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Research article

Pedal curves obtained from Frenet vector of a space curve and Smarandache curves belonging to these curves

Süleyman Şenyurt[1,*], Filiz Ertem Kaya[2] and Davut Canlı[1]

[1] Department of Mathematics, Ordu University, Ordu, 52200, Türkiye
[2] Department of Mathematics, Niğde Ömer Halisdemir University, Niğde, 51240, Türkiye

* **Correspondence:** Email: ssenyurt@odu.edu.tr; Tel: +905425920880.

Abstract: In this study, first the pedal curves as the geometric locus of perpendicular projections to the Frenet vectors of a space curve were defined and the Frenet vectors, curvature, and torsion of these pedal curves were calculated. Second, for each pedal curve, Smarandache curves were defined by taking the Frenet vectors as position vectors. Finally, the expressions of Frenet vectors, curvature, and torsion related to the main curves were obtained for each Smarandache curve. Thus, new curves were added to the curve family.

Keywords: pedal curves; Frenet apparatus; T-pedal curve; N-pedal curve; B-pedal curve; Smarandache curves
Mathematics Subject Classification: 53A04

1. Introduction

The existence of curves in nature is a natural phenomenon that has been extensively studied by scientists. Researchers have formulated theories to understand the characteristics of these curves by careful examinations and valuable analysis. Therefore, the theory of curves has played a significant role in the field of differential geometry, making it an intriguing area of research.

To gain further insights and theoretical knowledge, it is the best practice to establish an orthonormal system on a curve. By doing so, scientists can gather more information and delve deeper into the properties of the curve. For instance, if the torsion of a curve is found to be zero, it indicates that the curve is planar, which concludes that if the torsion is nonzero, it signifies that the curve is a space curve. Moreover, the behavior of a curve can also be determined by examining its harmonic curvature defined as the ratio of the curvature to torsion. If the harmonic curvature function is constant, then the curve is classified as a helix. A Salkowski curve, on the other hand, is special due to its constant curvature and nonconstant torsion [1].

In addition, there exist other special paired curves possessing some mathematical relations between them. Examples of such pairs of curves include involute-evolute curves, Bertrand and Mannheim curves, as well as Successor curves and Smarandache curves. These curves have been extensively studied, contributing to a wealth of knowledge in this field [1–5].

Another interesting aspect of curve analysis involves the geometric location of perpendicular projection points onto the tangent or normal vector of a curve from a point that does not lie on the curve. This location is defined as the pedal (or contra-pedal) curve (see Figure 1).

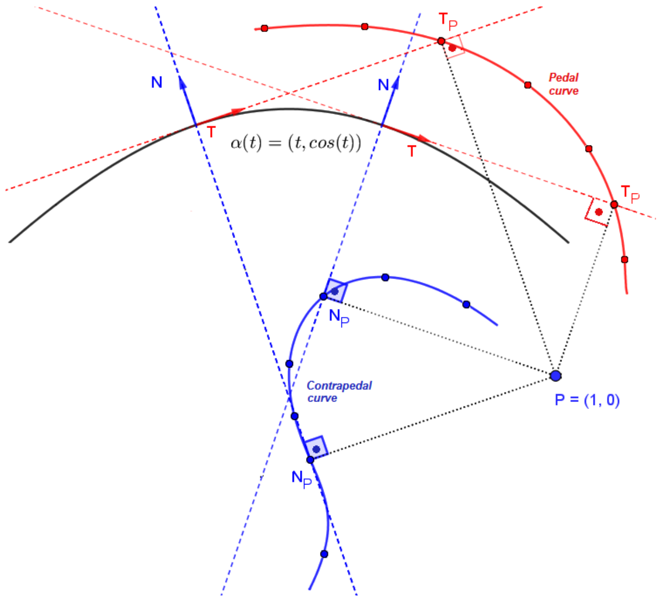


Figure 1. The construction steps of pedal and contra-pedal curves for cosine function.

Extensive research has been conducted on these types of curves, and numerous sources provide valuable insights into their properties [6–8]. The study of such curves has been conducted using various frames in different spaces. Researchers have explored these curves using different approaches and continue to make significant contributions to this field of study [9–13].

In this study, pedal curves belonging to the tangent, principal normal, and binormal vectors of a space curve are defined, and their Frenet vectors, curvature, and torsion functions are calculated. Next, Smarandache curves are defined by taking Frenet elements of each pedal curve as the position vectors. Finally, the corresponding Frenet apparatus are obtained and expressed in terms of the main curve. Thus, new curves are added to to the literature for the theory of curves. Let us recall the basic notions that will be used through the paper. For given a differentiable curve $\alpha(t)$, the formulae of Frenet vector fields and curvature functions are defined as in the followings:

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad N = B \wedge T = \frac{(\alpha' \wedge \alpha'') \wedge \alpha'}{\|\alpha' \wedge \alpha''\| \, \|\alpha'\|}, \quad B = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \tag{1.1}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \quad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}, \tag{1.2}$$

$$T' = v\kappa N, \quad N' = v\left(-\kappa T + \tau B\right), \quad B' = -v\tau N, \tag{1.3}$$

where $v = \|\alpha'\|$, and "$\wedge$" stands for the vector product operator [8, 14].

Definition 1.1. Let T_α denote the tangent vector of a regular curve α in $\mathbb{E}^2$. The geometric locus of the perpendicular projection of points onto a tangent vector from a given point $P \in \mathbb{E}^2$ that is not on the curve is called the pedal curve of the curve α [15].

Theorem 1.2. *[15] The pedal curve of a regular curve α according to the point P in $\mathbb{E}^2$ is given by the following equality:*

$$\alpha_P(t) = \alpha(t) + \langle P - \alpha(t), T_\alpha \rangle T_\alpha. \tag{1.4}$$

Definition 1.3. Let N_α be the normal vector of a regular curve α in $\mathbb{E}^2$. The geometric locus of perpendicular projection of points onto the normal vector from a given point $P \in \mathbb{E}^2$ that is not on the curve is called the contra-pedal curve of the curve α [15].

Theorem 1.4. *[15] The contra-pedal curve of a regular curve α according to the point P in $\mathbb{E}^2$ is given by the following equality:*

$$\alpha^{\perp}{}_P(t) = \alpha(t) + \langle P - \alpha(t), N_\alpha \rangle N_\alpha. \tag{1.5}$$

Example 1.5. According to the origin $O(0,0)$, the pedal and contra-pedal curves of an ellipse that is parameterized as $\alpha(t) = (2cos(t), sin(t))$ in $\mathbb{E}^2$ is given by the following relations (see Figure 2).

$$\alpha_P(t) = \left(\frac{2\cos t}{1+3\sin^2 t}, \frac{4\sin t}{1+3\sin^2 t}\right), \qquad \alpha^{\perp}{}_P(t) = \left(\frac{7\cos t\sin^2 t}{1+3\sin^2 t}, \frac{3\sin^3 t - \cos t \sin t}{1+3\sin^2 t}\right).$$

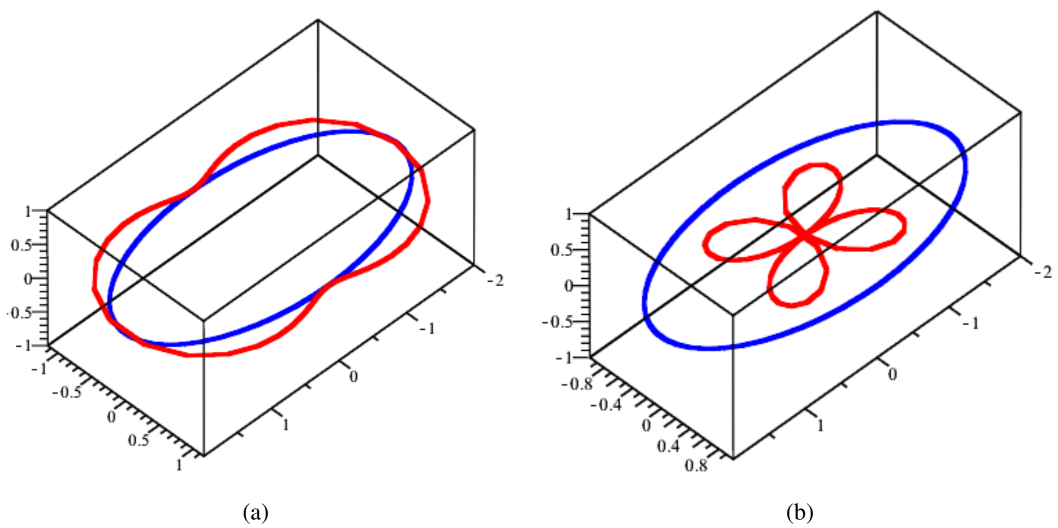


Figure 2. The pedal (a) and contra-pedal (b) curves (red) of the ellipse (blue) according to the origin $O(0,0)$ where $t \in [-\pi, \pi]$.

2. T-Pedal curve and Smarandache curves of the T-Pedal curve

Definition 2.1. Let T be the tangent vector of a given regular curve α in $\mathbb{E}^3$. The geometric locus of the perpendicular projection of points onto a tangent vector from a point $P \in \mathbb{E}^3$ that is not on the curve is called the $T - pedal$ curve of the curve α according to P.

Theorem 2.2. *The equation of the $T-$ pedal curve of a given regular curve α is as follows:*

$$\alpha_T(t) = \alpha(t) + \langle P - \alpha(t), T(t)\rangle T(t). \tag{2.1}$$

Proof. Let P' be a perpendicular projection point onto the tangent vector from the point P that is not on the curve α. The perpendicular projection vector $\overrightarrow{\alpha P'}$ is calculated by the following formula:

$$\overrightarrow{\alpha P'} = \frac{\langle \alpha P, \alpha' \rangle}{\|\alpha'\|^2}\overrightarrow{\alpha'}.$$

On the other hand, let $\alpha_T(t)$ be the geometric location of the point P'. According to this, we obtain following equations:

$$\begin{aligned}
\overrightarrow{\alpha P} = \overrightarrow{\alpha P'} + \overrightarrow{P'P} \Rightarrow \overrightarrow{\alpha P} &= \frac{\langle \alpha P, \alpha' \rangle}{\|\alpha'\|^2}\overrightarrow{\alpha'} + \overrightarrow{P'P} \\
\Rightarrow P' &= \alpha + \frac{\langle P - \alpha, \alpha' \rangle}{\|\alpha'\|^2}\overrightarrow{\alpha'} \\
\Rightarrow \alpha_T(t) &= \alpha(t) + \left\langle P - \alpha(t), \frac{\alpha'}{\|\alpha'\|} \right\rangle \frac{\alpha'}{\|\alpha'\|}.
\end{aligned}$$

When the tangent vector from the relation (1.1) is taken into consideration, the proof of the theorem is completed. □

With some subsequent algebraic operations, if $u(t) = \langle P - \alpha(t), T(t)\rangle$, then the relation (2.1) can be reduced to following:

$$\alpha_T(t) = \alpha(t) + u(t)T(t). \tag{2.2}$$

Specifically, if the point P is origin, then we have $u(t) = -\langle \alpha(t), T(t)\rangle$.

Theorem 2.3. *Let α_T be the $T-$ pedal curve of the given curve α with unit speed, and $\{T_1, N_1, B_1\}$ denotes the Frenet vectors of the $T-$ pedal curve of α. Then, among the Frenet vectors, the following relations exist:*

$$\begin{aligned}
T_1 =& \omega_1(1+u)T + \omega_1 u\kappa N, \\
N_1 =& -\eta_1\omega_1 u\kappa\left(\kappa(1+u)^2 + u^2\kappa^3 + (1+u)(u\kappa)' - uu'\kappa\right)T \\
&+ \eta_1\omega_1(1+u)\left(\kappa(1+u)^2 + u^2\kappa^3 + (1+u)(u\kappa)' - uu'\kappa\right)N \\
&+ \left(\eta_1\omega_1\tau(u\kappa)^3 + \eta_1\omega_1 u\kappa\tau(1+u)^2\right)B, \\
B_1 =& \eta_1\tau(u\kappa)^2T - \eta_1(1+u)u\kappa\tau N + \eta_1\left(\kappa(1+u)^2 + u^2\kappa^3 + (1+u)(u\kappa)' - uu'\kappa\right)B,
\end{aligned}$$

where

$$\begin{aligned}
\omega_1 &= \frac{1}{\sqrt{(1+u')^2 + (u\kappa)^2}}, \\
\eta_1 &= \frac{1}{\sqrt{\tau^2(u\kappa)^4 - (1+u')^2(u\kappa\tau)^2 + \left((1+u')^2\kappa + (1+u')(u\kappa)' - u\kappa(u'' - u\kappa^2)\right)^2}}.
\end{aligned}$$

Proof. By taking the necessary derivatives of the equality (2.2), we have

$$\begin{aligned}\alpha'_T &= (1+u')\,T + u\kappa N,\\ \alpha''_T &= \left(u''-u\kappa^2\right)T + \left((1+u')\,\kappa + (u\kappa)'\right)N + u\kappa\tau B,\\ \alpha'''_T &= \left(u''' - \left(u\kappa^2\right)' - (1+u')\,\kappa^2 - \kappa(u\kappa)'\right)T + \left(u''\kappa - u\kappa^3 - u\kappa\tau^2 + \left((1+u')\,\kappa + (u\kappa)'\right)'\right)N\\ &\quad + \left((1+u')\,\tau\kappa + \tau(u\kappa)' + (u\kappa\tau)'\right)B.\end{aligned} \tag{2.3}$$

