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Research article

Dynamic analysis and optimal control of rumor propagation models considering different education levels and hesitation mechanisms

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Abstract: The spread of rumors has an important impact on the production and life of human society. Moreover, in the process of rumor propagation, individuals with different educational levels show different degrees of trust and ability to spread rumors. Therefore, a new rumor propagation model was established, which considers the influence of education level on rumor propagation. Initially, the basic reproduction number of the model was calculated. Then, we analyzed the existence and stability of the rumor equilibrium point. Next, based on the principle of Pontryagin's maximum value, we obtained a control strategy, which effectively reduced the spread of rumors. Numerical simulations verified the results of theoretical analysis. The results showed that the higher the education level of the population, the slower the spread of rumors to a certain extent, but it could not prevent the spread of rumors. In addition, through the support of the government and the propaganda of the official media, strengthening education can improve people's education level to a certain extent, and then minimize the speed of rumor propagation.

Keywords: rumor propagation model; basic reproduction number; education level; stability; optimal control
Mathematics Subject Classification: 34D20, 49J15, 37D35

1. Introduction

Rumors are unconfirmed information that is spread through public or private channels [1]. Since ancient times, rumors have had a significant impact on the development of the country and society. Now, with the development of society, rumors can spread rapidly through platforms such as WeChat, QQ and Microblogging. This may lead to economic losses and psychological panic. Therefore, in today's society, it is of great practical significance to study the spread of rumors.

Due to the resemblance between the rumor propagation and infectious disease propagation models, we frequently apply the infectious disease propagation principle when examining the former. Initially, in 1965, Daley and Kendall assigned the rumor propagation predicament to a mathematical model, which they named the D-K model [2]. Subsequently, in 1973, Maki and Thomson refined the propagation law using the D-K model and introduced the M-T model [3].

After people's in-depth exploration and research on rumor propagation, many researchers have incorporated the forgetting mechanism [4–6], hesitation mechanism [7, 8] and counterattack mechanism [9], which is considered by early psychologists into the establishment of the rumor propagation model. Gu et al. [4] proposed a method to study the transmission process by considering factors of forgetting mechanism and memory mechanism. They used basic mathematical functions to characterize the evolution of forgetting and memory over time, and found that the saturation of rumor propagation is influenced by the "forgetting and remembering" mechanism, and may even lead to the end of rumor propagation under certain conditions. Zhao et al. [5] introduced the memory mechanism and forgetting mechanism into a homogeneous network and established a SHIR model to study the effects of propagation rate, forgetting rate and network average degree on rumor propagation. They found that the forgetting and memory mechanisms that occur in hibernators delay the end of the rumor and somewhat reduce the maximum rumor impact. Wang et al. [6] proposed a new SIR model, which mainly studied the influence of forget-recall mechanism and loss-benefit mechanism on rumor propagation in complex networks, considering that people would forget and lose interest in rumors during the propagation process. Xia et al. [7] added a new factor of hesitation mechanism to the classical SIR model and proposed a SEIR model. Hong et al. [8] added the factor of the actual situation of the investor network to the SEIR model, and then proposed a SE_2IR rumor propagation model with hesitation mechanism. Zan et al. [9] added a counterattack mechanism to the classical SIR model, and then proposed two new models: SCIR model and the upgraded SCIR model.

Currently, researchers primarily focus on various aspects including cost control [10], feedback mechanisms [11], multi-channel network interaction [12], and competitive information propagation [13]. Jain et al. [10] studied the optimal control of rumor propagation in uniformly mixed populations by introducing the thinker's influence delay. They established an SEI model. It has been concluded that the delay in influence of thinkers ought not to surpass the threshold for controlling the instability of social network systems. The system can be stabilized when the thinker has more time to react, and when experts in the media and interpersonal organizations disseminate information at the optimal speed. The optimal control system is capable of effectively suppressing the propagation of anti-news during emergency situations, at a minimized cost. Yao et al. [11] employed an epidemiological theory-based diffusion model to study the propagation of rumors amongst investors. The model incorporated feedback mechanisms and time-lag factors. The research shows that increasing the intensity of information supervision and the proportion coefficient related to the infected population in the short time lag is helpful to control the spread of rumors more effectively. Zhang et al. [12] proposed a new interaction model for rumor propagation and behavioural propagation in multiplexed networks-the $S_1I_1R_1 - S_2I_2R_2$ model. The results of their analysis and calculations indicated that the final size of the rumor spread was larger than the size of the behavioural spread. Nevertheless, coupled intensification results in the opposite range of final size changes. This shows that there is a complex interaction between rumor and behavior propagation in the multiplex network. Yang et al. [13] proposed a simple linear threshold model with competing generalised versions that takes into account the competition between rumor and truth in the same network. They employ three distinct heuristics to arrive at our conclusions: Both the diffusion dynamics-based approach (ContrId) and the centrality based approach (PageRank) possess the same computational complexity, with said complexity increasing linearly with the number of nodes n, rendering them scalable to large networks. Upon introducing the proximity effect, ContrId performs equally to MinGreedy, but at a significantly faster pace. These findings offer significant insights into the investigation of the rivalry between the propagation of rumors and the propagation of truth.

When discussing the influencing factors of rumor propagation, in addition to the factors discussed previously, some scholars such as Komi Afassinou [14] introduced the two key variables of population education rate and forgetting mechanism based on the classical SIR model. They demonstrated that the greater the proportion of educated individuals in a group, the smaller the eventual scale of rumor propagation. This highlights the role of education in the process of rumor propagation. Hu et al. [15] examined the impact of alterations in the proportion of wise individuals in a population on rumor propagation. They demonstrated that an increase in the proportion of wise individuals in a population has a systematic effect in resolving the issue of rumor propagation. Furthermore, it is important to consider the role of social media in rumor propagation. Furthermore, Li et al. [16] developed an enhanced rumor propagation model, with a particular focus on the impact of knowledge education and intervention strategies on the reduction of rumor propagation. Their results show that strengthening rumor recognition education and timely refutation of false information are very effective in controlling rumor propagation. In the above studies, Afassinou and Li et al. both involved the influence of education level on rumor propagation, but their classification of education level was relatively simple, only divided into the educated and the uneducated. Furthermore, the Hu et al. study only looked at the influence of the proportion of wise men on rumor propagation and did not classify the population. Therefore, based on the existing literature, we provide a more comprehensive analysis of the mechanism of rumor propagation. This study comprehensively considers several factors such as education level, hesitation mechanism and forgetting mechanism, and divides education level into three categories, namely less educated, more educated and uneducated categories. This comprehensive classification enables a more nuanced understanding of the role of groups with varying educational backgrounds in the process of rumor propagation. This, in turn, facilitates a more comprehensive understanding of the mechanisms underlying rumor propagation.

The rest of this paper is organized as follows: In Section 2, a rumor propagation model is constructed. In Section 3, the basic reproduction number and the rumor equilibrium point of the model are calculated, and the stability of the rumor equilibrium point is analyzed. In Section 4, the optimal control strategy is proposed using the Pontryagin's maximum principle. In Section 5, the feasibility of the conditions proposed above is verified by numerical simulation. In Section 6, we draw the conclusion of the paper.

2. Model building

We discuss the influence of individuals with different levels of education on rumor propagation. First of all, we divide the total population into three categories according to the level of education: the first category is the highly educated individuals; the second category is the less educated individuals; the third category is the uneducated individual. Then, according to the classical rumor propagation model, we use $N(t)$ to represent the change in the total number of people at any time t. Then, using the method of the chamber model, we divided the population into five categories: (1) Rumor unknowers (people who have never been exposed to rumors); (2) rumor hesitators with less education level (the less educated rumor unknowers contact the rumor, but do not directly believe the rumor, but think about the content of the spread); (3) rumor hesitators with higher education (the more educated rumor unknowers contact the rumors, but do not believe them straight away, but think about them); (4) rumor spreaders (people who believe and spread rumors); and (5) rumor immunizers (people who will not believe and will not spread rumors). The above five categories of people are respectively represented by $S(t), D(t), E(t), I(t), R(t)$.

In the society, the flow of people is a universal phenomenon. Therefore, the model established in this paper is based on the open system, and we make the following assumptions:

(1) Assume that individuals enter the rumor propagation system with a constant number of Λ. Then, consider that the population of each chamber has moved out of the system for specific reasons such as relocation, natural death, etc. Therefore, in order to facilitate the study, we assume that these five types of chambers have the same removal rate, which is defined as μ.

(2) When rumor unknowers with less education come into contact with rumors spreaders, the contact rate between them is defined as α_1; the proportion of rumor unknowers who received a less level of education in the total number of rumor unknowers was defined as β_1; moreover, when rumor unknowers with less education are contacted by rumors spreaders, generally speaking, these people are more likely to hesitate about the information spread by rumors spreaders than directly deny it. Therefore, we think that these people will transform into less educated rumor hesitators with a probability of $\alpha_1(1+\beta_1)$.

(3) When rumor unknowers with higher education come into contact with rumors spreaders, the contact rate between them is defined as α_2; the proportion of rumor unknowers who have received higher education in the total number of rumor unknowers is defined as β_2; furthermore, when rumor unknowers with higher education are contacted by rumor spreaders, generally speaking, these people tend to directly deny the rumor rather than hesitate to the information spread by rumor spreaders. Therefore, we believe that these people will transform into highly educated rumor hesitators with a probability of $\alpha_2(1-\beta_2)$.

(4) When uneducated rumor unknowers come into contact with rumor spreaders, generally speaking, these rumor unknowers will choose to directly believe the information spread by the rumor spreaders without their own thinking. We define the probability of these people turning into rumor spreaders as λ.

(5) In addition, rumor hesitators with less education and rumor hesitators with higher education will be removed from the system as mentioned above, and the following two situations may occur. In the first case, after thinking about it, they feel that the information they received before is correct and choose to continue believing and spreading it. Then, we define the probabilities of them turning into rumor spreaders as γ_1 and γ_2 respectively; in the second case, after thinking about it, they feel that the information they received before is wrong and choose not to continue believing and spreading it. Thus, we define the probabilities of them turning into rumor immunizers as ϕ_1 and ϕ_3, respectively. Furthermore, because of the rumor spreading process, the timeliness of the rumor

spreaders may lose interest in the rumor spreading process, and people have the characteristic of forgetting and the awakening of people's own consciousness (people find out that the information they believe is a rumor and then automatically become rumor immunizers) and other reasons, so that rumor spreaders will spontaneously become rumor immunizers. Let us define the probability of its transformation as ϕ_2 .

The parameters of model SDEIR are summarized in Table 1.

Table 1. Description of parameters in the model.

$S(t)$	the number of rumors unknowers at time t
$D(t)$	the number of rumor hesitators with less education at time t
$E(t)$	the number of rumor hesitators with higher educated at time t
$I(t)$	the number of rumor spreaders at time t
$R(t)$	the number of rumor immunizers at time t
Λ	the number of people who enter the rumor unknowers per unit time
μ	the probability of moving out of the rumor propagation system
α_1	the natural contact rate between less educated rumor unknowers and the rumor spreaders
β_1	the proportion of less educated rumor unknowers in the total number of rumor unknowers
$\alpha_1(1+\beta_1)$	the rate that $S(t)$ transforms into $D(t)$
α_2	the natural contact rate between higher educated rumor unknowers and rumor spreaders
β_2	the proportion of higher educated rumor unknowers in the total number of rumor unknowers
$\alpha_2(1-\beta_2)$	the rate that $S(t)$ transforms into $E(t)$
λ	the probability of uneducated rumor unknowers transforming into rumor spreaders
γ_1	the probability that $D(t)$ transforms into $I(t)$
γ_2	the probability that $E(t)$ transforms into $I(t)$
ϕ_1	the probability that $D(t)$ transforms into $R(t)$
ϕ_2	the probability that $I(t)$ transforms into $R(t)$
ϕ_3	the probability that $E(t)$ transforms into $R(t)$

On this basis, we drew the model flow chart, as shown in Figure 1, and constructed the dynamic equation, such as Eq (2.1).

