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Research article

Solvability and algorithm for Sylvester-type quaternion matrix equations with potential applications

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Abstract: This article explores Sylvester quaternion matrix equations and potential applications, which are important in fields such as control theory, graphics, sensitivity analysis, and three-dimensional rotations. Recognizing that the determination of solutions and computational methods for these equations is evolving, our study contributes to the area by establishing solvability conditions and providing explicit solution formulations using generalized inverses. We also introduce an algorithm that utilizes representations of quaternion Moore-Penrose inverses to improve computational efficiency. This algorithm is validated with a numerical example, demonstrating its practical utility. Additionally, our findings offer a generalized framework in which various existing results in the area can be viewed as specific instances, showing the breadth and applicability of our approach. Acknowledging the challenges in handling large systems, we propose future research focused on further improving algorithmic efficiency and expanding the applications to diverse algebraic structures. Overall, our research establishes the theoretical foundations necessary for solving Sylvester-type quaternion matrix equations and introduces a novel algorithmic solution to address their computational challenges, enhancing both the theoretical understanding and practical implementation of these complex equations.

Keywords: generalized inverses; Maple software; quaternion matrices; solvability conditions; Sylvester equations
Mathematics Subject Classification: 15A09, 15A24, 15A29, 65F05

1. Introduction

Matrix equations play a fundamental role in several fields, such as engineering, natural sciences, and physics [1–3]. In particular, Sylvester matrix equations have generated attention due to their applications in control theory, estimation theory, and sensitivity analysis, impacting topics such as system design [4], singular system control [5], and linear descriptor systems [6]. Moreover, in engineering, the relevance of generalized Sylvester matrix equations extends to various other applications, supported by foundational numerical methods for matrices [7]. Recent studies have focused on extensions and computations of the Moore-Penrose inverse (MPI), a concept of matrix theory that can be used to solve different types of these equations [8]. Growing interest in these complex matrix equations has led to the exploration of advanced algebraic systems, such as quaternions, which offer a broader framework for solving multidimensional problems.

The introduction of quaternionic mathematics has further diversified the applied mathematical field [9–12]. Quaternions are central in describing three-dimensional (3D) rotations and are widely utilized in computer graphics, navigation, and robotics, as well as in mechanics, quantum physics, and signal processing, extensively discussed in [13]. Recent advancements include explicit formulas and determinantal representations for quaternion matrix equations, which have allowed new approaches to address both standard and quaternion systems [14–16].

Contemporary research has focused on Sylvester-type matrix equations and quaternion matrices [17–24] creating a new framework. An approach to solving quaternion Sylvester equations using neural networks, with practical applications in controlling chaotic systems, was introduced in [25]. The advancements in this framework reflect the application of complex mathematical techniques in diverse systems, as seen in recent studies on time-varying multi-linear tensor equations [26] and algorithmic solutions for matrix equations [27]. These advancements include representations based on determinants and Cramer rules in quaternionic systems [28–34], alongside explorations into numerical solutions for quaternionic Riccati equations [20] and coupled quaternion matrix equations [21]. Recent interdisciplinary studies have further extended these mathematical frameworks to diverse fields, demonstrating significant applications in bio-economics, industrial systems, and software development best practices [35–37]. Additionally, recent investigations have focused on the domain of algorithmic approaches and solvability conditions for quaternion matrix equations, improving our comprehension in the mentioned framework.

Despite such advancements, analyses focusing on the solvability of quaternionic Sylvester matrix equations, particularly for multifaceted and highly complex systems, appear to be lacking, creating a gap in this solvability and in the application of these equations to fields such as high-dimensional data analysis [38], quantum mechanics, and robotics [39], where noncommutative operations are essential. Our research aims to bridge this gap, offering an in-depth theoretical and practical exploration of Sylvester-type quaternion matrix equations within this sophisticated multidimensional framework.

The motivation of this study stems from the need for mathematical approaches in the solvability of Sylvester-type quaternion matrix equations and their application to the high-dimensional data analysis and representation of complex physical phenomena. Sylvester equations in quaternions offer a promising methodology for handling essential noncommutative operations in areas such as quantum mechanics, 3D computer graphics, and robotics. The present work aims to contribute both to the theoretical development of quaternion matrix theory and computational strategies for complex systems.

The contributions of this article are twofold. First, we establish the necessary and sufficient conditions for solving Sylvester-type quaternion matrix equations, utilizing generalized inverses. This contribution of our work builds upon the research of various scholars [16, 28, 32, 33, 40–42]. Second, we develop a new algorithmic approach that addresses the computational challenges commonly associated with these equations, as demonstrated through numerical examples [22–24]. Our study aims to enrich theoretical understanding and provide computational tools for Sylvester-type quaternion matrix equations.

The remainder of this article is organized as follows. Section 2 discusses quaternion algebra and the specific problems addressed. In Section 3, we present our main results on solvability conditions. In Section 4, our algorithmic approach with a numerical example is detailed. Section 5 concludes with a summary of findings and future research directions.

2. Mathematical framework

This section begins with an exposition on quaternion algebra, necessary for addressing the more complex matrix structures and equations, such as Sylvester matrix equations and their general solutions, which form the basis of our study. It is important to note that, throughout this article, we deliberately follow the practice of reusing symbols for the coefficients of equations to indicate the consistent function of these matrices across various systems of matrix equations. This practice is intended to facilitate the reader understanding of the applicability of solution methods, solvability conditions, and structural relationships between the equations. While the matrices may not be identical in terms of their specific values in each instantiation, they are conceptually consistent in terms of their application and role within the equations presented.

2.1. Quaternion algebra

Quaternions are hypercomplex numbers that incorporate three distinct imaginary units, namely $\mathrm{i}, \mathrm{j}, \mathrm{k}$, each paired with real number coefficients. They enable the representation of rotations in 3D space and represent a four-dimensional construct. Let $\mathbb{R}$ denote the field of real numbers and $\mathbb{H}$ the quaternion algebra defined as

$$\mathbb{H} = \{a_0 + a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k} \mid \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1,\ a_0,\ a_1,\ a_2,\ a_3 \in \mathbb{R}\}. \tag{2.1}$$

The coefficients a_0, a_1, a_2, a_3 presented in the expression stated in (2.1) are all real numbers, indicating that while quaternions comprise imaginary numbers, their components are rooted in real numbers.

We denote the set of all $m \times n$ matrices over $\mathbb{H}$ by $\mathbb{H}^{m\times n}$. Quaternions extend into matrix algebra through matrices composed of quaternion elements. A quaternion matrix A of dimension $m\times n$, within $\mathbb{H}^{m\times n}$, encompasses elements from a quaternion algebra, allowing for complex spatial transformations and rotations to be represented in matrix form. This extension of quaternions into matrices facilitates advanced calculations in a 3D space and beyond.

For a quaternion matrix $A \in \mathbb{H}^{m\times n}$, the conjugate transpose is denoted by A^*. The MPI of A, represented as $A^\dagger$, satisfies the conditions $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$, and $(A^\dagger A)^* = A^\dagger A$. Also, we introduce the projection operators L_A and R_A for a quaternion matrix A as

$$L_A = I - A^\dagger A,\ R_A = I - AA^\dagger, \tag{2.2}$$

which are idempotent and Hermitian, satisfying $L_A = L_A^{\dagger} = L_A^{*} = L_A^{2}$ and $R_A = R_A^{*} = R_A^{2} = R_A^{\dagger}$. These operators are used in our subsequent analysis of quaternion matrix equations and denote the left and right projection operators applied to a matrix, reflecting operations that act on that matrix from the left and right sides, respectively. For example, L_A and R_A denote the left and right projection operators applied to matrix A, reflecting operations that act on A from the left and right sides, respectively.

The exploration of quaternion-related matrix equations in various applications, such as signal processing and system control, brings us to the Sylvester matrix equation, which is typically represented as

$$AX + XB = C, \tag{2.3}$$

where A, B, and C are known matrices, and X is the matrix to be determined, all of them being quaternion matrices. The equation stated in (2.3), a standard form of the Sylvester equation, has been the subject of extensive study [43], primarily due to its applications in control theory and linear systems.