Upon necessary algebraic operations that are performed, the following relations are obtained

$$\begin{aligned}\alpha'_T \wedge \alpha''_T &= \tau(u\kappa)^2 T - (1+u')\,u\kappa\tau N + \left((1+u')^2\kappa + (1+u')\,(u\kappa)' - u\kappa\left(u''-u\kappa^2\right)\right)B,\\ \det\left(\alpha'_T, \alpha''_T, \alpha'''_T\right) &= \tau(u\kappa)^2\left(u''' - \left(u\kappa^2\right)' - (1+u')\,\kappa^2 - \kappa(u\kappa)'\right)\\ &\quad - u\kappa\tau\,(1+u')\left(u''\kappa - u\kappa\left(\kappa^2+\tau^2\right) + \left((1+u')\,\kappa + (u\kappa)'\right)'\right)\\ &\quad + \left((1+u')^2\kappa + (1+u')\,(u\kappa)' - u\kappa\left(u''-u\kappa^2\right)\right)\left((1+u')\,\tau\kappa + \tau(u\kappa)' + (u\kappa\tau)'\right),\end{aligned} \tag{2.4}$$

$$\begin{aligned}\|\alpha'_T\| &= \sqrt{(1+u')^2 + (u\kappa)^2},\\ \|\alpha'_T \wedge \alpha''_T\| &= \sqrt{\tau^2(u\kappa)^4 - (1+u')^2(u\kappa\tau)^2 + \left((1+u')^2\kappa + (1+u')\,(u\kappa)' - u\kappa\,(u''-u\kappa^2)\right)^2}\,.\end{aligned} \tag{2.5}$$

By substituting the given equalities above into the relations at (1.1), the proof is completed. □

Theorem 2.4. *Let α_T be the $T-$pedal curve of the unit speed curve α, and let κ_1 and τ_1 denote the curvature and torsion functions for α_T, respectively. Then, the following relations exist among the curvatures:*

$$\begin{aligned}\kappa_1 &= \frac{\left(\tau^2(u\kappa)^4 - (1+u')^2(u\kappa\tau)^2 + \left((1+u')^2\kappa + (1+u')\,(u\kappa)' - u\kappa\left(u''-u\kappa^2\right)\right)^2\right)^{\frac{1}{2}}}{\left((1+u')^2 + (u\kappa)^2\right)^{\frac{3}{2}}},\\ \tau_1 &= \frac{\begin{array}{l}\tau(u\kappa)^2\left(u''' - \left(u\kappa^2\right)' - (1+u')\,\kappa^2 - \kappa(u\kappa)'\right) - u\kappa\tau\,(1+u')\left(u''\kappa - u\kappa\left(\kappa^2+\tau^2\right) + \left((1+u')\,\kappa + (u\kappa)'\right)'\right)\\ + \left((1+u')^2\kappa + (1+u')\,(u\kappa)' - u\kappa\left(u''-u\kappa^2\right)\right)\left((1+u')\,\tau\kappa + \tau(u\kappa)' + (u\kappa\tau)'\right)\end{array}}{\tau^2(u\kappa)^4 - (1+u')^2(u\kappa\tau)^2 + \left((1+u')^2\kappa + (1+u')\,(u\kappa)' - u\kappa\,(u''-u\kappa^2)\right)^2}.\end{aligned}$$

Proof. By substituting the given relations (2.4) and (2.5) into (1.2), the curvatures can be found, completing the proof. □

Corollary 2.5. *The following relations exist between the Frenet vectors of the $T-$pedal curve and their derivatives*

$$T_1' = \mu_1\kappa_1 N_1, \qquad N_1' = \mu_1\left(-\kappa_1 T_1 + \tau_1 B_1\right), \qquad B_1' = -\mu_1\tau_1 N_1, \tag{2.6}$$

where $\mu_1 = \|\alpha'_T\|$.

Definition 2.6. By taking the tangent and the principal normal vectors of the $T-pedal$ curve as position vectors, we define a regular curve called the T_1N_1 Smarandache curve as follows:

$$\alpha_1 = \frac{T_1 + N_1}{\sqrt{2}}. \tag{2.7}$$

Theorem 2.7. *Let T_{α_1}, N_{α_1}, and B_{α_1} be the Frenet vectors of the T_1N_1 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned}
T_{\alpha_1} &= \frac{-\kappa_1 T_1 + \kappa_1 N_1 + \tau_1 B_1}{\sqrt{2\kappa_1^2 + \tau_1^2}}, \\
N_{\alpha_1} &= B_{\alpha_1} \wedge T_{\alpha_1}, \\
B_{\alpha_1} &= \frac{-(\kappa_1 x_3 + \tau_1 x_2) T_1 + (\kappa_1 x_3 + \tau_1 x_1) N_1 - (\kappa_1 x_2 + \kappa_1 x_1) B_1}{\sqrt{(\kappa_1 x_3 + \tau_1 x_2)^2 + (\kappa_1 x_3 + \tau_1 x_1)^2 + (\kappa_1 x_2 + \kappa_1 x_1)^2}},
\end{aligned}$$

where $x_1 = -\frac{\mu_1^2\kappa_1^2 - (\mu_1\kappa_1)'}{\sqrt{2}}, \quad x_2 = \frac{(\mu_1\kappa_1)' - \mu_1^2\left(\kappa_1^2 + \tau_1^2\right)}{\sqrt{2}}, \quad x_3 = \frac{\mu_1^2\kappa_1\tau_1 + (\mu_1\tau_1)'}{\sqrt{2}}.$

Proof. The derivatives of the T_1N_1 curve up to the third degree are as given below.

$$\begin{aligned}
\alpha'_1 &= \frac{\mu_1(-\kappa_1 T_1 + \kappa_1 N_1 + \tau_1 B_1)}{\sqrt{2}}, \\
\alpha''_1 &= x_1 T_1 + x_2 N_1 + x_3 B_1, \\
\alpha'''_1 &= (x'_1 - \mu_1\kappa_1 x_2) T_1 + (x'_2 + \mu_1 x_1 - \mu_1 x_3) N_1 + (x'_3 + \mu_1\tau_1 x_2) B_1.
\end{aligned} \tag{2.8}$$

By taking the vectoral product and computing the determinants of first and second derivatives of the curve α given in equality (2.8), we get the equality (2.9) as below:

$$\begin{aligned}
\alpha'_1 \wedge \alpha''_1 &= -\frac{\mu_1(\kappa_1 x_3 + \tau_1 x_2)}{\sqrt{2}} T_1 + \frac{\mu_1(\kappa_1 x_3 + \tau_1 x_1)}{\sqrt{2}} N_1 - \frac{\mu_1(\kappa_1 x_2 + \kappa_1 x_1)}{\sqrt{2}} B_1, \\
\det(\alpha'_1, \alpha''_1, \alpha'''_1) &= \frac{\mu_1}{\sqrt{2}} \left(\begin{array}{l} (\kappa_1 x_3 + \tau_1 x_1)(x'_2 + \mu_1 x_1 - \mu_1 x_3) - (\kappa_1 x_3 + \tau_1 x_2)(x'_1 - \mu_1\kappa_1 x_2) \\ -(\kappa_1 x_2 + \kappa_1 x_1)(x'_3 + \mu_1\tau_1 x_2) \end{array} \right).
\end{aligned} \tag{2.9}$$

Moreover, by taking the norm of the first derivative of the curve α and the vectoral product of the first and second derivatives of α, we obtain the equality (2.10) as

$$\begin{aligned}
\|\alpha'_1\| &= \frac{\mu_1}{\sqrt{2}} \sqrt{2\kappa_1^2 + \tau_1^2}, \\
\|\alpha'_1 \wedge \alpha''_1\| &= \frac{\mu_1}{\sqrt{2}} \sqrt{(\kappa_1 x_3 + \tau_1 x_2)^2 + (\kappa_1 x_3 + \tau_1 x_1)^2 + (\kappa_1 x_2 + \kappa_1 x_1)^2}.
\end{aligned} \tag{2.10}$$

Finally, substituting the relations (2.8), (2.9), and (2.10) into (1.1) completes the proof. □

Theorem 2.8. *Let κ_{α_1} and τ_{α_1} denote the curvature and the torsion of the T_1N_1 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\alpha_1} = \frac{2\sqrt{(\kappa_1 x_3 + \tau_1 x_2)^2 + (\kappa_1 x_3 + \tau_1 x_1)^2 + (\kappa_1 x_2 + \kappa_1 x_1)^2}}{\mu_1^2\left(2\kappa_1^2 + \tau_1^2\right)\sqrt{2\kappa_1^2 + \tau_1^2}},$$

$$\tau_{\alpha_1} = \frac{\begin{array}{l}\sqrt{2}(\kappa_1 x_3 + \tau_1 x_1)(x'_2 + \mu_1 x_1 - \mu_1 x_3) - \sqrt{2}(\kappa_1 x_3 + \tau_1 x_2)(x'_1 - \mu_1 \kappa_1 x_2) \\ -\sqrt{2}(\kappa_1 x_2 + \kappa_1 x_1)(x'_3 + \mu_1 \tau_1 x_2)\end{array}}{\mu_1(\kappa_1 x_3 + \tau_1 x_2)^2 + \mu_1(\kappa_1 x_3 + \tau_1 x_1)^2 + \mu_1(\kappa_1 x_2 + \kappa_1 x_1)^2}.$$

Proof. By using (2.9) and (2.10) to substitute into (1.2), the proof is completed. □

Definition 2.9. By taking the tangent and the binormal vectors of the $T - pedal$ curve as position vectors, we define a regular curve called the T_1B_1 Smarandache curve as follows:

$$\alpha_2 = \frac{T_1 + B_1}{\sqrt{2}}. \tag{2.11}$$

Theorem 2.10. *Let T_{α_2}, N_{α_2}, and B_{α_2} be the Frenet vectors of the T_1B_1 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$T_{\alpha_2} = N_1, \qquad N_{\alpha_2} = \frac{-\kappa_1 T_1 + \tau_1 B_1}{\sqrt{\kappa_1^2 + \tau_1^2}}, \qquad B_{\alpha_2} = \frac{\tau_1 T_1 + \kappa_1 B_1}{\sqrt{\kappa_1^2 + \tau_1^2}}.$$

Proof. By taking the derivatives of (2.11), we first have

$$\begin{aligned}
\alpha'_2 &= \frac{\mu_1(\kappa_1 - \tau_1)N_1}{\sqrt{2}},\\
\alpha''_2 &= \frac{-\kappa_1\mu_1^2(\kappa_1 - \tau_1)T_1 + (\mu_1\kappa_1 - \mu_1\tau_1)'N_1 + \tau_1\mu_1^2(\kappa_1 - \tau_1)B_1}{\sqrt{2}},\\
\alpha'''_2 &= \frac{\begin{array}{l}\left(\left(-\kappa_1\mu_1^2(\kappa_1 - \tau_1)\right)' - \kappa_1\mu_1(\mu_1\kappa_1 - \mu_1\tau_1)''\right)T_1 + \left((\mu_1\kappa_1 - \mu_1\tau_1)'' - \mu_1^3(\kappa_1 - \tau_1)\left(\kappa_1^2 + \tau_1^2\right)\right)N_1 \\ + \left(\left(\tau_1\mu_1^2(\kappa_1 - \tau_1)\right)' + \tau_1\mu_1(\mu_1\kappa_1 - \mu_1\tau_1)''\right)B_1\end{array}}{\sqrt{2}}.
\end{aligned} \tag{2.12}$$

Further, by taking norms and having required vector products, we have

$$\begin{aligned}
\alpha'_2 \wedge \alpha''_2 &= \frac{\mu_1^3(\kappa_1 - \tau_1)^2(\tau_1 T_1 + \kappa_1 B_1)}{2},\\
\det(\alpha'_2, \alpha''_2, \alpha'''_2) &= \frac{\mu_1^5(\kappa_1 - \tau_1)^3(\kappa_1\tau_1' - \kappa_1'\tau_1)}{2\sqrt{2}},
\end{aligned} \tag{2.13}$$

and

$$\|\alpha'_2\| = \frac{\mu_1(\kappa_1 - \tau_1)}{\sqrt{2}}, \qquad \|\alpha'_2 \wedge \alpha''_2\| = \frac{\mu_1^3(\kappa_1 - \tau_1)^2\sqrt{\kappa_1^2 + \tau_1^2}}{2}. \tag{2.14}$$

If we substitute relations (2.12), (2.13), and (2.14) into (1.1), the proof is completed. □

Theorem 2.11. *Let κ_{α_2} and τ_{α_2} denote the curvature and the torsion of the T_1B_1 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\alpha_2} = \frac{\sqrt{2\kappa_1^2 + 2\tau_1^2}}{(\kappa_1 - \tau_1)}, \qquad \tau_{\alpha_2} = \frac{\sqrt{2}\,(\kappa_1 \tau_1{}' - \kappa_1{}' \tau_1)}{\mu_1\,(\kappa_1 - \tau_1)\left(\kappa_1^2 + \tau_1^2\right)}.$$

Proof. The proof is obvious by the substitution of (2.13) and (2.14) into (1.2). □