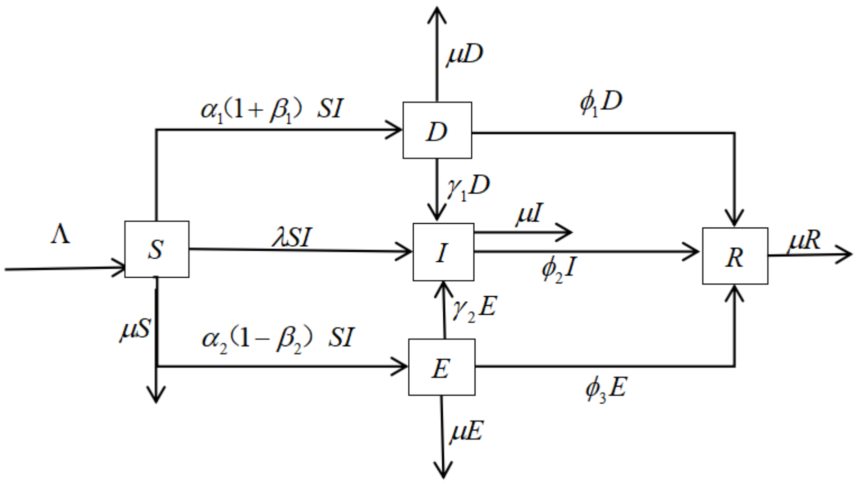

Figure 1. Consider the rumor propagation diagram of the education level.

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \alpha_1(1+\beta_1)SI - \lambda SI - \alpha_2(1-\beta_2)SI - \mu S \\ \frac{dD(t)}{dt} = \alpha_1(1+\beta_1)SI - \gamma_1 D - \phi_1 D - \mu D \\ \frac{dE(t)}{dt} = \alpha_2(1-\beta_2)SI - \gamma_2 E - \phi_3 E - \mu E \\ \frac{dI(t)}{dt} = \lambda SI + \gamma_1 D + \gamma_2 E - \phi_2 I - \mu I \\ \frac{dR(t)}{dt} = \phi_1 D + \phi_2 I + \phi_3 E - \mu R \end{cases} \tag{2.1}$$

where,

$$\Lambda, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \phi_1, \phi_2, \phi_3, \mu > 0; S(t) + D(t) + E(t) + I(t) + R(t) = N(t).$$

3. Model analysis

Lemma 3.1. The closed set $\Gamma = \left\{(S, D, E, I, R) \in R_+^5 | 0 \leq S + D + E + I + R \leq \frac{\Lambda}{\mu}\right\}$ is the positive definite invariant set of the model dynamics equation (2.1).

Proof of Lemma 3.1. According to the model dynamics equation, we can easily know $\frac{dN}{dt} = \Lambda - \mu N$. Thus, we can derive $N(t) = \frac{\Lambda}{\mu} - (\frac{\Lambda}{\mu} - N_0)e^{-\mu t}$, where $N(0) = N_0$ and when $t \longrightarrow \infty$, $\lim_{t\to\infty} N(t) = \frac{\Lambda}{\mu}$. This proves that the positive definite invariant set of the model dynamics equation (2.1) is $\Gamma = \left\{(S, D, E, I, R) \in R_+^5 | 0 \leq S + D + E + I + R \leq \frac{\Lambda}{\mu}\right\}$.

3.1. Basic reproduction number R_0

The basic reproduction number R_0 represents the number of people that a rumor spreader can infect during the average period of rumor propagation when all people are rumor unknowers [17]. In general, $R_0 = 1$ is used as a threshold to determine whether a rumor is dead or not. In other words, when $R_0 < 1$, the number of people a rumor spreader can infect during the average rumor propagation period is less than 1, which indicates that rumors will disappear in the future with time. When $R_0 > 1$, a rumor spreader can infect more than 1 person during the average rumor propagation period, indicating that rumors do not disappear over time, but will always exist. This paper uses the method in previous literature [18], that is, the next generation matrix method, to calculate the basic reproduction number R_0 of the model dynamic equation (2.1).

Let $X = (I, D, E, R, S)^T$, then the model dynamics equation (2.1) can be written as

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \mathcal{F}(\mathcal{X}) - \mathcal{V}(\mathcal{X}), \tag{3.1}$$

where,

$$\mathcal{F}(\mathcal{X}) = \begin{bmatrix} \lambda SI + \gamma_1 D + \gamma_2 E \\ \alpha_1(1+\beta_1)SI \\ \alpha_2(1-\beta_2)SI \\ 0 \\ 0 \end{bmatrix}, \mathcal{V}(\mathcal{X}) = \begin{bmatrix} (\phi_2+\mu)I \\ (\phi_1+\gamma_1+\mu)D \\ (\phi_3+\gamma_2+\mu)D \\ \mu R - \phi_1 D + \phi_2 I + \phi_3 E \\ \alpha_1(1+\beta_1)SI + \lambda SI + \alpha_2(1-\beta_2)SI + \mu S - \Lambda \end{bmatrix}. \tag{3.2}$$

Thus, we can get

$$f = \begin{bmatrix} \lambda\frac{\Lambda}{\mu} & \gamma_1 & \gamma_2 \\ \alpha_1(1+\beta_1)\frac{\Lambda}{\mu} & 0 & 0 \\ \alpha_2(1-\beta_2)\frac{\Lambda}{\mu} & 0 & 0 \end{bmatrix}, v = \begin{bmatrix} \mu+\phi_2 & 0 & 0 \\ 0 & \phi_1+\gamma_1+\mu & 0 \\ 0 & 0 & \phi_3+\gamma_2+\mu \end{bmatrix}. \tag{3.3}$$

And by calculation, we can get

$$v^{-1} = \begin{bmatrix} \frac{1}{\mu+\phi_2} & 0 & 0 \\ 0 & \frac{1}{\phi_1+\gamma_1+\mu} & 0 \\ 0 & 0 & \frac{1}{\phi_3+\gamma_2+\mu} \end{bmatrix}. \tag{3.4}$$

Then, fv^{-1} is used to represent the next generation matrix

$$fv^{-1} = \begin{bmatrix} \frac{\Lambda\lambda}{\mu(\mu+\phi_2)} & \frac{\gamma_1}{\mu+\gamma_1+\phi_1} & \frac{\gamma_2}{\mu+\gamma_2+\phi_3} \\ \frac{\alpha_1(1+\beta_1)\Lambda}{\mu(\mu+\phi_2)} & 0 & 0 \\ \frac{\alpha_2(1-\beta_2)\Lambda}{\mu(\mu+\phi_2)} & 0 & 0 \end{bmatrix}. \tag{3.5}$$

Therefore, according to reference [19], the basic reproduction number of model dynamic equation (2.1) is the spectral radius of matrix :

$$R_0 = \frac{\lambda\Lambda cd + \sqrt{(\lambda\Lambda cd)^2 + 4\mu ecd\Lambda(bc\gamma_2 + ad\gamma_1)}}{2\mu ecd}, \tag{3.6}$$

where,

$$\begin{cases} a = \alpha_1(1+\beta_1) \\ b = \alpha_2(1-\beta_2) \\ c = \phi_1+\gamma_1+\mu. \\ d = \phi_3+\gamma_2+\mu \\ e = \phi_2+\mu \end{cases} \tag{3.7}$$

3.2. Existence of equilibrium points

According to the model dynamics equation (2.1), we can find two equilibrium points, which are:

$$E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$$

and

$$E^* = (S^*, D^*, E^*, I^*, R^*)$$

$$S^* = \frac{ecd}{ad\gamma_1 + bc\gamma_2 + cd\lambda}; I^* = \frac{\Lambda(\lambda cd + ad\gamma_1 + bc\gamma_2) - \mu ecd}{ecd(a + d + \lambda)}$$

$$D^* = \frac{a\Lambda(\lambda cd + ad\gamma_1 + bc\gamma_2) - a\mu ecd}{c(\lambda cd + ad\gamma_1 + bc\gamma_2)(a + d + \lambda)}; E^* = \frac{b\Lambda(\lambda cd + ad\gamma_1 + bc\gamma_2) - b\mu ecd}{d(\lambda cd + ad\gamma_1 + bc\gamma_2)(a + d + \lambda)}$$

$$R^* = \frac{(\Lambda ad\gamma_1 + \Lambda bc\gamma_2 + \Lambda cd\lambda - cde\mu)(cd\lambda\phi_2 + ade\phi_1 + bce\phi_3 + ad\gamma_1\phi_2 + bc\gamma_2\phi_2)}{\mu ecd(\lambda cd + ad\gamma_1 + bc\gamma_2)(a + b + \lambda)}$$

where a, b, c, d, e satisfies Eq (3.7).

3.3. Stability of equilibrium points

Theorem 1. If $R_0 < 1$, $\frac{\Lambda}{\mu} < \frac{ecd}{\lambda cd+\gamma_1 ad+\gamma_2 bc}$, $c = d$ and $\gamma_1 a < \lambda c$ holds true, then the rumor-free equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ is locally asymptotically stable; If $\Lambda[\alpha_1(1 + \beta_1) + \alpha_2(1 - \beta_2) + \lambda] \leq \mu^2$ holds true, then the rumor-free equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ is globally asymptotically stable. Where a, b, c, d, e satisfies the above Eq (3.7).

Proof. The Jacobian matrix of system (2.1) at $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ is

$$J(E_0) = \begin{vmatrix} -\mu & 0 & 0 & -a\frac{\Lambda}{\mu} - \lambda\frac{\Lambda}{\mu} - b\frac{\Lambda}{\mu} & 0 \\ 0 & -c & 0 & a\frac{\Lambda}{\mu} & 0 \\ 0 & 0 & -d & b\frac{\Lambda}{\mu} & 0 \\ 0 & \gamma_1 & \gamma_2 & \lambda\frac{\Lambda}{\mu} - e & 0 \\ 0 & \phi_1 & \phi_3 & \phi_2 & -\mu \end{vmatrix}. \tag{3.8}$$

The characteristic equation of matrix $J(E_0)$ is

$$\begin{aligned} |\theta E - J(E_0)| &= \begin{vmatrix} \theta + \mu & 0 & 0 & a\frac{\Lambda}{\mu} + \lambda\frac{\Lambda}{\mu} + b\frac{\Lambda}{\mu} & 0 \\ 0 & \theta + c & 0 & -a\frac{\Lambda}{\mu} & 0 \\ 0 & 0 & \theta + d & -b\frac{\Lambda}{\mu} & 0 \\ 0 & -\gamma_1 & -\gamma_2 & \theta - \lambda\frac{\Lambda}{\mu} + e & 0 \\ 0 & -\phi_1 & -\phi_3 & -\phi_2 & \theta + \mu \end{vmatrix} \\ &= (\theta + \mu)^2(\theta^3 + A\theta^2 + B\theta + C) = 0 \end{aligned} \tag{3.9}$$

where,

$$\begin{cases} A = e + c + d - \lambda\frac{\Lambda}{\mu} \\ B = ec + ed + cd - (\lambda c + \gamma_1 a + \lambda d + \gamma_2 b)\frac{\Lambda}{\mu} \\ C = ecd - (\lambda cd + \gamma_1 ad + \gamma_2 bc)\frac{\Lambda}{\mu}. \end{cases} \tag{3.10}$$

