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Research article

Commuting Toeplitz operators on weighted harmonic Bergman spaces and hyponormality on the Bergman space of the punctured unit disk

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Abstract: We first describe commuting Toeplitz operators with harmonic symbols on weighted harmonic Bergman spaces. Then, a sufficient condition for hyponormality on weighted Bergman spaces of the punctured unit disk, when the analytic part of the symbol is a monomial, is shown.

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1. Introduction

Finding commuting Toeplitz operators is a question of interest for researchers working on Toeplitz operators. The most important results on this subject are from the work of Axler and Cuckovic [1] in the case of the Bergman space, and Choe and Lee in the case of the harmonic Bergman space [2]. The results of Axler and Cuckovic are generalized by the authors to the case of the weighted Bergman space [3]. In the first part, we generalize most of the results of Choe and Lee to the case of weighted harmonic Bergman spaces. In the second part, we give a sufficient condition for hyponormality on newly considered Bergman spaces on the punctured unit disk. Hyponormality has seen growing interest the last few decades. The first important result is the following.

Theorem 1.1. *Let f, g be bounded and analytic in the unit disk D with $f' \in H^2$. If $T_{f+\overline{g}}$ is hyponormal on the Bergman space, then $g' \in H^2$ and $|g'| \leq |f'|$ a.e in the unit circle.*

This can be found in [4]. Ahern and Cuckovic generalized this result in [5] by weakening the assumption on the derivative. Their result itself is generalized to weighted Bergman spaces in [6].

Theorem 1.2. *Let f, g be bounded and analytic in the unit disk D. Assume $f' \in H^2(I)$, where I is an open arc of the unit circle. If $T_{f+\overline{g}}$ is hyponormal on the weighted Bergman space $L^2_{a,\omega}$, where $\omega(r) = (\alpha+1)(1-r^2)^{\alpha}$, $\alpha > -1$, then, $g' \in H^2(I)$ and $|g'| \leq |f'|$ a.e on I.*

2. Commuting Toeplitz operators on the weighted harmonic Bergman space

Denote by D the unit disk and consider the Hilbert space $L^2(D, d\mu_\alpha)$ of measurable functions on D such that

$$(\alpha+1)\int_D |f(z)|^2(1-|z|^2)^\alpha \, dA(z) < \infty.$$

The closed subspace of $L^2(D, d\mu_\alpha)$ consisting of harmonic functions is denoted by L_h^2. If the weighted Bergman space is denoted by $L_{a,\omega}^2$ where $\omega(r) = (\alpha+1)(1-r^2)^\alpha$, then $L_h^2 = L_{a,\omega}^2 \oplus \overline{zL_{a,\omega}^2}$. As in the case of unweighted spaces ($\omega = 1$) [2], the reproducing kernel is given by

$$K(z,w) = \frac{1}{(1-z\overline{w})^{\alpha+2}} + \frac{1}{(1-\overline{z}w)^{\alpha+2}} - 1.$$

In what follows, we consider commuting Toeplitz operators on L_h^2 with harmonic symbols φ and ψ. We denote by P the projection of $L^2(D, d\mu_\alpha)$ onto $L_{a,\omega}^2$, and by Q the projection onto L_h^2. We have $Q(\varphi) = P(\varphi) + \overline{P(\overline{\varphi})} - P(\varphi)(0)$, where $P(\varphi)(0) = \int_D \varphi dA$. We give a generalization of some of the results in [2], shown in the case of the unweighted harmonic Bergman space, to the case of weighted harmonic Bergman spaces with radial weights $\omega(r) = (\alpha+1)(1-r^2)^\alpha$, where $\alpha > -1$. Our first main result is the following theorem.

Theorem 2.1. *Let φ and ψ be in L_h^2 and not both conjugate holomorphic. If $T_\varphi T_\psi = T_\psi T_\varphi$, then $\frac{\partial\psi}{\partial z} = c\frac{\partial\varphi}{\partial z}$ for some constant c.*

The proof is based on the properties of the projection. Here is a summary of the proof of Theorem 2.1. Set $\varphi = \varphi_1 + \overline{\varphi_2}$, $\psi = \psi_1 + \overline{\psi_2}$. Then,

$$T_\varphi T_\psi = T_\psi T_\varphi \Leftrightarrow T_{\varphi_1} T_{\overline{\psi_2}} + T_{\overline{\varphi_2}} T_{\psi_1} = T_{\overline{\psi_2}} T_{\varphi_1} + T_{\psi_1} T_{\overline{\varphi_2}}.$$

By evaluation at $\overline{w}$, and after making extensive use of some properties of the Bergman space projection we get $\varphi_1 P(\overline{w}\psi_1) = \psi_1 P(\overline{w}\varphi_1)$. By Proposition 2.2, this is reduced to $\psi_1 = c\varphi_1$.

We start by verifying these properties.

Proposition 2.2. *Let $\varphi \in L_{a,\omega}^2$. The projection onto $L_{a,\omega}^2$ satisfies the following properties:*

(i) $z^2 \frac{d}{dz} P\left((1-|w|^2)\varphi\right) + (\alpha+2) z P\left((1-|w|^2)\varphi\right) = (\alpha+1) z\varphi(z)$.

(ii) $z^2 \frac{d}{dz} P\left((1-|w|^2)\overline{\varphi}\right) + (\alpha+2) z P\left((1-|w|^2)\overline{\varphi}\right) = (\alpha+1) z\overline{\varphi}(0)$.

(iii) $P\left(|w|^2\varphi\right)(z) = \varphi(z) - \frac{(\alpha+1)}{z^{\alpha+2}} \int_0^z u^{\alpha+1}\varphi(u) du$.

(iv) $P\left(|w|^2\overline{\varphi}\right) = \overline{P\left(|w|^2\overline{\varphi}\right)(0)} = \frac{\alpha+1}{\alpha+2}\overline{\varphi}(0)$.