Definition 2.12. By taking the principal normal and the binormal vectors of the $T - pedal$ curve as position vectors, we define a regular curve called the N_1B_1 Smarandache curve as follows:

$$\alpha_3 = \frac{N_1 + B_1}{\sqrt{2}}. \tag{2.15}$$

Theorem 2.13. *Let T_{α_3}, N_{α_3}, and B_{α_3} be the Frenet vectors of the N_1B_1 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned}
T_{\alpha_3} &= \frac{-\kappa_1 T_1 - \tau_1 N_1 + \tau_1 B_1}{\sqrt{\kappa_1^2 + 2\tau_1^2}}, \\
N_{\alpha_3} &= B_{\alpha_3} \wedge T_{\alpha_3}, \\
B_{\alpha_3} &= \frac{-(\tau_1 y_3 + \tau_1 y_2)\,T_1 + (\kappa_1 y_3 + \tau_1 y_1)\,N_1 + (\tau_1 y_1 - \kappa_1 y_2)\,B_1}{\sqrt{(\tau_1 y_3 + \tau_1 y_2)^2 + (\kappa_1 y_3 + \tau_1 y_1)^2 + (\tau_1 y_1 - \kappa_1 y_2)^2}},
\end{aligned}$$

where $y_1 = \dfrac{\mu_1^2 \tau_1 \kappa_1 - (\mu_1 \kappa_1)'}{\sqrt{2}}, \quad y_2 = -\dfrac{\mu_1^2\left(\kappa_1^2 + \tau_1^2\right) + (\mu_1 \tau_1)'}{\sqrt{2}}, \quad y_3 = \dfrac{(\mu_1 \tau_1)' - \mu_1^2 \tau_1^2}{\sqrt{2}}.$

Proof. By taking the derivatives of (2.15), we have

$$\begin{aligned}
\alpha'_3 &= \frac{\mu_1\,(-\kappa_1 T_1 - \tau_1 N_1 + \tau_1 B_1)}{\sqrt{2}}, \\
\alpha''_3 &= y_1 T_1 + y_2 N_1 + y_3 B_1, \\
\alpha'''_3 &= (y'_1 - \mu_1 y_2 \kappa_1)\,T_1 + (y'_2 + \mu_1 y_1 \kappa_1 - \mu_1 y_3 \tau_1)\,N_1 + (y'_3 + \mu_1 y_2 \tau_1)\,B_1.
\end{aligned} \tag{2.16}$$

Moreover, we calculate the required vector products and the norms as

$$\begin{aligned}
\alpha'_3 \wedge \alpha''_3 &= \frac{\mu_1}{\sqrt{2}}\left((-(\tau_1 y_3 + \tau_1 y_2)\,T_1 + (\kappa_1 y_3 + \tau_1 y_1)\,N_1 + (\tau_1 y_1 - \kappa_1 y_2)\,B_1)\right), \\
\det(\alpha'_3, \alpha''_3, \alpha'''_3) &= \frac{\mu_1}{\sqrt{2}}\begin{pmatrix} (\tau_1 y_1 - \kappa_1 y_2)\,(y'_3 + \mu_1 y_2 \tau_1) - (\tau_1 y_3 + \tau_1 y_2)\,(y'_1 - \mu_1 y_2 \kappa_1) \\ + (y'_1 - \mu_1 y_2 \kappa_1)\,(y'_2 + \mu_1 y_1 \kappa_1 - \mu_1 y_3 \tau_1) \end{pmatrix}
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\|\alpha'_3\| &= \frac{\mu_1}{\sqrt{2}}\sqrt{\kappa_1^2 + 2\tau_1^2}, \\
\|\alpha'_3 \wedge \alpha''_3\| &= \frac{\mu_1}{\sqrt{2}}\sqrt{(\tau_1 y_3 + \tau_1 y_2)^2 + (\kappa_1 y_3 + \tau_1 y_1)^2 + (\tau_1 y_1 - \kappa_1 y_2)^2}.
\end{aligned} \tag{2.18}$$

When substituting relations (2.16), (2.17), and (2.18) into (1.1), the proof is completed. □

Theorem 2.14. *Let κ_{α_3} and τ_{α_3} denote the curvature and the torsion of the T_1B_1 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\alpha_3} = \frac{2\sqrt{(\tau_1 y_3 + \tau_1 y_2)^2 + (\kappa_1 y_3 + \tau_1 y_1)^2 + (\tau_1 y_1 - \kappa_1 y_2)^2}}{\mu_1^2\left(\kappa_1^2 + 2\tau_1^2\right)\sqrt{\kappa_1^2 + 2\tau_1^2}},$$

$$\tau_{\alpha_3} = \frac{\sqrt{2}\begin{pmatrix}(\tau_1 y_1 - \kappa_1 y_2)(y'_3 + \mu_1 y_2 \tau_1) - (\tau_1 y_3 + \tau_1 y_2)(y'_1 - \mu_1 y_2 \kappa_1) \\ + (y'_1 - \mu_1 y_2 \kappa_1)(y'_2 + \mu_1 y_1 \kappa_1 - \mu_1 y_3 \tau_1)\end{pmatrix}}{\mu_1\left((\tau_1 y_3 + \tau_1 y_2)^2 + (\kappa_1 y_3 + \tau_1 y_1)^2 + (\tau_1 y_1 - \kappa_1 y_2)^2\right)}.$$

Proof. The proof is done upon substituting the above relations (2.17) and (2.18) into (1.2). □

Definition 2.15. By taking the tangent and principal normal and binormal vectors of the $T - pedal$ curve as position vectors, we define a regular curve called the $T_1N_1B_1$ Smarandache curve as follows:

$$\alpha_4 = \frac{T_1 + N_1 + B_1}{\sqrt{3}}. \tag{2.19}$$

Theorem 2.16. *Let T_{α_4}, N_{α_4}, and B_{α_4} be the Frenet vectors of the $T_1N_1B_1$ Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned}
T_{\alpha_4} &= \frac{-\kappa_1 T_1 + (\kappa_1 - \tau_1) N_1 + \tau_1 B_1}{\sqrt{2\kappa_1^2 - 2\kappa_1\tau_1 + 2\tau_1^2}}, \\
N_{\alpha_4} &= B_{\alpha_4} \wedge T_{\alpha_4}, \\
B_{\alpha_4} &= \frac{(z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2) T_1 + (\tau_1 z_1 + \kappa_1 z_3) N_1 - (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2) B_1}{\sqrt{(z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2)^2 + (\tau_1 z_1 + \kappa_1 z_3)^2 + (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2)^2}},
\end{aligned}$$

where

$$z_1 = -\frac{(\mu_1\kappa_1)' + \mu_1^2\kappa_1(\kappa_1 - \tau_1)}{\sqrt{3}}, \qquad z_2 = \frac{(\mu_1\kappa_1 - \mu_1\tau_1)' - \mu_1^2\left(\kappa_1^2 + \tau_1^2\right)}{\sqrt{3}},$$

$$z_3 = \frac{(\mu_1\tau_1)' + \mu_1^2\tau_1(\kappa_1 - \tau_1)}{\sqrt{3}}.$$

Proof. The derivatives of (2.19) are

$$\begin{aligned}
\alpha'_4 &= \frac{\mu_1(-\kappa_1 T_1 + (\kappa_1 - \tau_1) N_1 + \tau_1 B_1)}{\sqrt{3}}, \\
\alpha''_4 &= z_1 T_1 + z_2 N_1 + z_3 B_1, \\
\alpha'''_4 &= (z'_1 - z_2\mu_1\kappa_1) T_1 + (z'_2 + z_1\mu_1\kappa_1 - z_3\mu_1\tau_1) N_1 + (z'_3 + z_2\mu_1\tau_1) B_1.
\end{aligned} \tag{2.20}$$

In addition, the required vector products and the norms are calculated as

$$\begin{aligned}
\alpha'_4 \wedge \alpha''_4 &= \frac{\mu_1}{\sqrt{3}}\left((z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2) T_1 + (\tau_1 z_1 + \kappa_1 z_3) N_1 - (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2) B_1\right), \\
\det(\alpha'_4, \alpha''_4, \alpha'''_4) &= \frac{\mu_1}{\sqrt{3}}\begin{pmatrix}(z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2)(z'_1 - z_2\mu_1\kappa_1) \\ + (\tau_1 z_1 + \kappa_1 z_3)(z'_2 + z_1\mu_1\kappa_1 - z_3\mu_1\tau_1) \\ - (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2)(z'_3 + z_2\mu_1\tau_1)\end{pmatrix},
\end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \|\alpha'_4\| &= \frac{\sqrt{6}\mu_1}{3}\sqrt{\kappa_1^2 - \kappa_1\tau_1 + \tau_1^2}, \\ \|\alpha'_4 \wedge \alpha''_4\| &= \frac{\mu_1}{\sqrt{3}}\sqrt{(z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2)^2 + (\tau_1 z_1 + \kappa_1 z_3)^2 + (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2)^2}\cdot \end{aligned} \tag{2.22}$$

By substituting relations (2.20), (2.21), and (2.22) into (1.1), the proof is completed. □

Theorem 2.17. *Let κ_{α_4} and τ_{α_4} denote the curvature and the torsion of the $T_1N_1B_1$ Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\begin{aligned} \kappa_{\alpha_4} &= \frac{3\sqrt{2}}{4}\frac{\sqrt{(z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2)^2 + (\tau_1 z_1 + \kappa_1 z_3)^2 + (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2)^2}}{\mu_1^2\left(\kappa_1^2 - \kappa_1\tau_1 + \tau_1^2\right)\sqrt{\kappa_1^2 - \kappa_1\tau_1 + \tau_1^2}}, \\ \tau_{\alpha_4} &= \frac{\sqrt{3}\begin{pmatrix} (z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2)(z'_1 - z_2\mu_1\kappa_1) + (\tau_1 z_1 + \kappa_1 z_3)(z'_2 + z_1\mu_1\kappa_1 - z_3\mu_1\tau_1) \\ -(\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2)(z'_3 + z_2\mu\tau_1) \end{pmatrix}}{\mu_1\left((z_3\kappa_1 - z_3\tau_1 - \tau_1 z_2)^2 + (\tau_1 z_1 + \kappa_1 z_3)^2 + (\kappa_1 z_1 - \tau_1 z_1 + \kappa_1 z_2)^2\right)}. \end{aligned}$$

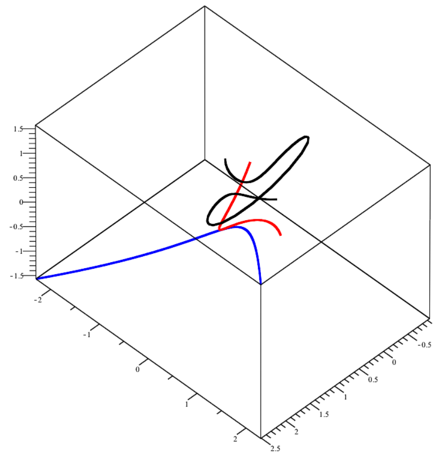
Proof. The proof is done upon substituting the above relations (2.21) and (2.22) into (1.2). □

Example 2.18. Let us consider the space curve $\gamma : [-\pi, \pi] \to E^3$ parameterized as $\gamma(t) = (cosh(s), sinh(s), s)$. Frenet vectors and the pedal curves according to the origin $O = (0, 0, 0)$ that correspond to each vector are given as follows:

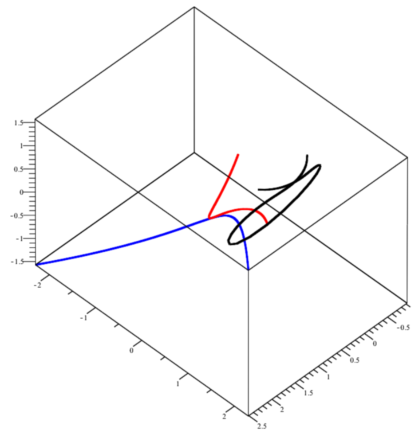
$$T = \frac{1}{\sqrt{2}}\left(\frac{sinh(s)}{cosh(s)}, 1, \frac{1}{cosh(s)}\right), \quad N = \left(\frac{1}{cosh(s)}, 0, -\frac{sinh(s)}{cosh(s)}\right), \quad B = \frac{1}{\sqrt{2}}\left(-\frac{sinh(s)}{cosh(s)}, 1, -\frac{1}{cosh(s)}\right),$$

$$\begin{aligned} T - Pedal \Rightarrow\ \alpha_T &= \left(\frac{2\cosh(s) - \sinh(s)\, s}{1 + \cosh(2s)}, \frac{-s}{2\cosh(s)}, \frac{s\cosh(2s) - \sinh(2s)}{1 + \cosh(2s)}\right), \\ N - Pedal \Rightarrow\ \alpha_N &= \left(\frac{\cosh(3s) - \cosh(s) + 4\sinh(s)\, s}{2(1 + \cosh(2s))}, \sinh(s), \frac{2s + \sinh(2s)}{1 + \cosh(2s)}\right), \\ B - Pedal \Rightarrow\ \alpha_B &= \left(\frac{\cosh(3s) + 3\cosh(s) - 2\sinh(s)\, s}{2(1 + \cosh(2s))}, \frac{\sinh(2s) + s}{2\cosh(s)}, \frac{s\cosh(2s)}{1 + \cosh(2s)}\right). \end{aligned}$$

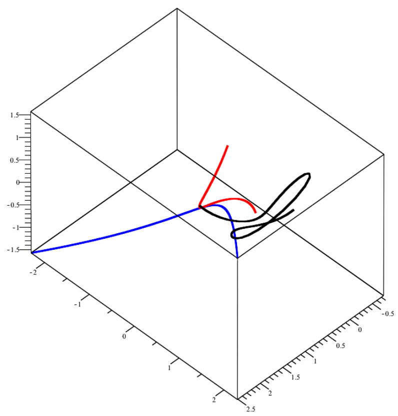
In Figure 3, four of the Smarandache curves of the $T-pedal$ curve according to the origin $O(0, 0, 0)$ are illustrated.