The solvability and general solution for a variation of the standard Sylvester equation presented in (2.3) have been explored in [44–46]. This equation involves an additional unknown quaternion matrix Y, which adds complexity to the problem, being particularly relevant in scenarios involving coupled or interconnected systems. The system defined by

$$A_1X_1 + Y_1B_1 = O_1,\ A_2X_2 + Y_1B_2 = O_2, \tag{2.4}$$

has been stated for its solvability and general solution [47, 48]. The system formulated in (2.4), an extended version of the Sylvester equation involving multidimensional matrices, encapsulates the complexity of quaternion-related matrix equations, where significant research has focused on analyzing its condition number [49] and establishing necessary and sufficient conditions for its solution [50].

Other studies include the analysis of mixed forms of these equations [51] and the development of solutions for their restricted versions [52]. Such studies address specific constraints and conditions within the mentioned systems, demonstrating the depth of study in this area. A notable example is the system of equations established in (2.4), which represents a pair of mixed generalized Sylvester matrix equations. Necessary and sufficient conditions for the solvability of this system, as well as its general solution, have been provided in [51]. Furthermore, general solutions for some mixed type generalized Sylvester matrix equations involving four variable matrices have been explored in [47].

Recent studies, as cited in [53–56], have expanded the discussion to include general solutions for various matrix equation systems, as the complex system given by

$$A_1X_1 + Y_1B_1 + F_1V_1G_1 = O_1,\ A_2X_2 + Y_2B_2 + F_2V_1G_2 = O_2, \tag{2.5}$$

considered in [50], offering a broader perspective on the solvability of complex matrix equations. In addition, the system stated as

$$\begin{aligned} &A_1X_1 = C_1,\ A_2X_2 = C_2,\ Y_1B_1 = C_3,\ Y_2B_2 = C_4,\ A_3V_1 = C_5,\ V_1B_3 = C_6,\\ &A_4X_1 + Y_1B_4 + F_1V_1G_1 = O_1,\ A_5X_2 + Y_2B_5 + F_2V_1G_2 = O_2, \end{aligned} \tag{2.6}$$

is considered in [14, 41], and its general solution has been obtained. More recently, the system established by

$$A_6X_1 + Y_1B_6 + D_1U_1E_1 + F_1V_1G_1 = O_3,\ A_7X_2 + Y_2B_7 + D_2U_2E_2 + F_2V_1G_2 = O_4, \tag{2.7}$$

was studied in [57] showing advances in understanding the consistency and solvability of increasingly complex matrix systems.

2.2. Sylvester matrix equations and general solutions

Building on the comprehensive examination of quaternion-related matrix equations and their roles in system control, signal processing, and related fields, our investigation extends into Sylvester matrix equations. These equations, historically crucial in contemporary applications, address the intricate noncommutative operations encountered in areas such as quantum mechanics, 3D computer graphics, and robotics. Highlighted in [58] and further detailed in [59], Sylvester equations underpin feedback mechanisms and perturbation theory, offering effective solutions to both theoretical and practical challenges across various domains.

Prompted by the above-mentioned aspects, our focus is on the derivation of general solutions for complex systems characterized by Sylvester-type quaternion matrix equations. This derivation motivates our exploration into a consistent system formulated as

$$\begin{aligned} &A_1X_1 = C_1,\ A_2X_2 = C_2,\ Y_1B_1 = C_3,\ Y_2B_2 = C_4,\ A_3U_1 = C_5,\\ &U_1B_3 = C_6,\ A_4U_2 = C_7,\ U_2B_4 = C_8,\ A_5V_1 = C_9,\ V_1B_5 = C_{10},\\ &A_6X_1 + Y_1B_6 + D_1U_1E_1 + F_1V_1G_1 = O_3,\ A_7X_2 + Y_2B_7 + D_2U_2E_2 + F_2V_1G_2 = O_4, \end{aligned} \tag{2.8}$$

where the matrices $A_1, \ldots, A_7$, $B_1, \ldots, B_7$, D_1, D_2, E_1, E_2, F_1, F_2 and G_1, G_2 work as coefficient matrices that interact with the unknown matrix variables X_1, Y_1, U_1, U_2, V_1, whereas the matrices $C_1, \ldots, C_{10}$ and O_3, O_4 serve as constant matrices representing the known terms or outcomes within the equations. A thorough understanding of the role and interaction of each coefficient and constant matrix is essential to comprehending the system overall behavior and for devising effective solutions.

The equations stated in (2.4)–(2.7) can be viewed as particular instances of the system presented in (2.8), illustrating its role in a broader mathematical context. The flowchart depicted in Figure 1 visually encapsulates these equations, offering a clear depiction of how each system progressively builds upon the previous one. We delineate the precise conditions that ensure the solvability of the system represented in (2.8), and construct a comprehensive framework for its general solution. This is achieved through an analytical approach, elaborated in the key lemmas that follow. The projection operators L_A and R_A introduced in (2.2), along with the MPI $A^\dagger$, play crucial roles in the analysis of quaternion matrix equations. These concepts are utilized in the following lemmas to explore the relations and solvability conditions of specific systems of matrix equations. The following lemma establishes a set of relations used in our analysis [60].

Lemma 2.1. *Let $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{m\times t}$, and $C \in \mathbb{H}^{l\times n}$. Then, the following relations hold:*

$$\operatorname{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right) - \operatorname{rank}(CL_A) = \operatorname{rank}(A), \quad \operatorname{rank}\left(\begin{bmatrix} A & B \end{bmatrix}\right) - \operatorname{rank}(R_BA) = \operatorname{rank}(B).$$

The next lemma addresses a system of matrix equations and conditions for its consistency [61].

Lemma 2.2. *Given the matrices $A \in \mathbb{H}^{m\times n}, B \in \mathbb{H}^{r\times s}, C \in \mathbb{H}^{m\times r}$, and $D \in \mathbb{H}^{n\times s}$, consider the unknown matrix $X \in \mathbb{H}^{n\times r}$. The system formulated as*

$$AX = C, \quad XB = D, \tag{2.9}$$

is consistent if, and only if, the conditions $R_AC = 0$, $DL_B = 0$, and $AC = DB$ are met. Under these conditions, the general solution to the system presented in (2.9) *is given by $X = A^\dagger C + L_ADB^\dagger + L_AUR_B$, where U is an arbitrary matrix with appropriate dimensions over $\mathbb{H}$.*