Proof. For (i) we have

$$f(z) = P\left((1-|w|^2)\varphi\right)(z) = (\alpha+1)\int_D \frac{\varphi(w)}{(1-z\overline{w})^{\alpha+2}}(1-|w|^2)^{\alpha+1} dA(w)$$

and, by differentiating under the integral sign, we get

$$f'(z) = (\alpha+1)(\alpha+2)\int_D \frac{\varphi(w)\overline{w}}{(1-z\overline{w})^{\alpha+3}}(1-|w|^2)^{\alpha+1}dA(w).$$

We deduce

$$\begin{aligned}(\alpha+2)zf(z)+z^2f'(z) &= (\alpha+1)(\alpha+2)\int_D \frac{\varphi(w)z}{(1-z\overline{w})^{\alpha+3}}(1-|w|^2)^{\alpha+1}dA(z)\\ &= (\alpha+1)z\varphi(z),\end{aligned}$$

where the last equality holds because $K_z(w) = \frac{1}{(1-z\overline{w})^{\alpha+3}}$ is the reproducing kernel of the weighted Bergman space with a weight $\omega_2(r) = (\alpha+2)(1-r^2)^{\alpha+1}dA(z)$ [7]. We obtain Eq (ii) in a similar manner. For Eq (iii), we formally solve the differential equation

$$f' + \frac{(\alpha+2)}{z}f = \frac{(\alpha+1)}{z}\varphi,$$

which gives

$$P\left((1-|w|^2)\varphi\right)(z) = \frac{(\alpha+1)}{z^{\alpha+2}}\int_0^z u^{\alpha+1}\varphi(u)\,du,$$

and

$$P\left(|w|^2\varphi\right)(z) = \varphi(z) - \frac{(\alpha+1)}{z^{\alpha+2}}\int_0^z u^{\alpha+1}\varphi(u)\,du.$$

Similarly, from (ii), by integration we get

$$P\left((1-|w|^2)\overline{\varphi}\right) = \frac{\alpha+1}{\alpha+2}\overline{\varphi(0)},$$

thus

$$P\left(|w|^2\overline{\varphi}\right) = \frac{1}{\alpha+2}\overline{\varphi(0)}.$$

□

This leads to the following properties of the projection.

Lemma 2.3. *Let $\varphi \in L^2_{a,\omega}$ satisfy $\varphi(0) = 0$. The following properties hold:*

1) $P(\overline{w}\varphi)(z) = \frac{\varphi(z)}{z} - \frac{\alpha+1}{z^{\alpha+2}}\int_0^z u^\alpha\varphi(u)\,du.$

2) $P(w\overline{\varphi})(z) = \frac{1}{\alpha+2}\overline{\varphi'(0)}.$

Proof. Since $\varphi(0) = 0$, $\varphi(z) = z\psi(z)$. From (iii) and (iv) of Proposition 2.2, we get

$$P(\overline{w}\varphi)(z) = P\left(|w|^2\psi\right)(z) = \frac{\varphi(z)}{z} - \frac{(\alpha+1)}{z^{\alpha+2}}\int_0^z u^\alpha\varphi(u)\,du,$$

$$P(w\overline{\varphi})(z) = P\left(|w|^2\overline{\psi}\right)(z) = \frac{1}{\alpha+2}\overline{\psi(0)} = \frac{1}{\alpha+2}\overline{\varphi'(0)}.$$

□

The following properties of the projection of $L^2(D, d\mu_\alpha)$ are easy to check. The first one is a consequence of the known property of Toeplitz operators $T_f T_g = T_{fg}$ on $L^2_{a,\omega}$ if g is analytic or f is conjugate analytic.

Lemma 2.4. *For φ and $\psi \in L^2_{a,\omega}$, we have the following equalities:*

(i) $P(\overline{\varphi} P(\overline{w}\psi))(z) = P(\overline{\varphi} \overline{w}\psi)(z)$.

(ii) $P\left(\varphi \overline{P(\overline{w}\psi}\right)(0) = P\left(\varphi w \overline{\psi}\right)(0)$.

Proof. We verify (ii):

$$\begin{aligned} P\left(\varphi \overline{P(\overline{w}\psi)}\right)(0) &= \left\langle P\left(\varphi \overline{P(\overline{w}\psi)}\right), 1\right\rangle = \left\langle \varphi \overline{P(\overline{w}\psi)}, 1\right\rangle = \left\langle \overline{P(\overline{w}\psi)}, \overline{\varphi}\right\rangle \\ &= \langle \varphi, P(\overline{w}\psi)\rangle = P\left(\varphi w \overline{\psi}\right)(0). \end{aligned}$$

□

This leads to the following theorem.

Theorem 2.5. *Let φ and ψ be in L^2_h. If $T_\varphi T_\psi = T_\psi T_\varphi$, then, there exists a constant λ such that $\frac{\partial \psi}{\partial z} = \lambda \frac{\partial \varphi}{\partial z}$.*

Proof. We have $\varphi = \varphi_1 + \overline{\varphi_2}$, and $\psi = \psi_1 + \overline{\psi_2}$, where φ_1, φ_2, ψ_1 and $\psi_2 \in L^2_{a,\omega}$. We may assume

$$\varphi_1(0) = \varphi_2(0) = \psi_1(0) = \psi_2(0) = 0.$$

We evaluate both sides of the equality $T_\varphi T_\psi = T_\psi T_\varphi$ at the function $\overline{w}$. We make use of the following equalities, where Lemmas 2.3 and 2.4 are used.

$$T_{\psi_1}(\overline{w}) = P(\overline{w}\psi_1) + \overline{P\left(w\overline{\psi_1}\right)} - P(\overline{w}\psi_1)(0) = P(\overline{w}\psi_1)$$

$$T_{\overline{\varphi_2}} T_{\psi_1}(\overline{w}) = P(\overline{\varphi_2}(P(\overline{w}\psi_1))) + \overline{P\left(\varphi_2 \overline{P(\overline{w}\psi_1)}\right)} - P(\overline{\varphi_2} P(\overline{w}\psi_1))(0)$$

$$T_{\varphi_1} T_{\psi_1}(\overline{w}) = \varphi_1 P(\overline{w}\psi_1)$$

$$T_{\varphi_1} T_{\overline{\psi_2}}(\overline{w}) = Q\left(\varphi_1 \overline{w}\overline{\psi_2}\right) = P\left(\varphi_1 \overline{w}\overline{\psi_2}\right) + \overline{P(\overline{\varphi_1} w \psi_2)} - P\left(\varphi_1 \overline{w}\overline{\psi_2}\right)(0)$$

$$T_{\overline{\varphi_2}} T_{\overline{\psi_2}}(\overline{w}) = \overline{\varphi_2}\overline{\psi_2}\overline{w}.$$