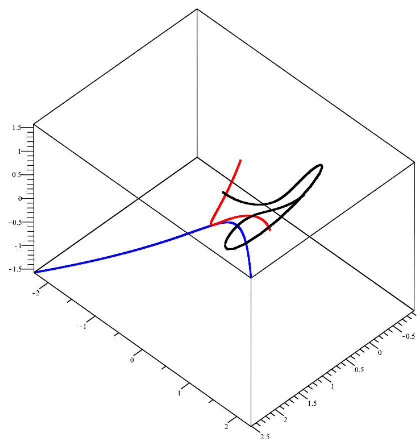
(a) T_1N_1- Smarancahe curve



(b) T_1B_1- Smarancahe curve



(c) N_1B_1- Smarancahe curve



(d) $T_1N_1B_1-$ Smarancahe curve

Figure 3. Smarandache curves (black) of the $T-pedal$ curve (red) of the curve $\gamma(t)$ (blue) according to the origin $O(0,0,0)$ where $t \in [-\pi, \pi]$.

3. N-Pedal curve and Smarandache curves of the N-Pedal curve

Definition 3.1. Let N be the principal normal vector of a given regular curve α in $\mathbb{E}^3$. The geometric locus of the perpendicular projection of points onto the normal vector from a point $P \in \mathbb{E}^3$ that is not on the curve is called the $N-pedal$ curve of the curve α according to P.

Theorem 3.2. *The equation of the $N-pedal$ curve of a given regular curve α is as follows:*

$$\alpha_N(t) = \alpha(t) + \langle P - \alpha(t), N(t)\rangle N(t). \tag{3.1}$$

Proof. Let P' be a perpendicular projection point onto the principal normal vector from the point P that is not on the curve α. The perpendicular projection vector $\overrightarrow{\alpha P'}$ is calculated by the following formula:

$$\overrightarrow{\alpha P'} = \frac{\langle \alpha P, (\alpha' \wedge \alpha'') \wedge \alpha' \rangle}{\|\alpha' \wedge \alpha''\|^2 \|\alpha'\|^2} \cdot \overrightarrow{(\alpha' \wedge \alpha'') \wedge \alpha'}.$$

Next, let $\alpha_N(t)$ be the geometric location of the point P'. According to this, we have following relations:

$$\begin{aligned}\overrightarrow{\alpha P} = \overrightarrow{\alpha P'} + \overrightarrow{P'P} \Rightarrow \overrightarrow{\alpha P} = \frac{\langle \alpha P, (\alpha' \wedge \alpha'') \wedge \alpha' \rangle}{\|\alpha' \wedge \alpha''\|^2 \|\alpha'\|^2} \cdot \overrightarrow{(\alpha' \wedge \alpha'') \wedge \alpha'} + \overrightarrow{P'P} \\ \Rightarrow P' = \alpha + \frac{\langle \alpha P, (\alpha' \wedge \alpha'') \wedge \alpha' \rangle}{\|\alpha' \wedge \alpha''\|^2 \|\alpha'\|^2} \cdot \overrightarrow{(\alpha' \wedge \alpha'') \wedge \alpha'} \\ \Rightarrow \alpha_B = \alpha + \left\langle \alpha P, \frac{(\alpha' \wedge \alpha'') \wedge \alpha'}{\|\alpha' \wedge \alpha''\| \, \|\alpha'\|} \right\rangle \cdot \frac{\overrightarrow{(\alpha' \wedge \alpha'') \wedge \alpha'}}{\|\alpha' \wedge \alpha''\| \, \|\alpha'\|}.\end{aligned}$$

When the principal normal vector from the relation (1.1) is taken into consideration, the proof of the theorem is completed. □

Moreover, if $\chi(t) = \langle P - \alpha(t), N(t) \rangle$, then the relation (3.1) can be written by following:

$$\alpha_N(t) = \alpha(t) + \chi(t) N(t), \tag{3.2}$$

and if the point P is specifically taken as origin, then we have $\chi(t) = -\langle \alpha(t), N(t) \rangle$.

Theorem 3.3. *Let α_N be the $N-$pedal curve of the given curve α with unit speed, and $\{T_2,\ N_2,\ B_2\}$ denotes the Frenet vectors of the $N-$pedal curve of α. Then, among the Frenet vectors, the following relations exist:*

$$\begin{aligned} T_2 =& \omega_2 (1 - \kappa\chi) T + \omega_2 \chi' N + \omega_2 \tau\chi B, \\ N_2 =& B_2 \wedge T_2, \\ B_2 =& \eta_2 \left(\chi' ((\tau\chi)' + \tau\chi') - \tau\chi \left(\kappa - \chi \left(\kappa^2 + \tau^2 \right) + \chi'' \right) \right) T \\ & - \eta_2 ((1 - \kappa\chi)((\tau\chi)' + \tau\chi') + \tau\chi((\kappa\chi)' + \kappa\chi')) N \\ & + \eta_2 \left((1 - \kappa\chi) \left(\kappa - \chi \left(\kappa^2 + \tau^2 \right) + \chi'' \right) + \chi' ((\kappa\chi)' + \kappa\chi') \right) B, \end{aligned}$$

where

$$\begin{aligned} \omega_2 =& \frac{1}{\sqrt{(1 - \kappa\chi)^2 + \chi'^2 + (\tau\chi)^2}}, \\ \eta_2 =& \frac{1}{\left(\begin{array}{l} \left(\chi' ((\tau\chi)' + \tau\chi') - \tau\chi \left(\kappa - \chi \left(\kappa^2 + \tau^2 \right) + \chi'' \right) \right)^2 + ((1 - \kappa\chi)((\tau\chi)' + \tau\chi') + \tau\chi((\kappa\chi)' + \kappa\chi'))^2 \\ + \left((1 - \kappa\chi) \left(\kappa - \chi \left(\kappa^2 + \tau^2 \right) + \chi'' \right) + \chi' ((\kappa\chi)' + \kappa\chi') \right)^2 \end{array} \right)^{\frac{1}{2}}}. \end{aligned}$$

Proof. By taking the derivatives of (3.2), we first have

$$\begin{aligned} \alpha'_N =& (1 - \kappa\chi) T + \chi' N + \tau\chi B, \\ \alpha''_N =& -((\kappa\chi)' + \kappa\chi') T + \left(\kappa - \chi \left(\kappa^2 + \tau^2 \right) + \chi'' \right) N + ((\tau\chi)' + \tau\chi') B, \\ \alpha'''_N =& -\left((\kappa\chi)'' + (\kappa\chi')' + \kappa^2 - \chi\kappa \left(\kappa^2 + \tau^2 \right) + \kappa\chi'' \right) T \\ & + \left(\left(\kappa - \chi \left(\kappa^2 + \tau^2 \right) + \chi'' \right)' - \kappa(\kappa\chi)' - \tau(\tau\chi)' - \chi' \left(\kappa^2 + \tau^2 \right) \right) N \\ & + \left(\kappa\tau - \chi\tau \left(\kappa^2 + \tau^2 \right) + \tau\chi'' + (\tau\chi)'' + (\tau\chi')' \right) B. \end{aligned} \tag{3.3}$$

Further, other necessary relations are obtained as

$$\begin{aligned}\alpha'_N \wedge \alpha''_N =& \left(\chi'\left((\tau\chi)' + \tau\chi'\right) - \tau\chi\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)\right)T \\ &- \left((1-\kappa\chi)\left((\tau\chi)' + \tau\chi'\right) + \tau\chi\left((\kappa\chi)' + \kappa\chi'\right)\right)N \\ &+ \left((1-\kappa\chi)\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right) + \chi'\left((\kappa\chi)' + \kappa\chi'\right)\right)B,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\det\left(\alpha'_N, \alpha''_N, \alpha'''_N\right) = \\ &- \left(\chi'\left((\tau\chi)' + \tau\chi'\right) - \tau\chi\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)\right)\left((\kappa\chi)'' + (\kappa\chi')' + \kappa^2 + \kappa\chi'' - \chi\kappa\left(\kappa^2+\tau^2\right)\right) \\ &- \left((1-\kappa\chi)\left((\tau\chi)' + \tau\chi'\right) + \tau\chi\left((\kappa\chi)' + \kappa\chi'\right)\right)\left(\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)' - \kappa(\kappa\chi)' - \tau(\tau\chi)' - \chi'\left(\kappa^2+\tau^2\right)\right) \\ &+ \left((1-\kappa\chi)\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right) + \chi'\left((\kappa\chi)' + \kappa\chi'\right)\right)\left(\kappa\tau - \chi\tau\left(\kappa^2+\tau^2\right) + \tau\chi'' + (\tau\chi)'' + (\tau\chi')'\right),\end{aligned}$$

and

$$\begin{aligned}\|\alpha'_N\| &= \sqrt{(1-\kappa\chi)^2 + \chi'^2 + (\tau\chi)^2}, \\ \|\alpha'_N \wedge \alpha''_N\| &= \begin{pmatrix} \left(\chi'\left((\tau\chi)' + \tau\chi'\right) - \tau\chi\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)\right)^2 \\ +\left((1-\kappa\chi)\left((\tau\chi)' + \tau\chi'\right) + \tau\chi\left((\kappa\chi)' + \kappa\chi'\right)\right)^2 \\ +\left((1-\kappa\chi)\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right) + \chi'\left((\kappa\chi)' + \kappa\chi'\right)\right)^2 \end{pmatrix}^{\frac{1}{2}}.\end{aligned} \tag{3.5}$$

By substituting equalities (3.3), (3.4), and (3.5) into the relations at (1.1), the proof is completed. □

Theorem 3.4. *Let α_N, be the $N-$ pedal curve of the unit speed curve α, and let κ_2 and τ_2 denote the curvature and torsion functions for α_N, respectively. Then, the following relations exist among the curvatures:*

$$\kappa_2 = \frac{\begin{pmatrix} \left(\chi'\left((\tau\chi)' + \tau\chi'\right) - \tau\chi\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)\right)^2 + \left((1-\kappa\chi)\left((\tau\chi)' + \tau\chi'\right) + \tau\chi\left((\kappa\chi)' + \kappa\chi'\right)\right)^2 \\ +\left((1-\kappa\chi)\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right) + \chi'\left((\kappa\chi)' + \kappa\chi'\right)\right)^2 \end{pmatrix}^{\frac{1}{2}}}{\left((1-\kappa\chi)^2 + \chi'^2 + (\tau\chi)^2\right)\sqrt{(1-\kappa\chi)^2 + \chi'^2 + (\tau\chi)^2}},$$

$$\tau_2 = \frac{\begin{array}{l} -\left(\chi'\left((\tau\chi)' + \tau\chi'\right) - \tau\chi\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)\right)\left((\kappa\chi)'' + (\kappa\chi')' + \kappa^2 + \kappa\chi'' - \chi\kappa\left(\kappa^2+\tau^2\right)\right) \\ -\left((1-\kappa\chi)\left((\tau\chi)' + \tau\chi'\right) + \tau\chi\left((\kappa\chi)' + \kappa\chi'\right)\right)\left(\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)' - \kappa(\kappa\chi)' - \tau(\tau\chi)' - \chi'\left(\kappa^2+\tau^2\right)\right) \\ +\left((1-\kappa\chi)\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right) + \chi'\left((\kappa\chi)' + \kappa\chi'\right)\right)\left(\kappa\tau - \chi\tau\left(\kappa^2+\tau^2\right) + \tau\chi'' + (\tau\chi)'' + (\tau\chi')'\right) \end{array}}{\begin{array}{c} \left(\chi'\left((\tau\chi)' + \tau\chi'\right) - \tau\chi\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right)\right)^2 + \left((1-\kappa\chi)\left((\tau\chi)' + \tau\chi'\right) + \tau\chi\left((\kappa\chi)' + \kappa\chi'\right)\right)^2 \\ +\left((1-\kappa\chi)\left(\kappa - \chi\left(\kappa^2+\tau^2\right) + \chi''\right) + \chi'\left((\kappa\chi)' + \kappa\chi'\right)\right)^2 \end{array}}.$$

Proof. By substituting (3.4) and (3.5) into (1.2), the proof is completed. □

Corollary 3.5. *The following relations exist between the Frenet vectors of the $N-$ pedal curve and their derivatives*

$$T_2' = \mu_2\kappa_2 N_2, \qquad N_2' = \mu_2\left(-\kappa_2 T_2 + \tau_2 B_2\right), \qquad B_3' = -\mu_2\tau_2 N_2, \tag{3.6}$$

where $\mu_2 = \|\alpha'_N\|$.

Definition 3.6. By taking the tangent and the principal normal vectors of the $N-pedal$ curve as position vectors, we define a regular curve called the T_2N_2 Smarandache curve as follows:

$$\beta_1 = \frac{T_2 + N_2}{\sqrt{2}}. \tag{3.7}$$

Theorem 3.7. *Let T_{β_1}, N_{β_1}, and B_{β_1} be the Frenet vectors of the T_2N_2 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned}
T_{\beta_1} =& \frac{-\kappa_2 T_2 + \kappa_2 N_2 + \tau_2 B_2}{\sqrt{2\kappa_2^2 + \tau_2^2}},\\
N_{\beta_1} =& B_{\beta_1} \wedge T_{\beta_1},\\
B_{\beta_1} =& \frac{-\left(\kappa_2 x_6 + \tau_2 x_5\right) T_2 + \left(\kappa_2 x_6 + \tau_2 x_4\right) N_2 - \left(\kappa_2 x_5 + \kappa_2 x_4\right) B_2}{\sqrt{\left(\kappa_2 x_6 + \tau_2 x_5\right)^2 + \left(\kappa_2 x_6 + \tau_2 x_4\right)^2 + \left(\kappa_2 x_5 + \kappa_2 x_4\right)^2}},
\end{aligned}$$

where $x_4 = -\frac{(\mu_2\kappa_2)' + \mu_2^2\kappa_2^2}{\sqrt{2}}$, $x_5 = \frac{(\mu_2\kappa_2)' - \mu_2^2\left(\kappa_2^2 + \tau_2^2\right)}{\sqrt{2}}$, $x_6 = \frac{(\mu_2\tau_2)' + \mu_2^2\kappa_2\tau_2}{\sqrt{2}}$.