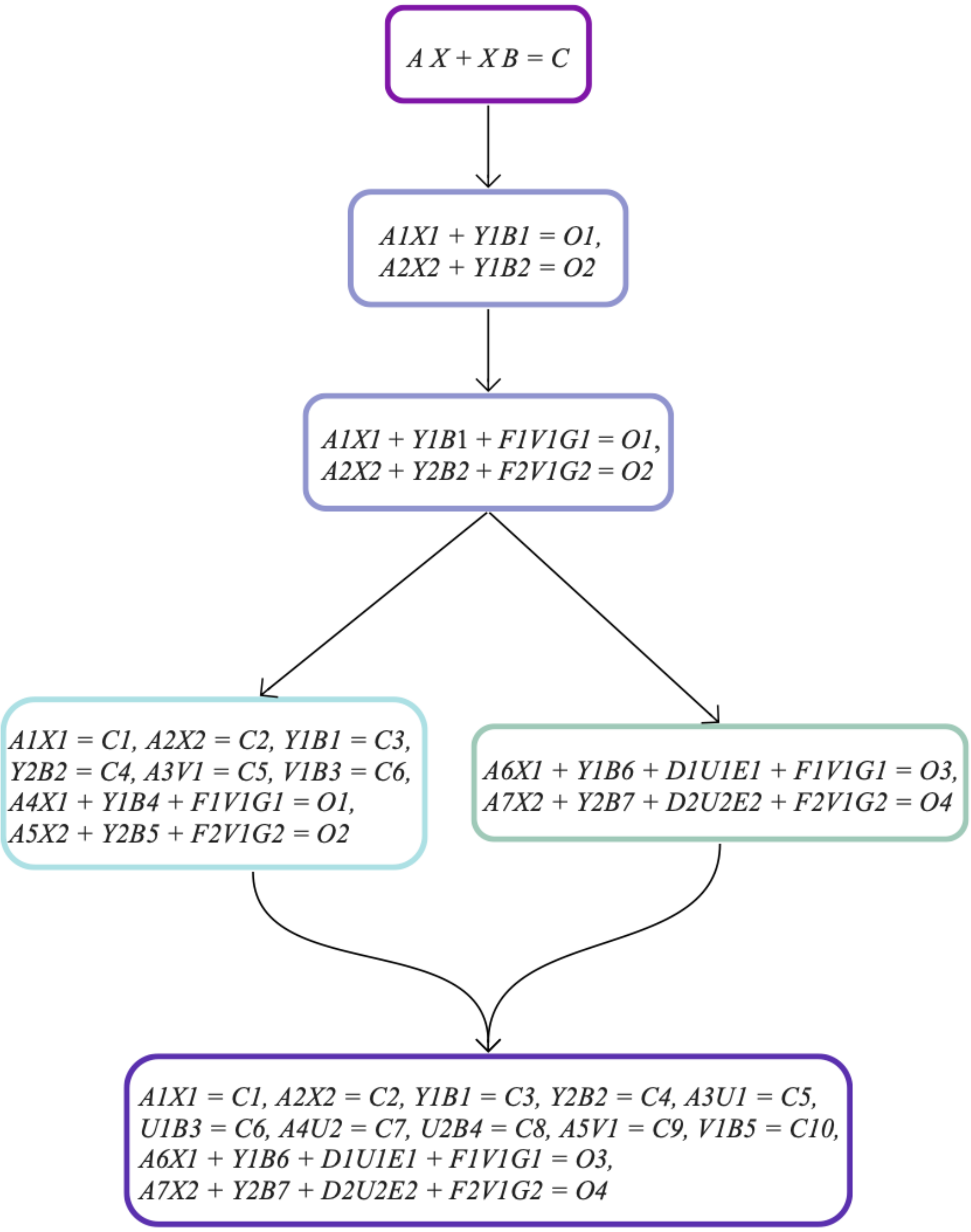

Figure 1. Flowchart illustrating the progression and relationships between the equations.

The subsequent lemma, as detailed in [62], focuses on specific matrix relationships and their equivalences.

Lemma 2.3. *Consider the matrices* A, B, C, D, E, F, *and* G *over* $\mathbb{H}$ *with suitable dimentions. Define the new matrices* $P = R_AC$, $Q = DL_B$, $R = R_AE$, $S = FL_B$, $T = R_AF$, $U = GL_B$, *and* $V = CL_F$, *where* $R_A = I - AA^{\dagger}$ *and* $L_B = I - B^{\dagger}B$ *denote the right and left projection operators for matrices* A *and* B, *respectively. Then, the following statements are equivalent:*

(i) The equation given by $AO + VB + CWD + EYZ = G$ *has a solution.*
(ii) The conditions $R_TR_AV = 0$, $VL_BL_U = 0$, $R_AVL_D = 0$, *and* $R_CVL_B = 0$ *are satisfied.*

(iii) The rank conditions are stated as

$$\operatorname{rank}\begin{pmatrix} G & E & C & A \\ B & 0 & 0 & 0 \end{pmatrix} = \operatorname{rank}(B) + \operatorname{rank}(E\ C\ A),$$

$$\operatorname{rank}\begin{pmatrix} G & A \\ D & 0 \\ F & 0 \\ B & 0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} D \\ F \\ B \end{pmatrix} + \operatorname{rank}(A),$$

$$\operatorname{rank}\begin{pmatrix} G & C & A \\ F & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = \operatorname{rank}(A\ C) + \operatorname{rank}\begin{pmatrix} F \\ B \end{pmatrix},$$

$$\operatorname{rank}\begin{pmatrix} G & E & A \\ D & 0 & 0 \\ B & 0 & 0 \end{pmatrix} = \operatorname{rank}(A\ E) + \operatorname{rank}\begin{pmatrix} D \\ B \end{pmatrix}.$$

Under these conditions, the general solution to the equation system is specified as

$$\begin{aligned}
O &= A^{\dagger}(G - EWD - EYZ) - A^{\dagger}S_7B + L_AS_6,\\
V &= R_A(G - EWD - EYZ)B^{\dagger} + AA^{\dagger}S_7 + S_8R_B,\\
W &= P^{\dagger}GB^{\dagger} - P^{\dagger}CS^{\dagger}GB^{\dagger} - P^{\dagger}OT^{\dagger}GE^{\dagger}DB^{\dagger} - P^{\dagger}OS_2R_UDB^{\dagger} + L_AS_4 + S_5R_B,\\
Z &= S^{\dagger}GD^{\dagger} + O^{\dagger}OT^{\dagger}GE^{\dagger} + L_FL_OS_1 + L_FS_2R_U + S_3R_D,
\end{aligned}$$

where $S_1, \dots, S_8$ are arbitrary matrices of appropriate dimensions over $\mathbb{H}$.

In this section, we have addressed the outlined objectives by establishing a detailed mathematical framework for the solution of the system formulated in (2.8). Our approach, integrating generalized inverses, offers an alternative to traditional iterative methods.

3. Main results and theoretical contributions

This section presents and proves the main theorem that emerged from our investigation, along with related corollaries and their uses, demonstrating the wide range and flexibility of the theorem.

3.1. Main result

Next, we state the main theorem which establishes the necessary and sufficient conditions for the consistency of the system described in (2.8), within the quaternion algebra $\mathbb{H}$. The formulation and analysis of this theorem rely on the utilization of projection operators L_A and R_A, as detailed in the system presented in (2.2), offering a methodological advantage in dissecting the complexity of the system under study.

Theorem 3.1. *Consider the matrices $A_1, \dots, A_7$, $B_1, \dots, B_7$, $C_1, \dots, C_{10}$, D_1, D_2, E_1, E_2, F_1, F_2, G_1, G_2, and O_1, O_2 of appropriate dimensions over $\mathbb{H}$, as well as the transformations and relations between them defined as*

$$
\begin{aligned}
A_8 &= A_5L_{A_1},\ B_8 = R_{B_1}B_5,\ D_3 = D_1L_{A_3},\ E_3 = R_{B_3}E_1,\ F_3 = F_1L_{A_7}, X_1 = A_1^{\dagger}C_1, Y_1 = C_3B_1^{\dagger},\\
G_3 &= R_{B_7}G_1,\ D_4 = R_{A_8}D_3,\ E_4 = E_3L_{B_8}, U_1 = A_3^{\dagger}C_5 + LA_3C_6B_3^{\dagger},\ F_4 = R_{A_8}F_3,\ G_4 = G_3L_{B_8},\\
O_3 &= O_1 - (A_5A_1^{\dagger}C_1 - C_3B_1^{\dagger}B_5 - D_1A_3^{\dagger}C_5E_1 - D_1L_{A_3}C_6B_3^{\dagger}E_1 - F_1A_7^{\dagger}C_9G_1 - F_1L_{A_8}C_{10}B_7^{\dagger}G_1),\\
O_4 &= R_{A_8}O_3L_{B_8},\ M_1 = R_{D_4}F_4,\ N_1 = G_4L_{E_4},\ S_1 = F_4L_{M_1}, V_1 = A_7^{\dagger}C_9 + LA_7C_{10}B_7^{\dagger},\\
A_9 &= A_6L_{A_2},\ B_9 = R_{B_2}B_6,\ D_6 = D_2L_{A_4},\ E_6 = R_{B_4}E_2,\ F_6 = F_2L_{A_7},\ G_6 = R_{B_7}G_2,\\
O_5 &= O_2 - (A_6A_2^{\dagger}C_2 - C_4B_2^{\dagger}B_6 - D_2A_4^{\dagger}C_7E_2 - D_2L_{A_4}C_8B_4^{\dagger}E_2 - F_2A_7^{\dagger}C_9G_2 - F_2L_{A_7}C_{10}B_7^{\dagger}G_2),\\
D_7 &= R_{A_9}D_6,\ E_7 = E_6L_{B_9},\ F_7 = R_{A_9}F_6,\ G_7 = G_6L_{B_9}, X_2 = A_2^{\dagger}C_2, Y_2 = C_4B_2^{\dagger},\\
O_6 &= R_{A_9}O_5L_{B_9},\ M_2 = R_{D_7}F_7,\ N_2 = G_7L_{E_7},\ S_2 = F_7L_{M_2}, U_2 = A_4^{\dagger}C_7 + L_{A_2}C_8B_4^{\dagger},\\
A &= \begin{bmatrix} L_{M_1}L_{S_1} & L_{M_2}L_{S_2} \end{bmatrix},\ B = \begin{bmatrix} R_{G_4} \\ R_{G_7} \end{bmatrix},\ T_9 = -T_4,\ N_3 = G_8L_{E_8},\ S_3 = F_8L_{M_3},\\
W_1 &= M_2^{\dagger}O_6G_7^{\dagger} + S_2^{\dagger}S_2F_7^{\dagger}O_6N_2^{\dagger},\ W_2 = M_1^{\dagger}O_4G_4^{\dagger} + S_1^{\dagger}S_1F_4^{\dagger}O_4N_1^{\dagger},\ M_3 = R_{D_8}F_8,\\
W_3 &= W_1 - W_2,\ D_8 = R_AL_{M_1},\ E_8 = R_{N_1}L_B,\ F_8 = R_AL_{M_2},\ G_8 = R_{N_2}L_B,\ O_8 = R_AW_3L_B. \qquad (3.1)
\end{aligned}
$$

