Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 19997–20013.
DOI: 10.3934/math.2024975
Received: 22 April 2024
Revised: 11 June 2024
Accepted: 13 June 2024
Published: 20 June 2024

Research article

On the oscillation of fourth-order canonical differential equation with several delays

Mohammed Ahmed Alomair[1,*] **and Ali Muhib**[2,3,*]

[1] Department of Quantitative Methods, School of Business, King Faisal University, Al-Ahsa 31982, Saudi Arabia
[2] Department of Mathematics, Faculty of Applied and Educational Sciences, Al-Nadera, Ibb University, Ibb, Yemen
[3] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

* **Correspondence:** Email: ma.alomair@kfu.edu.sa, muhib39@yahoo.com.

Abstract: This study is concerned with investigating the oscillatory properties of a general class of neutral differential equations. Neutral equations are characterized by being rich in both practical and theoretical aspects. We obtain criteria that guarantee the oscillation of solutions to a fourth-order neutral differential equation with multiple delays. Considering the canonical case, we obtain some new relations and inequalities that help in obtaining improved criteria. We use the reduction method to relate the oscillation of the studied equation to a first-order equation. We apply the results to a special case. Through this application, we evaluated the efficiency of the new results in the oscillation test compared to previous results in the literature.

Keywords: canonical form; fourth-order; oscillation criteria
Mathematics Subject Classification: 34C10, 34K11

1. Introduction

The main objective of the paper is to investigate the qualitative behavior of a differential equation

$$\left(b(\vartheta)\left((x(\vartheta)+g(\vartheta)x(\tau(\vartheta)))'''\right)^{\alpha}\right)' + \sum_{i=1}^{\eta}\phi_i(\vartheta)\,x^{\beta}(\delta_i(\vartheta)) = 0, \tag{1.1}$$

where $\vartheta \geq \vartheta_0 > 0$ and η is a positive integer.

Let us define the corresponding function υ for the solution x as follows:

$$\upsilon(\vartheta) = x(\vartheta) + g(\vartheta)\,x(\tau(\vartheta)). \tag{1.2}$$

Furthermore, we make the following supposition:

(M1) α and β are quotients of odd positive integers;

(M2) $b \in C([\vartheta_0, \infty), (0, \infty))$, $b'(\vartheta) \geq 0$ and

$$\int_{\vartheta_0}^{\vartheta} \frac{1}{b^{1/\alpha}(\zeta)} d\zeta \to \infty \text{ as } \vartheta \to \infty; \tag{1.3}$$

(M3) $\tau, \delta_i \in C([\vartheta_0, \infty), \mathbb{R})$, $\delta_i(\vartheta) \leq \vartheta$, $\tau(\vartheta) \leq \vartheta$, $\tau'(\vartheta) > 0$, and $\lim_{\vartheta \to \infty} \tau(\vartheta) = \lim_{\vartheta \to \infty} \delta_i(\vartheta) = \infty$, $i = 1, 2, ...\eta$;

(M4) $\phi_i, g \in C([\vartheta_0, \infty), \mathbb{R})$, $g(\vartheta) > 0$, $\phi_i \geq 0$ and ϕ_i is not identically zero for large ϑ, $i = 1, 2, ...\eta$;

(M5) there exists a constant $\varepsilon \in (0, 1)$ such that

$$\lim_{\vartheta \to \infty} \left(\frac{\vartheta}{\tau(\vartheta)} \right)^{3/\varepsilon} \frac{1}{g(\vartheta)} = 0. \tag{1.4}$$

By a solution of (1.1), we mean a nontrivial function $x \in C([\vartheta_x, \infty), \mathbb{R})$, $\vartheta_x \geq \vartheta_0$, which has the properties $\upsilon(\vartheta) \in C^3([\vartheta_x, \infty), \mathbb{R})$, $b(\vartheta)(\upsilon'''(\vartheta))^{\alpha} \in C^1([\vartheta_x, \infty), \mathbb{R})$ and x satisfies (1.1) on $[\vartheta_x, \infty)$. We focus in our study on the solutions that satisfy $\sup\{|x(\vartheta)| : \vartheta \geq \vartheta_a\} > 0$, for every $\vartheta_a \geq \vartheta_x$, and we assume that (1.1) possesses such solutions. Such a solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. An equation is called oscillatory if all of its solutions are oscillatory.

We frequently see repetitive movements used to illustrate different mechanical actions that occur in nature. In other words, oscillations abound in our universe. Examples of oscillatory motions include a ship moving up and down on the waves and a pendulum swing. Finding new necessary conditions for the oscillation or nonoscillation of the solutions of neutral functional differential equations, which is a component of so-called dynamical systems, has become increasingly important in recent years. As far as physicists and engineers are concerned, understanding and managing oscillations in various systems is their primary objective. Oscillation is a phenomenon that is observed in a variety of fields, including biology and economics, in addition to physics and mechanics.

One of the important differential equation branching problems is the oscillatory behavior of ordinary differential equations. The oscillatory problems to the wings of the plane can be modeled by the oscillatory problems of ordinary differential equations. Delay differential equations with deviating arguments introduce an additional layer of complexity by incorporating time delays into the modeling process. Unlike ordinary differential equations, these equations account for the influence of both current and past values of variables. In fact, differential equations with deviating arguments are widely used in physics, engineering, biology, economics, and more, making them an indispensable tool for understanding and predicting the behavior of complex phenomena, see [8, 9, 13, 15, 21, 23, 26, 27]).

Bills and Schoenberg [12] examined certain oscillatory outcomes for a self-adjoint system of second-order equations. The oscillatory behavior of solutions to various classes of differential equations with a linear neutral term has been extensively studied in recent decades. However, there are few results about the oscillation of differential equations with nonlinear neutral terms; see, for example, Agarwal et al. and Grace and Graef [3, 14, 16].

Many researchers have studied the oscillatory properties of even-order differential equations on a larger scale than their odd-order counterparts. Different techniques and methods have been used to study the oscillation of different types of even-order differential equations. Illustrative and additional information can be found in references [4, 11, 17–20, 25]. Specifically, we provide some detail.

Bazighifan and Ahmad [10] investigated the qualitative behavior of an even-order advanced differential equation

$$\left(b(\vartheta)\left(x^{(n-1)}(\vartheta)\right)^{\alpha}\right)' + \sum_{i=1}^{\eta} \phi_i(\vartheta) f(x(\delta_i(\vartheta))) = 0, \tag{1.5}$$

where $\phi_i(\vartheta) \geq 0$, $\delta_i(\vartheta) \geq \vartheta$, and $f(x)/x^{\beta} \geq k > 0$ for $x \neq 0$. They established sufficient conditions for oscillation of (1.5) by utilizing the theory of comparison with first-order and second-order delay equations, as well as the Riccati substitution technique.

The oscillatory behavior of the differential equations

$$(b(\vartheta)((x(\vartheta) + g(\vartheta)x(\tau(\vartheta)))')^{\alpha})' + \sum_{i=1}^{\eta} \phi_i(\vartheta) x^{\beta}(\delta_i(\vartheta)) = 0 \tag{1.6}$$

was taken into consideration by Abdelnaser et al. [1]. By introducing a new set of criteria, the researchers were able to prove that all solutions to Eq (1.6) oscillate.

Muhib et al. [22] considered a class of neutral differential equations

$$\left(b(\vartheta)\left(\left(x(\vartheta) + \rho(\vartheta)x^{\hbar}(\sigma_1(\vartheta)) + g(\vartheta)x^{\gamma}(\tau(\vartheta))\right)^{(n-1)}\right)^{\alpha}\right)' + f(\vartheta, x(\delta(\vartheta))) = 0. \tag{1.7}$$

They used Riccati transformations to present new conditions for the oscillation of (1.7), where $\rho(\vartheta) \geq 0$, $g(\vartheta) \geq 0$, γ and $\hbar$ are ratios of odd positive integers with $\gamma \geq 1$, $0 < \hbar < 1$, and $f \in C([\vartheta_0, \infty) \times \mathbb{R}, \mathbb{R})$ and there exists $\phi \in C([\vartheta_0, \infty), (0, \infty))$ such that $|f(\vartheta, x)| \geq \phi(\vartheta)|x|^{\beta}$. Below, we present one of the results in [22].