Proof. The derivatives of (3.7) up to the third degree are as given below

$$\begin{aligned}
\beta_1{}' =& \frac{\mu_2\left(-\kappa_2 T_2 + \kappa_2 N_2 + \tau_2 B_2\right)}{\sqrt{2}},\\
\beta_1{}'' =& x_4 T_2 + x_5 N_2 + x_6 B_2,\\
\beta_1{}''' =& \left(x'_4 - \mu_2\kappa_2 x_2\right) T_2 + \left(x'_5 + \mu_2 x_4 - \mu_2 x_6\right) N_2 + \left(x'_6 + \mu_2\tau_2 x_5\right) B_2 .
\end{aligned} \tag{3.8}$$

By doing the necessary algebra and by taking the required norms, we have

$$\begin{aligned}
\beta_1{}' \wedge \beta_1{}'' =& -\frac{\mu_2\left(\kappa_2 x_6 + \tau_2 x_5\right)}{\sqrt{2}} T_2 + \frac{\mu_2\left(\kappa_2 x_6 + \tau_2 x_4\right)}{\sqrt{2}} N_2 - \frac{\mu_2\left(\kappa_2 x_5 + \kappa_2 x_4\right)}{\sqrt{2}} B_2,\\
\det\left(\beta_1{}', \beta_1{}'', \beta_1{}'''\right) =& \frac{\mu_2}{\sqrt{2}}\left(\begin{array}{l} \left(\kappa_2 x_6 + \tau_3 x_4\right)\left(x'_5 + \mu_2 x_4 - \mu_2 x_6\right) - \left(\kappa_2 x_6 + \tau_2 x_5\right)\left(x'_4 - \mu_2\kappa_2 x_5\right) \\ -\left(\kappa_2 x_5 + \kappa_2 x_4\right)\left(x'_6 + \mu_2\tau_2 x_5\right) \end{array}\right),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\|\beta_1{}'\| =& \frac{\mu_2}{\sqrt{2}}\sqrt{2\kappa_2^2 + \tau_2^2},\\
\|\beta_1{}' \wedge \beta_1{}''\| =& \frac{\mu_2}{\sqrt{2}}\sqrt{\left(\kappa_2 x_6 + \tau_2 x_5\right)^2 + \left(\kappa_2 x_6 + \tau_2 x_4\right)^2 + \left(\kappa_2 x_5 + \kappa_2 x_4\right)^2} .
\end{aligned} \tag{3.10}$$

Substituting the relations (3.8), (3.9), and (3.10) into (1.1) completes the proof. □

Theorem 3.8. *Let κ_{β_1} and τ_{β_1} denote the curvature and the torsion of the T_2N_2 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\begin{aligned}
\kappa_{\beta_1} =& \frac{2\sqrt{\left(\kappa_2 x_6 + \tau_2 x_5\right)^2 + \left(\kappa_2 x_6 + \tau_2 x_4\right)^2 + \left(\kappa_2 x_5 + \kappa_2 x_4\right)^2}}{\mu_2^2\left(2\kappa_2^2 + \tau_2^2\right)\sqrt{2\kappa_2^2 + \tau_2^2}},\\
\tau_{\beta_1} =& \frac{\sqrt{2}\left(\left(\kappa_2 x_6 + \tau_3 x_4\right)\left(x'_5 + \mu_2 x_4 - \mu_2 x_6\right) - \left(\kappa_2 x_6 + \tau_2 x_5\right)\left(x'_4 - \mu_2\kappa_2 x_5\right) - \left(\kappa_2 x_5 + \kappa_2 x_4\right)\left(x'_6 + \mu_2\tau_2 x_5\right)\right)}{\mu_2\left(\left(\kappa_2 x_6 + \tau_2 x_5\right)^2 + \left(\kappa_2 x_6 + \tau_2 x_4\right)^2 + \left(\kappa_2 x_5 + \kappa_2 x_4\right)^2\right)} .
\end{aligned}$$

Proof. By using (3.9) and (3.10) to substitute into (1.2), the proof is completed. □

Definition 3.9. By taking the tangent and the binormal vectors of the $N-pedal$ curve as position vectors, we define a regular curve called the T_2B_2 Smarandache curve as follows:

$$\beta_2 = \frac{T_2 + B_2}{\sqrt{2}}. \tag{3.11}$$

Theorem 3.10. *Let T_{β_2}, N_{β_2}, and B_{β_2} be the Frenet vectors of the T_2B_2 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$T_{\beta_2} = N_2, \qquad N_{\beta_2} = \frac{-\kappa_2 T_2 + \tau_2 B_2}{\sqrt{\kappa_2^2 + \tau_2^2}}, \qquad B_{\beta_2} = \frac{\tau_2 T_2 + \kappa_2 B_2}{\sqrt{\kappa_2^2 + \tau_2^2}}\cdot$$

Proof. By taking the derivatives of (3.11), we have

$$\begin{aligned}
\beta'_2 &= \frac{\mu_2(\kappa_2 - \tau_2) N_2}{\sqrt{2}},\\
\beta''_2 &= \frac{\mu_2^2(\kappa_2 - \tau_2)(-\kappa_2 T_2 + \tau_2 B_2)}{\sqrt{2}},\\
\beta'''_2 &= \frac{-\left(\mu_2^2\kappa_2^2 - \mu_2^2\kappa_2\tau_2\right)' T_2 + \left(\mu_2^2\kappa_2\tau_2 - \mu_2^2\tau_2^2\right)' B_2 + \mu_2^3\left(-\kappa_2^3 + \tau_2^3 + \kappa_2^2\tau_2 - \tau_2^2\kappa_2\right) N_2}{\sqrt{2}}.
\end{aligned} \tag{3.12}$$

Further, by taking norms and having required vector products, we have

$$\begin{aligned}
\beta'_2 \wedge \beta''_2 &= \frac{\mu_2^3(\kappa_2 - \tau_2)^2(\tau_2 T_2 + \kappa_2 B_2)}{2},\\
\det(\beta'_2, \beta''_2, \beta'''_2) &= \frac{\mu_2^5(\kappa_2 - \tau_2)^3(\kappa_2\tau_2{}' - \tau_2\kappa_2{}')}{2\sqrt{2}},
\end{aligned} \tag{3.13}$$

and

$$\left\|\beta'_2\right\| = \frac{\mu_2(\kappa_2 - \tau_2)}{\sqrt{2}}, \qquad \left\|\beta'_2 \wedge \beta''_2\right\| = \frac{\mu_2^3(\kappa_2 - \tau_2)^2\sqrt{\kappa_2^2 + \tau_2^2}}{2}. \tag{3.14}$$

If we substitute relations (3.12), (3.13), and (3.14) into (1.1), the proof is completed. □

Theorem 3.11. *Let κ_{β_2} and τ_{β_2} denote the curvature and the torsion of the T_2B_2 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\beta_2} = \frac{\sqrt{2\kappa_2^2 + 2\tau_2^2}}{(\kappa_2 - \tau_2)}, \qquad \tau_{\beta_2} = \frac{\sqrt{2}(\kappa_2\tau_2{}' - \tau_2\kappa_2{}')}{\mu_2(\kappa_2 - \tau_2)\left(\kappa_2^2 + \tau_2^2\right)}.$$

Proof. The proof is clear by the substitution of (3.13) and (3.14) into (1.2). □

Definition 3.12. By taking the principal normal and the binormal vectors of the $N-pedal$ curve as position vectors, we define a regular curve called the N_2B_2 Smarandache curve as follows:

$$\beta_3 = \frac{N_2 + B_2}{\sqrt{2}}. \tag{3.15}$$

Theorem 3.13. *Let T_{β_3}, N_{β_3}, and B_{β_3} be the Frenet vectors of the N_2B_2 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned}
T_{\beta_3} =& \frac{-\kappa_2 T_2 - \tau_2 N_2 + \tau_2 B_2}{\sqrt{\kappa_2^2 + 2\tau_2^2}}, \\
N_{\beta_3} =& B_{\beta_3} \wedge T_{\beta_3}, \\
B_{\beta_3} =& \frac{-(\tau_2 y_6 + \tau_2 y_5) T_2 + (\kappa_2 y_6 + \tau_2 y_4) N_2 + (\tau_2 y_4 - \kappa_2 y_5) B_2}{\sqrt{(\tau_2 y_6 + \tau_2 y_5)^2 + (\kappa_2 y_6 + \tau_2 y_4)^2 + (\tau_2 y_4 - \kappa_2 y_5)^2}},
\end{aligned}$$

where $y_4 = \frac{\mu_2^2 \tau_2 \kappa_2 - (\mu_2 \kappa_2)'}{\sqrt{2}}, \quad y_5 = -\frac{\mu_2^2 \left(\kappa_2^2 + \tau_2^2\right) + (\mu_2 \tau_2)'}{\sqrt{2}}, \quad y_6 = \frac{(\mu_2 \tau_2)' - \mu_2^2 \tau_2^2}{\sqrt{2}}.$

Proof. By taking the derivatives of (3.15), we have

$$\begin{aligned}
\beta'_3 =& \frac{\mu_2 (-\kappa_2 T_2 - \tau_2 N_2 + \tau_2 B_2)}{\sqrt{2}}, \\
\beta''_3 =& y_4 T_2 + y_5 N_2 + y_6 B_2, \\
\beta'''_3 =& (y'_4 - \mu_2 y_4 \kappa_2) T_2 + (y'_5 + \mu_2 y_4 \kappa_2 - \mu_2 y_6 \tau_2) N_2 + (y'_6 + \mu_2 y_5 \tau_2) B_2.
\end{aligned} \tag{3.16}$$

Moreover, we calculate the required vector products and the norms as

$$\begin{aligned}
\beta'_3 \wedge \beta''_3 =& \frac{\mu_2}{\sqrt{2}} ((-(\tau_2 y_6 + \tau_2 y_5) T_2 + (\kappa_2 y_6 + \tau_2 y_4) N_2 + (\tau_2 y_4 - \kappa_2 y_5) B_2)), \\
\det(\beta'_3, \beta''_3, \beta'''_3) =& \frac{\mu_2}{\sqrt{2}} \begin{pmatrix} (\tau_2 y_4 - \kappa_2 y_5)(y'_6 + \mu_2 y_5 \tau_2) - (\tau_2 y_6 + \tau_2 y_5)(y'_4 - \mu_2 y_5 \kappa_2) \\ + (y'_4 - \mu_2 y_5 \kappa_2)(y'_5 + \mu_2 y_4 \kappa_2 - \mu_2 y_6 \tau_2) \end{pmatrix},
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\left\|\beta'_3\right\| =& \frac{\mu_2}{\sqrt{2}} \sqrt{\kappa_2^2 + 2\tau_2^2}, \\
\left\|\beta'_3 \wedge \beta''_3\right\| =& \frac{\mu_2}{\sqrt{2}} \sqrt{(\tau_2 y_6 + \tau_2 y_5)^2 + (\kappa_2 y_6 + \tau_2 y_4)^2 + (\tau_2 y_4 - \kappa_2 y_5)^2}.
\end{aligned} \tag{3.18}$$

When substituting relations (3.16), (3.17), and (3.18) into (1.1), the proof is completed. □

Theorem 3.14. *Let κ_{β_3} and τ_{β_3} denote the curvature and the torsion of the T_2B_2 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\begin{aligned}
\kappa_{\beta_3} =& \frac{2\sqrt{(\tau_2 y_6 + \tau_2 y_5)^2 + (\kappa_2 y_6 + \tau_2 y_4)^2 + (\tau_2 y_4 - \kappa_2 y_5)^2}}{\mu_2^2 \left(\kappa_2^2 + 2\tau_2^2\right) \sqrt{\kappa_2^2 + 2\tau_2^2}}, \\
\tau_{\beta_3} =& \frac{\sqrt{2} \begin{pmatrix} (\tau_2 y_4 - \kappa_2 y_5)(y'_6 + \mu_2 y_5 \tau_2) - (\tau_2 y_6 + \tau_2 y_5)(y'_4 - \mu_2 y_5 \kappa_2) \\ + (y'_4 - \mu_2 y_5 \kappa_2)(y'_5 + \mu_2 y_4 \kappa_2 - \mu_2 y_6 \tau_2) \end{pmatrix}}{\mu_2 \left((\tau_2 y_6 + \tau_2 y_5)^2 + (\kappa_2 y_6 + \tau_2 y_4)^2 + (\tau_2 y_4 - \kappa_2 y_5)^2\right)}.
\end{aligned}$$

Proof. The proof is done upon substituting the above relations (3.17) and (3.18) into (1.2). □

