.0000, 0.1172, 0.1505, 0.3321, 0.5119, 4.4760, 72.3055, which implies that X is the Hermitian solution of the matrix equality $C^*XC \geq D$ s.t. $A^*XA = B$.

5. Conclusions

In this paper, we have established the necessary and sufficient conditions (see (2.1), (3.7), (3.23), and (2.1), (3.11), and (3.27)) for the Hermitian solution of (1.1), and achieve the explicit representation of the general Hermitian solution by the SD and the GSVD when the stated conditions are satisfied. One numerical example verifies the correctness of the introduced method.

Author contributions

Yinlan Chen: Conceptualization, Methodology, Project administration, Supervision, Writing-review & editing; Wenting Duan: Investigation, Software, Validation, Writing-original draft, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in this article.

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Conflict of interest

The authors declare no conflicts of interest in this article.

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