This gives us the eigenvalues $\theta_1 = \theta_2 = -\mu$ and the simplified characteristic equation

$$\theta^3 + A\theta^2 + B\theta + C = 0. \tag{3.11}$$

We then construct a cubic polynomial using a_0, a_1, a_2, a_3 instead of the coefficients to find the other eigenvalues of Eq (3.11). We can then rewrite Eq (3.11) as

$$a_0\theta^3 + a_1\theta^2 + a_2\theta + a_3 = 0 \tag{3.12}$$

where,

$$\begin{aligned} a_0 &= 1 \\ a_1 &= A = e + c + d - \lambda\frac{\Lambda}{\mu} \\ a_2 &= B = ec + ed + cd - (\lambda c + \gamma_1 a + \lambda d + \gamma_2 b)\frac{\Lambda}{\mu} \\ a_3 &= C = ecd - (\lambda cd + \gamma_1 ad + \gamma_2 bc)\frac{\Lambda}{\mu} \end{aligned}$$

and

$$\begin{aligned} & a_2a_1 - a_3a_0 \\ & = (\lambda^2 c + \lambda\gamma_1 a + \lambda\gamma_2 b + \lambda^2 d)(\frac{\Lambda}{\mu})^2 \\ & - (2\lambda cd + 2\lambda ec + 2\lambda ed + \lambda c^2 + \lambda d^2 + \gamma_1 ae + \gamma_1 ac + \gamma_2 be + \gamma_2 bd)\frac{\Lambda}{\mu} \\ & + (2ecd + e^2c + ec^2 + e^2d + ed^2 + c^2d + cd^2). \end{aligned} \tag{3.13}$$

The conditions for local asymptotic stability of the rumor-free equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ based on the Routh-Hurwitz criterion [20] are as follows: (1) $a_n > 0, n = 0, 1, 2, 3$ (2) $a_2a_1 - a_3a_0 > 0$. Through calculation, it can be concluded that when the condition of $\frac{\Lambda}{\mu} < \frac{ecd}{\lambda cd+\gamma_1 ad+\gamma_2 bc}, c = d$ and $\gamma_1 a < \lambda c$ is met, the rumor-free equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ is locally asymptotically stable.

The above is the proof of the local asymptotic stability of the rumor-free equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$, followed by the proof of the global asymptotic stability.

We construct a Lyapunov function [21] $L(t) = D(t) + E(t) + I(t) + R(t)$. Then, we can get $L'(t)$,

$$\begin{aligned} L'(t) &= D'(t) + E'(t) + I'(t) + R'(t) \\ &= \alpha_1(1 + \beta_1)SI - \gamma_1 D - \phi_1 D - \mu D \\ &+ \alpha_2(1 - \beta_2)SI - \gamma_2 E - \phi_3 E - \mu E \\ &+ \lambda SI + \gamma_1 D + \gamma_2 E - \phi_2 I - \mu I \\ &+ \phi_1 D + \phi_2 I + \phi_3 E - \mu R \\ &= \alpha_1(1 + \beta_1)SI + \alpha_2(1 - \beta_2)SI + \lambda SI - \mu I - \mu D - \mu E - \mu R \\ &= \{[\alpha_1(1 + \beta_1) + \alpha_2(1 - \beta_2) + \lambda]S - \mu\} I - \mu(D + E + R). \end{aligned} \tag{3.14}$$

We know $S \leq \frac{\Lambda}{\mu}$ from Lemma 1. Obviously, if $\Lambda[\alpha_1(1+\beta_1)+\alpha_2(1-\beta_2)+\lambda] \leq \mu^2$ and $S \leq \frac{\Lambda}{\mu}$, then $L'(t) \leq 0$ is always true.

Furthermore, $L'(t) = 0$ is true if and only if $S(t) = S_0, D(t) = E(t) = I(t) = R(t) = 0$. From system (2.1) we know that when $L'(t) = 0$, E_0 is the only solution in Γ. Therefore, based on Lyapunov function and LaSalle invariance principle [22], it can be shown that every solution of the system (2.1) tends to E_0 infinitely for $t \longrightarrow \infty$. Therefore, the rumor-free existence equilibrium point $E_0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ of system (2.1) is globally asymptotically stable. □

Theorem 2. If $R_0 > 1$, $P_3P_2 - P_4P_1 > 0$ and $P_3P_2P_1 - P_4P_1^2 - P_3^2 > 0$, then the rumor existence equilibrium point $E^* = (S^*, D^*, E^*, I^*, R^*)$ is locally asymptotically stable; If $R_0 > 1$, then the rumor existence equilibrium point $E^* = (S^*, D^*, E^*, I^*, R^*)$ is globally asymptotically stable. Where a, b, c, d, e satisfies the above Eq (3.7).

Proof. The Jacobian matrix of system (2.1) at $E^* = (S^*, D^*, E^*, I^*, R^*)$ is

$$J(E^*) = \begin{bmatrix} -aI^* - \lambda I^* - bI^* - \mu & 0 & 0 & -aS^* - \lambda S^* - bS^* & 0 \\ aI^* & -c & 0 & aS^* & 0 \\ bI^* & 0 & -d & bS^* & 0 \\ \lambda I^* & \gamma_1 & \gamma_2 & \lambda S^* - e & 0 \\ 0 & \phi_1 & \phi_3 & \phi_2 & -\mu \end{bmatrix}. \tag{3.15}$$

The characteristic equation of matrix $J(E^*)$ is

$$\begin{aligned} |\theta E - J(E^*)| &= \begin{vmatrix} \theta + aI^* + \lambda I^* + bI^* + \mu & 0 & 0 & aS^* + \lambda S^* + bS^* & 0 \\ -aI^* & \theta + c & 0 & -aS^* & 0 \\ -bI^* & 0 & \theta + d & -bS^* & 0 \\ -\lambda I^* & -\gamma_1 & -\gamma_2 & \theta - \lambda S^* + e & 0 \\ 0 & -\phi_1 & -\phi_3 & -\phi_2 & \theta + \mu \end{vmatrix} \\ &= (\theta + \mu)(\theta^4 + P_1\theta^3 + P_2\theta^2 + P_3\theta + P_4) = 0, \end{aligned} \tag{3.16}$$

where,

$$\begin{cases} P_1 = (e + c + d) - \dfrac{\lambda ecd}{\lambda cd + \gamma_1 ad + \gamma_2 bc} + \dfrac{\Lambda(\lambda cd + \gamma_1 ad + \gamma_2 bc)}{ecd} \\ P_2 = (ec + ed + cd) - \dfrac{(\lambda c + \gamma_1 a + \gamma_2 b + \lambda\mu + \lambda d)ecd}{\lambda cd + \gamma_1 ad + \gamma_2 bc} + \dfrac{\Lambda(e + c + d)(\lambda cd + \gamma_1 ad + \gamma_2 bc)}{ecd} \\ P_3 = \dfrac{\Lambda(ec + cd + ed)(\lambda cd + \gamma_1 ad + \gamma_2 bc)}{ecd} - \dfrac{(\lambda c + \gamma_1 a + \gamma_2 b + \lambda d)\mu ecd}{\lambda cd + \gamma_1 ad + \gamma_2 bc} \\ P_4 = \Lambda(\lambda cd + \gamma_1 ad + \gamma_2 bc) - \mu ecd. \end{cases}$$

From this, the eigenvalues $\theta_1 = -\mu < 0$ and the simplified characteristic equation $\theta^4 + P_1\theta^3 + P_2\theta^2 + P_3\theta + P_4 = 0$ can be obtained.

According to the simplified characteristic equation above, with the help of Routh-Hurwitz criterion, it can be seen that all the coefficients of the characteristic equation are positive numbers, and the values of each coefficient in the first column of table Routh-Hurwitz are also greater than 0, that is, the conditions of local asymptotic stability are satisfied [23]. Therefore, we can judge that the condition of local stability of the model is $R_0 > 1$, $P_3P_2 - P_4P_1 > 0$ and $P_3P_2P_1 - P_4P_1^2 - P_3^2 > 0$ when the equilibrium point $E^* = (S^*, D^*, E^*, I^*, R^*)$ exists.

The above is the proof of the local asymptotic stability of the rumor equilibrium point $E^* = (S^*, D^*, E^*, I^*, R^*)$, followed by the proof of the global asymptotic stability.

First, we construct a Lyapunov function $W(t) = [(S(t) - S^*) + (D(t) - D^*) + (E(t) - E^*) + (I(t) - I^*) + (R(t) - R^*)]^2$. Then, you get $W'(t)$,

$$\begin{aligned} W'(t) &= 2[(S(t) - S^*) + (D(t) - D^*) + (E(t) - E^*) + (I(t) - I^*) \\ &\quad + (R(t) - R^*)][S'(t) + D'(t) + E'(t) + I'(t) + R'(t)] \\ &= 2[(S(t) - S^*) + (D(t) - D^*) + (E(t) - E^*) + (I(t) - I^*) + (R(t) - R^*)] \\ &\quad (\Lambda - \mu S - \mu D - \mu E - \mu I - \mu R). \end{aligned} \tag{3.17}$$

From the above discussion, it can be seen that the equilibrium point $E^* = (S^*, D^*, E^*, I^*, R^*)$ of rumors exists. Therefore, $\Lambda - \mu S^* - \mu D^* - \mu E^* - \mu I^* - \mu R^* = 0$. In other words, $\Lambda = \mu S^* + \mu D^* + \mu E^* + \mu I^* + \mu R^*$. So, Eq (3.17) can be calculated as

$$\begin{aligned} W'(t) =& 2[(S(t) - S^*) + (D(t) - D^*) + (E(t) - E^*) + (I(t) - I^*) + (R(t) - R^*)] \\ & [\mu S^* + \mu D^* + \mu E^* + \mu I^* + \mu R^* - \mu S - \mu D - \mu E - \mu I - \mu R] \\ =& - 2\mu[(S - S^*) + (D - D^*) + (E - E^*) + (I - I^*) + (R - R^*)]^2 \leq 0. \end{aligned} \tag{3.18}$$

Furthermore, $W'(t) = 0$ is true if and only if $S(t) = S^*, D(t) = D^*, E(t) = E^*, I(t) = I^*, R(t) = R^*$. Therefore, based on Lyapunov function and LaSalle invariance principle, it can be obtained that the rumor existence equilibrium point $E^* = (S^*, D^*, E^*, I^*, R^*)$ of system (2.1) is globally asymptotically stable. □

4. Optimal control

The model developed in this study is predicated on the social environment with the aim of preventing the propagation of misinformation. Optimal control is a well-known concept involving the purposeful exertion of control over systems and processes in a variety of contexts, such as production and life operations, to achieve specific performance targets [24]. Given the alignment between the objectives of this study and the optimal control theory, the model is analyzed in this section using established methods from existing literature [25].

First, it is assumed that the model discussed in this paper is operating in a social communication environment that is closed.

Second, our purpose of this paper is to reduce the number of rumor hesitators and rumor spreaders, while increasing the number of rumor immunizers. To achieve this goal, we controlled for the following five parameters: The natural contact rate α_1 between rumor unknowers with higher education and rumor spreaders, and the natural contact rate α_2 between rumor unknowers with less education and rumor spreaders. The proportion β_1 of rumor unknowers with less education level and the proportion β_2 of rumor unknowers with higher education level, as well as the probability λ of transforming uneducated rumor unknowers into rumor spreaders.

Next, we carry out the following analysis:

(1) We convert the five parameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda$ in the model into control variables $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t)$ and $\lambda(t)$.

(2) In this model, control variables satisfy $0 \leq \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \lambda(t) \leq 1$. When $\alpha_1(t) = 0, \alpha_2(t) = 0, \beta_1(t) = 0, \beta_2(t) = 1, \lambda(t) = 0$, we can conclude that rumor unknowers did not transform into rumor spreaders and rumor hesitators, but directly transformed into rumor immunizers people. At this point, the desired result of the control has been achieved, which means that the control measure has been a hundred percent effective; When $\alpha_1(t) = 1, \alpha_2(t) = 1, \beta_1(t) = 1, \beta_2(t) = 0, \lambda(t) = 1$, we can conclude that the rumor unknowers all transform into rumor hesitators, and then all transform into rumor spreaders, which means that the control measures are completely ineffective.