Then, the following statements are equivalent:

(i) The system represented in (2.8) *is consistent.*

(ii) This system satisfies the conditions stated as

$$
\begin{gathered}
R_{A_1}C_1 = 0, C_3L_{B_1} = 0, R_{A_3}C_5 = 0, C_6L_{B_3} = 0, A_3C_6 = C_5B_3, R_{A_4}C_7 = 0, C_8L_{B_4} = 0,\\
A_4C_8 = C_7B_4, R_{A_2}C_2 = 0, C_4L_{B_2} = 0, R_{A_7}C_9 = 0, C_{10}L_{B_7} = 0, A_7C_{10} = C_9B_7, \qquad (3.2)\\
R_{D_4}O_4L_{G_4} = 0, R_{D_7}O_6L_{G_7} = 0, R_{D_8}O_8L_{G_8} = 0, \qquad (3.3)\\
R_{F_4}O_4L_{E_4} = 0, O_4L_{E_4}L_{N_1} = 0, R_{M_1}R_{D_4}O_4 = 0, R_{F_7}O_6L_{E_7} = 0, O_6L_{E_7}L_{N_2} = 0,\\
R_{M_2}R_{D_7}O_6 = 0, R_{F_8}O_8L_{E_8} = 0, O_8L_{E_8}L_{N_3} = 0, R_{M_3}R_{D_8}O_8 = 0. \qquad (3.4)
\end{gathered}
$$

(iii) The conditions among the matrices defined in (ii) are established as

$$
\begin{aligned}
&\operatorname{rank}\begin{bmatrix} A_1 & C_1 \end{bmatrix} = \operatorname{rank}(A_1), \quad \operatorname{rank}\begin{bmatrix} C_3 \\ B_1 \end{bmatrix} = \operatorname{rank}(B_1),\\
&\operatorname{rank}\begin{bmatrix} A_2 & C_2 \end{bmatrix} = \operatorname{rank}(A_2), \quad \operatorname{rank}\begin{bmatrix} C_4 \\ B_2 \end{bmatrix} = \operatorname{rank}(B_2),\\
&\operatorname{rank}\begin{bmatrix} A_3 & C_5 \end{bmatrix} = \operatorname{rank}(A_3), \quad \operatorname{rank}\begin{bmatrix} C_6 \\ B_3 \end{bmatrix} = \operatorname{rank}(B_3),\\
&\operatorname{rank}\begin{bmatrix} A_4 & C_7 \end{bmatrix} = \operatorname{rank}(A_4), \quad \operatorname{rank}\begin{bmatrix} C_8 \\ B_4 \end{bmatrix} = \operatorname{rank}(B_4),\\
&\operatorname{rank}\begin{bmatrix} A_8 & C_9 \end{bmatrix} = \operatorname{rank}(A_7), \quad \operatorname{rank}\begin{bmatrix} C_{10} \\ B_7 \end{bmatrix} = \operatorname{rank}(B_7), \qquad (3.5)\\
&A_3C_6 = C_5B_3, \quad A_4C_8 = C_7B_4, \quad A_7C_{10} = C_9B_7,
\end{aligned}
$$

$$\operatorname{rank}\begin{bmatrix} O_1 & D_1 & A_5 & F_1C_{10} & C_3 \\ G_1 & 0 & 0 & B_7 & 0 \\ B_5 & 0 & 0 & 0 & B_1 \\ C_5E_1 & A_3 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_5 & D_1 \\ A_1 & 0 \\ 0 & A_3 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} G_1 & B_7 & 0 \\ B_5 & 0 & B_1 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_1 & F_1 & A_5 & D_1C_6 & C_3 \\ E_1 & 0 & 0 & B_3 & 0 \\ B_5 & 0 & 0 & 0 & B_1 \\ C_9G_1 & A_7 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_5 & F_1 \\ 0 & A_7 \\ A_1 & 0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} E_1 & B_3 & 0 \\ B_5 & 0 & B_1 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_1 & A_5 & F_1C_{10} & D_1C_6 & C_3 \\ G_1 & 0 & B_7 & 0 & 0 \\ E_1 & 0 & 0 & B_3 & 0 \\ B_5 & 0 & 0 & 0 & B_1 \\ C_1 & A_1 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_5 \\ A_1 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} G_1 & B_7 & 0 & 0 \\ E_1 & 0 & B_3 & 0 \\ B_5 & 0 & 0 & B_1 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_1 & F_1 & D_1 & A_5 & C_3 \\ B_5 & 0 & 0 & 0 & B_1 \\ C_9G_1 & A_7 & 0 & 0 & 0 \\ C_5G_1 & 0 & A_3 & 0 & 0 \\ C_1 & 0 & 0 & A_1 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_5 & B_1 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} F_1 & D_1 & A_5 \\ A_7 & 0 & 0 \\ 0 & A_3 & 0 \\ 0 & 0 & A_1 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_2 & D_2 & A_6 & F_2C_{10} & C_4 \\ G_2 & 0 & 0 & B_7 & 0 \\ B_6 & 0 & 0 & 0 & B_2 \\ 0 & A_4 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_6 & D_2 \\ 0 & A_4 \\ A_2 & 0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} G_2 & B_7 & 0 \\ B_6 & 0 & B_2 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_2 & F_2 & A_6 & D_2C_8 & C_4 \\ E_2 & 0 & 0 & B_4 & 0 \\ B_6 & 0 & 0 & 0 & B_2 \\ C_9G_2 & A_7 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_6 & F_2 \\ 0 & A_7 \\ A_2 & 0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} E_2 & B_4 & 0 \\ B_6 & 0 & B_2 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_2 & A_6 & F_2C_{10} & D_2C_8 & C_4 \\ G_2 & 0 & B_7 & 0 & 0 \\ E_2 & 0 & 0 & B_4 & 0 \\ B_6 & 0 & 0 & 0 & B_2 \\ C_2 & A_2 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_6 \\ A_2 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} G_2 & B_7 & 0 & 0 \\ E_2 & 0 & B_4 & 0 \\ B_6 & 0 & 0 & B_2 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} O_2 & F_2 & D_2 & A_6 & C_4 \\ B_6 & 0 & 0 & 0 & B_2 \\ C_9G_2 & A_7 & 0 & 0 & 0 \\ C_7E_2 & 0 & A_4 & 0 & 0 \\ 0 & 0 & 0 & A_2 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_6 & B_2 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} F_2 & D_2 & A_6 \\ A_7 & 0 & 0 \\ 0 & A_4 & 0 \\ 0 & 0 & A_2 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} 0 & -G_2 & G_1 & 0 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & -G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 \\ -F_1 & 0 & 0 & O_1 & D_1 & A_5 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 \\ F_2 & O_2 & 0 & 0 & 0 & 0 & A_6 & 0 & 0 & D_2C_8 & C_4 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_4 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 \\ A_7 & C_9 & 0 & C_5E_1 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} G_2 & G_1 & 0 & 0 & 0 & 0 & B_7 & 0 & 0 \\ 0 & -G_1 & G_2 & B_7 & 0 & 0 & 0 & 0 & 0 \\ E_2 & 0 & 0 & 0 & 0 & 0 & 0 & B_4 & 0 \\ 0 & B_5 & 0 & 0 & B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_6 & 0 & 0 & B_2 & 0 & 0 & 0 \\ B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} -F_1 & D_1 & 0 & A_5 \\ F_2 & 0 & A_6 & 0 \\ A_7 & 0 & 0 & 0 \\ 0 & A_3 & 0 & 0 \\ 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & A_1 \end{bmatrix},$$