Similarly, we compute

$$T_{\overline{\psi_2}} T_{\varphi_1}(\overline{w}) = P\left(\overline{\psi_2}(P(\overline{w}\varphi_1))\right) + \overline{P\left(\psi_2 \overline{P(\overline{w}\varphi)}\right)} - P\left(\overline{\psi_2} P(\overline{w}\varphi_1)\right)(0)$$

$$T_{\psi_1} T_{\overline{\varphi_2}}(\overline{w}) = Q(\psi_1 \overline{w\varphi_2}) = P(\psi_1 \overline{w\varphi_2}) + \overline{P(\overline{\psi_1} w \varphi_2)} - P(\psi_1 \overline{w\varphi_2})(0)$$

$$T_{\psi_1} T_{\varphi_1}(\overline{w}) = \psi_1 P(\overline{w}\varphi_1)$$

$$T_{\overline{\psi_2}} T_{\overline{\varphi_2}}(\overline{w}) = \overline{\varphi_2}\overline{\psi_2}\overline{w}.$$

Thus, the equality

$$T_\varphi T_\psi(\overline{w}) = T_\psi T_\varphi(\overline{w})$$

leads to

$$\varphi_1 P(\overline{w}\psi_1) + \overline{P(\overline{\varphi_1} w\psi_2)} + \overline{P\left(\varphi_2 \overline{P(\overline{w}\psi_1)}\right)} = \psi_1 P(\overline{w}\varphi_1) + \overline{P(\overline{\psi_1} w\varphi_2)} + \overline{P\left(\psi_2 \overline{P(\overline{w}\varphi_1)}\right)}. \tag{2.1}$$

We deduce that

$$\varphi_1 P(\overline{w}\psi_1) = \psi_1 P(\overline{w}\varphi_1),$$

which, by Proposition 2.2, gives

$$\varphi_1(z).\left(\frac{\psi_1(z)}{z} - \frac{\alpha+1}{z^{\alpha+2}}\int_0^z u^\alpha \psi_1(u)du\right) = \psi_1(z).\left(\frac{\varphi_1(z)}{z} - \frac{\alpha+1}{z^{\alpha+2}}\int_0^z u^\alpha \varphi_1(u)\, du\right),$$

which can be written as

$$\Phi_1'(z)\Psi_1(z) = \Phi_1(z)\Psi_1'(z),$$

where

$$\Phi_1(z) = \int_0^z u^\alpha \varphi_1(u)du), \quad \Psi_1(z) = \int_0^z u^\alpha \psi_1(u)du.$$

This shows that

$$\Psi_1(z) = c\Phi_1(z),$$

and, consequently,

$$\psi_1 = c\varphi_1$$

for some constant c. □

Corollary 2.6. *Let φ, ψ be in $L^2_{a,\omega}$. The equality $T_\varphi T_\psi = T_\psi T_\varphi$ holds if and only if $\psi = c\varphi + d$ for some constants c and d.*

Our next main result is based on the following lemmas.

Lemma 2.7. *Let φ, ψ be in $L^2_{a,\omega}$. Then we have the following identity:*

$$(\alpha+1)\left(\int_D \varphi(w)\overline{\psi}(w)(1-|w|^2)^\alpha dA(w)\right) = \int_D \left((\alpha+2)\varphi(w)\overline{\psi(w)} + \overline{w\psi'(w)}\right)(1-|w|^2)^{\alpha+1} dA(w).$$

Proof. Use power series and the fact that, for φ, ψ in $L^2_{a,\omega}$, with $\varphi = \sum_{n\geq 0}\varphi_n z^n$, $\psi = \sum_{n\geq 0}\psi_n z^n$, we have the inner product property [7]

$$(\alpha+1)\left(\int_D \varphi(w)\overline{\psi}(w)(1-|w|^2)^\alpha dA(w)\right) = \sum_{n\geq 0}\frac{n!\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)}\varphi_n\overline{\psi_n}.$$

□

Lemma 2.8. *Let φ, ψ be in $L^2_{a,\omega}$ and satisfy $\varphi(0) = \psi(0) = 0$. If $T_\varphi T_{\overline{\psi}} = T_{\overline{\psi}} T_\varphi$ on L^2_h, then the following equality holds*

$$\int_D \varphi(w)\overline{\Psi(w)w^{n+\alpha}}(1-|w|^2)^{\alpha+1} dA(w) = 0$$

for any $n \in \mathbb{N}$, where $\Psi(w) = \frac{\alpha+1}{w^{\alpha+1}}\int_0^w u^\alpha \psi(u)du$.

Proof. From $T_\varphi T_{\overline{\psi}} = T_{\overline{\psi}} T_\varphi$, we deduce $T_\psi T_{\overline{\varphi}} = T_{\overline{\varphi}} T_\psi$, which, by evaluation at $\overline{w}$ and from Eq (2.1) by taking $\varphi = \varphi_2$, $\varphi_1 = 0$, $\psi = \psi_1$, and $\psi_2 = 0$ gives

$$P\left(\varphi w\overline{\psi}\right) = P\left(\varphi\overline{P(\overline{w}\psi)}\right).$$

This gives

$$\left\langle \varphi, P(\psi\overline{w})w^{n+\alpha+1}\right\rangle = \left\langle P\left(\varphi\overline{P\left(\overline{w}\psi\right)}\right), w^{n+\alpha+1}\right\rangle = \left\langle P\left(\varphi w\overline{\psi}\right), w^{n+\alpha+1}\right\rangle = \left\langle \varphi w, \psi w^{n+\alpha+1}\right\rangle, \quad n \geq 0.$$