Definition 3.15. By taking the tangent and principal normal and binormal vectors of the $N-pedal$ curve as position vectors, we define a regular curve called the $T_2N_2B_2$ Smarandache curve as follows:

$$\beta_4 = \frac{T_2 + N_2 + B_2}{\sqrt{3}}. \tag{3.19}$$

Theorem 3.16. *Let T_{β_4}, N_{β_4} and B_{β_4} be the Frenet vectors of the $T_2N_2B_2$ Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned}
T_{\beta_4} &= \frac{-\kappa_2 T_2 + (\kappa_2 - \tau_2) N_2 + \tau_2 B_2}{\sqrt{2\kappa_2^2 - 2\kappa_2\tau_2 + 2\tau_2^2}},\\
N_{\beta_4} &= B_{\beta_4} \wedge T_{\beta_4},\\
B_{\beta_4} &= \frac{(z_6\kappa_2 - z_6\tau_2 - \tau_2 z_5) T_2 + (\tau_2 z_4 + \kappa_2 z_6) N_2 - (\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5) B_2}{\sqrt{(z_6\kappa_2 - z_6\tau_2 - \tau_2 z_5)^2 + (\tau_2 z_4 + \kappa_2 z_6)^2 + (\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5)^2}},
\end{aligned}$$

where

$$z_4 = -\frac{(\mu_2\kappa_2)' + \mu_2^2\kappa_2(\kappa_2 - \tau_2)}{\sqrt{3}}, \qquad z_5 = \frac{(\mu_2\kappa_2 - \mu_2\tau_2)' - \mu_2^2\left(\kappa_2^2 + \tau_2^2\right)}{\sqrt{3}},$$
$$z_6 = \frac{(\mu_2\tau_2)' + \mu_2^2\tau_2(\kappa_2 - \tau_2)}{\sqrt{3}}.$$

Proof. The derivatives of (3.19) are

$$\begin{aligned}
\beta'_4 &= \frac{\mu_2(-\kappa_2 T_2 + (\kappa_2 - \tau_2) N_2 + \tau_2 B_2)}{\sqrt{3}},\\
\beta''_4 &= z_4 T_2 + z_5 N_2 + z_6 B_2,\\
\beta'''_4 &= (z'_4 - z_5\mu_2\kappa_2) T_2 + (z'_5 + z_4\mu_2\kappa_2 - z_6\mu_2\tau_2) N_2 + (z'_6 + z_5\mu_2\tau_2) B_2.
\end{aligned} \tag{3.20}$$

In addition, the required vector products and the norms are calculated as

$$\begin{aligned}
\beta'_4 \wedge \beta''_4 &= \frac{\mu_2}{\sqrt{3}}\left((z_6\kappa_2 - z_6\tau_2 - z_5\tau_2) T_2 + (\tau_2 z_4 + \kappa_2 z_6) N_2 - (\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5) B_2\right),\\
\det(\beta'_4, \beta''_4, \beta'''_4) &= \frac{\mu_2}{\sqrt{3}}\begin{pmatrix} (\kappa_2 z_6 - \tau_2 z_6 - \tau_2 z_5)(z'_4 - z_5\mu_2\kappa_2) \\ +(\tau_2 z_4 + \kappa_2 z_6)(z'_5 + z_4\mu_2\kappa_2 - z_6\mu_2\tau_2) \\ -(\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5)(z'_6 + z_5\mu_2\tau_2) \end{pmatrix},
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
\left\|\beta'_4\right\| &= \frac{\sqrt{2}\mu_2}{\sqrt{3}}\sqrt{\kappa_2^2 - \kappa_2\tau_2 + \tau_2^2},\\
\left\|\beta'_4 \wedge \beta''_4\right\| &= \frac{\mu_2}{\sqrt{3}}\sqrt{(z_6\kappa_2 - z_6\tau_2 - z_5\tau_2)^2 + (\tau_2 z_4 + \kappa_2 z_6)^2 + (\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5)^2}.
\end{aligned} \tag{3.22}$$

By substituting relations (3.20), (3.21), and (3.22) into (1.1), the proof is completed. □

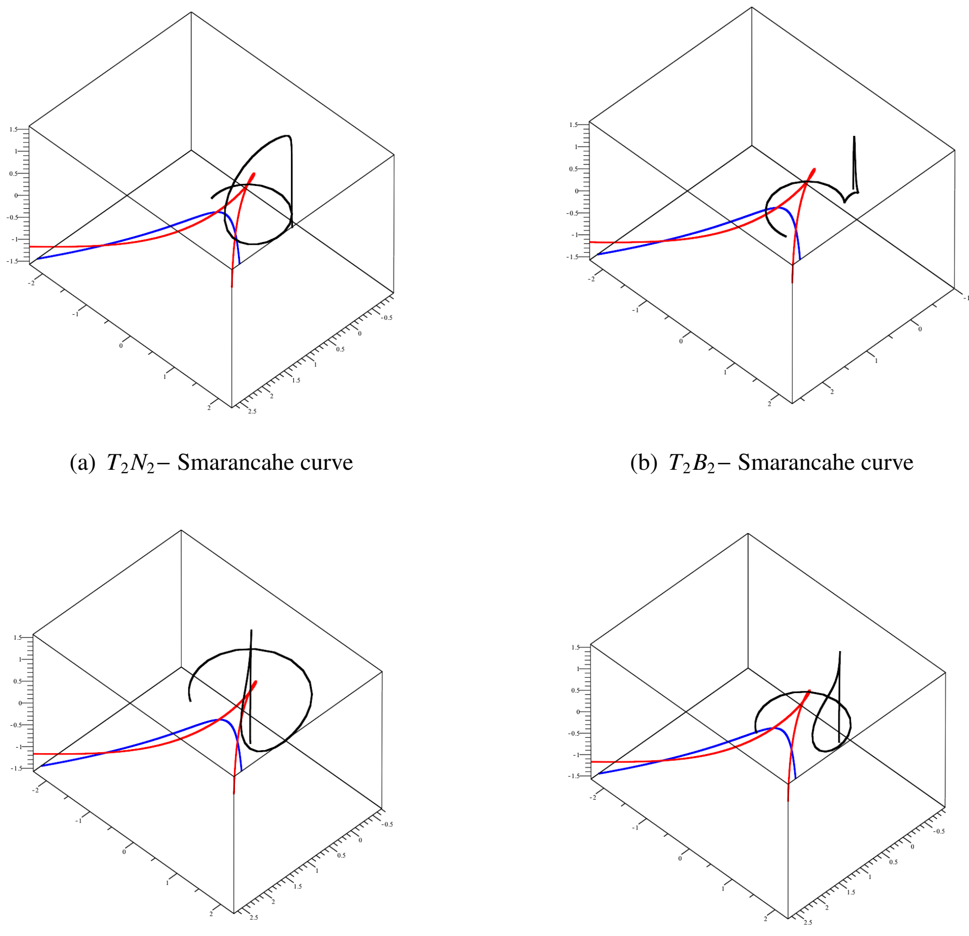
Theorem 3.17. *Let κ_{β_4} and τ_{β_4} denote the curvature and the torsion of the $T_2N_2B_2$ Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\beta_4} = \frac{\sqrt{(z_6\kappa_2 - z_6\tau_2 - z_5\tau_2)^2 + (\tau_2 z_4 + \kappa_2 z_6)^2 + (\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5)^2}}{\sqrt{2}\mu_2\left(\kappa_2^2 - \kappa_2\tau_2 + \tau_2^2\right)\sqrt{\kappa_2^2 - \kappa_2\tau_2 + \tau_2^2}},$$

$$\tau_{\beta_4} = \frac{\sqrt{3}\begin{pmatrix}(\kappa_2 z_6 - \tau_2 z_6 - \tau_2 z_5)(z'_4 - z_5\mu_2\kappa_2) + (\tau_2 z_4 + \kappa_2 z_6)(z'_5 + z_4\mu_2\kappa_2 - z_6\mu_2\tau_2) \\ -(\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5)(z'_6 + z_5\mu_2\tau_2)\end{pmatrix}}{\mu_2\left((z_6\kappa_2 - z_6\tau_2 - z_5\tau_2)^2 + (\tau_2 z_4 + \kappa_2 z_6)^2 + (\kappa_2 z_4 - \tau_2 z_4 + \kappa_2 z_5)^2\right)}.$$

Proof. The proof is done upon substituting the above relations (3.21) and (3.22) into (1.2). □

By recalling Example 2.18, Smarandache curves of the $N - pedal$ curve according to the origin $O(0, 0, 0)$ are illustrated in Figure 4.



(a) T_2N_2- Smarancahe curve (b) T_2B_2- Smarancahe curve

(c) N_2B_2- Smarancahe curve (d) $T_2N_2B_2-$ Smarancahe curve

Figure 4. Smarandache curves (black) of the $N - pedal$ curve (red) of the curve $\gamma(t)$ (blue) according to the origin $O(0, 0, 0)$ where $t \in [-\pi, \pi]$.

4. B-Pedal curve and Smarandache curves of the B-Pedal curve

Definition 4.1. Let B be the binormal vector of a given regular curve α in $\mathbb{E}^3$. The geometric locus of the perpendicular projection of points onto a binormal vector from a point $P \in \mathbb{E}^3$ that is not on the curve is called the $B-$ *pedal* curve of the curve α according to P.

Theorem 4.2. *The equation of the $B-$ pedal curve of a given regular curve α is as follows:*

$$\alpha_B(t) = \alpha(t) + \langle P - \alpha(t), B(t)\rangle B(t). \tag{4.1}$$

Proof. Let P' be a perpendicular projection point onto the principal normal vector from the point P that is not on the curve α. The perpendicular projection vector $\overrightarrow{\alpha P'}$ is calculated by the following formula:

$$\overrightarrow{\alpha P'} = \frac{\langle \alpha P, \alpha' \wedge \alpha''\rangle}{\|\alpha' \wedge \alpha''\|^2}\overrightarrow{\alpha' \wedge \alpha''}.$$

Next, let $\alpha_B(t)$ be the geometric location of the point P'. According to this, we have following relations:

$$\begin{aligned}\overrightarrow{\alpha P} = \overrightarrow{\alpha P'} + \overrightarrow{P'P} \Rightarrow \overrightarrow{\alpha P} &= \frac{\langle \alpha P, \alpha' \wedge \alpha''\rangle}{\|\alpha' \wedge \alpha''\|^2}\cdot\overrightarrow{\alpha' \wedge \alpha''} + \overrightarrow{P'P}\\ \Rightarrow P' &= \alpha + \frac{\langle \alpha P, \alpha' \wedge \alpha''\rangle}{\|\alpha' \wedge \alpha''\|^2}\cdot\overrightarrow{\alpha' \wedge \alpha''}\\ \Rightarrow \alpha_B &= \alpha + \left\langle \alpha P, \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}\right\rangle \cdot \frac{\overrightarrow{\alpha' \wedge \alpha''}}{\|\alpha' \wedge \alpha''\|}.\end{aligned}$$

When the principal normal vector from the relation (1.1) is taken into consideration, the proof of the theorem is completed. □

Further, if $\xi(t) = \langle P - \alpha(t), B(t)\rangle$, then the relation (4.1) can be written by following:

$$\alpha_B(t) = \alpha(t) + \xi(t) B(t), \tag{4.2}$$

and if the point P is specifically taken as origin, then we have $\xi(t) = -\langle \alpha(t), B(t)\rangle$.

Theorem 4.3. *Let α_B be the $B-$ pedal curve of the given curve α with unit speed, and $\{T_3,\ N_3,\ B_3\}$ denotes the Frenet vectors of the $B-$ pedal curve of α. Then, among the Frenet vectors, the following relations exist:*

$$\begin{aligned}T_3 &= \omega_3 T - \omega_3\tau N + \omega_3\xi' B,\\ N_3 &= B_3 \wedge T_3,\\ B_3 &= -\eta_3\left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)T + \eta_3\left(\kappa\tau\xi' - \xi'' + \tau^2\right)N + \eta_3\left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right)B,\end{aligned}$$

where

$$\begin{aligned}\omega_3 &= \frac{1}{\sqrt{1 + \tau^2 + \xi'^2}},\\ \eta_3 &= \frac{1}{\sqrt{\left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)^2 + (\kappa\tau\xi' - \xi'' + \tau^2)^2 + (\kappa - \tau' - \xi'\tau + \kappa\tau^2)^2}}.\end{aligned}$$

Proof. By taking the derivatives of (4.2), we first have

$$\begin{aligned} \alpha'_B &= T - \tau N + \xi' B, \\ \alpha''_B &= \kappa\tau T + \left(\kappa - \tau' - \xi'\tau\right) N + \left(\xi'' - \tau^2\right) B, \\ \alpha'''_B &= \left((\kappa\tau)' - \kappa\left(\kappa - \tau' - \xi'\tau\right)\right) T + \left(\kappa^2\tau - \tau^3 + \tau\xi'' + \left(\kappa - \tau' - \xi'\tau\right)'\right) N \\ &\quad + \left(\left(\xi'' - \tau^2\right)' + \kappa\tau - \tau\tau' - \xi'\tau^2\right) B. \end{aligned} \tag{4.3}$$

Further, other necessary relations are obtained as

$$\begin{aligned} \alpha'_B \wedge \alpha''_B &= -\left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right) T + \left(\kappa\tau\xi' - \xi'' + \tau^2\right) N + \left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right) B, \\ \det\left(\alpha'_B, \alpha''_B, \alpha'''_B\right) &= \left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right)\left(\left(\xi'' - \tau^2\right)' + \kappa\tau - \tau\tau' - \xi'\tau^2\right) \\ &\quad + \left(\kappa\tau\xi' - \xi'' + \tau^2\right)\left(\kappa^2\tau - \tau^3 + \tau\xi'' + \left(\kappa - \tau' - \xi'\tau\right)'\right) \\ &\quad - \left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)\left((\kappa\tau)' - \kappa^2 + \tau'\kappa + \xi'\tau\kappa\right), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \|\alpha'_B\| &= \sqrt{1 + \tau^2 + \xi'^2}, \\ \|\alpha'_B \wedge \alpha''_B\| &= \sqrt{\left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)^2 + \left(\kappa\tau\xi' - \xi'' + \tau^2\right)^2 + \left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right)^2}. \end{aligned} \tag{4.5}$$