Theorem 1.1. *Assume that*

$$\lim_{\vartheta\to\infty} g(\vartheta)\left(\vartheta^{n-2}\int_{\vartheta_0}^{\vartheta}\frac{1}{b^{1/\alpha}(\zeta)}\mathrm{d}\zeta\right)^{\gamma-1} = \lim_{\vartheta\to\infty}\rho(\vartheta) = 0 \tag{1.8}$$

holds. If

$$\liminf_{\vartheta\to\infty}\frac{1}{\varphi(\vartheta)}\int_{\vartheta}^{\infty}\alpha\lambda N\delta^{n-2}(\zeta)\delta'(\zeta)b^{-1/\alpha}(\zeta)\varphi^{(\alpha+1)/\alpha}(\zeta)\mathrm{d}\zeta > \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}},$$

for all $\lambda \in (0, 1)$ *and* $N > 0$*, then (1.7) is oscillatory, where*

$$\varphi(\vartheta) = \epsilon^{\beta}\int_{\vartheta}^{\infty}\phi(u)\Omega(u)\mathrm{d}u$$

and

$$\Omega(\vartheta) = \begin{cases} k_1^{\beta-\alpha} & \text{if } \beta \geq \alpha, \\ k_2^{\beta-\alpha}\left(\vartheta^{n-2}\right)^{\beta-\alpha}\left(\int_{\vartheta_1}^{\vartheta}\frac{1}{b^{1/\ell}(\zeta)}\mathrm{d}\zeta\right)^{\beta-\alpha} & \text{if } \beta < \alpha, \end{cases}$$

for some $\epsilon \in (0, 1)$ *and* $k_1, k_2 > 0$.

Agarwal et al. [2] studied the oscillation of a neutral differential equation

$$(x(\vartheta) + g(\vartheta)\, x(\tau(\vartheta)))^{(n)} + \phi(\vartheta)\, x(\delta(\vartheta)) = 0, \tag{1.9}$$

where n is even and $g(\vartheta) \geq 0$. They established some sufficient conditions for oscillation of (1.9) using the Riccati transformation technique. Below, we present one of the results in [2].

Theorem 1.2. *Let $n \geq 4$ be even (M3), (M4), and*

$$\delta(\vartheta) \leq \tau(\vartheta), \quad 1 - \frac{\Upsilon^{n-1}(\vartheta)}{g(\tau^{-1}(\tau^{-1}(\vartheta)))} \geq 0$$

hold. If the equation

$$\left(\frac{(n-2)!}{\lambda_0 \vartheta^{n-2}} x'(\vartheta)\right)' + \phi(\vartheta)\, g_*(\delta(\vartheta)) \frac{\left(\tau^{-1}(\delta(\vartheta))\right)^{n-1}}{\vartheta^{n-1}} x(\vartheta) = 0$$

is oscillatory for some constant $\lambda_0 \in (0,1)$, and the equation

$$x''(\vartheta) + \frac{\int_{\vartheta}^{\infty} (s-\vartheta)^{n-3} \phi(s)\, g^*(\delta(s)) \frac{\tau^{-1}(\delta(s))}{s} \mathrm{d}s}{(n-3)!} x(\vartheta) = 0$$

is oscillatory, then (1.9) is oscillatory, where

$$g^*(\vartheta) = \frac{1}{g(\tau^{-1}(\vartheta))}\left(1 - \frac{\Upsilon(\vartheta)}{g(\tau^{-1}(\tau^{-1}(\vartheta)))}\right),$$

$$g_*(\vartheta) = \frac{1}{g(\tau^{-1}(\vartheta))}\left(1 - \frac{\Upsilon^{n-1}(\vartheta)}{g(\tau^{-1}(\tau^{-1}(\vartheta)))}\right)$$

and

$$\Upsilon(\vartheta) = \frac{\tau^{-1}\left(\tau^{-1}(\vartheta)\right)}{\tau^{-1}(\vartheta)}.$$

The objective of this paper is to provide new results of oscillation (1.1) in canonical form, which would improve and extend some previous literature. In addition, an example is given that shows the applicability of the results we obtained.

The following notation will be used in the remaining sections of this work:

$$\delta(\vartheta) = \min\{\delta_i(\vartheta) : \ i = 1, 2, ..., \eta\}.$$

2. Preliminary lemmas

For the proof of our main results, we need to give the following lemmas:

Lemma 2.1. *[5] Let $f \in C^n([\vartheta_0, \infty), (0, \infty))$, the derivative $f^{(n)}(\vartheta)$ is of fixed sign and not identically zero on a subray of $[\vartheta_0, \infty)$, and there exists a $\vartheta_x \geq \vartheta_0$ such that $f^{(n-1)}(\vartheta) f^{(n)}(\vartheta) \leq 0$ for all $\vartheta \geq \vartheta_1$. If $\lim_{\vartheta \to \infty} f(\vartheta) \neq 0$, then for every $\lambda \in (0,1)$ there exists a $\vartheta_\lambda \geq \vartheta_1$ such that*

$$|f(\vartheta)| \geq \frac{\lambda}{(n-1)!} \vartheta^{n-1} \left|f^{(n-1)}(\vartheta)\right|, \tag{2.1}$$

for all $\vartheta \geq \vartheta_\lambda$.

Lemma 2.2. *[7, Lemma 1] Let the function f satisfy $f^{(n)}(\vartheta) > 0,\ n = 1, 2, ..., \kappa$, and $f^{(\kappa+1)}(\vartheta) \leq 0$ eventually. Then, for every $\varepsilon \in (0, 1)$,*

$$\frac{f(\vartheta)}{f'(\vartheta)} \geq \varepsilon \frac{\vartheta}{\kappa}, \tag{2.2}$$

eventually.

Lemma 2.3. *[6] Let x be a positive solution of (1.1), and (1.3) holds. Then, $(b(\vartheta)(\upsilon'''(\vartheta))^{\alpha})' < 0$, we also find that there exist two potential cases eventually, which are as follows:*

$$\begin{aligned} Case\ (\mathbf{1}) &: \quad \upsilon(\vartheta) > 0,\ \upsilon'(\vartheta) > 0,\ \upsilon''(\vartheta) > 0,\ \upsilon'''(\vartheta) > 0,\ \upsilon^{(4)}(\vartheta) \leq 0, \\ Case\ (\mathbf{2}) &: \quad \upsilon(\vartheta) > 0,\ \upsilon'(\vartheta) > 0,\ \upsilon''(\vartheta) < 0,\ \upsilon'''(\vartheta) > 0,\ \upsilon^{(4)}(\vartheta) \leq 0. \end{aligned}$$

Lemma 2.4. *[24] If y is a positive and strictly decreasing solution of the integral inequality*

$$y(\vartheta) \geq \int_{\vartheta}^{\infty} \frac{(\nu - \vartheta)^{n-1}}{(n-1)!} f(\nu, y(g_1(\nu)), y(g_2(\nu)), ..., y(g_m(\nu)))\, \mathrm{d}\nu,$$

then there exists a positive solution $x(\vartheta)$ of the differential equation

$$(-1)^n x^{(n)}(\vartheta) = f(\vartheta, x(g_1(\vartheta)), x(g_2(\vartheta)), ..., x(g_m(\vartheta))),\ ,\vartheta \geq \vartheta_0$$

being such that $x(\vartheta) \leq y(\vartheta)$ for all large ϑ and satisfying $\lim_{\vartheta\to\infty} x^{(i)}(\vartheta) = 0$ monotonically $(i = 1, 2, ..., n-1)$, where f is a continuous function defined on $[\vartheta_0, \infty) \times [0, \infty)^m$ and $g_j(\vartheta)$ are continuous functions on the interval $[\vartheta_0, \infty)$ such that

$$\lim_{\vartheta\to\infty} g_j(\vartheta) = \infty \quad (j = 1, 2, ..., m).$$

The function $f = f(\vartheta, u_1, u_2, ..., u_m)$ is assumed to be increasing in each of $u_1, u_2, ..., u_m$. Moreover, it is supposed that f is positive on $[\vartheta_0, \infty) \times [0, \infty)^m$ and that the delays $\vartheta - g_j(\vartheta)$ are positive for $\vartheta \geq \vartheta_0$, i.e., $g_j(\vartheta) < \vartheta$ for every $\vartheta \geq \vartheta_0$ and $(j = 1, 2, ..., m)$.