Using the identity $wP\left(\psi\overline{w}\right) = \psi - \Psi$, we get

$$(\alpha+1)\int_D \varphi(w)\overline{(\psi(w) - \Psi(w))w^{n+\alpha}}(1-|w|^2)^\alpha dA(w) = (\alpha+1)\int_D \varphi(w)\overline{\psi(w)w^{n+\alpha}}|w|^2(1-|w|^2)^\alpha dA(w),$$

which can be written as

$$(\alpha+1)\int_D \varphi(w)\overline{\psi(w)w^{n+\alpha}}(1-|w|^2)^{\alpha+1} dA(w) = (\alpha+1)\int_D \varphi(w)\overline{\Psi(w)w^{n+\alpha}}(1-|w|^2)^\alpha dA(w). \tag{2.2}$$

Using Lemma 2.7, we get

$$\begin{aligned}(\alpha+1)\int_D \varphi(w)\overline{\Psi(w)w^{n+\alpha}}(1-|w|^2)^\alpha dA(w) &= \int_D \varphi(w)\left((\alpha+2)\overline{\Psi(w)w^{n+\alpha}} + \overline{w}\overline{(\Psi(w)w^{n+\alpha})'}\right)(1-|w|^2)^{\alpha+1}dA(w) \\ &= \int \varphi(w)\left((n+\alpha+1)\overline{\Psi(w)} + (\alpha+1)\overline{\psi(w)}\right)\overline{w^{n+\alpha}}(1-|w|^2)^{\alpha+1}dA(w).\end{aligned}$$

After a simplification using Eq (2.2), we deduce

$$\int_D \varphi(w)\overline{(\Psi(w)w^{n+\alpha})'}(1-|w|^2)^{\alpha+1}dA(w) = 0.$$

□

This leads to the second main result. The proof, being similar to the case of the unweighted harmonic Bergman space [2], is omitted.

Theorem 2.9. *Assume $f \in L^2_h$. Then, T_f is normal if and only if $f(D)$ is contained in a line in the complex plane. If $f \in L^2_{a,\omega}$, then T_f is normal if and only if f is constant.*

3. Hyponormality on the weighted Bergman space of the punctured unit disk

Let m be a nonnegative integer, β be in $(m-1, m]$, and $\alpha > -1$ be a real number. Let $d\mu_{\alpha,\beta}(z) = \frac{1}{B(\alpha+1,\beta+1)}|z|^{2\beta}(1-|z|^2)^\alpha dA(z)$ be the normalized Lebesgue measure on the unit disk D. The space $L^2(D, d\mu_{\alpha,\beta})$ is the Hilbert space of measurable functions f on D such that

$$\|f\|^2 = \int_D |f(z)|^2 d\mu_{\alpha,\beta}(z) < \infty.$$

We consider holomorphic functions g on $D^* = D - \{0\}$ that satisfy

$$\int_{D^*} |g(z)|^2 d\mu_{\alpha,\beta}(z) < \infty.$$

In [8], it is shown that this space, denoted by $A^2_{\alpha,\beta}$, is a closed Hilbert space of $L^2\left(D, d\mu_{\alpha,\beta}\right)$. If $f = \sum_{n\geq -m} a_n z^n$ and $g = \sum_{n\geq -m} b_n z^n$, then the inner product is given by

$$\langle f, g\rangle = \sum_{n\geq -m} \frac{B(\alpha+1, \beta+n+1)}{B(\alpha+1, \beta+1)} a_n \overline{b_n},$$

and its natural orthonormal basis is $\left\{e_n(z) = \sqrt{\frac{B(\alpha+1,\beta+1)}{B(\alpha+1,\beta+n+1)}} z^n,\ n \geq -m\right\}$. We set $\rho_n = \frac{B(\alpha+1,\beta+n+1)}{B(\alpha+1,\beta+1)}$, $n \geq -m$, and recall that a Toeplitz operator T_φ, for φ bounded measurable on D, is defined on $A^2_{\alpha,\beta}$ by $T_\varphi(f) = P(\varphi f)$, where P is the orthogonal projection onto $A^2_{\alpha,\beta}$. Hankel opertaors on $A^2_{\alpha,\beta}$ are defined by $H_\varphi(f) = (I - P)(\varphi f)$. The following properties of Toeplitz operators, known in the case of the weighted Bergman space A^2_α ($\beta = 0$), hold also on $A^2_{\alpha,\beta}$.

Proposition 3.1. *[8] Let φ and ψ be bounded measurable on D. The following properties hold:*

(i) $T_{\varphi+\psi} = T_\varphi + T_\psi$.

(ii) $\left(T_\varphi\right)^* = T_{\overline{\varphi}}$.

(iii) $T_\psi T_\varphi = T_{\varphi\psi}$ if φ analytic or ψ is conjugate analytic on D.

A bounded operator S on a Hilbert space is hyponormal if $S^*S - SS^* \geq 0$. In what follows, we consider hyponormality of Toeplitz operator $T_{z^q+\overline{\psi}}$ on $A^2_{\alpha,\beta}$ where ψ is a polynomial. We show a sufficient condition in this case. As in the case of the classical unweighted Bergman space $A^2_{0,0}$ [4], hyponormality is expressed in various equivalent forms which are listed in the following proposition.

Proposition 3.2. *[4] The following statements are equivalent:*

(a) $T_{\varphi+\overline{\psi}}$ is hyponormal.

(b) $T_{\overline{\psi}} T_\psi - T_\psi T_{\overline{\psi}} \leq T_{\overline{\varphi}} T_\varphi - T_\varphi T_{\overline{\varphi}}$.

(c) $\left(H_{\overline{\psi}}\right)^* H_{\overline{\psi}} \leq \left(H_{\overline{\varphi}}\right)^* H_{\overline{\varphi}}$.

(d) $H_{\overline{\psi}} = C H_{\overline{\varphi}}$, where C is bounded of norm less than or equal to one.