From the above statement, we set up an objective function, defined as follows:

$$\begin{aligned} J(\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda) = \int_0^{t_f} & e^{-\alpha t}[D(t) + E(t) + I(t) + \frac{c_1}{2}\alpha_1^2(t) \\ & + \frac{c_2}{2}\alpha_2^2(t) + \frac{c_3}{2}\beta_1^2(t) + \frac{c_4}{2}\beta_2^2(t) + \frac{c_5}{2}\lambda^2(t)] \end{aligned} \tag{4.1}$$

and satisfy the following state system:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \alpha_1(t)(1 + \beta_1(t))SI - \lambda(t)SI - \alpha_2(t)(1 - \beta_2(t))SI - \mu S \\ \frac{dD(t)}{dt} = \alpha_1(t)(1 + \beta_1(t))SI - \gamma_1 D - \phi_1 D - \mu D \\ \frac{dE(t)}{dt} = \alpha_2(t)(1 - \beta_2(t))SI - \gamma_2 E - \phi_3 E - \mu E \\ \frac{dI(t)}{dt} = \lambda(t)SI + \gamma_1 D + \gamma_2 E - \phi_2 I - \mu I \\ \frac{dR(t)}{dt} = \phi_1 D + \phi_2 I + \phi_3 E - \mu R. \end{cases} \tag{4.2}$$

The initial conditions for satisfying formula (4.2) are as follows:

$$S(0) = S_0 \geq 0, D(0) = D_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0 \tag{4.3}$$

where,

$$\begin{aligned} & \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \lambda(t) \in U \triangleq [\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda|(\alpha_1(t), \\ & \alpha_2(t), \beta_1(t), \beta_2(t), \lambda(t)), 0 \leq \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \lambda(t) \leq 1, \forall t \in [0, t_f]], \end{aligned} \tag{4.4}$$

and U is the admissible control set. The control time interval is between 0 and t_f. c_1, c_2, c_3, c_4, c_5 is a positive weight coefficient indicating the prevention intensity and importance of the five control measures.

4.1. Existence of optimal control

In this paper, we demonstrate the existence of an optimal control policy in the above optimal control problem by applying the following theorem.

Theorem 4.1. There exists an optimal control for $(\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, \lambda^*) \in U$ such that the function is established as follows [26]:

$$J(\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, \lambda^*) = min\{J(\alpha_1^*, \alpha_2^*, \beta_1^*, \beta_2^*, \lambda^*), \alpha_1^*(t), \alpha_2^*(t), \beta_1^*(t), \beta_2^*(t), \lambda^*(t) \in U\} \tag{4.5}$$

Proof. In order to prove the existence of optimal control, it is only necessary to verify the following conditions [27, 28].

(1) The control admissible set and the set of state variables are non-empty.

(2) The control allows set U to be convex and closed.

(3) The integrand in the objective functional is convex in the control admissible set U.

(4) The right end of the state system is linear with respect to state variables and control variables.

(5) The existence of constants $d_1, d_2 > 0$ and $\varepsilon > 1$ makes the integrand expression in the target functional.

Let $X = (S(t), D(t), E(t), I(t), R(t))^T$ and

$$L(t, X(t), \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \lambda(t)) \triangleq e^{-\alpha t}[D(t) + E(t) + I(t) + \frac{c_1}{2}\alpha_1^2(t) + \frac{c_2}{2}\alpha_2^2(t) + \frac{c_3}{2}\beta_1^2(t) + \frac{c_4}{2}\beta_2^2(t) + \frac{c_5}{2}\lambda^2(t))] \tag{4.6}$$

satisfy

$$L(t, \alpha_1, \alpha_2, \beta_1, \beta_2, \lambda) \geq d_1(|\alpha_1|^2 + |\alpha_2|^2 + |\beta_1|^2 + |\beta_2|^2 + |\lambda|^2)^{\frac{\varepsilon}{2}} - d_2. \tag{4.7}$$

Conditions (1)–(3) are obviously true, only the last two need to be verified. From Lemma 1

$$max\{S(t), D(t), E(t), I(t), R(t)\} \leq \frac{\Lambda}{\mu}. \tag{4.8}$$

The following inequality can be obtained:

$$\begin{aligned} &S^{'} \leq \Lambda, D^{'} \leq \alpha_1(1 + \beta_1)SI, E^{'} \leq \alpha_2(1 - \beta_2)SI, I^{'} \leq \gamma_1 D + \lambda SI + \gamma_2 E, \\ &R^{'} \leq \phi_1 D + \phi_2 E + \phi_3 R. \end{aligned} \tag{4.9}$$

Thus, the fourth condition holds. If we take $d_1 = \frac{e^{-\alpha t_0}}{2} min\{c_1, c_2, c_3, c_4, c_5\}$ and $\forall d_2 \in R^+$, and $\varepsilon = 2$, then the fifth condition holds. □

4.2. Optimal control expression

According to the optimal control expression given in 4.1, the optimal control system is obtained. To do this, define the following enlarge Hamiltonian operator L with a penalty term [29]:

$$\begin{aligned} L = {} & e^{-\alpha t}[D(t) + E(t) + I(t) + \frac{c_1}{2}\alpha_1^2(t) + \frac{c_2}{2}\alpha_2^2(t) + \frac{c_3}{2}\beta_1^2(t) + \frac{c_4}{2}\beta_2^2(t) + \frac{c_5}{2}\lambda^2(t))] \\ & + \eta_1(t)[\Lambda - \alpha_1(t)(1 + \beta_1(t))SI - \lambda(t)SI - \alpha_2(t)(1 - \beta_2(t))SI - \mu S] \\ & + \eta_2(t)[\alpha_1(t)(1 + \beta_1(t))SI - \gamma_1 D - \phi_1 D - \mu D] \\ & + \eta_3(t)[\alpha_2(t)(1 - \beta_2(t))SI - \gamma_2 E - \phi_3 E - \mu E] \\ & + \eta_4(t)[\lambda(t)SI + \gamma_1 D + \gamma_2 E - \phi_2 I - \mu I] \\ & + \eta_5(t)[\phi_1 D + \phi_2 I + \phi_3 E - \mu R] \\ & - \omega_{11}(t)\alpha_1(t) - \omega_{12}(t)(1 - \alpha_1(t)) - \omega_{21}(t)\alpha_2(t) - \omega_{22}(t)(1 - \alpha_2(t)) \\ & - \omega_{31}(t)\beta_1(t) - \omega_{32}(t)(1 - \beta_1(t)) - \omega_{41}(t)\beta_2(t) - \omega_{42}(t)(1 - \beta_2(t)) \\ & - \omega_{51}(t)\lambda(t) - \omega_{52}(t)(1 - \lambda(t)), \end{aligned} \tag{4.10}$$

where $\omega_{ij} \geq 0(i, j = 1, 2, 3, 4, 5)$ is the penalty operator, it satisfies $\omega_{11}(t)\alpha_1(t) = \omega_{12}(1 - \alpha_1(t)) = 0$, at optimal control α_1^*;

$\omega_{21}(t)\alpha_2(t) = \omega_{22}(1 - \alpha_2(t)) = 0$, at optimal control α_2^*;

$\omega_{31}(t)\beta_1(t) = \omega_{32}(1 - \beta_1(t)) = 0$, at optimal control β_1^*;

$\omega_{41}(t)\beta_2(t) = \omega_{42}(1 - \beta_2(t)) = 0$, at optimal control β_2^*;

$\omega_{51}(t)\lambda(t) = \omega_{52}(1 - \lambda(t)) = 0$, at optimal control λ^*.

Theorem 4.2. Given the optimal control pair $(\alpha_1^*(t), \alpha_2^*(t), \beta_1^*(t), \beta_2^*(t), \lambda^*(t))$ and solution $S(t), D(t), E(t), I(t), R(t)$ of Eq (2.1) of state system , there is a costate variable $\eta_i(t), i = 1, 2, 3, 4, 5$, which satisfies the equation:

$$\begin{cases} \eta_{1'} = \eta_1(t)[\alpha_1(t)(1 + \beta_1(t))I + \lambda(t)I + \alpha_2(t)(1 - \beta_2(t))I + \mu] \\ \qquad - \eta_2(t)\alpha_1(t)(1 + \beta_1(t))I - \eta_3(t)\alpha_2(t)(1 - \beta_2(t))I - \eta_4(t)\lambda(t)I \\ \eta_{2'} = - e^{-\alpha t} + \eta_2(t)(\phi_1 + \gamma_1 + \mu) - \eta_4(t)\gamma_1 - \eta_5(t)\phi_1 \\ \eta_{3'} = - e^{-\alpha t} + \eta_3(t)(\phi_3 + \gamma_2 + \mu) - \eta_4(t)\gamma_2 - \eta_5(t)\phi_3 \\ \eta_{4'} = - e^{-\alpha t}\eta_1(t)[\alpha_1(t)(1 + \beta_1(t))S + \lambda(t)S + \alpha_2(t)(1 - \beta_2(t))S] \\ \qquad - \eta_2(t)\alpha_1(t)(1 + \beta_1(t))S - \eta_3(t)\alpha_2(t)(1 - \beta_2(t))S \\ \qquad - \eta_4(t)(\lambda(t)S - \mu - \phi_2) - \eta_5(t)\phi_2 \\ \eta_{5'} = \eta_5(t)\mu \end{cases} \tag{4.11}$$

and terminal conditions

$$\eta_i(t_f) = 0, i = 1, 2, 3, 4, 5. \tag{4.12}$$

Moreover, the expression for optimal control $(\alpha_1^*(t), \alpha_2^*(t), \beta_1^*(t), \beta_2^*(t), \lambda^*(t))$ is expressed as follows:

$$\begin{cases} \alpha_1^*(t) = min(1, max(0, \frac{\eta_1^*(t)(1+\beta_1^*(t))S^*(t)I^*(t) - \eta_2^*(t)(1+\beta_1^*(t))S^*(t)I^*(t))}{e^{-\alpha t}c_1})) \\ \alpha_2^*(t) = min(1, max(0, \frac{\eta_1^*(t)(1-\beta_2^*(t))S^*(t)I^*(t) - \eta_3^*(t)(1-\beta_2^*(t))S^*(t)I^*(t))}{e^{-\alpha t}c_2})) \\ \beta_1^*(t) = min(1, max(0, \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t) - \eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_3})) \\ \beta_2^*(t) = min(1, max(0, \frac{\eta_1^*(t)\alpha_2^*(t)S^*(t)I^*(t) + \eta_3^*(t)\alpha_2^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_4})) \\ \lambda^*(t) = min(1, max(0, \frac{\eta_1^*(t)S^*(t)I^*(t) - \eta_4^*(t)S^*(t)I^*(t)}{e^{-\alpha t}c_5})) \end{cases} \tag{4.13}$$

Proof. Based on Pontriagin's maximum principle [30], partial derivatives of the Hamiltonian operator L with respect to each state variable give the following costate system $\eta_1^{'} = -\frac{\partial L}{\partial S}, \eta_2^{'} = -\frac{\partial L}{\partial D}, \eta_3^{'} = -\frac{\partial L}{\partial E}, \eta_4^{'} = -\frac{\partial L}{\partial I}, \eta_5^{'} = -\frac{\partial L}{\partial R}$, and the final value condition $\eta_i(t_f) = 0, i = 1, 2, 3, 4, 5$.