$$\operatorname{rank}\begin{bmatrix} 0 & -G_1 & G_1 & 0 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & -G_1 & 0 & G_2 & 0 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 \\ -F_2 & 0 & 0 & O_2 & D_2 & A_6 & 0 & 0 & F_2C_{10} & 0 & 0 & 0 & C_4 \\ F_1 & O_1 & 0 & 0 & 0 & 0 & A_5 & 0 & 0 & -D_1C_6 & -C_3 & 0 & 0 \\ 0 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 \\ 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 \\ A_7 & -C_9G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_7E_2 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_2 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -C_1 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \operatorname{rank}\begin{bmatrix} G_1 & G_1 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & -G_1 & G_2 & 0 & B_7 & 0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 \\ 0 & B_5 & 0 & 0 & 0 & B_1 & 0 & 0 & 0 \\ 0 & 0 & B_6 & 0 & 0 & 0 & B_2 & 0 & 0 \\ B_5 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} -F_2 & D_2 & 0 & A_6 \\ F_1 & 0 & A_5 & 0 \\ A_7 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{bmatrix},$$

$$\mathrm{rank}\begin{bmatrix} 0 & G_2 & G_1 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_2 & 0 & G_1 & 0 & 0 & 0 & B_7 & 0 & 0 & 0 & 0 & 0 \\ -F_1 & 0 & O_1 & 0 & A_5 & 0 & 0 & 0 & 0 & D_1C_6 & 0 & C_3 & 0 \\ F_2 & O_2 & 0 & 0 & 0 & A_6 & 0 & 0 & D_2C_8 & C_4 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & B_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_1 & 0 & 0 & 0 & 0 & 0 & 0 & B_3 & 0 & 0 & 0 \\ 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0 \\ 0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 \\ A_7 & 0 & C_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=\mathrm{rank}\begin{bmatrix} G_1 & G_2 & B_7 & 0 \\ B_5 & 0 & 0 & B_1 \\ 0 & B_6 & 0 & 0 \end{bmatrix} + \mathrm{rank}\begin{bmatrix} -F_1 & A_5 & 0 \\ F_2 & 0 & A_6 \\ A_7 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix},$$

$$\mathrm{rank}\begin{bmatrix} 0 & G_1 & G_2 & 0 & 0 & 0 & 0 & B_7 & 0 & C_4 \\ -F_2 & 0 & O_2 & D_2 & 0 & A_6 & 0 & 0 & 0 & 0 \\ F_1 & O_1 & 0 & 0 & D_1 & 0 & A_5 & 0 & C_3 & 0 \\ 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0 \\ 0 & 0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 \\ A_7 & 0 & -C_9E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_7E_2 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_5E_1 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_2 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 & 0 & 0 & A_1 & 0 & 0 & 0 \end{bmatrix}$$

$$=\mathrm{rank}\begin{bmatrix} -F_2 & D_2 & 0 & A_6 & 0 \\ F_1 & 0 & D_1 & 0 & A_5 \\ A_7 & 0 & 0 & 0 & 0 \\ 0 & A_4 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 & A_1 \end{bmatrix} + \mathrm{rank}\begin{bmatrix} G_1 & G_2 & B_7 & 0 \\ B_5 & 0 & 0 & B_1 \\ 0 & B_6 & 0 & 0 \end{bmatrix}. \tag{3.6}$$

When these conditions are met, the system formulated in (2.8) *admits a general solution that can be expressed as*

$$\begin{aligned} X_1 &= A_1^{\dagger}C_1 + L_{A_1}W_1,\ Y_1 = C_3B_1^{\dagger} + W_3R_{B_1},\ U_1 = A_3^{\dagger}C_5 + L_{A_3}C_6B_3^{\dagger} + L_{A_3}W_5R_{B_3}, \\ V_1 &= A_7^{\dagger}C_9 + L_{A_7}C_{10}B_7^{\dagger} + L_{A_7}W_7R_{B_7},\ X_2 = A_2^{\dagger}C_2 + L_{A_2}W_2, \\ Y_2 &= C_4B_2^{\dagger} + W_4R_{B_2},\ U_2 = A_4^{\dagger}C_7 + L_{A_4}C_8B_4^{\dagger} + L_{A_4}W_6R_{B_4}. \end{aligned} \tag{3.7}$$

Additionally, the auxiliary variables used in these solutions are defined by

$$
\begin{aligned}
W_1 &= A_7^{\dagger}(O_3 - D_3W_5E_3 - F_3W_7G_3) - A_7^{\dagger}Z_1B_7 + L_{A_7}Z_2,\\
W_3 &= R_{A_7}(O_3 - D_3W_5E_3 - F_3W_7G_3)B_7^{\dagger} + A_7A_7^{\dagger}Z_1 + Z_3R_{B_7},\\
W_5 &= D_4^{\dagger}O_4E_4^{\dagger} - D_4^{\dagger}F_4M_1^{\dagger}O_4E_4^{\dagger} - D_4^{\dagger}S_1F_4^{\dagger}O_4N_1^{\dagger}G_4E_4^{\dagger} - D_4^{\dagger}S_1Z_4R_{N_1}G_4E_4^{\dagger} + L_{D_4}Z_5 + Z_6R_{E_4},\\
W_7 &= M_1^{\dagger}O_4G_4^{\dagger} + S_1^{\dagger}S_1F_4^{\dagger}O_4N_1^{\dagger} + L_{M_1}L_{S_1}Z_7 + L_{M_1}Z_4R_{N_1} + Z_8R_{G_4}.
\end{aligned}
$$

For W_7 an alternative formulation is $W_7 = M_2^{\dagger}O_6G_7^{\dagger} + S_2^{\dagger}S_2F_7^{\dagger}O_6N_2^{\dagger} + L_{M_2}L_{S_1}T_7 + L_{M_2}T_4R_{N_2} + T_8R_{G_7}$. Continuing with the definitions of other auxiliary variables, we have that

$$
\begin{aligned}
W_2 &= A_8^{\dagger}(O_5 - D_6W_6E_6 - F_6W_7G_6) - A_8^{\dagger}T_1B_8 + L_{A_8}T_2,\\
W_4 &= R_{A_8}(O_5 - D_6W_6E_6 - F_6W_7G_6)B_8^{\dagger} + A_8A_8^{\dagger}T_1 + T_3R_{B_8},\\
W_6 &= D_7^{\dagger}O_6E_7^{\dagger} - D_7^{\dagger}F_7M_2^{\dagger}O_6E_8^{\dagger} - D_7^{\dagger}S_2F_7^{\dagger}O_6N_2^{\dagger}G_7E_7^{\dagger} - D_7^{\dagger}S_2T_4R_{N_2}G_7E_7^{\dagger} + L_{D_7}T_5 + T_6R_{E_7},\\
Z_7 &= \begin{bmatrix} I & 0 \end{bmatrix}[A^{\dagger}(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) - A^{\dagger}K_1B + L_AK_2],\\
T_7 &= \begin{bmatrix} 0 & I \end{bmatrix}[A^{\dagger}(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) - A^{\dagger}K_1B + L_AK_2],\\
Z_8 &= [R_A(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) + AA^{\dagger}K_1 + K_3R_B]\begin{bmatrix} I \\ 0 \end{bmatrix},\\
T_8 &= [R_A(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) + AA^{\dagger}K_1 + K_3R_B]\begin{bmatrix} 0 \\ I \end{bmatrix},\\
Z_4 &= D_8^{\dagger}O_8E_8^{\dagger} - D_8^{\dagger}F_8M_3^{\dagger}O_8E_8^{\dagger} - D_8^{\dagger}S_3F_8^{\dagger}O_8N_3^{\dagger}G_8E_8^{\dagger} - D_8^{\dagger}S_3K_4R_{N_3}G_8E_8^{\dagger} + L_{D_8}K_5 + K_6R_{E_8},\\
T_9 &= M_3^{\dagger}O_8G_8^{\dagger} + S_3^{\dagger}S_3F_8^{\dagger}O_8N_3^{\dagger} + L_{M_3}L_{S_3}K_7 + L_{M_3}K_4R_{N_3} + K_8R_{G_8}.
\end{aligned}
$$

The auxiliary variables $Z_2, \ldots, Z_8, T_1, \ldots, T_8$, and $K_1, \ldots, K_8$ play a central role in defining the primary variables of the solution $W_1, \ldots, W_7$. While they do not appear directly in the final expression of the solution, these auxiliary variables are essential in formulating the necessary conditions and ensuring that the solution conforms to the structure of the equation system and quaternion algebra. The variables $Z_2, \ldots, Z_8, T_1, \ldots, T_8$, $K_1, \ldots, K_8$, and $W_1, \ldots, W_7$ have dimensions conforming to the algebraic operations involved and are defined over $\mathbb{H}$.