3.1. The main result

We consider the case $\alpha = 3$. Let l and q be two integers such that $q \geq m + 1$. Denote by N the smallest integer such that $N \geq \max\left(6q + 1, 8, \frac{1}{2}(q^2 - 1)\right)$. Our main result is the following theorem.

Theorem 3.3. *Let $(\lambda_l)_{N\leq l\leq N_1}$ be any finite sequence of complex numbers such that $\sum_{N\leq l\leq N_1} |\lambda_l| \leq 1$, and set $g = \sum_{N\leq l\leq N_1} \lambda_l \frac{q}{l} z^l$. Then, the operator $T_{z^q+\overline{g}}$ is hyponormal on $A^2_{\alpha,\beta}$ i.e $T_{\overline{g}} T_g - T_g T_{\overline{g}} \leq T_{\overline{z}^q} T_{z^q} - T_{z^q} T_{\overline{z}^q}$.*

In the case $\alpha = 3$, we have

$$\rho_n = \frac{(\beta+4)(\beta+3))(\beta+2)(\beta+1)}{(\beta+n+4)(\beta+n+3)(\beta+n+2)(\beta+n+1)}.$$

We start by computing the matrix of $T_{\overline{z}^q} T_{z^q} - T_{z^q} T_{\overline{z}^q}$ in the orthonomal basis $\{e_n,\ n \geq -m\}$.

Lemma 3.4. *The matrix of $T_{\overline{z}^q}T_{z^q} - T_{z^q}T_{\overline{z}^q}$ is diagonal and is given by*

$$\eta_{i,i} = \begin{cases} \frac{\rho_{q+i}}{\rho_i}, & \text{if} \quad i < q-m; \\ \frac{\rho_{q+i}}{\rho_i} - \frac{\rho_i}{\rho_{i-q}}, & \text{if} \quad i \geq q-m. \end{cases}$$

Proof. We clearly have

$$\left\langle T_{\overline{z}^q}T_{z^q}e_i, e_j\right\rangle = 0 \quad \text{if} \quad i \neq j, \quad \text{and} \quad \langle T_{\overline{z}^q}T_{z^q}e_i, e_i\rangle = \frac{\rho_{q+i}}{\rho_i}.$$

Since

$$\left\langle P(\overline{z}^q z^i), z^k\right\rangle = 0 \quad \text{if} \quad k \neq i-q,$$

necessarily,

$$P(\overline{z}^q z^i) = cz^{i-q},$$

and

$$\left\langle P(\overline{z}^q z^i), z^{i-q}\right\rangle = \left\langle cz^{i-q}, z^{i-q}\right\rangle = c\rho_{i-q}$$

$$\left\langle P(\overline{z}^q z^i), z^{i-q}\right\rangle = \rho_i.$$

We deduce $c = \frac{\rho_i}{\rho_{i-q}}$, and

$$\langle T_{\overline{z}^q}e_i, T_{\overline{z}^q}e_i\rangle = \begin{cases} \frac{\rho_i}{\rho_{i-q}} & \text{if} \quad i \geq q-m; \\ 0 & \text{if} \quad i < q-m. \end{cases}$$

□

This leads to the following proposition on which our main result is based.

Proposition 3.5. *Let l and q be two integers with $q \geq m+1$, $l \geq \max\left(6q+1, \frac{1}{2}(q^2-1), 8\right)$. Then, $T_{z^q+\delta\overline{z}^l}$ is hyponormal if and only if $|\delta| \leq \frac{q}{l}$.*

Proof. Hyponormality is equivalent to the following three inequalities:

$$|\delta|^2\frac{\rho_{l+i}}{\rho_i} \leq \frac{\rho_{q+i}}{\rho_i}, \quad i < q-m. \tag{3.1}$$

$$|\delta|^2\frac{\rho_{l+i}}{\rho_i} \leq \frac{\rho_{q+i}}{\rho_i} - \frac{\rho_i}{\rho_{i-q}}, \quad q-m \leq i < l-m. \tag{3.2}$$

$$|\delta|^2\left(\frac{\rho_{l+i}}{\rho_i} - \frac{\rho_i}{\rho_{i-l}}\right) \leq \frac{\rho_{q+i}}{\rho_i} - \frac{\rho_i}{\rho_{i-q}}, \quad i \geq l-m. \tag{3.3}$$

The first inequality (3.1) takes the form

$$|\delta|^2 \leq \min\left\{\frac{(\beta+l+i+4)(\beta+l+i+3)(\beta+l+i+2)(\beta+l+i+1)}{(\beta+q+i+4)(\beta+q+i+3)(\beta+q+i+2)(\beta+q+i+1)}, \ i < q-m\right\}.$$

Since the right hand side of this inequality decreases with i, inequality (3.1) is equivalent to

$$|\delta| \leq \sqrt{\frac{(\beta+l+q-m+3)(\beta+l+q-m+2)(\beta+l+q-m+1)(\beta+l+q-m)}{(\beta+2q-m+3)(\beta+2q-m+2)(\beta+2q-m+1)(\beta+2q-m)}}.$$

Inequality (3.2) can be written as

$$|\delta|^2 \leq \frac{\rho_{i+q}\rho_{i-q} - \rho_{i^2}}{\rho_{l+i}\rho_{i-q}}, \quad q - m \leq i < l - m.$$

A computation reduces the above inequality to

$$|\delta|^2 \leq \min\left\{Q_{l,q,i}R_{q,i}, \quad q - m \leq i < l - m\right\},$$

where

$$Q_{l,q,i} = \frac{(\beta + l + i + 4)(\beta + l + i + 3)(\beta + l + i + 2)(\beta + l + i + 1)}{(\beta + q + i + 4)(\beta + q + i + 3)(\beta + q + i + 2)(\beta + q + i + 1)},$$

and

$$R_{q,i} = 1 - (1 - \frac{q^2}{(\beta + i + 4)^2})(1 - \frac{q^2}{(\beta + i + 3)^2})(1 - \frac{q^2}{(\beta + i + 2)^2})(1 - \frac{q^2}{(\beta + i + 1)^2}). \tag{3.4}$$