In order to obtain the necessary conditions for optimality, the partial derivative of the operator L with respect to the control variable $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \lambda(t)$ is obtained separately. Subsequently, the value should be set to zero and get

$$\begin{cases} \frac{\partial L}{\partial \alpha_1} = e^{-\alpha t}c_1\alpha_1(t) - \eta_1(t)(1 + \beta_1(t))SI + \eta_2(t)(1 + \beta_1(t))SI - \omega_{11}(t) + \omega_{12}(t) = 0 \\ \frac{\partial L}{\partial \alpha_2} = e^{-\alpha t}c_2\alpha_2(t) - \eta_1(t)(1 - \beta_2(t))SI + \eta_3(t)(1 - \beta_2(t))SI - \omega_{21}(t) + \omega_{22}(t) = 0 \\ \frac{\partial L}{\partial \beta_1} = e^{-\alpha t}c_3\beta_1(t) - \eta_1(t)\alpha_1(t)SI + \eta_2(t)\alpha_1(t)SI - \omega_{31}(t) + \omega_{32}(t) = 0 \\ \frac{\partial L}{\partial \beta_2} = e^{-\alpha t}c_4\beta_2(t) + \eta_1(t)\alpha_2(t)SI - \eta_3(t)\alpha_2(t)SI - \omega_{41}(t) + \omega_{42}(t) = 0 \\ \frac{\partial L}{\partial \lambda} = e^{-\alpha t}c_5\lambda_1(t) - \eta_1(t)SI + \eta_4(t)SI - \omega_{51}(t) + \omega_{52}(t) = 0. \end{cases} \tag{4.14}$$

To solve it, the expression of optimal control can be obtained in the following form:

$$\begin{cases} \alpha_1^*(t) = \frac{\eta_1^*(t)(1+\beta_1^*(t))S^*(t)I^*(t)-\eta_2^*(t)(1+\beta_1^*(t))S^*(t)I^*(t))+\omega_{11}(t)-\omega_{12}(t)}{e^{-\alpha t}c_1} \\ \alpha_2^*(t) = \frac{\eta_1^*(t)(1-\beta_2^*(t))S^*(t)I^*(t)-\eta_3^*(t)(1-\beta_2^*(t))S^*(t)I^*(t))+\omega_{21}(t)-\omega_{22}(t)}{e^{-\alpha t}c_2} \\ \beta_1^*(t) = \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))+\omega_{31}(t)-\omega_{32}(t)}{e^{-\alpha t}c_3} \\ \beta_2^*(t) = \frac{\eta_1^*(t)\alpha_2^*(t)S^*(t)I^*(t)+\eta_3^*(t)\alpha_2^*(t)S^*(t)I^*(t))+\omega_{41}(t)-\omega_{42}(t)}{e^{-\alpha t}c_4} \\ \lambda^*(t) = \frac{\eta_1^*(t)S^*(t)I^*(t)-\eta_4^*(t)S^*(t)I^*(t)+\omega_{51}(t)-\omega_{52}(t)}{e^{-\alpha t}c_5} \end{cases} . \tag{4.15}$$

In order to obtain the concrete expression of optimal control $\eta_i(t)(i = 1,2,3,4,5)$ without penalty term $\omega_{ij}(i = 1,2,3,4,5; j = 1,2)$, the standard practice of reference [31] is adopted. Taking $\beta_1^*(t)$ as an example, the specific process is divided into three situations for implementation:

(1) In case 1, when $\left\{t|0 < \beta_1^*(t) < 1\right\}$ is satisfied, let $\omega_{31}(t) = \omega_{32}(t) = 0$.

Therefore, the optimal control is $\beta_1^*(t) = \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_3}$.

(2) In case 2, when $\left\{t|\beta_1^*(t) = 1\right\}$ is satisfied, let $\omega_{31}(t) = 0$.

Therefore, $1 = \beta_1^*(t) = \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))-\omega_{32}(t)}{e^{-\alpha t}c_3}$.

This means that when $\omega_{32}(t) \geq 0$, $\frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_3} \geq 1$.

(3) In case 3, when $\left\{t|\beta_1^*(t) = 0\right\}$ is satisfied, let $\omega_{32}(t) = 0$.

Therefore, $0 = \beta_1^*(t) = \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))+\omega_{31}(t)}{e^{-\alpha t}c_3}$.

This means that when $\omega_{32}(t) \geq 0$, $\frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_3} \leq 0$.

Combining the above three cases, the expression of optimal control $\beta_1^*(t)$ can be obtained: $\beta_1^*(t) = min(1, max(0, \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_3}))$.

Similar to the discussion, the corresponding expressions for optimal control $\alpha_1^*(t), \alpha_2^*(t), \beta_1^*(t), \beta_2^*(t)$ and $\lambda^*(t)$ can be obtained:

$$\begin{cases} \alpha_1^*(t) = min(1, max(0, \frac{\eta_1^*(t)(1+\beta_1^*(t))S^*(t)I^*(t)-\eta_2^*(t)(1+\beta_1^*(t))S^*(t)I^*(t))}{e^{-\alpha t}c_1})) \\ \alpha_2^*(t) = min(1, max(0, \frac{\eta_1^*(t)(1-\beta_2^*(t))S^*(t)I^*(t)-\eta_3^*(t)(1-\beta_2^*(t))S^*(t)I^*(t))}{e^{-\alpha t}c_2})) \\ \beta_1^*(t) = min(1, max(0, \frac{\eta_1^*(t)\alpha_1^*(t)S^*(t)I^*(t)-\eta_2^*(t)\alpha_1^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_3})) \\ \beta_2^*(t) = min(1, max(0, \frac{\eta_1^*(t)\alpha_2^*(t)S^*(t)I^*(t)+\eta_3^*(t)\alpha_2^*(t)S^*(t)I^*(t))}{e^{-\alpha t}c_4})) \\ \lambda^*(t) = min(1, max(0, \frac{\eta_1^*(t)S^*(t)I^*(t)-\eta_4^*(t)S^*(t)I^*(t)}{e^{-\alpha t}c_5})) \end{cases} .$$

It is worth noting that the optimal control system comprises the state system (4.1) and its initial condition (4.3), the costate system (4.11) and its terminal condition (4.12), and the optimal control form expression (4.13). Every optimal control must fulfill this complex optimal control system structure.

The following is the optimal control system for future simulations:

$$
\begin{cases}
\dfrac{dS(t)}{dt} = \Lambda - \alpha_1(t)(1+\beta_1(t))SI - \lambda(t)SI - \alpha_2(t)(1-\beta_2(t))SI - \mu S, \\
\dfrac{dD(t)}{dt} = \alpha_1(t)(1+\beta_1(t))SI - \gamma_1 D - \phi_1 D - \mu D, \\
\dfrac{dE(t)}{dt} = \alpha_2(t)(1-\beta_2(t))SI - \gamma_2 E - \phi_3 E - \mu E, \\
\dfrac{dI(t)}{dt} = \lambda(t)SI + \gamma_1 D + \gamma_2 E - \phi_2 I - \mu I, \\
\dfrac{dR(t)}{dt} = \phi_1 D + \phi_2 I + \phi_3 E - \mu R, \\
\eta_{1'} = \eta_1(t)[\alpha_1(t)(1+\beta_1(t))I + \lambda(t)I + \alpha_2(t)(1-\beta_2(t))I + \mu] \\
\quad - \eta_2(t)\alpha_1(t)(1+\beta_1(t))I - \eta_3(t)\alpha_2(t)(1-\beta_2(t))I - \eta_4(t)\lambda(t)I, \\
\eta_{2'} = -e^{-\alpha t} + \eta_2(t)(\phi_1 + \gamma_1 + \mu) - \eta_4(t)\gamma_1 - \eta_5(t)\phi_1, \\
\eta_{3'} = -e^{-\alpha t} + \eta_3(t)(\phi_3 + \gamma_2 + \mu) - \eta_4(t)\gamma_2 - \eta_5(t)\phi_3, \\
\eta_{4'} = -e^{-\alpha t}\eta_1(t)[\alpha_1(t)(1+\beta_1(t))S + \lambda(t)S + \alpha_2(t)(1-\beta_2(t))S] \\
\quad - \eta_2(t)\alpha_1(t)(1+\beta_1(t))S - \eta_3(t)\alpha_2(t)(1-\beta_2(t))S \\
\quad - \eta_4(t)(\lambda(t)S - \mu - \phi_2) - \eta_5(t)\phi_2, \\
\eta_{5'} = \eta_5(t)\mu, \\
S(0) = S_0 \geq 0, D(0) = D_0 \geq 0, E(0) = E_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0, \\
\eta_i(t_f) = 0, i = 1,2,3,4,5.
\end{cases}
$$

5. Numerical simulation

To ensure the accuracy of the aforementioned conclusions, we adopted established methodologies from existing literature [23] and employed Matlab as a calculation tool for numerical simulation of said conclusions. We initially introduced hesitators and their corresponding parameters to the fundamental SIR model. Numerical simulation graphs were drawn to illustrate changes in population density over time in SIR and SDEIR models. Results indicated that including hesitators and their corresponding parameters in the SIR model could moderately reduce the speed of rumor propagation. The impact of each parameter on chamber population density was charted by varying their values. Finally, the model includes optimal control conditions to numerically simulate and confirm the accuracy of the aforementioned conclusions. Additionally, understanding how to manage the spread of rumors in real-life holds considerable practical significance.

5.1. Comparison of results between SIR model and SDEIR model

To facilitate a more precise comparison between the outcomes of SIR and SDEIR models, we initially constructed an SIR model, adopting the parameters specified in this paper. The model's flow chart is presented below (see Figure 2):

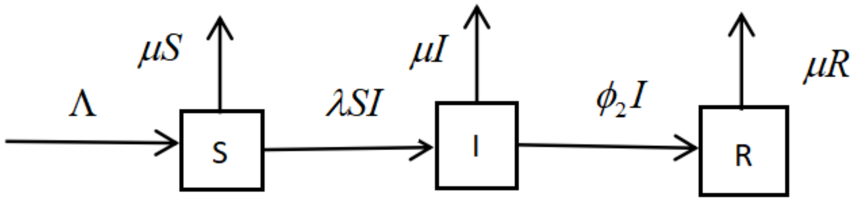

Figure 2. Flow chart of SIR model.

The model dynamics equation is as follows:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \lambda S I - \mu S \\ \frac{dI}{dt} = \lambda S I - \phi_2 I - \mu I \\ \frac{dR}{dt} = \phi_2 I - \mu R. \end{cases}$$

In the year 2020, there have been a number of rumors circulating in the city of Anshan regarding the potential prevention of COVID-19 infection. These rumors include the suggestion that alcohol consumption may offer protection against the virus, that drinking garlic water may prevent infection, and that the virus may only spread at certain times. In order to simulate these rumors, we initially conducted a demographic analysis of the city. A survey of the Anshan City population revealed that the total number of educated individuals is 3170657[*], of which 439634[*]have completed university studies, 497423[*] have completed high school, 170689894[*] have completed junior high school, and 526706[*]have completed primary school. The total number of uneducated individuals, or those who are illiterate, is 19911[*]. The average duration of the formation period for medical and health rumors is 2–3 days[†], the average duration of the outbreak period is 5–6 days[†], the average duration of the decline period is 2–3 days[†], and the duration of the rumor is mostly 8 days[†]. Therefore, $\frac{1}{\lambda} = 5$ is selected, namely a transmission rate of $\lambda = 0.2$, assuming a number of people entering the system $\Lambda = 50$, an immunisation rate $\phi_2 = 0.65$ and a removal rate $\mu = 0.3$.

Figure 3 shows the change of population density over time in the SIR model. Among them, $\Lambda = 50, \phi_2 = 0.65, \mu = 0.3, \lambda = 0.3$.