3. Main results

We now present the main results of this paper.

Theorem 3.1. *Assume that $\beta \geq 1$ and that there exists a positive function $\mu \in C^1([\vartheta_0, \infty), \mathbb{R})$ such that*

$$\mu(\vartheta) < \tau(\vartheta),\ \mu(\vartheta) \leq \delta_i(\vartheta),\ \mu'(\vartheta) > 0 \text{ and } \lim_{\vartheta\to\infty} \mu(\vartheta) = \infty. \tag{3.1}$$

If

$$(b(\vartheta)(\upsilon'''(\vartheta))^{\alpha})' + \epsilon_1^{\beta} c^{\beta-1} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))} \right) \upsilon(q(\vartheta)) \leq 0 \tag{3.2}$$

and

$$(b(\vartheta)(\upsilon'''(\vartheta))^{\alpha})' + \epsilon_2^{\beta} c^{\beta-1} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))} \right) \upsilon(q(\vartheta)) \leq 0 \tag{3.3}$$

have no positive solutions, then every solution of (1.1) is oscillatory, where $q(\vartheta) = \tau^{-1}(\mu(\vartheta))$, $c > 0$ and $\epsilon_1, \epsilon_2 \in (0, 1)$.

Proof. Assume that Eq (1.1) has a nonoscillatory solution $x(\vartheta)$, say $x(\vartheta) > 0$, $x(\delta(\vartheta)) > 0$, and $x(\tau(\vartheta)) > 0$ for $\vartheta \geq \vartheta_1 \geq \vartheta_0$. From (1.2), we find

$$x(\vartheta) = \frac{\upsilon\left(\tau^{-1}(\vartheta)\right) - x\left(\tau^{-1}(\vartheta)\right)}{g(\tau^{-1}(\vartheta))}$$

and so

$$x(\vartheta) \geq \frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{g(\tau^{-1}(\vartheta))} - \frac{\upsilon\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right)}{g(\tau^{-1}(\vartheta))\, g(\tau^{-1}(\tau^{-1}(\vartheta)))}. \tag{3.4}$$

We first consider what Case (1) holds. Since $\kappa = 3$, in view of (2.2), for every $\varepsilon \in (0,1)$, we get

$$\frac{\upsilon(\vartheta)}{\upsilon'(\vartheta)} \geq \varepsilon\frac{\vartheta}{\kappa} \geq \varepsilon\frac{\vartheta}{3}, \tag{3.5}$$

now,

$$\begin{aligned} \left(\frac{\upsilon(\vartheta)}{\vartheta^{3/\varepsilon}}\right)' &= \frac{\varepsilon\vartheta^{3/\varepsilon}\upsilon'(\vartheta) - 3\upsilon(\vartheta)\,\vartheta^{(3/\varepsilon)-1}}{\varepsilon\vartheta^{2(3/\varepsilon)}} \\ &= \frac{\varepsilon\vartheta\upsilon'(\vartheta) - 3\upsilon(\vartheta)}{\varepsilon\vartheta^{(3/\varepsilon)+1}}, \end{aligned} \tag{3.6}$$

using (3.5), we find

$$\left(\frac{\upsilon(\vartheta)}{\vartheta^{3/\varepsilon}}\right)' = \frac{\varepsilon\vartheta\upsilon'(\vartheta) - 3\upsilon(\vartheta)}{\varepsilon\vartheta^{(3/\varepsilon)+1}} \leq 0. \tag{3.7}$$

Since $\tau(\vartheta) \leq \vartheta$ and $\tau'(\vartheta) > 0$, $\left(\tau^{-1}(\vartheta)\right)' > 0$ and furthermore $\vartheta \leq \tau^{-1}(\vartheta)$. Thus,

$$\tau^{-1}(\vartheta) \leq \tau^{-1}\left(\tau^{-1}(\vartheta)\right). \tag{3.8}$$

By using (3.7) and (3.8), it follows that

$$\frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{(\tau^{-1}(\vartheta))^{3/\varepsilon}} \geq \frac{\upsilon\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right)}{(\tau^{-1}(\tau^{-1}(\vartheta)))^{3/\varepsilon}}$$

and so

$$\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right)^{3/\varepsilon}\upsilon\left(\tau^{-1}(\vartheta)\right) \geq \left(\tau^{-1}(\vartheta)\right)^{3/\varepsilon}\upsilon\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right). \tag{3.9}$$

From (3.4) and (3.9), we find

$$\begin{aligned} x(\vartheta) &\geq \frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{g(\tau^{-1}(\vartheta))} - \left(\frac{\tau^{-1}\left(\tau^{-1}(\vartheta)\right)}{\tau^{-1}(\vartheta)}\right)^{3/\varepsilon}\frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{g(\tau^{-1}(\vartheta))\, g(\tau^{-1}(\tau^{-1}(\vartheta)))} \\ &\geq \frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{g(\tau^{-1}(\vartheta))}\left(1 - \left(\frac{\tau^{-1}\left(\tau^{-1}(\vartheta)\right)}{\tau^{-1}(\vartheta)}\right)^{3/\varepsilon}\frac{1}{g(\tau^{-1}(\tau^{-1}(\vartheta)))}\right). \end{aligned} \tag{3.10}$$

From (M5), there exists a $\epsilon_1 \in (0,1)$ such that

$$\left(\frac{\tau^{-1}\left(\tau^{-1}(\vartheta)\right)}{\tau^{-1}(\vartheta)}\right)^{3/\varepsilon}\frac{1}{g(\tau^{-1}(\tau^{-1}(\vartheta)))} \leq 1 - \epsilon_1.$$

Using the above inequality in (3.10) gives

$$x(\vartheta) \geq \frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{g\left(\tau^{-1}(\vartheta)\right)}\epsilon_1. \tag{3.11}$$

From (1.1) and (3.11), we have

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \sum_{i=1}^{\eta}\phi_i(\vartheta)\frac{\upsilon^{\beta}\left(\tau^{-1}(\delta_i(\vartheta))\right)}{g^{\beta}\left(\tau^{-1}(\delta_i(\vartheta))\right)}\epsilon_1^{\beta} \leq 0$$

and so

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_1^{\beta}\upsilon^{\beta}\left(\tau^{-1}(\delta(\vartheta))\right)\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}(\delta_i(\vartheta))\right)} \leq 0. \tag{3.12}$$

In view of the fact that $\mu(\vartheta) \leq \delta(\vartheta)$ and $\upsilon'(\vartheta) > 0$, inequality (3.12) becomes

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_1^{\beta}\upsilon^{\beta}\left(\tau^{-1}(\mu(\vartheta))\right)\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}(\delta_i(\vartheta))\right)} \leq 0. \tag{3.13}$$

Since $\upsilon(\vartheta) > 0$ and $\upsilon'(\vartheta) > 0$, there exists a constant $c > 0$ such that

$$\upsilon(\vartheta) \geq c. \tag{3.14}$$

From (3.13), (3.14), and $\beta \geq 1$, we find the following differential inequality:

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_1^{\beta}c^{\beta-1}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}(\delta_i(\vartheta))\right)}\right)\upsilon(q(\vartheta)) \leq 0, \tag{3.15}$$

has a positive solution υ. This implies that (3.2) also has a positive solution, which contradicts our assumption.