As mentioned before, $Q_{l,q,i}$ is a decreasing function of i and it is not difficult to see that $R_{q,i}$ is also decreasing. Thus, we get that inequality (3.2) is equivalent to

$$|\delta| \leq \sqrt{Q_{l,q,l-m-1}R_{q,l-m-1}},$$

or, equivalently,

$$|\delta| \leq \sqrt{\frac{(\beta + 2l - m + 3)(\beta + 2l - m + 2)(\beta + 2l - m + 1)(\beta + 2l - m)R_{q,l-m-1}}{(\beta + q + l - m + 3)(\beta + q + l - m + 2)(\beta + q + l - m + 1)(\beta + q + l - m)}}.$$

Since $Q_{l,q,i}$ is decreasing, $Q_{l,q,l-m-1} \leq Q_{l,q,q-m-1}$. Thus, we have

$$\sqrt{Q_{l,q,l-m-1}R_{q,l-m-1}} \leq \sqrt{Q_{l,q,q-m-1}}.$$

That is, if inequality (3.2) is satisfied, so is inequality (3.1). Inequality (3.3) takes the form

$$|\delta|^2 \leq \frac{\rho_{i-l}}{\rho_{i-q}}\frac{\rho_{i+q}\rho_{i-q} - \rho_i^2}{\rho_{i+l}\rho_{i-l} - \rho_i^2}, \quad i \geq l - m.$$

To simplify notation, set $A_i := (\beta + i + 1)$. A computation leads to the following inequality for $i \geq l - m$:

$$|\delta|^2 \leq \frac{A_{i+3}^2A_{i+2}^2A_{i+1}^2A_i^2 - (A_{i+3}^2 - q^2)((A_{i+2}^2 - q^2)(A_{i+1}^2 - q^2)(A_i^2 - q^2)}{A_{i+3}^2A_{i+2}^2A_{i+1}^2A_i^2 - (A_{i+3}^2 - l^2)((A_{i+2}^2 - l^2)(A_{i+1}^2 - l^2)(A_i^2 - l^2)}Q_{l,q,i}.$$

After simplification of

$$F(i) = \frac{A_{i+3}^2A_{i+2}^2A_{i+1}^2A_i^2 - (A_{i+3}^2 - q^2)((A_{i+2}^2 - q^2)(A_{i+1}^2 - q^2)(A_i^2 - q^2)}{A_{i+3}^2A_{i+2}^2A_{i+1}^2A_i^2 - (A_{i+3}^2 - l^2)((A_{i+2}^2 - l^2)(A_{i+1}^2 - l^2)(A_i^2 - l^2)},$$

we get

$$F(i) = \frac{q^2F_1(i) - q^4F_2(i) + q^6F_3(i) - q^8}{l^2F_1(i) - l^4F_2(i) + l^6F_3(i) - l^8},$$

where the functions F_1, F_2, and F_3 are as follows:

$$F_1(i) = A_{i+3}^2 A_{i+2}^2 A_{i+1}^2 + A_{i+3}^2 A_{i+2}^2 A_i^2 + A_{i+2}^2 A_{i+1}^2 A_i^2 + A_{i+3}^2 A_{i+1}^2 A_i^2;$$

$$F_2(i) = A_{i+3}^2 A_{i+2}^2 + A_{i+3}^2 A_{i+1}^2 + A_{i+3}^2 A_i^2 + A_{i+2}^2 A_{i+1}^2 + A_{i+2}^2 A_i^2 + A_{i+1}^2 A_i^2;$$

$$F_3(i) = A_{i+3}^2 + A_{i+2}^2 + A_{i+1}^2 + A_i^2.$$

As seen before, $Q_{l,q,i}$ is a decreasing function of i. We will show that $F(i)$ is also a decreasing function. A computation gives

$$F'(i) = \frac{E_1(i) + E_2(i) + E_3(i) + E_4(i) + E_5(i) + E_6(i)}{(l^2 F_1(i) - l^4 F_2(i) + l^6 F_3(i) - l^8)^2},$$

where we have

$$E_1(i) = l^2 q^2 (q^6 - l^6) F_1'(i);$$

$$E_2(i) = l^4 q^4 (l^2 + q^2)(l^2 - q^2) F_2'(i);$$

$$E_3(i) = l^6 q^6 (q^2 - l^2) F_3'(i);$$

$$E_4(i) = l^2 q^2 (q^2 - l^2)(F_1'(i) F_2(i) - F_1(i) F_2'(i));$$

$$E_5(i) = l^2 q^2 (l^2 + q^2)(l^2 - q^2)(F_1'(i) F_3(i) - F_1(i) F_3'(i));$$

$$E_6(i) = l^4 q^4 (q^2 - l^2)(F_2'(i) F_3(i) - F_2(i) F_3'(i)).$$

It suffices to verify the inequalities

$$E_1(i) + E_2(i) \leq 0, \quad i \geq l - m; \tag{3.5}$$

$$E_4(i) + E_5(i) \leq 0, \quad i \geq l - m. \tag{3.6}$$

Note $l - m \geq q - m + 1 \geq 2$. We show inequality (3.5) first. It takes the form

$$l^2 q^2 (l^2 + q^2) F_2'(i) \leq (l^4 + l^2 q^2 + q^4) F_1'(i).$$

From the expressions of $F_1(i)$ and $F_2(i)$, we see that, since $\beta + i + 1 \geq q$ for $i \geq l - m$,

$$q^2 F_2'(i) \leq F_1'(i),$$

and thus inequality (3.5) holds. Inequality (3.6) can be reduced to

$$(l^2 + q^2)(F_1'(i) F_3(i) - F_1(i) F_3'(i)) \leq F_1'(i) F_2(i) - F_1(i) F_2'(i). \tag{3.7}$$