[*]http://tjj.anshan.gov.cn/html/TJJ/202106/0162398536049853.html

[†]Ren Qun. A Study on the Law of Rumors in Domeestic Network-An Empirical Study Based on 100 Hot Stops of Rumors Since 2016[D], Shandong University, China, 2017

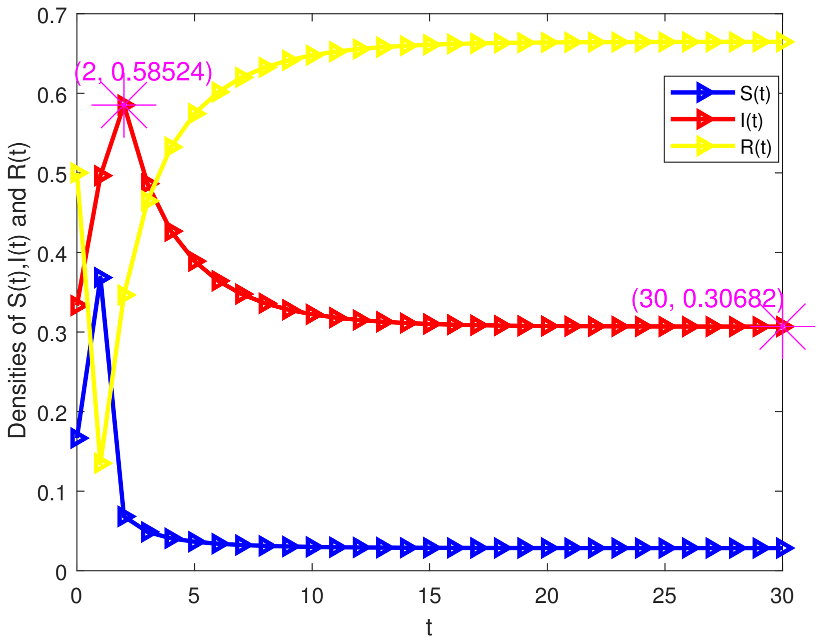

Figure 3. Changes of crowd density over time in the SIR model.

Figure 4 illustrates the population density variation in the SDEIR model with the inclusion of two chambers of rumor hesitators possessing diverse educational backgrounds on the basis of the SIR model. The results depict the changes over time, among them, $\Lambda = 50, \alpha_1 = 0.1, \alpha_2 = 0.13, \beta_1 = 0.15, \beta_2 = 0.33, \gamma_1 = 0.23, \gamma_2 = 0.27, \phi_1 = 0.25, \phi_2 = 0.65, \phi_3 = 0.21, \mu = 0.3, \lambda = 0.3$.

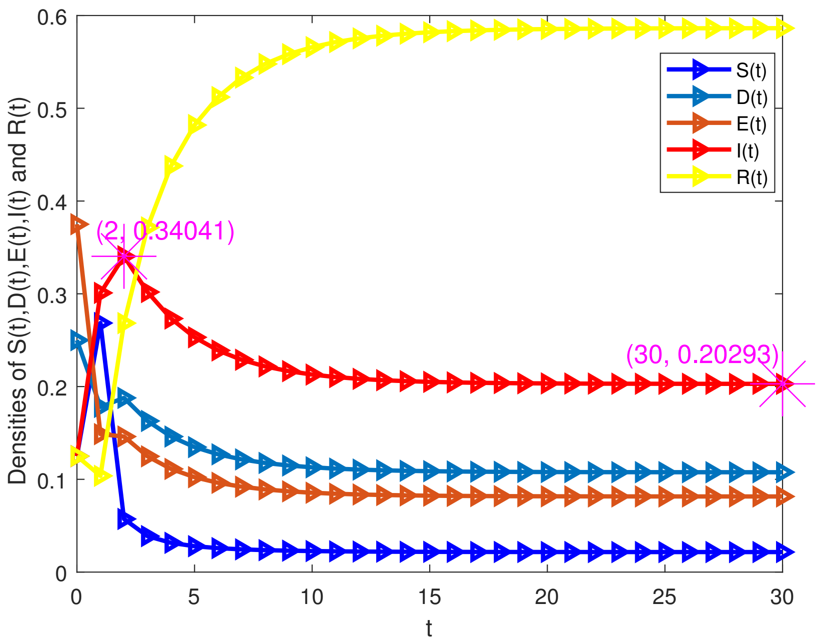

Figure 4. Changes of crowd density over time in the SDEIR model.

By comparing Figures 3 and 4, it can be seen that the scale of rumor outbreaks (the peak of the spreader curve) decreases significantly after adding the level of education under the condition of ensuring that the original parameter remains unchanged, (the peak of the number of spreaders in Figure 3 reaches 0.58524, and then decreases, and finally it is stabilized at 0.30682; the peak of the number of spreaders in Figure 4 reaches 0.34041, and then decreases, and finally it is stable at 0.20293). This means that, when people are educated, people will think when facing rumors, which reduces the scale of rumor outbreaks to a certain extent. By comparing the results of these two models, the practical significance of the model constructed in this paper can be visualized.

5.2. Numerical simulation of system stability

Let $\Lambda = 1, \alpha_1 = 0.1, \alpha_2 = 0.13, \beta_1 = 0.15, \beta_2 = 0.33, \gamma_1 = 0.23, \gamma_2 = 0.27, \phi_1 = 0.25, \phi_2 = 0.85, \phi_3 = 0.21, \mu = 0.3, \lambda = 0.23$, calculate $R_0 = 0.8781 < 1$, and the setting of parameter values satisfies the conditions of Theorem 1, then rumor-free equilibrium point E_0 is stable under different initial conditions.

As illustrated in Figure 5, over time, the number of $D(t), E(t), I(t), R(t)$ will approach zero, while the number of $S(t)$ will approach one. In other words, as time passes, the number of $S(t)$ will eventually equal the total number of people in the rumor system. Consequently, as time continues to elapse, rumors will eventually cease to exist.

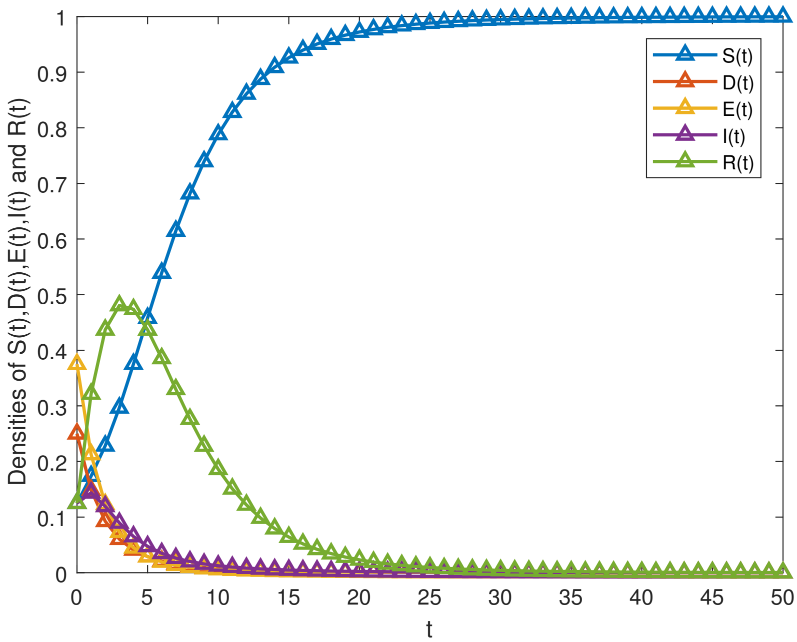

Figure 5. Stability of equilibrium E_0.

Let $\Lambda = 1, \alpha_1 = 0.5, \alpha_2 = 0.6, \beta_1 = 0.15, \beta_2 = 0.33, \gamma_1 = 0.23, \gamma_2 = 0.27, \phi_1 = 0.25, \phi_2 = 0.85, \phi_3 = 0.21, \mu = 0.3, \lambda = 0.23$. $R_0 = 1.3364 > 1$ is calculated, and the setting of parameter values satisfies the conditions of Theorem 2. At this time, Figure 6 clearly shows that the equilibrium point E^* of rumor propagation is locally asymptotically stable under this set of parameters.

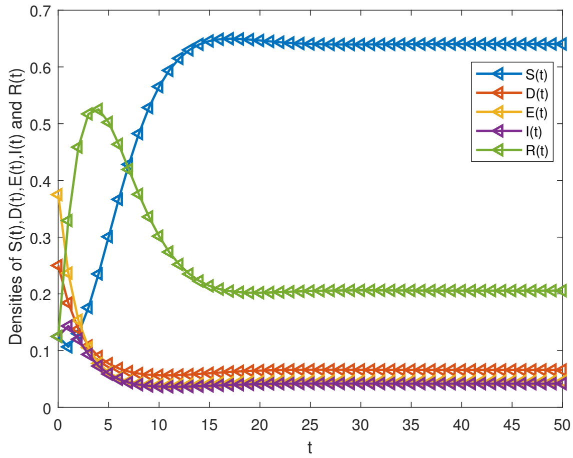

Figure 6. Stability of equilibrium point E^*.

As can be seen in Figure 6, in the early stage of rumor spreading, rumor unknowers begin to receive different levels of education and then start to change their views on rumors. As a result, the number of $D(t), E(t), R(t)$ will drop sharply in a short period of time and then level off, and the number of $R(t)$ will rise sharply and reach a peak in a short period of time, and then begin to drop and eventually level off. The above trend shows that over time, rumors can be controlled, but they do not disappear and will continue to exist.

In order to discuss the effect of different values of parameters in the system (2.1) on rumor propagation, the following Table 2 is plotted using the uniform distribution assumption. Based on the data in Table 2, the changes in the number of cabin corresponding to the parameter changes are plotted separately and conclusions are drawn.

Table 2. The individual compartments and values of the individual parameters.

Compartments and parameters	value
$S(t)$	100
$D(t)$	50
$E(t)$	50
$I(t)$	50
$R(t)$	50
Λ	100
μ	0.3
α_1	0.1
β_1	0.15
α_2	0.13
β_2	0.33
λ	0.23
γ_1	0.23
γ_2	0.27
ϕ_1	0.25
ϕ_2	0.85
ϕ_3	0.21