Now, we consider what Case (2) holds. Since $\kappa = 1$, in view of (2.2), for every $\varepsilon \in (0,1)$, we get

$$\frac{\upsilon(\vartheta)}{\upsilon'(\vartheta)} \geq \varepsilon\frac{\vartheta}{1}, \tag{3.16}$$

from which we see that

$$\begin{aligned}\left(\frac{\upsilon(\vartheta)}{\vartheta^{1/\varepsilon}}\right)' &= \frac{\varepsilon\vartheta^{1/\varepsilon}\upsilon'(\vartheta) - \upsilon(\vartheta)\vartheta^{(1/\varepsilon)-1}}{\varepsilon\vartheta^{2/\varepsilon}} \\ &= \frac{\varepsilon\vartheta\upsilon'(\vartheta) - \upsilon(\vartheta)}{\varepsilon\vartheta^{1+(1/\varepsilon)}} \leq 0.\end{aligned} \tag{3.17}$$

By (3.8) and (3.17),

$$\left(\tau^{-1}(\vartheta)\right)^{1/\varepsilon}\upsilon\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right) \leq \left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right)^{1/\varepsilon}\upsilon\left(\tau^{-1}(\vartheta)\right). \tag{3.18}$$

Combining (3.4) and (3.18), we obtain

$$x(\vartheta) \geq \frac{\upsilon\left(\tau^{-1}(\vartheta)\right)}{g\left(\tau^{-1}(\vartheta)\right)}\left(1 - \left(\frac{\tau^{-1}\left(\tau^{-1}(\vartheta)\right)}{\tau^{-1}(\vartheta)}\right)^{1/\varepsilon}\frac{1}{g\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right)}\right). \tag{3.19}$$

From (M5), there exists a $\epsilon_2 \in (0,1)$ such that

$$\left(\frac{\tau^{-1}\left(\tau^{-1}(\vartheta)\right)}{\tau^{-1}(\vartheta)}\right)^{1/\varepsilon}\frac{1}{g\left(\tau^{-1}\left(\tau^{-1}(\vartheta)\right)\right)} \leq 1-\epsilon_2,$$

and using this in (3.19) implies

$$x(\vartheta) \geq \frac{\epsilon_2 \upsilon\left(\tau^{-1}(\vartheta)\right)}{g\left(\tau^{-1}(\vartheta)\right)}. \tag{3.20}$$

Using (3.20) in (1.1) yields

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \sum_{i=1}^{\eta}\phi_i(\vartheta)\frac{\epsilon_2^{\beta}\upsilon^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)} \leq 0$$

and so

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_2^{\beta}\upsilon^{\beta}\left(\tau^{-1}\left(\delta(\vartheta)\right)\right)\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)} \leq 0. \tag{3.21}$$

Since $\mu(\vartheta) \leq \delta(\vartheta)$ and $\upsilon'(\vartheta) > 0$, then (3.21) becomes

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_2^{\beta}\upsilon^{\beta}\left(\tau^{-1}\left(\mu(\vartheta)\right)\right)\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)} \leq 0. \tag{3.22}$$

In view of (3.14) and $\beta \geq 1$, we find the following differential inequality:

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_2^{\beta}c^{\beta-1}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}\right)\upsilon(q(\vartheta)) \leq 0, \tag{3.23}$$

has a positive solution υ. This implies that (3.3) also has a positive solution, which contradicts our assumption. The proof is now complete. □

Theorem 3.2. *Assume that $\beta < 1$ and there exists a positive function $\mu \in C^1([\vartheta_0,\infty),\mathbb{R})$ such that (3.1) holds. If*

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_1^{\beta}d_1^{\beta-1}\left(q^{3/\varepsilon}(\vartheta)\right)^{\beta-1}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}\right)\upsilon(q(\vartheta)) \leq 0 \tag{3.24}$$

and

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_2^{\beta}d_2^{\beta-1}\left(q^{1/\varepsilon}(\vartheta)\right)^{\beta-1}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}\right)\upsilon(q(\vartheta)) \leq 0, \tag{3.25}$$

have no positive solutions, then every solution of (1.1) is oscillatory, where $q(\vartheta) = \tau^{-1}(\mu(\vartheta))$, $d_1, d_2 > 0$ and $\epsilon_1, \epsilon_2 \in (0,1)$.

Proof. Assume that Eq (1.1) has a nonoscillatory solution $x(\vartheta)$, say $x(\vartheta) > 0$, $x(\delta(\vartheta)) > 0$, and $x(\tau(\vartheta)) > 0$ for $\vartheta \geq \vartheta_1 \geq \vartheta_0$.

We first consider what Case (1) holds. By performing the same steps as in the proof of Theorem 3.1, we arrive at (3.7) and (3.13). By (3.7), there exists a constant $d_1 > 0$ such that

$$\frac{\upsilon(\vartheta)}{\vartheta^{3/\varepsilon}} \leq d_1$$

and so

$$\upsilon(\vartheta) \leq d_1 \vartheta^{3/\varepsilon}. \tag{3.26}$$

Using (3.26) in (3.13) and applying the fact that $\beta < 1$ yields

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_1^{\beta} d_1^{\beta-1}\left(q^{3/\varepsilon}(\vartheta)\right)^{\beta-1}\left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}\right)\upsilon(q(\vartheta)) \leq 0. \tag{3.27}$$

That is, (3.24) has a positive solution, a contradiction.

Now, we consider what Case (2) holds. By performing the same steps as in the proof of Theorem 3.1, we arrive at (3.17) and (3.22). By (3.17), there exists a constant $d_2 > 0$ such that

$$\frac{\upsilon(\vartheta)}{\vartheta^{1/\varepsilon}} \leq d_2$$

and so

$$\upsilon(\vartheta) \leq d_2 \vartheta^{1/\varepsilon}. \tag{3.28}$$

Using (3.28) in (3.22) and applying the fact that $\beta < 1$ yields

$$\left(b(\vartheta)\left(\upsilon'''(\vartheta)\right)^{\alpha}\right)' + \epsilon_2^{\beta} d_2^{\beta-1}\left(q^{1/\varepsilon}(\vartheta)\right)^{\beta-1}\left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}\right)\upsilon(q(\vartheta)) \leq 0. \tag{3.29}$$

That is, (3.25) has a positive solution, a contradiction. The proof is now complete. □

Theorem 3.3. *Assume that $\beta \geq 1$ and there exists a positive function $\mu \in C^1([\vartheta_0, \infty), \mathbb{R})$ such that (3.1) holds. If*

$$y'(\vartheta) + \frac{\lambda}{3!}\epsilon_1^{\beta} c^{\beta-1} \frac{q^3(\vartheta)}{b^{1/\alpha}(q(\vartheta))}\left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\vartheta)\right)\right)}\right) y^{1/\alpha}(q(\vartheta)) = 0 \tag{3.30}$$

and

$$\omega'(\vartheta) + \frac{\epsilon_2^{\beta/\alpha} c^{(\beta-1)/\alpha}}{\varepsilon^{-1/\alpha} q^{-1/\alpha}(\vartheta)}\left(\int_{\vartheta}^{\infty} \frac{\left(\int_u^{\infty}\left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^{\beta}\left(\tau^{-1}\left(\delta_i(\zeta)\right)\right)}\right) \mathrm{d}\zeta\right)^{1/\alpha}}{b^{1/\alpha}(u)} \mathrm{d}u\right)\omega^{1/\alpha}(q(\vartheta)) = 0 \tag{3.31}$$

are oscillatory, for some constants $\lambda, \varepsilon \in (0,1)$, then every solution of (1.1) is oscillatory, where $q(\vartheta) = \tau^{-1}(\mu(\vartheta))$, $c > 0$ and $\epsilon_1, \epsilon_2 \in (0,1)$.