By substituting the Eqs (4.3), (4.4), and (4.5) into the relations at (1.1), the proof is completed. □

Theorem 4.4. *Let α_B be the $B-$ pedal curve of the unit speed curve α, and let κ_3 and τ_3 denote the curvature and the torsion functions for α_B, respectively. Then, the following relations exist among the curvatures*

$$\kappa_3 = \frac{\sqrt{\left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)^2 + \left(\kappa\tau\xi' - \xi'' + \tau^2\right)^2 + \left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right)^2}}{\left(1 + \tau^2 + \xi'^2\right)\sqrt{1 + \tau^2 + \xi'^2}},$$

$$\tau_3 = \frac{\begin{array}{c}\left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right)\left(\left(\xi'' - \tau^2\right)' + \kappa\tau - \tau\tau' - \xi'\tau^2\right) \\ + \left(\kappa\tau\xi' - \xi'' + \tau^2\right)\left(\kappa^2\tau - \tau^3 + \tau\xi'' + \left(\kappa - \tau' - \xi'\tau\right)'\right) \\ - \left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)\left((\kappa\tau)' - \kappa^2 + \tau'\kappa + \xi'\tau\kappa\right)\end{array}}{\left(\tau\xi'' - \tau^3 + \xi'\kappa - \xi'\tau' - \xi'^2\tau\right)^2 + \left(\kappa\tau\xi' - \xi'' + \tau^2\right)^2 + \left(\kappa - \tau' - \xi'\tau + \kappa\tau^2\right)^2}.$$

Proof. By substituting (4.4) and (4.5) into (1.2), the proof is completed. □

Corollary 4.5. *The following relations exist between the Frenet vectors of the $B-$ pedal curve and their derivatives:*

$$T'_3 = \mu_3\kappa_3 N_3, \qquad N'_3 = \mu_3\left(-\kappa_3 T_3 + \tau_3 B_3\right), \qquad B'_3 = -\mu_3\tau_3 N_3, \tag{4.6}$$

where $\mu_3 = \|\alpha'_B\|$.

Definition 4.6. By taking the tangent and the principal normal vectors of the $B-pedal$ curve as position vectors, we define a regular curve called the T_3N_3 Smarandache curve as follows:

$$\delta_1 = \frac{T_3 + N_3}{\sqrt{2}}. \tag{4.7}$$

Theorem 4.7. *Let T_{δ_1}, N_{δ_1}, and B_{δ_1} be the Frenet vectors of the T_3N_3 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned} T_{\delta_1} &= \frac{-\kappa_3 T_3 + \kappa_3 N_3 + \tau_3 B_3}{\sqrt{2\kappa_3^2 + \tau_3^2}}, \\ N_{\delta_1} &= B_{\delta_1} \wedge T_{\delta_1}, \\ B_{\delta_1} &= \frac{-\left(\kappa_3 x_9 + \tau_3 x_8\right) T_3 + \left(\kappa_3 x_9 + \tau_3 x_7\right) N_3 - \left(\kappa_3 x_8 + \kappa_3 x_7\right) B_3}{\sqrt{\left(\kappa_3 x_9 + \tau_3 x_8\right)^2 + \left(\kappa_3 x_9 + \tau_3 x_7\right)^2 + \left(\kappa_3 x_8 + \kappa_3 x_7\right)^2}}, \end{aligned}$$

where $x_7 = -\dfrac{(\mu_3\kappa_3)' + \mu_3^2\kappa_3^2}{\sqrt{2}}, \quad x_8 = \dfrac{(\mu_3\kappa_3)' - \mu_3^2\left(\kappa_3^2 + \tau_3^2\right)}{\sqrt{2}}, \quad x_9 = \dfrac{(\mu_3\tau_3)' + \mu_3^2\kappa_3\tau_3}{\sqrt{2}}.$

Proof. The derivatives of (4.7) up to the third degree are as given below.

$$\begin{aligned} \delta_1' &= \frac{\mu_3\left(-\kappa_3 T_3 + \kappa_3 N_3 + \tau_3 B_3\right)}{\sqrt{2}}, \\ \delta_1'' &= x_7 T_3 + x_8 N_3 + x_9 B_3, \\ \delta_1''' &= \left(x'_7 - \mu_3\kappa_3 x_8\right) T_3 + \left(x'_8 + \mu_3 x_7 - \mu_3 x_9\right) N_3 + \left(x'_9 + \mu_3\tau_3 x_8\right) B_3. \end{aligned} \tag{4.8}$$

By doing the necessary algebra and by taking the required norms, we have

$$\begin{aligned} \delta_1' \wedge \delta_1'' &= \frac{-1}{\sqrt{2}}\left(\mu_3\left(\kappa_3 x_9 + \tau_3 x_8\right) T_3 - \mu_3\left(\kappa_3 x_9 + \tau_3 x_7\right) N_3 + \mu_3\left(\kappa_3 x_8 + \kappa_3 x_7\right) B_3\right), \\ \det\left(\delta_1', \delta_1'', \delta_1'''\right) &= \frac{\mu_3}{\sqrt{2}}\left(\begin{array}{l} \left(\kappa_3 x_9 + \tau_3 x_7\right)\left(x'_8 + \mu_3 x_7 - \mu_3 x_9\right) \\ -\left(\kappa_3 x_9 + \tau_3 x_8\right)\left(x'_7 - \mu_3\kappa_3 x_8\right) - \left(\kappa_3 x_8 + \kappa_3 x_7\right)\left(x'_9 + \mu_3\tau_3 x_8\right) \end{array}\right), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \|\delta_1'\| &= \frac{\mu_3}{\sqrt{2}}\sqrt{2\kappa_3^2 + \tau_3^2}, \\ \|\delta_1' \wedge \delta_1''\| &= \frac{\mu_3}{\sqrt{2}}\sqrt{\left(\kappa_3 x_6 + \tau_3 x_5\right)^2 + \left(\kappa_3 x_6 + \tau_3 x_4\right)^2 + \left(\kappa_3 x_5 + \kappa_3 x_4\right)^2}. \end{aligned} \tag{4.10}$$

Substituting the relations (4.8), (4.9), and (4.10) into (1.1) completes the proof. □

Theorem 4.8. *Let κ_{δ_1} and τ_{δ_1} denote the curvature and the torsion of the T_3N_3 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\begin{aligned} \kappa_{\delta_1} &= \frac{2\sqrt{\left(\kappa_3 x_9 + \tau_3 x_8\right)^2 + \left(\kappa_3 x_9 + \tau_3 x_7\right)^2 + \left(\kappa_3 x_8 + \kappa_3 x_7\right)^2}}{\mu_3^2\left(2\kappa_3^2 + \tau_3^2\right)\sqrt{2\kappa_3^2 + \tau_3^2}}, \\ \tau_{\delta_1} &= \frac{\sqrt{2}\left(\begin{array}{l} \left(\kappa_3 x_9 + \tau_3 x_7\right)\left(x'_8 + \mu_3 x_7 - \mu_3 x_9\right) - \left(\kappa_3 x_9 + \tau_3 x_8\right)\left(x'_7 - \mu_3\kappa_3 x_8\right) \\ -\left(\kappa_3 x_8 + \kappa_3 x_7\right)\left(x'_9 + \mu_3\tau_3 x_8\right) \end{array}\right)}{\mu_3(\kappa_3 x_9 + \tau_3 x_8)^2 + \mu_3(\kappa_3 x_9 + \tau_3 x_7)^2 + \mu_3(\kappa_3 x_8 + \kappa_3 x_7)^2}. \end{aligned}$$

Proof. By using (4.9) and (4.10) to substitute into (1.2), the proof is completed. □

Definition 4.9. By taking the tangent and the binormal vectors of the $B-pedal$ curve as position vectors, we define a regular curve called the T_3B_3 Smarandache curve as follows:

$$\delta_2 = \frac{T_3 + B_3}{\sqrt{2}}. \tag{4.11}$$

Theorem 4.10. *Let T_{δ_2}, N_{δ_2}, and B_{δ_2} be the Frenet vectors of the T_3B_3 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$T_{\delta_2} = N_3, \qquad N_{\delta_2} = \frac{-\kappa_3 T_3 + \tau_3 B_3}{\sqrt{\kappa_3^2 + \tau_3^2}}, \qquad B_{\delta_2} = \frac{\tau_3 T_3 + \kappa_3 B_3}{\sqrt{\kappa_3^2 + \tau_3^2}}.$$

Proof. By taking the derivatives of (4.11), we have

$$\begin{aligned}
\delta'_2 &= \frac{\mu_3 (\kappa_3 - \tau_3) N_3}{\sqrt{2}}, \\
\delta''_2 &= \frac{\mu_3^2 (\kappa_3 - \tau_3)(-\kappa_3 T_3 + \tau_3 B_3)}{\sqrt{2}}, \\
\delta'''_2 &= \frac{\begin{array}{l} -\left(\mu_3^2\kappa_3^2 - \mu_3^2\kappa_3\tau_3\right)' T_3 + \left(-\mu_3^3\kappa_3^3 + \mu_3^3\tau_3^3 + \mu_3^3\kappa_3^2\tau_3 - \mu_3^3\tau_3^2\kappa_3\right) N_3 \\ +\left(\mu_3^2\kappa_3\tau_3 - \mu_3^2\tau_3^2\right)' B_3 \end{array}}{\sqrt{2}}.
\end{aligned} \tag{4.12}$$

Further, by taking norms and having required vector products, we have

$$\begin{aligned}
\delta'_2 \wedge \delta''_2 &= \frac{\mu_3^3(\kappa_3 - \tau_3)^2 (\tau_3 T_3 + \kappa_3 B_3)}{2}, \\
\det(\delta'_2, \delta''_2, \delta'''_2) &= \frac{\mu_3^5(\kappa_3 - \tau_3)^3 (\kappa_3\tau_3' - \tau_3\kappa_3')}{2\sqrt{2}},
\end{aligned} \tag{4.13}$$

and

$$\|\delta'_2\| = \frac{\mu_3 (\kappa_3 - \tau_3)}{\sqrt{2}}, \qquad \|\delta'_2 \wedge \delta''_2\| = \frac{\mu_3^3(\kappa_3 - \tau_3)^2 \sqrt{\kappa_3^2 + \tau_3^2}}{2}. \tag{4.14}$$

If we substitute relations (4.12), (4.13), and (4.14) into (1.1), the proof is completed. □

Theorem 4.11. *Let κ_{δ_2} and τ_{δ_2} denote the curvature and the torsion of the T_3B_3 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\delta_2} = \frac{\sqrt{2\kappa_3^2 + 2\tau_3^2}}{(\kappa_3 - \tau_3)}, \qquad \tau_{\delta_2} = \frac{\sqrt{2}(\kappa_3\tau_3' - \tau_3\kappa_3')}{\mu_3(\kappa_3 - \tau_3)\left(\kappa_3^2 + \tau_3^2\right)}.$$

Proof. The proof is clear by the substitution of (4.13) and (4.14) into (1.2). □

Definition 4.12. By taking the principal normal and the binormal vectors of the $B-pedal$ curve as position vectors, we define a regular curve called the N_3B_3 Smarandache curve as follows:

$$\delta_3 = \frac{N_3 + B_3}{\sqrt{2}}. \tag{4.15}$$

Theorem 4.13. *Let T_{δ_3}, N_{δ_3}, and B_{δ_3} be the Frenet vectors of the N_3B_3 Smarandache curve. The relations among Frenet vectors are given as follows:*

$$\begin{aligned} T_{\delta_3} =& \frac{-\kappa_3 T_3 - \tau_3 N_3 + \tau_3 B_3}{\sqrt{\kappa_3^2 + 2\tau_3^2}}, \\ N_{\delta_3} =& B_{\delta_3} \wedge T_{\delta_3}, \\ B_{\delta_3} =& \frac{-(\tau_3 y_9 + \tau_3 y_8) T_3 + (\kappa_3 y_9 + \tau_3 y_7) N_3 + (\tau_3 y_7 - \kappa_3 y_8) B_3}{\sqrt{(\tau_3 y_9 + \tau_3 y_8)^2 + (\kappa_3 y_9 + \tau_3 y_7)^2 + (\tau_3 y_7 - \kappa_3 y_8)^2}}, \end{aligned}$$

where $y_7 = \dfrac{\mu_3^2 \tau_3 \kappa_3 - (\mu_3\kappa_3)'}{\sqrt{2}}$, $\quad y_8 = -\dfrac{\mu_3^2\left(\kappa_3^2 + \tau_3^2\right) + (\mu_3\tau_3)'}{\sqrt{2}}$, $\quad y_9 = \dfrac{(\mu_3\tau_3)' - \mu_3^2\tau_3^2}{\sqrt{2}}$.