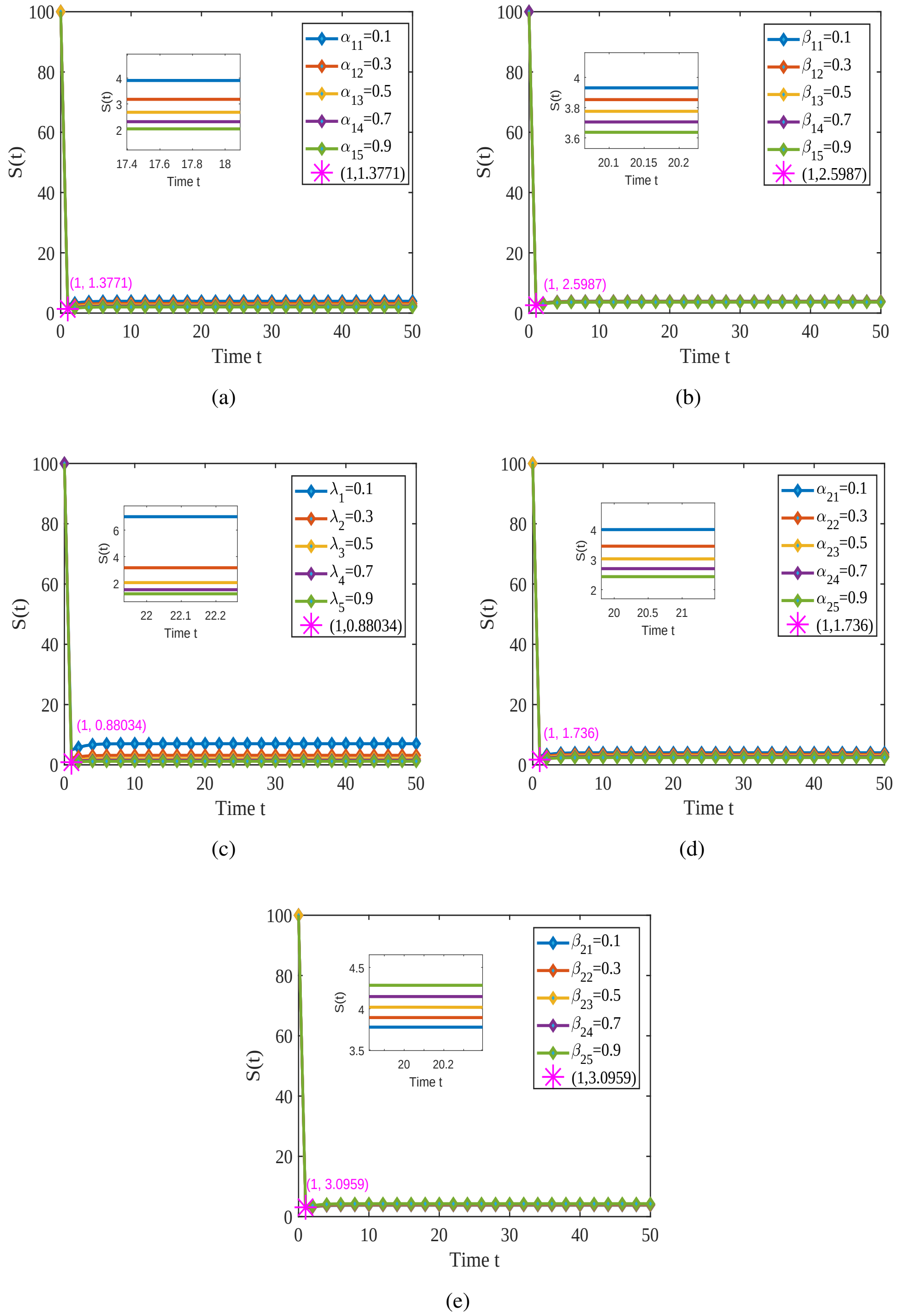

Figure 7. Variation in the number of $S(t)$ under different parameter conditions ((a) is the number of $S(t)$ under different values of parameter α_1; (b) is the number of $S(t)$ under different values of parameter β_1; (c) is the number of $S(t)$ under different values of parameter λ; (d) is the number of $S(t)$ under different values of parameter α_2; (e) is the number of $S(t)$ under different values of parameter β_2)

Figure 7 illustrates the variation in the number of $S(t)$ as a function of different values of parameters $\alpha_1, \beta_1, \lambda, \alpha_2$ and β_2, respectively. It can be observed that parameters $\alpha_1, \beta_1, \lambda, \alpha_2$ and $S(t)$ exhibit a negative correlation, whereas parameters β_2 and $S(t)$ exhibit a positive correlation. Consequently, in order to increase the number of $S(t)$, it is necessary to decrease the value of parameter $\alpha_1, \beta_1, \lambda, \alpha_2$ and increase the value of parameter β_2.

Figure 8 illustrates the variation in the number of $D(t)$ as a function of different values of parameters $\alpha_1, \beta_1, \gamma_1$ and ϕ_1. It can be observed that parameters α_1, β_1 and $D(t)$ exhibit a positive correlation, while parameters γ_1, ϕ_1 and $D(t)$ exhibit a negative correlation. Consequently, in order to reduce the number of $D(t)$, it is necessary to decrease the value of parameter α_1, β_1 and increase the value of parameter γ_1, ϕ_1.

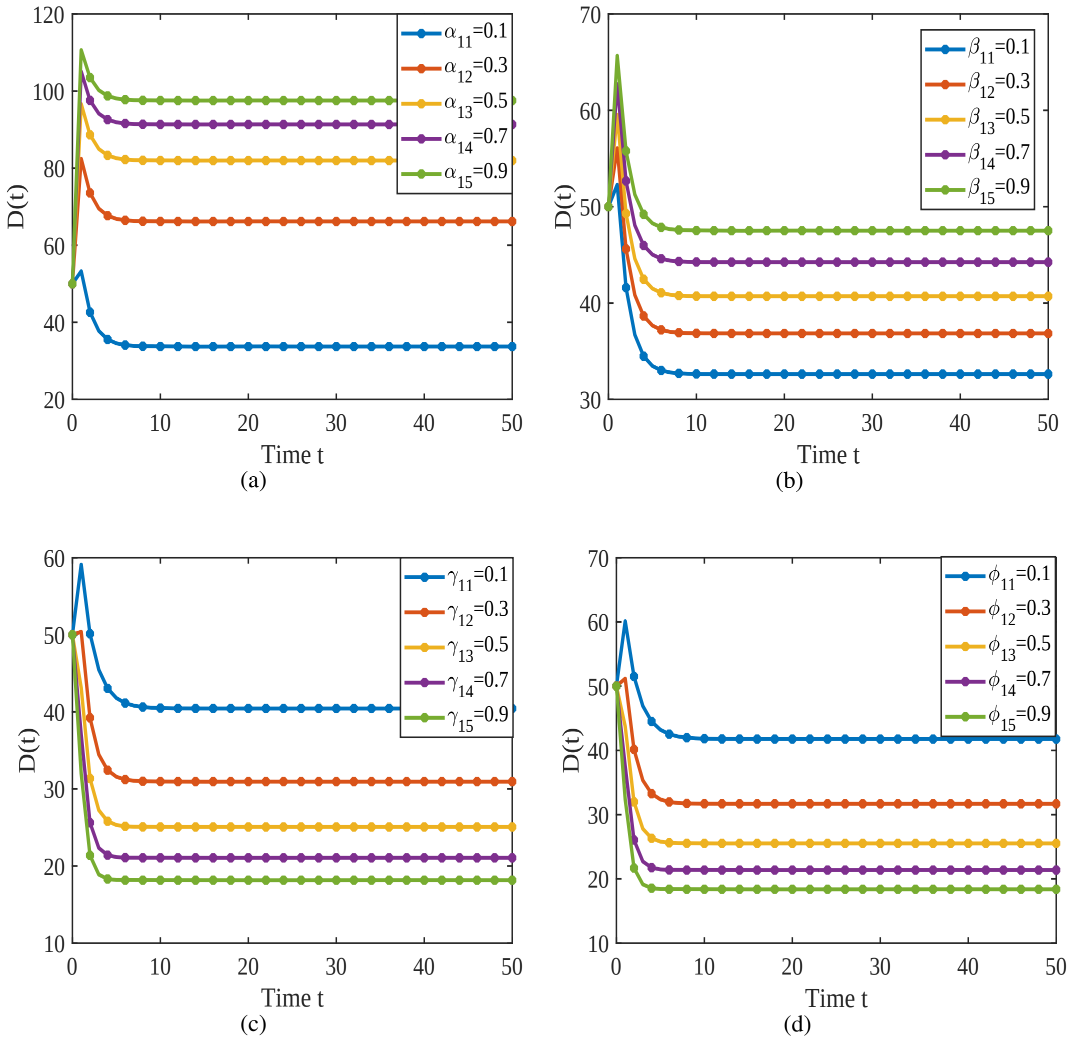

Figure 8. Variation in the number of $D(t)$ under different parameter conditions ((a) is the number of $D(t)$ under different values of parameter α_1; (b) is the number of $D(t)$ under different values of parameter β_1; (c) is the number of $D(t)$ under different values of parameter γ_1; (d) is the number of $D(t)$ under different values of parameter ϕ_1.

Figure 9 illustrates the variation in the number of $E(t)$ as a function of different values of parameters $\alpha_2, \beta_2, \gamma_2$ and ϕ_3. It can be observed that parameters α_2 and $E(t)$ exhibit a positive correlation, while parameters $\beta_2, \gamma_2, \phi_3$ and $E(t)$ exhibit a negative correlation. Consequently, in order to reduce the number of $E(t)$, it is necessary to decrease the value of parameter α_2 and increase the value of parameter $\beta_2, \gamma_2, \phi_3$.

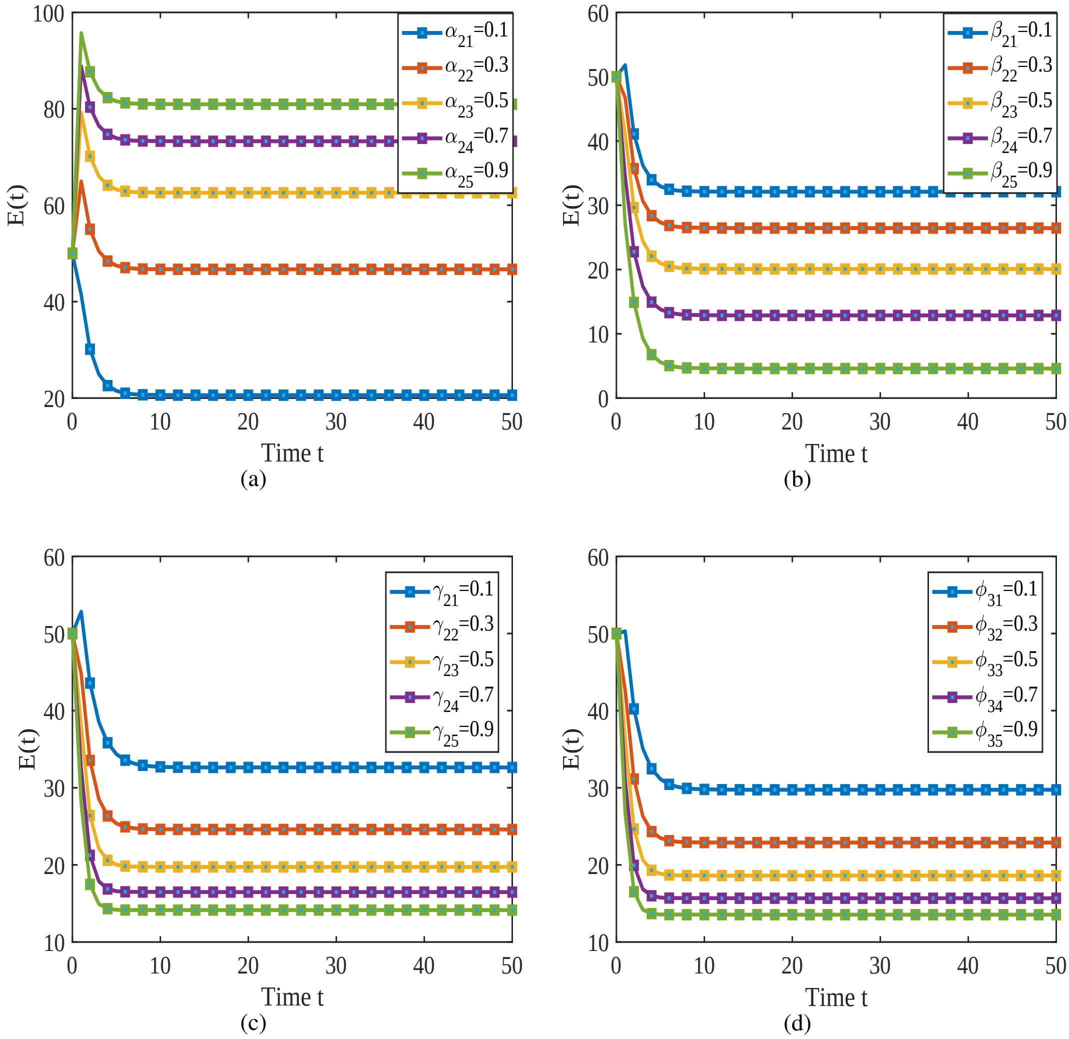

Figure 9. Variation in the number of $E(t)$ under different parameter conditions ((a) is the number of $E(t)$ under different values of parameter α_2; (b) is the number of $E(t)$ under different values of parameter β_2; (c) is the number of $E(t)$ under different values of parameter γ_2; (d) is the number of $E(t)$ under different values of parameter ϕ_3.