An elementary computation gives

$$F_1'(i) F_3(i) - F_1(i) F_3'(i) = \Phi_1(i) + \Phi_2(i) + \Phi_3(i) + \Phi_4(i) + \Psi(i),$$

where

$$\Phi_1(i) = P_1^{2,3}(i) + P_1^{2,4}(i) + P_1^{3,4}(i);$$

$$P_1^{2,3}(i) = 2(\beta + i + 1)\left((\beta + i + 2)^4 (\beta + i + 3)^2 + (\beta + i + 2)^2 (\beta + i + 3)^4\right);$$

$$P_1^{2,4}(i) = 2(\beta + i + 1)\left((\beta + i + 2)^4(\beta + i + 4)^2 + (\beta + i + 2)^2(\beta + i + 4)^4\right);$$

and

$$P_1^{3,4}(i) = 2(\beta + i + 1)\left((\beta + i + 3)^4(\beta + i + 4)^2 + (\beta + i + 3)^2(\beta + i + 4)^4\right).$$

The function $\Phi_2(i)$ is defined by

$$\Phi_2(i) = P_2^{1,3}(i) + P_2^{1,4}(i) + P_2^{3,4}(i)$$

where

$$P_2^{1,3}(i) = 2(\beta + i + 2)\left((\beta + i + 1)^4(\beta + i + 3)^2 + (\beta + i + 1)^2(\beta + i + 3)^4\right).$$

$P_2^{1,4}$ and $P_2^{3,4}$ are defined in a similar manner.
The functions Φ_3 and Φ_4 are defined in a similar way. The function Ψ is given by

$$\Psi(i) = \Psi_1(i) + \Psi_2(i) + \Psi_3(i) + \Psi_4(i)$$

with

$$\Psi_1(i) = 4(\beta + i + 1)(\beta + i + 2)^2(\beta + i + 3)^2(\beta + i + 4)^2,$$

and

$$\Psi_2(i) = 4(\beta + i + 2)(\beta + i + 1)^2(\beta + i + 3)^2(\beta + i + 4)^2,$$

and similar definitions for Ψ_3 and Ψ_4. Next, we simplify the right hand side of inequality (3.7). A tedious, but elementary, computation gives

$$F_1'(i)F_2(i) - F_1(i)F_2'(i) = \Delta(i) + \Omega(i),$$

where

$$\Delta(i) = \Delta_1(i) + \Delta_2(i) + \Delta_3(i) + \Delta_4(i)$$

with

$$\Delta_1(i) = 2(\beta + i + 1)\left[(\beta + i + 2)^4(\beta + i + 3)^4 + (\beta + i + 2)^4(\beta + i + 4)^4 + (\beta + i + 3)^4(\beta + i + 4)^4\right];$$

$$\Delta_2(i) = 2(\beta + i + 2)\left[(\beta + i + 1)^4(\beta + i + 3)^4 + (\beta + i + 1)^4(\beta + i + 4)^4 + (\beta + i + 3)^4(\beta + i + 4)^4\right];$$

and similar definitions for $\Delta_3(i)$ and $\Delta_4(i)$. The function $\Omega(i)$ is given by

$$\Omega(i) = \Omega_1(i) + \Omega_2(i) + \Omega_3(i) + \Omega_4(i)$$

with $\Omega_1(i)$ and $\Omega_2(i)$ defined by

$$\Omega_1(i) = \Omega_{1,2}(i) + \Omega_{1,3}(i) + \Omega_{1,4}(i),$$

where

$$\Omega_{1,2}(i) = 2(\beta + i + 1)\left((\beta + i + 2)^4(\beta + i + 3)^2(\beta + i + 4)^2\right);$$

$$\Omega_{1,3}(i) = 2(\beta + i + 1)\left((\beta + i + 3)^4(\beta + i + 2)^2(\beta + i + 4)^2\right);$$

and $\Omega_{1,4}(i)$ defined in a similar way. We show

$$(l^2+q^2)\Phi_1(i) \leq \Delta_1(i),$$

and, since $l \geq \frac{1}{2}(q^2-1)$,

$$(l^2+q^2)(\beta+i+2)^4(\beta+i+3)^2 \leq (\beta+i+2)^4(\beta+i+3)^4,$$

and similarly

$$(l^2+q^2)(\beta+i+2)^2(\beta+i+3)^4 \leq (\beta+i+2)^4(\beta+i+3)^4.$$

Thus,

$$(l^2+q^2)P_1^{2,3} \leq 2(\beta+i+1)(\beta+i+2)^4(\beta+i+3)^4.$$

Similar inequalities lead to

$$(l^2+q^2)\Phi_1(i) \leq \Delta_1(i),$$

and we have

$$(l^2+q^2)\Phi_2(i) \leq \Delta_2(i), \quad (l^2+q^2)\Phi_3(i) \leq \Delta_3(i), \quad (l^2+q^2)\Phi_4(i) \leq \Delta_4(i)$$

for $i \geq l-m$. Next, we show $(l^2+q^2)\Psi_1(i) \leq \Omega_1(i)$ for $i \geq l-m$. We clearly have

$$(l^2+q^2)\Psi_1(i) \leq 6(\beta+i+1)((\beta+i+2)^4(\beta+i+3)^2(\beta+i+4)^2 \leq \Omega_1(i).$$

Similarly, we have

$$(l^2+q^2)\Psi_2(i) \leq \Omega_2(i), \quad (l^2+q^2)\Psi_3(i) \leq \Omega_3(i), \quad (l^2+q^2)\Psi_4(i) \leq \Omega_4(i).$$

We deduce that inequality (3.6) holds and that $F(i)$ is decreasing. Thus, inequality (3.3) holds if and only if $|\delta| \leq \sqrt{\lim_{i\to\infty} F(i)} = \frac{q}{l}$. We finally show that

$$\frac{q^2}{l^2} \leq \frac{(\beta+2l-m+3)(\beta+2l-m+2)(\beta+2l-m+1)(\beta+2l-m)R_{q,l-m-1}}{(\beta+l+q-m+3)(\beta+l+q-m+2)(\beta+l+q-m+1)(\beta+l+q-m)}. \tag{3.8}$$

From inequality (3.7), we have

$$R_{q,l-m-1} = 1-(1-\frac{q^2}{(\beta+l-m+3)^2})(1-\frac{q^2}{(\beta+l-m+2)^2})(1-\frac{q^2}{(\beta+l-m+1)^2})(1-\frac{q^2}{(\beta+l-m)^2}).$$