Proof. Assume that Eq (1.1) has a nonoscillatory solution $x(\vartheta)$, say $x(\vartheta) > 0$, $x(\delta(\vartheta)) > 0$, and $x(\tau(\vartheta)) > 0$ for $\vartheta \geq \vartheta_1 \geq \vartheta_0$.

We first consider what Case (1) holds. By performing the same steps as in the proof of Theorem 3.1, we arrive at (3.15). Since $\upsilon(\vartheta) > 0$ and $\upsilon'(\vartheta) > 0$, we have $\lim_{\vartheta \to \infty} \upsilon(\vartheta) \neq 0$. Thus, by Lemma 2.1, we obtain

$$\upsilon(\vartheta) \geq \frac{\lambda}{3!}\vartheta^3 \upsilon'''(\vartheta), \tag{3.32}$$

from which we see that

$$\upsilon(q(\vartheta)) \geq \frac{\lambda}{3!}q^3(\vartheta)\,\upsilon'''(q(\vartheta)). \tag{3.33}$$

Using (3.33) in (3.15) yields

$$(b(\vartheta)(\upsilon'''(\vartheta))^{\alpha})' + \frac{\lambda}{3!}\epsilon_1^{\beta}c^{\beta-1}q^3(\vartheta)\left(\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))}\right)\upsilon'''(q(\vartheta)) \leq 0,$$

With $y(\vartheta) = b(\vartheta)(\upsilon'''(\vartheta))^{\alpha}$, we find $y(\vartheta)$ is a positive solution of the differential inequality

$$y'(\vartheta) + \frac{\lambda}{3!}\epsilon_1^{\beta}c^{\beta-1}\frac{q^3(\vartheta)}{b^{1/\alpha}(q(\vartheta))}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))}\right)y^{1/\alpha}(q(\vartheta)) \leq 0. \tag{3.34}$$

It follows from Lemma 2.4 that the differential equation (3.30) also has a positive solution for all $\lambda \in (0,1)$, but this contradicts our assumption on (3.30).

Now, we consider what Case (2) holds. By performing the same steps as in the proof of Theorem 3.1, we arrive at (3.16) and (3.23). Integrating (3.23) from ϑ to ∞ gives

$$(\upsilon'''(\vartheta))^{\alpha} \geq \epsilon_2^{\beta}c^{\beta-1}\frac{\upsilon(q(\vartheta))}{b(\vartheta)}\int_{\vartheta}^{\infty}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta$$

and so

$$\upsilon'''(\vartheta) \geq \epsilon_2^{\beta/\alpha}c^{(\beta-1)/\alpha}\frac{\upsilon^{1/\alpha}(q(\vartheta))}{b^{1/\alpha}(\vartheta)}\left(\int_{\vartheta}^{\infty}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta\right)^{1/\alpha}. \tag{3.35}$$

Integrating (3.35) from ϑ to ∞, we have

$$-\upsilon''(\vartheta) \geq \epsilon_2^{\beta/\alpha}c^{(\beta-1)/\alpha}\left(\int_{\vartheta}^{\infty}\frac{\left(\int_u^{\infty}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta\right)^{1/\alpha}}{b^{1/\alpha}(u)}\mathrm{d}u\right)\upsilon^{1/\alpha}(q(\vartheta))$$

and so

$$\upsilon''(\vartheta) + \epsilon_2^{\beta/\alpha}c^{(\beta-1)/\alpha}\left(\int_{\vartheta}^{\infty}\frac{\left(\int_u^{\infty}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta\right)^{1/\alpha}}{b^{1/\alpha}(u)}\mathrm{d}u\right)\upsilon^{1/\alpha}(q(\vartheta)) \leq 0. \tag{3.36}$$

Using (3.16) in (3.36) yields

$$\upsilon''(\vartheta) + \frac{\epsilon_2^{\beta/\alpha}c^{(\beta-1)/\alpha}}{\varepsilon^{-1/\alpha}q^{-1/\alpha}(\vartheta)}\left(\int_{\vartheta}^{\infty}\frac{\left(\int_u^{\infty}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta\right)^{1/\alpha}}{b^{1/\alpha}(u)}\mathrm{d}u\right)(\upsilon'(q(\vartheta)))^{1/\alpha} \leq 0. \tag{3.37}$$

With $\omega(\vartheta) = \upsilon'(\vartheta)$, we see that $\omega(\vartheta)$ is a positive solution of the differential inequality

$$\omega'(\vartheta) + \frac{\epsilon_2^{\beta/\alpha}c^{(\beta-1)/\alpha}}{\varepsilon^{-1/\alpha}q^{-1/\alpha}(\vartheta)}\left(\int_{\vartheta}^{\infty}\frac{\left(\int_u^{\infty}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta\right)^{1/\alpha}}{b^{1/\alpha}(u)}\mathrm{d}u\right)\omega^{1/\alpha}(q(\vartheta)) \leq 0, \tag{3.38}$$

for every $\varepsilon \in (0,1)$. We finalize the proof using the same method as outlined in Case (1). The proof is now complete. □

Corollary 3.1. *Let $\alpha = 1$ and $\beta \geq 1$ hold. Assume further that there exists a positive function $\mu \in C^1([\vartheta_0, \infty), \mathbb{R})$ such that (3.1) holds. If*

$$\lim_{\vartheta\to\infty} \int_{q(\vartheta)}^{\vartheta} \frac{q^3(\zeta)}{b(q(\zeta))} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^\beta(\tau^{-1}(\delta_i(\zeta)))} \right) \mathrm{d}\zeta = \infty \tag{3.39}$$

and

$$\lim_{\vartheta\to\infty} \int_{q(\vartheta)}^{\vartheta} q(\zeta) \left(\int_{\zeta}^{\infty} \frac{\left(\int_u^\infty \left(\sum_{i=1}^{\eta} \frac{\phi_i(\varrho)}{g^\beta(\tau^{-1}(\delta_i(\varrho)))} \right) \mathrm{d}\varrho \right)}{b(u)} \mathrm{d}u \right) \mathrm{d}\zeta = \infty, \tag{3.40}$$

then every solution of (1.1) is oscillatory, where $q(\vartheta) = \tau^{-1}(\mu(\vartheta))$.

Proof. We first consider what Case (1) holds. By performing the same steps as in the proof of Theorem 3.3, we arrive at (3.34). Integrating (3.34) from $q(\vartheta)$ to ϑ and then using $\alpha = 1$ and the fact that y is a decreasing function, we see that

$$\int_{q(\vartheta)}^{\vartheta} y'(\zeta)\,\mathrm{d}\zeta \leq -\frac{\lambda}{3!}\epsilon_1^\beta c^{\beta-1} y(q(\zeta)) \int_{q(\vartheta)}^{\vartheta} \frac{q^3(\zeta)}{b(q(\zeta))} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^\beta(\tau^{-1}(\delta_i(\zeta)))} \right) \mathrm{d}\zeta$$

and so

$$-y(q(\vartheta)) \leq -\frac{\lambda}{3!}\epsilon_1^\beta c^{\beta-1} y(q(\vartheta)) \int_{q(\vartheta)}^{\vartheta} \frac{q^3(\zeta)}{b(q(\zeta))} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^\beta(\tau^{-1}(\delta_i(\zeta)))} \right) \mathrm{d}\zeta,$$

this can be expressed as follows:

$$\frac{3!}{\lambda\epsilon_1^\beta c^{\beta-1}} \geq \int_{q(\vartheta)}^{\vartheta} \frac{q^3(\zeta)}{b(q(\zeta))} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^\beta(\tau^{-1}(\delta_i(\zeta)))} \right) \mathrm{d}\zeta,$$

so this contradicts (3.39).