Figure 10 illustrates the variation in the number of $I(t)$ as a function of different values of parameters $\lambda, \gamma_1, \gamma_2$ and ϕ_2. It can be observed that parameters $\lambda, \gamma_1, \gamma_2$ and $I(t)$ exhibit a positive correlation, while parameters ϕ_2 and $I(t)$ exhibit a negative correlation. Consequently, in order to reduce the number of $I(t)$, it is necessary to decrease the value of parameter $\lambda, \gamma_1, \gamma_2$ and increase the value of parameter ϕ_2.

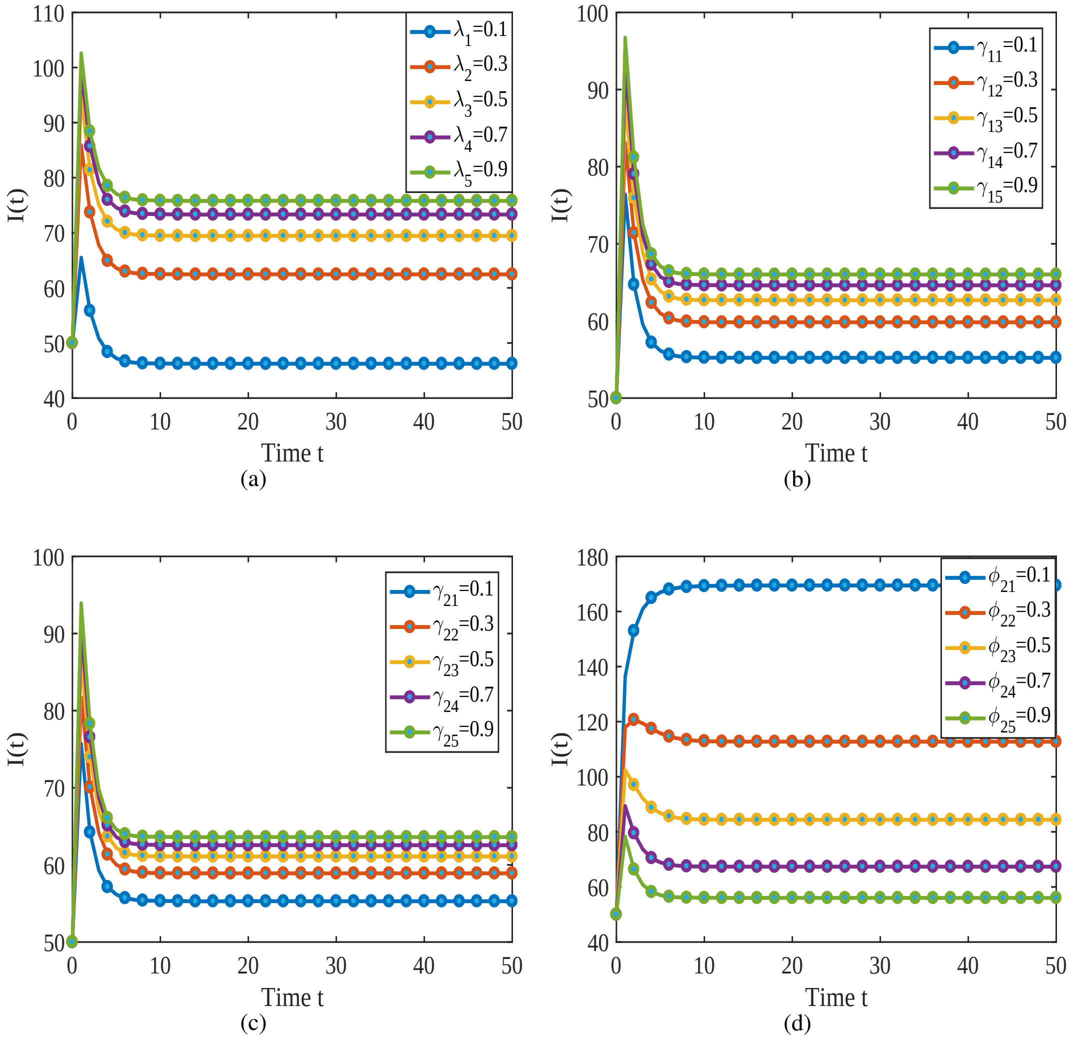

Figure 10. Variation in the number of $I(t)$ under different parameter conditions ((a) is the number of $I(t)$ under different values of parameter λ; (b) is the number of $I(t)$ under different values of parameter γ_1; (c) is the number of $I(t)$ under different values of parameter γ_2; (d) is the number of $I(t)$ under different values of parameter ϕ_2.

Figure 11 illustrates the variation in the number of $R(t)$ as a function of different values of parameters ϕ_1, ϕ_2 and ϕ_3. It can be observed that parameters ϕ_1, ϕ_2, ϕ_3 and $R(t)$ exhibit a positive correlation. Consequently, in order to increase the number of $R(t)$, it is necessary to increase the value of parameter ϕ_1, ϕ_2, ϕ_3 .

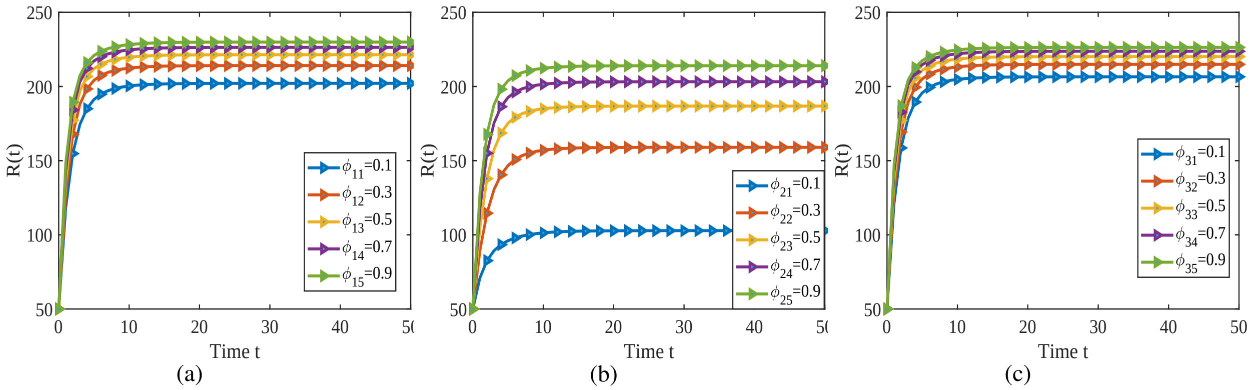

Figure 11. Variation in the number of $R(t)$ under different parameter conditions ((a) is the number of $R(t)$ under different values of parameter ϕ_1; (b) is the number of $R(t)$ under different values of parameter ϕ_2; (c) is the number of $R(t)$ under different values of parameter ϕ_3.

In conclusion, in order to reduce the spread of rumors, it is necessary to decrease the natural contact rate between rumor unknowers and rumor spreaders, increase the percentage of rumor unknowers with a higher level of education, and decrease the transmission rate and increase the immunity rate. Consequently, in order to mitigate the spread of rumors, the following section will continue to detail the most effective method of controlling the spread of rumors by manipulating parameter $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda$ values.

5.3. Numerical simulation of optimal control

In this section, by controlling the above five parameters, that is, $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda$, Figure 12 is drawn.

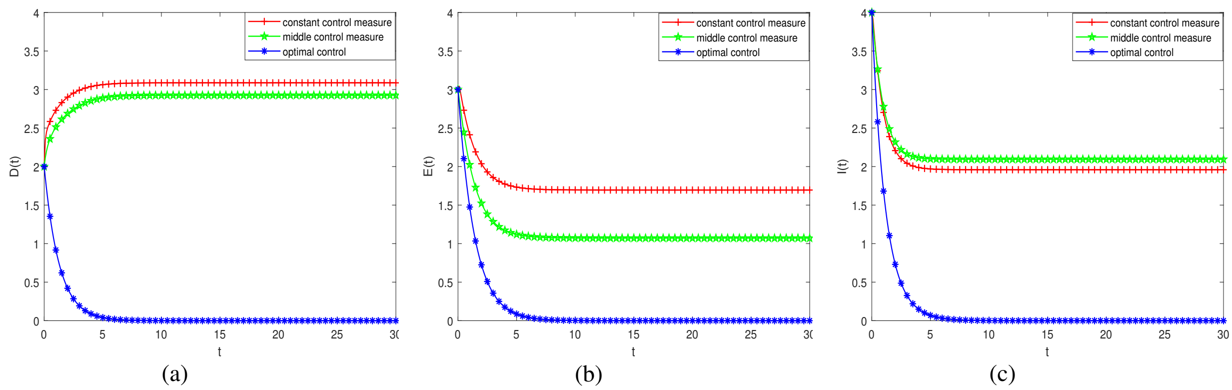

Figure 12. The change of the density of $D(t), E(t), I(t)$ with time under different strategies.

Because optimal control mainly controls the population density of three chambers: Rumor hesitators with less education, rumor hesitators with higher education, and rumor spreaders. Therefore, during the drawing process, we only display $D(t)$, $E(t)$ and $I(t)$.

In Figure 12,

(a) optimal control: $\alpha_1 = 0, \alpha_2 = 0, \beta_1 = 0, \beta_2 = 1, \lambda = 0$

(b) middle control: $\alpha_1 = 0.5, \alpha_2 = 0.5, \beta_1 = 0.5, \beta_2 = 0.5, \lambda = 0.5$

(c) constant control: $\alpha_1 = 1, \alpha_2 = 1, \beta_1 = 1, \beta_2 = 0, \lambda = 1$.

After observing Figure 12, it becomes clear that optimal control is more effective than middle control or constant control at all. With optimal control, the number of less-educated rumor hesitators, higher-educated rumor hesitators, and rumor spreaders can all be reduced to zero, resulting in successful rumor control.

Combined with real-life scenarios, specific measures can be employed to regulate the five key parameters outlined in the article, thus controlling the spread of rumors. The government and official media are recommended to reinforce their supervision of online information propagation, which could effectively decrease the natural contact rate between rumor spreaders and rumor unknowers. Furthermore, the government and schools could enhance the education of those rumors unknowers and uplift people's education level to limit the propagation of rumors.

6. Conclusions

We studied the influence of education level on rumor propagation, established the SDEIR model, analyzed the basic reproduction number, rumor equilibrium point and stability, optimized the control strategy, and verified the theoretical results through numerical simulation.

The main conclusions of this paper are as follows:

(1) When $R_0 < 1$, the rumor will disappear from the system over time; when $R_0 > 1$, the rumor will not disappear in the future, but will reach a stable state.

(2) Based on the classical SIR rumor propagation model, the scale of rumor outbreak is significantly reduced after the education level is introduced.

(3) Reducing the contact rate of $I(t)$ and $S(t)$ with different levels of education, increasing the proportion of $S(t)$ with higher education level, reducing the transmission rate and improving the immunization rate can inhibit the spread of rumors to varying degrees.

(4) When controlling parameter $\alpha_1(t) = 0, \alpha_2(t) = 0, \beta_1(t) = 0, \beta_2(t) = 1, \lambda(t) = 0$, the optimal control effect is achieved and the purpose of controlling rumor propagation is realized.

(5) Although improving people's education level can reduce the spread of rumors to a certain extent, it cannot completely stop the spread of rumors.

The spread of rumors is a complex social phenomenon, and further research and comprehensive measures are needed to effectively deal with the spread of rumors. In fact, in addition to the changes between different chambers considered in this paper, $S(t)$ may be directly transformed into $R(t)$, which also has an important impact on the process of rumor propagation. In addition to government and state media, other media have the potential to change the way rumors spread. However, these cases are not detailed in this article. It is hoped that future studies can comprehensively consider the influence of various factors and provide more effective control measures and prevention strategies to meet the challenges brought by rumor propagation.

Author contributions

Hongshuang Wang: Writing-original draft and writing-review & editing; Sida Kang: Writing-original draft; Yuhan Hu: Supervision and writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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