Proof. By taking the derivatives of (4.15), we have

$$\begin{aligned} \delta'_3 =& \frac{\mu_3 (-\kappa_3 T_3 - \tau_3 N_3 + \tau_3 B_3)}{\sqrt{2}}, \\ \delta''_3 =& y_7 T_3 + y_8 N_3 + y_9 B_3, \\ \delta'''_3 =& (y'_7 - \mu_3 y_8 \kappa_3) T_3 + (y'_8 + \mu_3 y_7 \kappa_3 - \mu_3 y_9 \tau_3) N_3 + (y'_9 + \mu_3 y_8 \tau_3) B_3. \end{aligned} \tag{4.16}$$

Moreover, we calculate the required vector products and the norms as

$$\begin{aligned} \delta'_3 \wedge \delta''_3 =& \frac{\mu_3}{\sqrt{2}} \left(\left(-(\tau_3 y_9 + \tau_3 y_8) T_3 + (\kappa_3 y_9 + \tau_3 y_7) N_3 + (\tau_3 y_7 - \kappa_3 y_2) B_3 \right) \right), \\ \det(\delta'_3, \delta''_3, \delta'''_3) =& \frac{\mu_3}{\sqrt{2}} \begin{pmatrix} (\tau_3 y_7 - \kappa_3 y_8)(y'_9 + \mu_3 y_8 \tau_3) - (\tau_3 y_9 + \tau_3 y_8)(y'_7 - \mu_3 y_8 \kappa_3) \\ + (y'_7 - \mu_3 y_8 \kappa_3)(y'_8 + \mu_3 y_7 \kappa_3 - \mu_3 y_9 \tau_3) \end{pmatrix}, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \|\delta'_3\| =& \frac{\mu_3}{\sqrt{2}} \sqrt{\kappa_3^2 + 2\tau_3^2}, \\ \|\delta'_3 \wedge \delta''_3\| =& \frac{\mu_3}{\sqrt{2}} \sqrt{(\tau_3 y_9 + \tau_3 y_8)^2 + (\kappa_3 y_9 + \tau_3 y_7)^2 + (\tau_3 y_7 - \kappa_3 y_8)^2}. \end{aligned} \tag{4.18}$$

When substituting relations (4.16), (4.17), and (4.18) into (1.1), the proof is completed. □

Theorem 4.14. *Let κ_{δ_3} and τ_{δ_3} denote the curvature and the torsion of the T_3B_3 Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\begin{aligned} \kappa_{\delta_3} =& \frac{2\sqrt{(\tau_3 y_9 + \tau_3 y_8)^2 + (\kappa_3 y_9 + \tau_3 y_7)^2 + (\tau_3 y_7 - \kappa_3 y_8)^2}}{\mu_3^2 \left(\kappa_3^2 + 2\tau_3^2\right) \sqrt{\kappa_3^2 + 2\tau_1^2}}, \\ \tau_{\delta_3} =& \frac{\sqrt{2} \begin{pmatrix} (\tau_3 y_7 - \kappa_3 y_8)(y'_9 + \mu_3 y_8 \tau_3) - (\tau_3 y_9 + \tau_3 y_8)(y'_7 - \mu_3 y_8 \kappa_3) \\ + (y'_7 - \mu_3 y_8 \kappa_3)(y'_8 + \mu_3 y_7 \kappa_3 - \mu_3 y_9 \tau_3) \end{pmatrix}}{\mu_3 \left((\tau_3 y_9 + \tau_3 y_8)^2 + (\kappa_3 y_9 + \tau_3 y_7)^2 + (\tau_3 y_7 - \kappa_3 y_8)^2 \right)}. \end{aligned}$$

Proof. The proof is done upon substituting the above relations (4.17) and (4.18) into (1.2). □

Definition 4.15. By taking the tangent and principal normal and binormal vectors of the $B-pedal$ curve as position vectors, we define a regular curve called the $T_3N_3B_3$ Smarandache curve as follows:

$$\delta_4 = \frac{T_3 + N_3 + B_3}{\sqrt{3}}. \tag{4.19}$$

Theorem 4.16. *Let T_{δ_4}, N_{δ_4}, and B_{δ_4} be the Frenet vectors of the $T_3N_3B_3$ Smarandache curve. The relations among Frenet vectors are given as follows:*

$$T_{\delta_4} = \frac{-\kappa_3 T_3 + (\kappa_3 - \tau_3) N_3 + \tau_3 B_3}{\sqrt{2\kappa_3^2 - 2\kappa_3\tau_3 + 2\tau_3^2}}, \quad N_{\delta_4} = B_{\delta_4} \wedge T_{\delta_4},$$
$$B_{\delta_4} = \frac{(z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8) T_3 + (\tau_3 z_7 + \kappa_3 z_9) N_3 - (\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8) B_3}{\sqrt{(z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8)^2 + (\tau_3 z_7 + \kappa_3 z_9)^2 + (\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8)^2}},$$

where

$$z_7 = -\frac{(\mu_3\kappa_3)' + \mu_3^2\kappa_3(\kappa_3 - \tau_3)}{\sqrt{3}}, \qquad z_8 = \frac{(\mu_3\kappa_3 - \mu_3\tau_3)' - \mu_3^2\left(\kappa_3^2 + \tau_3^2\right)}{\sqrt{3}}, \qquad z_9 = \frac{(\mu_3\tau_1)' + \mu_3^2\tau_3(\kappa_3 - \tau_3)}{\sqrt{3}}.$$

Proof. The derivatives of (4.19) are

$$\begin{aligned} \delta'_4 &= \frac{\mu_3(-\kappa_3 T_3 + (\kappa_3 - \tau_3) N_3 + \tau_3 B_3)}{\sqrt{3}}, \quad \delta''_4 = z_7 T_3 + z_8 N_3 + z_9 B_3, \\ \delta'''_4 &= (z'_7 - z_8\mu_3\kappa_3) T_3 + (z'_8 + z_7\mu_3\kappa_3 - z_9\mu_3\tau_3) N_3 + (z'_9 + z_8\mu_3\tau_3) B_3. \end{aligned} \tag{4.20}$$

In addition, the required vector products and the norms are calculated as

$$\begin{aligned} \delta'_4 \wedge \delta''_4 &= \frac{\mu_3}{\sqrt{3}}\left((z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8) T_3 + (\tau_3 z_7 + \kappa_3 z_9) N_3 - (\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8) B_3\right), \\ \det(\delta'_4, \delta''_4, \delta'''_4) &= \frac{\mu_3}{\sqrt{3}}\left(\begin{array}{l} (z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8)(z'_7 - z_8\mu_3\kappa_3) \\ +(\tau_3 z_7 + \kappa_3 z_9)(z'_8 + z_7\mu_3\kappa_3 - z_9\mu_3\tau_3) \\ -(\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8)(z'_9 + z_8\mu_3\tau_3) \end{array}\right), \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \|\delta'_4\| &= \frac{\sqrt{2}\mu_3}{\sqrt{3}}\sqrt{\kappa_3^2 - \kappa_3\tau_3 + \tau_3^2}, \\ \|\delta'_4 \wedge \delta''_4\| &= \frac{\mu_3}{\sqrt{3}}\sqrt{(z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8)^2 + (\tau_3 z_7 + \kappa_3 z_9)^2 + (\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8)^2}. \end{aligned} \tag{4.22}$$

By substituting relations (4.20), (4.21), and (4.22) into (1.1), the proof is completed. □

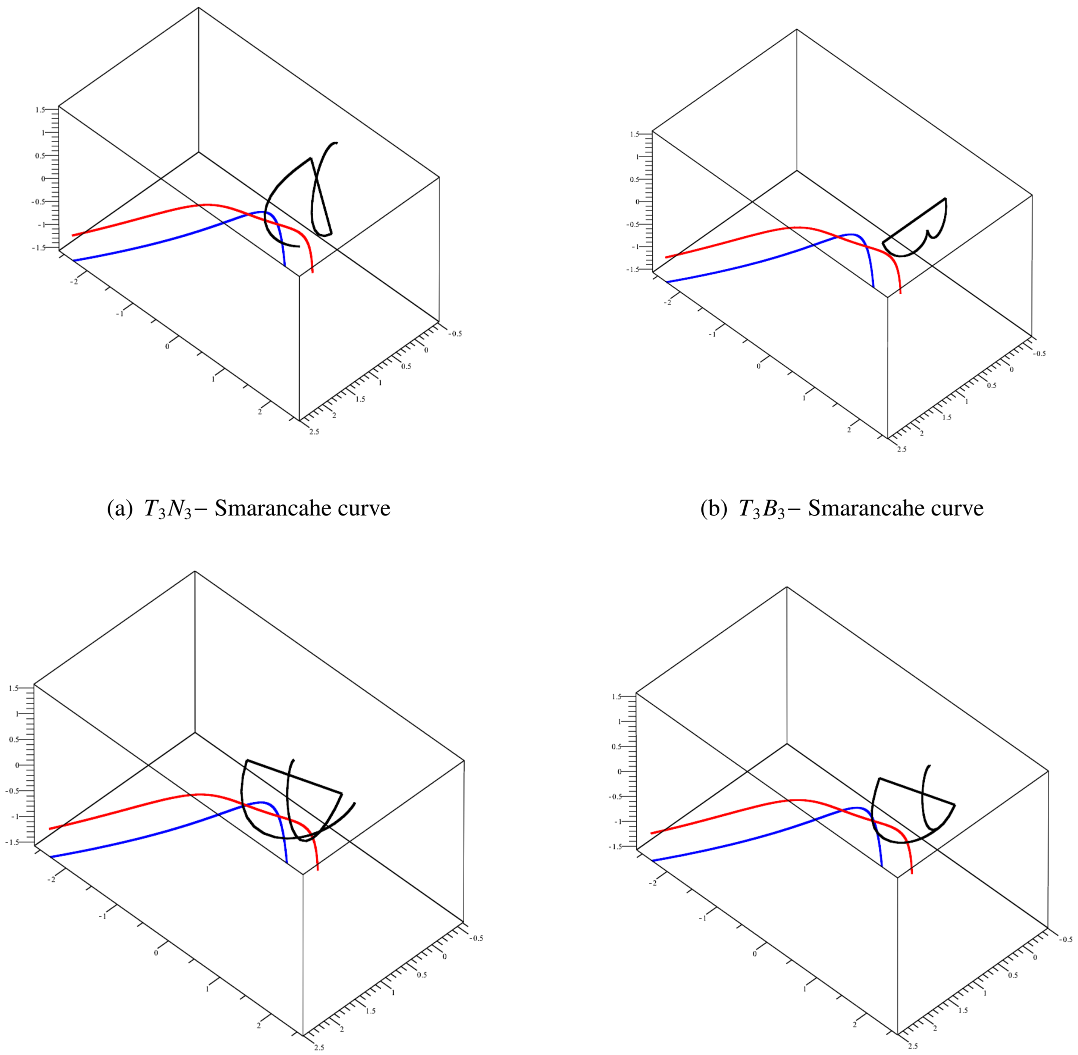
Theorem 4.17. *Let κ_{δ_4} and τ_{δ_4} denote the curvature and the torsion of the $T_3N_3B_3$ Smarandache curve, respectively. The following relations exist among the curvatures as*

$$\kappa_{\delta_4} = \frac{\sqrt{(z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8)^2 + (\tau_3 z_7 + \kappa_3 z_9)^2 + (\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8)^2}}{\sqrt{2}\mu_3\left(\kappa_3^2 - \kappa_3\tau_3 + \tau_3^2\right)\sqrt{\kappa_3^2 - \kappa_3\tau_3 + \tau_3^2}},$$

$$\tau_{\delta_4} = \frac{\sqrt{3}\begin{pmatrix} (z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8)(z'_7 - z_8\mu_3\kappa_3) + (\tau_3 z_7 + \kappa_3 z_9)(z'_8 + z_7\mu_3\kappa_3 - z_9\mu_3\tau_3) \\ -(\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8)(z'_9 + z_8\mu_3\tau_3) \end{pmatrix}}{\mu_3\left((z_9\kappa_3 - z_9\tau_3 - \tau_3 z_8)^2 + (\tau_3 z_7 + \kappa_3 z_9)^2 + (\kappa_3 z_7 - \tau_3 z_7 + \kappa_3 z_8)^2\right)}.$$

Proof. The proof is done upon substituting the above relations (4.21) and (4.22) into (1.2). □

By recalling Example 2.18, Smarandache curves of the $B-pedal$ curve according to the origin $O(0,0,0)$ are illustrated in Figure 5.



(a) T_3N_3- Smarancahe curve

(b) T_3B_3- Smarancahe curve

(c) N_3B_3- Smarancahe curve

(d) $T_3N_3B_3-$ Smarancahe curve

Figure 5. Smarandache curves (black) of the $B-pedal$ curve (red) of the curve $\gamma(t)$ (blue) according to the origin $O(0,0,0)$ where $t \in [-\pi, \pi]$.

5. Conclusions

In this study, first we obtain the pedal curves drawn by the geometric locus of the perpendicular projection of points onto a tangent, principal normal, and binormal vectors of a space curve from the origin, and their Frenet vectors, curvature, and torsion functions are calculated. After these calculations, three of the pedal curves (T-pedal, N-pedal, B-pedal curves) are obtained. Second, we get the Smarandache curves defined by taking Frenet elements of each pedal curve as the position vectors. So, we obtain twelve new curves. Therefore, a set of new curves is contributed to the literature of the theory of curves. By taking a different point from the origin, numerous sequences of different new curves can be found to add more curves to the area.

Author contributions

Süleyman Şenyurt: Methodology, Writing–Original draft preparation, Supervision, Formal analysis, Resources; Filiz Ertem Kaya: Investigation, Conceptualization, Validation, Writing, Reviewing, Editing; Davut Canlı: Investigation, Formal analysis, Software, Validation, Visualization. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest in this paper.

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