Proof. To state equivalence between (i) and (ii), we reform the expression given in (2.8) as

$$A_1X_1 = C_1, Y_1B_1 = C_3, A_3U_1 = C_5, U_1B_3 = C_6, A_5V_1 = C_9, V_1B_5 = C_{10}, A_5X_1 + Y_1B_5 + D_1U_1E_1 + F_1V_1G_1 = O_1, \quad (3.8)$$

$$A_2X_2 = C_2, Y_2B_2 = C_4,\ A_4U_2 = C_7,\ U_2B_4 = C_8,\ A_6X_2 + Y_2B_6 + D_2U_2E_2 + F_2V_1G_2 = O_2. \quad (3.9)$$

By Lemma 2.2, the general solution to $A_1X_1 = C_1$, $Y_1B_1 = C_3$, $A_3U_1 = C_5$, $U_1B_3 = C_6$, $A_5V_1 = C_9$, and $V_1B_5 = C_{10}$ is presented as

$$X_1 = A_1^{\dagger}C_1 + L_{A_1}W_1, \quad Y_1 = C_3B_1^{\dagger} + W_3R_{B_1}, \quad (3.10)$$

$$U_1 = A_3^{\dagger}C_5 + L_{A_3}C_6B_3^{\dagger} + L_{A_3}W_5R_{B_3}, \quad V_1 = A_7^{\dagger}C_9 + L_{A_7}C_{10}B_7^{\dagger} + L_{A_7}W_7R_{B_7}, \quad (3.11)$$

where $W_1,\ W_3,\ W_5$, and W_7 are matrices with dimensions suitable for the given operations.

After substituting the equations formulated in (3.10) and (3.11) into the expression stated in (3.8) and performing simplifications, we arrive at

$$A_7W_1 + W_3B_7 + D_3W_5E_3 + F_3W_7G_3 = O_3. \tag{3.12}$$

According to Lemma 2.3, the expression stated in (3.12) is solvable if, and only if, $R_{D_4}O_4L_{G_4} = 0$, $R_{F_4}O_4L_{E_4} = 0$, $O_4L_{E_4}L_{N_1} = 0$, and $R_{M_1}R_{D_4}O_4 = 0$. Then, the corresponding general solution of the equation established in (3.12) can be presented as

$$\begin{aligned}
W_1 &= A_7^{\dagger}(O_3 - D_3W_5E_3 - F_3W_7G_3) - A_7^{\dagger}Z_1B_7 + L_{A_7}Z_2,\\
W_3 &= R_{A_7}(O_3 - D_3W_5E_3 - F_3W_7G_3)B_7^{\dagger} + A_7A_7^{\dagger}Z_1 + Z_3R_{B_7},\\
W_5 &= D_4^{\dagger}O_4E_4^{\dagger} - D_4^{\dagger}F_4M_1^{\dagger}O_4E_4^{\dagger} - D_4^{\dagger}S_1F_4^{\dagger}O_4N_1^{\dagger}G_4E_4^{\dagger} - D_4^{\dagger}S_1Z_4R_{N_1}G_4E_4^{\dagger} + L_{D_4}Z_5 + Z_6R_{E_4},\\
W_7 &= M_1^{\dagger}O_4G_4^{\dagger} + S_1^{\dagger}S_1F_4^{\dagger}O_4N_1^{\dagger} + L_{M_1}L_{S_1}Z_7 + L_{M_1}Z_4R_{N_1} + Z_8R_{G_4}.
\end{aligned} \tag{3.13}$$

In each case, the matrices $Z_1, \ldots, Z_8$ can be chosen arbitrarily having dimensions compatible with the quaternion algebra $\mathbb{H}$. Continuing in the same vein as in Lemma 2.2, the general solutions to the equations $A_2X_2 = C_2$, $Y_2B_2 = C_4$, $A_4U_2 = C_7$, and $U_2B_4 = C_8$ can be expressed as

$$\begin{aligned}
X_2 &= A_2^{\dagger}C_2 + L_{A_2}W_2,\\
Y_2 &= C_4B_2^{\dagger} + W_4R_{B_2},\\
U_2 &= A_4^{\dagger}C_7 + L_{A_4}C_8B_4^{\dagger} + L_{A_4}W_6R_{B_4},
\end{aligned} \tag{3.14}$$

where W_2, W_4, and W_6 serve as free matrices with dimensions that conform to the relevant equation.

By substituting the equations defined in (3.11)–(3.14) into the expression stated in (3.9) and then simplifying, we attain that

$$A_8W_2 + W_4B_8 + D_6W_6E_6 + F_6W_7G_6 = O_5. \tag{3.15}$$

According to Lemma 2.3, the formula defined in (3.15) is solvable if, and only if, $R_{D_7}O_6L_{G_7} = 0$, $R_{F_7}O_6L_{E_7} = 0$, $O_6L_{E_7}L_{N_2} = 0$, and $R_{M_2}R_{D_7}O_6 = 0$. Then, the general solution to the equation stated in (3.15) is given by

$$\begin{aligned}
W_2 &= A_8^{\dagger}(O_5 - D_6W_6E_6 - F_6W_7G_6) - A_8^{\dagger}T_1B_8 + L_{A_8}T_2,\\
W_4 &= R_{A_8}(O_5 - D_6W_6E_6 - F_6W_7G_6)B_8^{\dagger} + A_8A_8^{\dagger}T_1 + T_3R_{B_8},\\
W_6 &= D_7^{\dagger}O_6E_7^{\dagger} - D_7^{\dagger}F_7M_2^{\dagger}O_6E_8^{\dagger} - D_7^{\dagger}S_2F_7^{\dagger}O_6N_2^{\dagger}G_7E_7^{\dagger} - D_7^{\dagger}S_2T_4R_{N_2}G_7E_7^{\dagger} + L_{D_7}T_5 + T_6R_{E_7},\\
W_7 &= M_2^{\dagger}O_6G_7^{\dagger} + S_2^{\dagger}S_2F_7^{\dagger}O_6N_2^{\dagger} + L_{M_2}L_{S_1}T_7 + L_{M_2}T_4R_{N_2} + T_8R_{G_7},
\end{aligned} \tag{3.16}$$

where $T_1, \ldots, T_8$ are arbitrary matrices of suitable dimensions over $\mathbb{H}$.

Upon comparing the equations established in (3.13) and (3.16) and performing further simplifications, we obtain

$$A\begin{bmatrix} Z_7 \\ T_7 \end{bmatrix} + \begin{bmatrix} Z_8 & T_8 \end{bmatrix}B + L_{M_1}Z_4R_{N_1} + L_{M_2}T_9R_{N_2} = W_3. \tag{3.17}$$

According to Lemma 2.3, the expression given by (3.17) is solvable if, and only if, the conditions $R_{D_8}O_8L_{G_8} = 0, R_{F_8}O_8L_{E_8} = 0, O_8L_{E_8}L_{N_3} = 0, R_{M_3}R_{D_8}O_8 = 0$ hold. Hence, the general solution to the equation stated in (3.17) is given by

$$\begin{aligned}
Z_7 &= \begin{bmatrix} I & 0 \end{bmatrix}[A^\dagger(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) - A^\dagger K_1B + L_AK_2],\\
T_7 &= \begin{bmatrix} 0 & I \end{bmatrix}[A^\dagger(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) - A^\dagger K_1B + L_AK_2],\\
Z_8 &= [R_A(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) + AA^\dagger K_1 + K_3R_B]\begin{bmatrix} I \\ 0 \end{bmatrix},\\
T_8 &= [R_A(W_3 - L_{M_1}Z_4R_{N_1} - L_{M_2}T_9R_{N_2}) + AA^\dagger K_1 + K_3R_B]\begin{bmatrix} 0 \\ I \end{bmatrix},\\
Z_4 &= D_8^\dagger O_8E_8^\dagger - D_8^\dagger F_8M_3^\dagger O_8E_8^\dagger - D_8^\dagger S_3F_8^\dagger O_8N_3^\dagger G_8E_8^\dagger - D_8^\dagger S_3K_4R_{N_3}G_8E_8^\dagger + L_{D_8}K_5 + K_6R_{E_8},\\
T_9 &= M_3^\dagger O_8G_8^\dagger + S_3^\dagger S_3F_8^\dagger O_8N_3^\dagger + L_{M_3}L_{S_3}K_7 + L_{M_3}K_4R_{N_3} + K_8R_{G_8}.
\end{aligned}$$

To demonstrate the equivalence between (ii) and (iii), we aim to establish the equivalence between the set of conditions given in (3.2)–(3.4) and the equalities presented in (3.5) and (3.6). For brevity, we focus on representative rank equalities, whereas the remaining can be proven analogously. Specifically, the conditions defined in (3.2) straightforwardly correspond to the equalities stated in (3.5). Moreover, we demonstrate that the equalities in the formula presented in (3.3) are congruent to their corresponding equalities defined in (3.6).