Now, we consider what Case (2) holds. By performing the same steps as in the proof of Theorem 3.3, we arrive at (3.38). Integrating (3.38) from $q(\vartheta)$ to ϑ and then using $\alpha = 1$ and the fact that ω is a decreasing function, we see that

$$\int_{q(\vartheta)}^{\vartheta} \omega'(\zeta)\,\mathrm{d}\zeta \leq -\varepsilon\epsilon_2^\beta c^{(\beta-1)} \omega(q(\vartheta)) \int_{q(\vartheta)}^{\vartheta} q(\zeta) \left(\int_{\zeta}^{\infty} \frac{\left(\int_u^\infty \left(\sum_{i=1}^{\eta} \frac{\phi_i(\varrho)}{g^\beta(\tau^{-1}(\delta_i(\varrho)))} \right) \mathrm{d}\varrho \right)}{b(u)} \mathrm{d}u \right) \mathrm{d}\zeta$$

and so

$$-\omega(q(\vartheta)) \leq -\varepsilon\epsilon_2^\beta c^{(\beta-1)} \omega(q(\vartheta)) \int_{q(\vartheta)}^{\vartheta} q(\zeta) \left(\int_{\zeta}^{\infty} \frac{\left(\int_u^\infty \left(\sum_{i=1}^{\eta} \frac{\phi_i(\varrho)}{g^\beta(\tau^{-1}(\delta_i(\varrho)))} \right) \mathrm{d}\varrho \right)}{b(u)} \mathrm{d}u \right) \mathrm{d}\zeta,$$

this can be expressed as follows:

$$\frac{1}{\varepsilon\epsilon_2^\beta c^{(\beta-1)}} \geq \int_{q(\vartheta)}^{\vartheta} q(\zeta) \left(\int_{\zeta}^{\infty} \frac{\left(\int_u^\infty \left(\sum_{i=1}^{\eta} \frac{\phi_i(\varrho)}{g^\beta(\tau^{-1}(\delta_i(\varrho)))} \right) \mathrm{d}\varrho \right)}{b(u)} \mathrm{d}u \right) \mathrm{d}\zeta,$$

which contradicts (3.40). The proof is now complete. □

Theorem 3.4. *Assume that $\beta < 1$ and there exists a positive function $\mu \in C^1([\vartheta_0, \infty), \mathbb{R})$ such that (3.1) holds. If*

$$y'(\vartheta) + \frac{\epsilon_1^{\beta} d_1^{\beta-1} \lambda}{3!} \frac{q^{(3(\beta-1)/\varepsilon)+3}(\vartheta)}{b^{1/\alpha}(q(\vartheta))} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))} \right) y^{1/\alpha}(q(\vartheta)) = 0 \tag{3.41}$$

and

$$\omega'(\vartheta) + \frac{\epsilon_2^{\beta/\alpha} d_2^{(\beta-1)/\alpha}}{\varepsilon^{-1/\alpha} q^{-1/\alpha}(\vartheta)} \left(\int_{\vartheta}^{\infty} \left(\frac{\int_{\zeta}^{\infty} \left(q^{1/\varepsilon}(u) \right)^{\beta-1} \left(\sum_{i=1}^{\eta} \frac{\phi_i(u)}{g^{\beta}(\tau^{-1}(\delta_i(u)))} \right) \mathrm{d}u}{b(\zeta)} \right)^{1/\alpha} \mathrm{d}\zeta \right) \omega^{1/\alpha}(q(\vartheta)) = 0 \tag{3.42}$$

are oscillatory for some constants $\lambda, \varepsilon \in (0,1)$, then every solution of (1.1) is oscillatory, where $q(\vartheta) = \tau^{-1}(\mu(\vartheta))$, $d_1, d_2 > 0$ and $\epsilon_1, \epsilon_2 \in (0,1)$.

Proof. Assume that Eq (1.1) has a nonoscillatory solution $x(\vartheta)$, say $x(\vartheta) > 0$, $x(\delta(\vartheta)) > 0$, and $x(\tau(\vartheta)) > 0$ for $\vartheta \geq \vartheta_1 \geq \vartheta_0$.

We first consider what Case (1) holds. By performing the same steps as in the proof of Theorem 3.2, we arrive at (3.27). Since $\upsilon(\vartheta) > 0$ and $\upsilon'(\vartheta) > 0$, we have $\lim_{\vartheta\to\infty} \upsilon(\vartheta) \neq 0$ and so by Lemma 2.1, we find (3.32) holds. Using (3.32) in (3.27) gives

$$(b(\vartheta)(\upsilon'''(\vartheta))^{\alpha})' + \frac{\epsilon_1^{\beta} d_1^{\beta-1} \lambda}{3!} q^{(3(\beta-1)/\varepsilon)+3}(\vartheta) \left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))} \right) \upsilon'''(q(\vartheta)) \leq 0.$$

With $y(\vartheta) = b(\vartheta)(\upsilon'''(\vartheta))^{\alpha}$, we see that $y(\vartheta)$ is a positive solution of the differential inequality

$$y'(\vartheta) + \frac{\epsilon_1^{\beta} d_1^{\beta-1} \lambda}{3!} \frac{q^{(3(\beta-1)/\varepsilon)+3}(\vartheta)}{b^{1/\alpha}(q(\vartheta))} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\vartheta)}{g^{\beta}(\tau^{-1}(\delta_i(\vartheta)))} \right) y^{1/\alpha}(q(\vartheta)) \leq 0. \tag{3.43}$$

It follows from Lemma 2.4 that the differential equation (3.41) also has a positive solution for all $\lambda_1 \in (0,1)$, but this contradicts our assumption on (3.41).

Now, we consider what Case (2) holds. Then again (3.16) holds for every $\varepsilon \in (0,1)$. By performing the same steps as in the proof of Theorem 3.2, we arrive at (3.29). Integrating (3.29) from ϑ to ∞, we obtain

$$-b(\vartheta)(\upsilon'''(\vartheta))^{\alpha} + \epsilon_2^{\beta} d_2^{\beta-1} \upsilon(q(\vartheta)) \int_{\vartheta}^{\infty} \left(q^{1/\varepsilon}(\zeta) \right)^{\beta-1} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))} \right) \mathrm{d}\zeta \leq 0$$

and so

$$\upsilon'''(\vartheta) \geq \epsilon_2^{\beta/\alpha} d_2^{(\beta-1)/\alpha} \left(\frac{1}{b(\vartheta)} \int_{\vartheta}^{\infty} \left(q^{1/\varepsilon}(\zeta) \right)^{\beta-1} \left(\sum_{i=1}^{\eta} \frac{\phi_i(\zeta)}{g^{\beta}(\tau^{-1}(\delta_i(\zeta)))} \right) \mathrm{d}\zeta \right)^{1/\alpha} \upsilon^{1/\alpha}(q(\vartheta)). \tag{3.44}$$

Integrating (3.44) from ϑ to ∞, we obtain

$$-\upsilon''(\vartheta) \geq \epsilon_2^{\beta/\alpha} d_2^{(\beta-1)/\alpha} \upsilon^{1/\alpha}(q(\vartheta)) \int_{\vartheta}^{\infty} \left(\frac{\int_{\zeta}^{\infty} \left(q^{1/\varepsilon}(u) \right)^{\beta-1} \left(\sum_{i=1}^{\eta} \frac{\phi_i(u)}{g^{\beta}(\tau^{-1}(\delta_i(u)))} \right) \mathrm{d}u}{b(\zeta)} \right)^{1/\alpha} \mathrm{d}\zeta$$

and so

$$\upsilon''(\vartheta)+\epsilon_2^{\beta/\alpha}d_2^{(\beta-1)/\alpha}\left(\int_\vartheta^\infty\left(\frac{\int_\zeta^\infty\left(q^{1/\varepsilon}(u)\right)^{\beta-1}\left(\sum_{i=1}^\eta\frac{\phi_i(u)}{g^\beta\left(\tau^{-1}(\delta_i(u))\right)}\right)\mathrm{d}u}{b(\zeta)}\right)^{1/\alpha}\mathrm{d}\zeta\right)\upsilon^{1/\alpha}(q(\vartheta))\leq 0. \tag{3.45}$$