To simplify notation, we set $B_i := (\beta+i+3)(\beta+i+2)(\beta+i+1)(\beta+i)$. Inequality (3.8) takes the form

$$1 \leq \frac{l^2 B_{2l-m}(B_{l-m}^2 - B_{l-m+q}B_{l-m-q})}{q^2 B_{l-m+q}B_{l-m}^2},$$

and we have

$$B_{l-m}^2 - B_{l-m+q}B_{l-m-q} = q^2F_1(l-m-1) - q^4F_2(l-m-1) + q^6F_3(l-m-1) - q^8,$$

where

$$F_1(l-m-1) = A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2 + A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m-1}^2 + A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2 + A_{l-m+2}^2 A_{l-m}^2 A_{l-m-1}^2$$

and, with $A_i := \beta + i + 1$ as before,

$$F_2(l-m-1) = A_{l-m}^2 A_{l-m-1}^2 + A_{l-m+1}^2 A_{l-m-1}^2 + A_{l-m+1}^2 A_{l-m}^2 + A_{l-m+2}^2 A_{l-m+1}^2 + A_{l-m+2}^2 A_{l-m}^2 + A_{l-m+2}^2 A_{l-m-1}^2,$$

and

$$F_3(l-m-1) = A_{l-m-1}^2 + A_{l-m}^2 + A_{l-m+1}^2 + A_{l-m+2}^2.$$

Since

$$B_{l-m}^2 = A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2,$$

we get

$$\frac{F_1(l-m-1)}{B_{l-m}^2} = \frac{1}{A_{l-m-1}^2} + \frac{1}{A_{l-m}^2} + \frac{1}{A_{l-m+1}^2} + \frac{1}{A_{l-m+2}^2}$$

$$\frac{F_2(l-m-1)}{B_{l-m}^2} = \frac{1}{A_{l-m-1}^2 A_{l-m}^2} + \frac{1}{A_{l-m-1}^2 A_{l-m+1}^2} + \frac{1}{A_{l-m-1}^2 A_{l-m+2}^2} + \frac{1}{A_{l-m}^2 A_{l-m+1}^2} + \frac{1}{A_{l-m}^2 A_{l-m+2}^2} + \frac{1}{A_{l-m+1}^2 A_{l-m+2}^2}$$

$$\frac{F_3(l-m-1)}{B_{l-m}^2} = \frac{1}{A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2} + \frac{1}{A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2} + \frac{1}{A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m-1}^2} + \frac{1}{A_{l-m+2}^2 A_{l-m}^2 A_{l-m-1}^2}.$$

We see that inequality (3.8) is equivalent to

$$\frac{l^2(B_{l-m}^2 - B_{l-m+q}B_{l-m-q})}{q^2 B_{l-m}^2} \geq \frac{4l^2}{A_{l-m+2}^2} - \frac{6q^2l^2}{A_{l-m-1}^2 A_{l-m}^2} + \frac{4q^4l^2}{A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2} - \frac{q^6 l^2}{A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2}. \quad (3.9)$$

We show that if $l \geq N$, then

$$\frac{4l^2}{A_{l-m+2}^2} - \frac{6q^2l^2}{A_{l-m-1}^2 A_{l-m}^2} + \frac{4q^4l^2}{A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2} - \frac{q^6 l^2}{A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2} \geq 1.$$

We then rewrite this last inequality as

$$4l^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2 - 6q^2 l^2 A_{l-m+2}^2 A_{l-m+1}^2 + 4q^4 l^2 A_{l-m-1}^2 - q^6 l^2 \geq A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2.$$

Since $\beta \in (m-1, m]$, by assumptions on l we have

$$2l^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2 \geq A_{l-m+2}^2 A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2.$$

Thus, inequality (3.9) is satisfied if

$$2A_{l-m+1}^2 A_{l-m}^2 A_{l-m-1}^2 - 6q^2 A_{l-m+2}^2 A_{l-m+1}^2 + 4q^4 A_{l-m-1}^2 - q^6 \geq 0.$$

Since by assumptions on l and q,

$$4q^4 A_{l-m-1}^2 \geq q^6,$$

it is enough to have

$$A_{l-m}^2 A_{l-m-1}^2 - 3q^2 A_{l-m+2}^2 \geq 0,$$

and this is satisfied by assumptions on l. Hence, inequality (3.9) holds. We conclude that if $q \geq m+1$ and $l \geq \max\left(6q+1, \frac{1}{2}(q^2-1), 8\right)$, $T_{z^q+\delta\bar{z}^l}$ is hyponormal if and only if $|\delta| \leq \frac{q}{l}$. The proof is complete. $\square$

It is easy to see that if a and b are complex numbers satisfying $|a| + |b| \leq 1$, and if $T_{\varphi+\overline{\psi_1}}$ and $T_{\varphi+\overline{\psi_2}}$ are hyponormal then, $T_{\varphi+a\overline{\psi_1}+b\overline{\psi_2}}$ is hyponormal. This leads to our main result, where N is defined as earlier.

Theorem 3.6. *Let* $(\lambda_l)_{N\leq l\leq N_1}$ *be any finite sequence of complex numbers such that* $\sum\limits_{N\leq l\leq N_1} |\lambda_l| \leq 1$, *and set* $g = \sum\limits_{N\leq l\leq N_1} \lambda_l \frac{q}{l} z^l$. *Then, we have* $T_{\overline{g}}T_g - T_gT_{\overline{g}} \leq T_{\overline{z}^q}T_{z^q} - T_{z^q}T_{\overline{z}^q}$ *on* $A^2_{\alpha,\beta}$.

4. Conclusions

The other results of Choe and Lee [2] can be generalized and this could be addressed in a future work. Hyponormality on the punctured disk is a very recent subject of interest and much remains to be done on this. We hope this paves the way for further works.

Author contributions

All authors contributed equally and significantly to the study conception and design, material preparation, data collection and analysis. The first draft of the manuscript was written by H. Sadraoui, and all authors read and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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