Next, employing Lemma 2.1 and applying elementary row operations to $\mathrm{rank}(R_{D_4}O_4L_{E_4})$, we find that

$$\begin{aligned}
&\mathrm{rank}\begin{bmatrix} O_4 & D_4 \\ G_4 & 0 \end{bmatrix} - \mathrm{rank}(D_4) - \mathrm{rank}(G_4)\\
=&\mathrm{rank}\begin{bmatrix} R_{A_7}O_3L_{B_7} & R_{A_7}D_3 \\ G_3L_{B_7} & 0 \end{bmatrix} - \mathrm{rank}(R_{A_7}D_3) - \mathrm{rank}(G_3L_{B_7})\\
=&\mathrm{rank}\begin{bmatrix} O_3 & D_3 & A_7 \\ G_3 & 0 & 0 \\ B_7 & 0 & 0 \end{bmatrix} - \mathrm{rank}\begin{bmatrix} D_3 & A_7 \end{bmatrix} - \mathrm{rank}\begin{bmatrix} G_3 \\ B_7 \end{bmatrix}\\
=&\mathrm{rank}\begin{bmatrix} O_1 - A_5X_{01} - Y_{01}B_5 - D_1U_{01}E_1 - F_1V_{01}G_1 & D_1L_{A_3} & A_5L_{A_1} \\ R_{B_{15}}G_1 & 0 & 0 \\ R_{B_1}B_5 & 0 & 0 \end{bmatrix}\\
&- \mathrm{rank}\begin{bmatrix} A_5L_{A_1} & D_1L_{A_3} \end{bmatrix} - \mathrm{rank}\begin{bmatrix} R_{B_{15}}G_1 \\ R_{B_1}B_5 \end{bmatrix}\\
=&\mathrm{rank}\begin{bmatrix} O_1 & D_1 & A_5 & F_1C_{10} & C_3 \\ G_1 & 0 & 0 & B_{15} & 0 \\ B_5 & 0 & 0 & 0 & B_1 \\ C_5E_1 & A_3 & 0 & 0 & 0 \\ C_1 & 0 & A_1 & 0 & 0 \end{bmatrix} - \mathrm{rank}\begin{bmatrix} A_5 & D_1 \\ A_1 & 0 \\ 0 & A_3 \end{bmatrix} - \mathrm{rank}\begin{bmatrix} G_1 & B_{15} & 0 \\ B_5 & 0 & B_1 \end{bmatrix}.
\end{aligned}$$

Consequently, $\mathrm{rank}(R_{D_4}O_4L_{E_4}) = 0$ coincides with the fourteenth rank equality stated in (3.6).

Following the same reasoning, we have that

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} O_6 & D_7 \\ G_7 & 0 \end{bmatrix} - \operatorname{rank}(D_7) - \operatorname{rank}(G_7)\\
=&\operatorname{rank}\begin{bmatrix} R_{A_8}O_5L_{B_8} & R_{A_8}D_6 \\ G_6L_{B_8} & 0 \end{bmatrix} - \operatorname{rank}(R_{A_8}D_3) - \operatorname{rank}(G_6L_{B_8})\\
=&\operatorname{rank}\begin{bmatrix} O_5 & D_6 & A_8 \\ G_6 & 0 & 0 \\ B_8 & 0 & 0 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} D_6 & A_8 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} G_6 \\ B_8 \end{bmatrix}\\
=&\operatorname{rank}\begin{bmatrix} O_2 - A_6X_{02} - Y_{02}B_6 - D_2U_{02}E_2 - F_2V_{01}G_2 & D_2L_{A_4} & A_6L_{A_2} \\ R_{B_{15}}G_2 & 0 & 0 \\ R_{B_2}B_6 & 0 & 0 \end{bmatrix}\\
&- \operatorname{rank}\begin{bmatrix} A_5L_{A_1} & D_1L_{A_3} \end{bmatrix} - \operatorname{rank}\begin{bmatrix} R_{B_{15}}G_2 \\ R_{B_2}B_6 \end{bmatrix}\\
=&\operatorname{rank}\begin{bmatrix} O_2 & D_2 & A_6 & F_2C_{10} & C_4 \\ G_2 & 0 & 0 & B_{15} & 0 \\ B_6 & 0 & 0 & 0 & B_2 \\ 0 & A_4 & 0 & 0 & 0 \\ C_2 & 0 & A_2 & 0 & 0 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} A_6 & D_2 \\ A_2 & 0 \\ 0 & A_4 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} G_2 & B_{15} & 0 \\ B_6 & 0 & B_2 \end{bmatrix}.
\end{aligned}$$

Consequently, $\operatorname{rank}(R_{D_7}O_6L_{G_7}) = 0$ aligns with the eighteenth rank equality in the expression given in (3.6). Similarly, $\operatorname{rank}(R_{D_8}O_8L_{G_8})$ corresponds to

$$\begin{aligned}
&\operatorname{rank}\begin{bmatrix} O_8 & D_8 \\ G_8 & 0 \end{bmatrix} - \operatorname{rank}(D_8) - \operatorname{rank}(G_8)\\
=&\operatorname{rank}\begin{bmatrix} R_AW_3L_B & R_AL_{M_1} \\ R_{N_2}L_B & 0 \end{bmatrix} - \operatorname{rank}(R_AL_{M_1}) - \operatorname{rank}(R_{N_2}L_B)\\
=&\operatorname{rank}\begin{bmatrix} W_2 - W_1 & L_{M_1} & A \\ R_{N_2} & 0 & 0 \\ B & 0 & 0 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} L_{M_1} & A \end{bmatrix} - \operatorname{rank}\begin{bmatrix} R_{N_2} \\ B \end{bmatrix}\\
=&\operatorname{rank}\begin{bmatrix} W_2 - W_1 & L_{M_1} & A \\ R_{N_2} & 0 & 0 \\ B & 0 & 0 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} L_{M_1} & A \end{bmatrix} - \operatorname{rank}\begin{bmatrix} R_{N_2} \\ B \end{bmatrix}\\
=&\operatorname{rank}\begin{bmatrix} W_2 - W_1 & L_{M_1} & L_{M_2}L_{S_2} \\ R_{N_2} & 0 & 0 \\ R_{G_1} & 0 & 0 \\ R_{G_7} & 0 & 0 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} L_{M_1} & L_{M_2}L_{S_2} \end{bmatrix} - \operatorname{rank}\begin{bmatrix} R_{N_2} \\ R_{G_4} \\ R_{G_7} \end{bmatrix}\\
=&\operatorname{rank}\begin{bmatrix} W_2 - W_1 & I & L_{M_2} & 0 & 0 & 0 \\ I & 0 & 0 & N_2 & 0 & 0 \\ I & 0 & 0 & 0 & G_4 & 0 \\ I & 0 & 0 & 0 & 0 & G_7 \\ 0 & M_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & S_2 & 0 & 0 & 0 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} I & L_{M_2} \\ M_1 & 0 \\ 0 & S_2 \end{bmatrix} - \operatorname{rank}\begin{bmatrix} I & N_2 & 0 & 0 \\ I & G_4 & 0 & 0 \\ I & 0 & G_7 & 0 \end{bmatrix}
\end{aligned}$$