With $\omega(\vartheta)=\upsilon'(\vartheta)$ and using (3.16) in (3.45), we see that $\omega(\vartheta)$ is a positive solution of

$$\omega'(\vartheta)+\frac{\epsilon_2^{\beta/\alpha}d_2^{(\beta-1)/\alpha}}{\varepsilon^{-1/\alpha}q^{-1/\alpha}(\vartheta)}\left(\int_\vartheta^\infty\left(\frac{\int_\zeta^\infty\left(q^{1/\varepsilon}(u)\right)^{\beta-1}\left(\sum_{i=1}^\eta\frac{\phi_i(u)}{g^\beta\left(\tau^{-1}(\delta_i(u))\right)}\right)\mathrm{d}u}{b(\zeta)}\right)^{1/\alpha}\mathrm{d}\zeta\right)\omega^{1/\alpha}(q(\vartheta))\leq 0.$$

We finalize the proof using the same method as outlined in Case (1). The proof is now complete. □

Corollary 3.2. *Let $\alpha=1$ and $\beta<1$ hold. Assume further that there exists a positive function $\mu\in C^1([\vartheta_0,\infty),\mathbb{R})$ such that (3.1) holds. If*

$$\lim_{\vartheta\to\infty}\int_{q(\vartheta)}^{\vartheta}\frac{q^{(3(\beta-1)/\varepsilon)+3}(\zeta)}{b(q(\zeta))}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^\beta\left(\tau^{-1}(\delta_i(\zeta))\right)}\right)\mathrm{d}\zeta=\infty \tag{3.46}$$

and

$$\lim_{\vartheta\to\infty}\int_{q(\vartheta)}^{\vartheta}q(\varrho)\left(\int_\varrho^\infty\left(\frac{1}{b(\zeta)}\int_\zeta^\infty\left(q^{1/\varepsilon}(u)\right)^{\beta-1}\left(\sum_{i=1}^{\eta}\frac{\phi_i(u)}{g^\beta\left(\tau^{-1}(\delta_i(u))\right)}\right)\mathrm{d}u\right)\mathrm{d}\zeta\right)\mathrm{d}\varrho=\infty, \tag{3.47}$$

then every solution of (1.1) is oscillatory, where $q(\vartheta)=\tau^{-1}(\mu(\vartheta))$.

Proof. The details of the proof are omitted as they are similar to those of Corollary 3.1. □

We provide the following example to demonstrate our results:

Example 3.1. *Consider the neutral differential equation*

$$\left(x(\vartheta)+\vartheta x\left(\frac{\vartheta}{A_1}\right)\right)^{(4)}+\frac{\phi_0}{\vartheta^2}x\left(\frac{\vartheta}{A_2}\right)+\frac{\phi_0}{\vartheta^2}x\left(\frac{\vartheta}{A_3}\right)=0,\ \vartheta\geq 1, \tag{3.48}$$

It is easy to verify that

$$\int_{\vartheta_0}^\infty\frac{1}{b^{1/\alpha}(s)}\mathrm{d}s=\infty.$$

Choosing $\mu(\vartheta)=\vartheta/A_4$ and $A=\max\{A_2,A_3\}$, where $A_4>A_1$, $A_4\geq A$ and $A_1,A_2,A_3,A_4>1$, then (3.1) holds. We also find that

$$\tau^{-1}(\vartheta)=A_1\vartheta,\ q(\vartheta)=\frac{A_1}{A_4}\vartheta,\ \tau^{-1}(\delta_1(\vartheta))=\frac{A_1}{A_2}\vartheta\text{ and }\tau^{-1}(\delta_2(\vartheta))=\frac{A_1}{A_3}\vartheta.$$

Let $\varepsilon=1/4$; we see that (1.4) holds.

Now, it is easy to check that the condition (3.39) is satisfied, where

$$\begin{aligned}\lim_{\vartheta\to\infty}\int_{q(\vartheta)}^{\vartheta}\frac{q^3(\zeta)}{b(q(\zeta))}\left(\sum_{i=1}^{\eta}\frac{\phi_i(\zeta)}{g^\beta(\tau^{-1}(\delta_i(\zeta)))}\right)\mathrm{d}\zeta\\ &= \lim_{\vartheta\to\infty}\int_{A_1\vartheta/A_4}^{\vartheta}\left(\frac{A_1}{A_4}s\right)^3\left(\frac{\phi_0}{s^2}\left(\frac{A_1}{A_2}s\right)^{-1}+\frac{\phi_0}{s^2}\left(\frac{A_1}{A_3}s\right)^{-1}\right)\mathrm{d}s\\ &= \lim_{\vartheta\to\infty}\int_{A_1\vartheta/A_4}^{\vartheta}\phi_0\left(\frac{A_1}{A_4}\right)^3\left(\left(\frac{A_1}{A_2}\right)^{-1}+\left(\frac{A_1}{A_3}\right)^{-1}\right)\mathrm{d}s=\infty.\end{aligned}$$

Moreover, we find that condition (3.40) is satisfied, where

$$\begin{aligned}\lim_{\vartheta\to\infty}\int_{q(\vartheta)}^{\vartheta}q(\zeta)\left(\int_\zeta^\infty\frac{1}{b(u)}\left(\int_u^\infty\left(\sum_{i=1}^{\eta}\frac{\phi_i(\varrho)}{g^\beta(\tau^{-1}(\delta_i(\varrho)))}\right)\mathrm{d}\varrho\right)\mathrm{d}u\right)\mathrm{d}\zeta\\ &= \lim_{\vartheta\to\infty}\int_{A_1\vartheta/A_4}^{\vartheta}\frac{A_1}{A_4}\zeta\left(\int_\zeta^\infty\left(\int_u^\infty\left(\frac{\phi_0}{\varrho^3}\left(\left(\frac{A_1}{A_2}\right)^{-1}+\left(\frac{A_1}{A_3}\right)^{-1}\right)\right)\mathrm{d}\varrho\right)\mathrm{d}u\right)\mathrm{d}\zeta\\ &= \lim_{\vartheta\to\infty}\int_{A_1\vartheta/A_4}^{\vartheta}\frac{A_1}{A_4}\zeta\left(\int_\zeta^\infty\left(\frac{\phi_0}{2u^2}\left(\left(\frac{A_1}{A_2}\right)^{-1}+\left(\frac{A_1}{A_3}\right)^{-1}\right)\right)\mathrm{d}u\right)\mathrm{d}\zeta\\ &= \lim_{\vartheta\to\infty}\int_{A_1\vartheta/A_4}^{\vartheta}\frac{A_1}{A_4}\zeta\left(\frac{\phi_0}{2\zeta}\left(\left(\frac{A_1}{A_2}\right)^{-1}+\left(\frac{A_1}{A_3}\right)^{-1}\right)\right)\mathrm{d}\zeta=\infty.\end{aligned}$$

Thus, using Corollary 3.1, every solution of (3.48) is oscillatory.

Remark 3.1. *Consider the neutral differential equation*

$$\left(x(\vartheta)+\vartheta x\left(\frac{\vartheta}{3}\right)\right)^{(4)}+\frac{\phi_0}{\vartheta^2}x\left(\frac{\vartheta}{2}\right)=0,\ \vartheta\geq 1, \tag{3.49}$$

as a special case of Eq (3.48), we see that Theorem 1.1 cannot be applied to (3.49) since

$$\lim_{\vartheta\to\infty}g(\vartheta)\left(\vartheta^{n-2}\int_{\vartheta_0}^{\vartheta}\frac{1}{b^{1/\alpha}(\zeta)}\mathrm{d}\zeta\right)^{\gamma-1}\neq 0,$$

accordingly, Theorem 1.1 fails to study the oscillation of (3.49).

Also, we see that Theorem 1.2 cannot be applied to (3.49) since $\delta(\vartheta)=\vartheta/2$ is greater than $\tau(\vartheta)=\vartheta/3$ for $\vartheta\geq 1$. Accordingly, Theorem 1.2 fails to study the oscillation of (3.49).

Now, by using Corollary 3.1 and choosing $\mu(\vartheta)=\vartheta/4$ and $\varepsilon=1/4$, we see that the conditions (3.39) and (3.40) are satisfied and therefore, Eq (3.49) is oscillatory.

4. Conclusions

By using comparison principles, we analyze the asymptotic behavior of solutions to a class of fourth-order neutral differential equations. We have obtained some new oscillation results for (1.1) in the case where $\int_\vartheta^\infty 1/b^{1/\alpha}(\zeta)\,\mathrm{d}\zeta=\infty$. These results ensure that all solutions to the studied equation are oscillatory, and they also improve and extend some results from previous studies. It will be of interest to investigate the higher-order differential equations of the form

$$\left(b(\vartheta)\left(\left(x(\vartheta)+g(\vartheta)\,x(\tau(\vartheta))\right)^{(n-1)}\right)^{\alpha}\right)'+\sum_{i=1}^{\eta}\phi_i(\vartheta)\,x^{\beta}(\delta_i(\vartheta))=0,$$

where $n \geq 4$ is an even natural number.

Author contributions

The authors contributed equally to this work. Both of the authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors acknowledge the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia, under project Grant No. 6018.

Conflict of interest

The authors declare no conflicts of interest.

References

1. A. Abdelnaser, O. Moaaz, C. Cesarano, S. Askar, E. M. Elabbasy, Oscillation test for second-order differential equations with several delays, *Symmetry*, **15** (2023), 452. https://doi.org/10.3390/sym15020452
2. R. P. Agarwal, M. Bohner, T. Li, C. Zhang, A new approach in the study of oscillatory behavior of even-order neutral delay differential equations, *Appl. Math. Comput.*, **225** (2013), 787–794. https://doi.org/10.1016/j.amc.2013.09.037
3. R. P. Agarwal, M. Bohner, T. Li, C. Zhang, Oscillation of second-order differential equations with a sublinear neutral term, *Carpathian J. Math.*, **30** (2014), 1–6. https://doi.org/10.37193/CJM.2014.01.01
4. R. P. Agarwal, S. R. Grace, D. O'Regan, Oscillation criteria for certain nth-order differential equations with deviating arguments, *J. Math. Anal. Appl.*, **262** (2001), 601–622. https://doi.org/10.1006/jmaa.2001.7571
5. R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation theory for difference and functional differential equations*, Springer Dordrecht, 2000. https://doi.org/10.1007/978-94-015-9401-1

6. S. Althubiti, I. Aldawish, J. Awrejcewicz, O. Bazighifan, New oscillation results of even-order Emden-Fowler neutral differential equations, *Symmetry*, **13** (2021), 2177. https://doi.org/10.3390/sym13112177

7. B. Baculíková, J. Džurina, On certain inequalities and their applications in the oscillation theory, *Adv. Differ. Equ.*, **165** (2013), 165. https://doi.org/10.1186/1687-1847-2013-165

8. D. D. Bainov, V. A. Petrov, V. S. Proytcheva, Existence and asymptotic behavior of nonoscillatory solutions of second-order neutral differential equations with "maxima", *J. Comput. Appl. Math.*, **83** (1997), 237–249. https://doi.org/10.1016/S0377-0427(97)00105-2

9. M. Bartušek, Z. Došlá, Oscillation of third-order differential equation with damping term, *Czech. Math. J.*, **65** (2015), 301–316. https://doi.org/10.1007/s10587-015-0176-3

10. O. Bazighifan, H. Ahmad, Asymptotic behavior of solutions of even-order advanced differential equations, *Math. Probl. Eng.*, **2020** (2020), 8041857. https://doi.org/10.1155/2020/8041857

11. O. Bazighifan, I. Dassios, Riccati technique and asymptotic behavior of fourth-order advanced differential equations, *Mathematics*, **8** (2020), 590. https://doi.org/10.3390/math8040590

12. G. A. Bliss, I. J. Schoenberg, On separation, comparison and oscillation theorems for self-adjoint systems of linear second-order differential equations, *Amer. J. Math.*, 53 (1931), 781–800. https://doi.org/10.2307/2371226

13. J. L. Chern, W. C. Lian, C. C. Yeh, Oscillation criteria for second-order half-linear differential equations with functional arguments, *Publ. Math. Debrecen*, **48** (1996), 209–216.

14. S. R. Grace, J. R. Graef, Oscillatory behavior of second-order nonlinear differential equations with a sublinear neutral term, *Math. Model. Anal.*, **23** (2018), 217–226. https://doi.org/10.3846/mma.2018.014

15. S. R. Grace, Oscillation criteria for third-order nonlinear delay differential equations with damping, *Opuscula Math.*, **35** (2015), 485–497. https://doi.org/10.7494/OpMath.2015.35.4.485

16. J. R. Graef, S. R. Grace, E. Tunç, Oscillatory behavior of even-order nonlinear differential equations with a sublinear neutral term, *Opuscula Math.*, **39** (2019), 39–47. https://doi.org/10.7494/OpMath.2019.39.1.39

17. K. Kamo, H. Usami, Nonlinear oscillations of fourth-order quasilinear ordinary differential equations, *Acta Math. Hungar.*, **132** (2011), 207–222. https://doi.org/10.1007/s10474-011-0127-x

18. T. Li, Y. V. Rogovchenko, On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations, *Appl. Math. Lett.*, **67** (2017), 53–59. https://doi.org/10.1016/j.aml.2016.11.007

19. O. Moaaz, C. Cesarano, A. Muhib, Some new oscillation results for fourth-order neutral differential equations, *Eur. J. Pure Appl. Math.*, **13** (2020), 185–199. https://doi.org/10.29020/nybg.ejpam.v13i2.3654

20. O. Moaaz, C. Park, A. Muhib, O. Bazighifan, Oscillation criteria for a class of even-order neutral delay differential equations, *J. Appl. Math. Comput.*, **63** (2020), 607–617. https://doi.org/10.1007/s12190-020-01331-w

21. A. Muhib, T. Abdeljawad, O. Moaaz, E. M. Elabbasy, Oscillatory properties of odd-order delay differential equations with distribution deviating arguments, *Appl. Sci.*, **10** (2020), 5952. https://doi.org/10.3390/app10175952

22. A. Muhib, E. M. Elabbasy, O. Moaaz, New oscillation criteria for differential equations with sublinear and superlinear neutral terms, *Turkish J. Math.*, **45** (2021), 919–928. https://doi.org/10.3906/mat-2012-11

23. A. Muhib, On oscillation of second-order noncanonical neutral differential equations, *J. Inequal. Appl.*, **2021** (2021), 79. https://doi.org/10.1186/s13660-021-02595-x

24. C. G. Philos, On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays, *Arch. Math.*, **36** (1981), 168–178. https://doi.org/10.1007/BF01223686

25. T. Tanigawa, Oscillation criteria for a class of higher-order nonlinear differential equations, *Mem. Differ. Equations Math. Phys.*, **37** (2006), 137–152.

26. E. Thandapani, V. Ganesan, Classifiction of solutions of second-order neutral delay differential equations with "maxima", *Inter. J. Diff. Equ. Appl.*, **11** (2012), 145–155.

27. E. Tunç, Oscillatory and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments, *Electron. J. Differ. Equations,* 2017.

AIMS Press