$$
=\text{rank}\begin{bmatrix}
0 & -G_2 & G_1 & 0 & 0 & 0 & 0 & B_{15} & 0 & 0 & 0 & 0 & 0\\
0 & -G_2 & 0 & G_1 & 0 & 0 & 0 & 0 & B_{15} & 0 & 0 & 0 & 0\\
-F_1 & 0 & 0 & O_1 & D_1 & A_5 & 0 & 0 & 0 & 0 & 0 & 0 & C_3\\
F_2 & O_2 & 0 & 0 & 0 & 0 & A_6 & 0 & 0 & D_2C_8 & C_4 & 0 & 0\\
0 & E_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_4 & 0 & 0 & 0\\
0 & B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2 & 0 & 0\\
0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1 & 0\\
0 & 0 & 0 & B_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_1\\
A_{15} & C_9 & 0 & C_5E_1 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & C_1 & 0 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & C_2 & 0 & 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
- \text{rank}\begin{bmatrix}
G_2 & G_1 & 0 & 0 & 0 & 0 & B_{15} & 0 & 0\\
0 & G_1 & G_2 & B_{15} & 0 & 0 & 0 & 0 & 0\\
E_2 & 0 & 0 & 0 & 0 & 0 & 0 & B_4 & 0\\
0 & B_5 & 0 & 0 & B_1 & 0 & 0 & 0 & 0\\
0 & 0 & B_6 & 0 & 0 & B_2 & 0 & 0 & 0\\
B_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_2
\end{bmatrix}
- \text{rank}\begin{bmatrix}
-F_1 & D_1 & 0 & A_5\\
F_2 & 0 & A_6 & 0\\
A_{15} & 0 & 0 & 0\\
0 & A_3 & 0 & 0\\
0 & 0 & A_2 & 0\\
0 & 0 & 0 & A_1
\end{bmatrix}.
$$

Thus, $\text{rank}(R_{D_8}O_8L_{G_8}) = 0$ is the twenty-second rank equality given in (3.6) completing the proof. □

3.2. Special cases and further implications

We proceed to explore specific cases of the system stated in (2.8). These cases validate the robustness of Theorem 3.1 and extend its range of application. If $A_1, \ldots, A_6$, $B_1, \ldots, B_6$, and $C_1, \ldots, C_{10}$ are null, then the expression established in (2.5) can be considered a particular case of the formula presented in (2.8). This results in the following corollary.

Corollary 3.1. *Given* A_5, A_6, B_5, B_6, F_1, F_2, G_1, G_2, O_1, *and* O_2 *matrices over* $\mathbb{H}$, *assign*

$$A_3 = R_{A_5}F_1,\ B_3 = G_1L_{B_5},\ C_3 = R_{A_5}O_1L_{B_5},\ B_4 = R_{B_3}G_2,\ C_4 = O_2 - F_2A_3^{\dagger}C_3B_3^{\dagger}G_2,$$
$$A = R_{A_6}A_4,\ B = G_2L_{B_6},\ C = R_{A_6}F_2,\ D = B_4L_{B_6},\ E = R_{A_6}C_4L_{B_6},\ F = R_AC,\ G = DL_B,\ H = CL_F.$$

Then, the system represented in (2.5) *is consistent if, and only if, the following conditions are met: (i)* $R_{A_3}C_3 = 0$, *(ii)* $C_3L_{B_3} = 0$, *(iii)* $R_FR_AE = 0$, *(iv)* $EL_BL_G = 0$, $R_AEL_D = 0$, *and (v)* $R_CEL_B = 0$. *Under these conditions, the general solution to the equations presented in* (2.5) *is given by*

$$X_1 = A_5^{\dagger}(O_1 - F_1V_1G_1) - A_5^{\dagger}T_1B_5 + L_{A_5}T_2,\ Y_1 = R_{A_5}(O_1 - F_1V_1G_1) + A_5A_5^{\dagger}T_1 + T_3R_{B_5},$$
$$V_1 = A_3^{\dagger}C_3B_3^{\dagger} + L_{A_3}W_1 + W_2R_{B_3},\ X_2 = A_6^{\dagger}(C_4 - A_4W_1G_2 - F_2W_2B_4) - A_6^{\dagger}T_4B_6 + L_{A_6}T_5,$$
$$Y_2 = R_{A_6}(C_4 - A_4W_1G_2 - F_2W_2B_4)B_6^{\dagger} + A_6A_6^{\dagger}T_4 + T_6R_{B_6},$$

with $W_1 = A^{\dagger}EB^{\dagger} - A^{\dagger}CF^{\dagger}EB^{\dagger} - A^{\dagger}HC^{\dagger}EG^{\dagger}DB^{\dagger} - A^{\dagger}HZ_2R_GDB^{\dagger} + L_AZ_3 + Z_4R_B$ *and* $W_2 = F^{\dagger}ED^{\dagger} + H^{\dagger}HC^{\dagger}EG^{\dagger} + L_FL_HZ_5 + L_FZ_2R_G + Z_6R_D$, *where* $T_1, \ldots, T_6$ *and* $Z_2, \ldots, Z_6$ *are matrices with dimensions appropriate for compatibility over* $\mathbb{H}$.

Corollary 3.1 aligns with key findings from [50] and shows the versatility of our main theorem. Now, we explore another case that offers further insights on the wide-ranging effectiveness of our approach.

Corollary 3.2. *Suppose that $A_5, A_6, B_5, B_6, O_1, O_2$ are given and the solution to the equation stated in* (2.4) *is presented by $A = R_{B_5}$, $B = R_{A_6}A_5$, $C = B_6L_A$, $D = R_{A_6}(O_2 - R_{A_5}O_1B_5^{\dagger}B_6)L_A$. Then, the equation formulated in* (2.4) *has a solution if, and only if, the following conditions are reached: $R_{A_5}O_1L_{B_5} = 0$, $R_BD = 0$, and $DL_C = 0$. Under these conditions, the solution of the equation established in* (2.4) *is expressed as $X_1 = A_5^{\dagger}C_1 - W_1B_5 + L_{A_5}W_2$, $Y_1 = R_{A_5}O_1B_5^{\dagger} + A_5W_1 + W_3R_{B_5}$, and $X_2 = A_6^{\dagger}(O_2 - R_{A_5}O_1B_5^{\dagger}B_6 - A_5W_1B_6) - W_4A + L_{A_6}W_5$, with $W_1 = B^{\dagger}DC^{\dagger} + L_BW_6 + W_7R_C$ and $W_3 = R_{A_6}(O_2 - RA_5C_5B_5^{\dagger}B_6 - A_5W_1B_6)A^{\dagger} + A_6W_4 + W_8R_A$, where $W_1, \ldots, W_8$ are free matrices with compatible dimensions over $\mathbb{H}$.*

Corollary 3.2, which is based on the research presented in [47], demonstrates the depth and flexibility of the applications of Theorem 3.1. We now transition to another scenario that considers simplifications in the matrix configurations as represented in (2.8) and is in line with the solution approaches documented in [51].

Corollary 3.3. *Suppose that A_5, A_6, B_5, B_6, and O_1, O_2 are given matrices defined as $A = R_{(A_6A_5)}A_6$, $B = R_{B_5}L_{B_6}$, and $C = R_{(A_6A_5)}(O_2 - A_6R_{A_5}O_1B_5^{\dagger})L_{B_6}$. Then, the equation given in* (2.4) *is solvable if, and only if, the following conditions are met: $R_{A_5}O_1L_{B_5} = 0$, $R_AC = 0$, and $CL_B = 0$. Under these conditions, the solution to the equations presented in* (2.4) *is stated as $X_1 = A_5^{\dagger}O_1 - W_1B_5 + L_{A_5}W_2$, $Y_1 = R_{A_5}O_1B_5^{\dagger} + A_5W_1 + W_3R_{B_5}$, and $X_2 = R_{(A_6A_5)}(O_2 - A_6R_{A_5}O_1B_5^{\dagger} - A_6W_3R_{B_5})B_6^{\dagger} + A_6A_5W_4 + W_5R_{B_6}$, with $W_3 = A^{\dagger}CB^{\dagger} + L_AW_6 + W_7R_B$, $W_1 = (A_6A_5)^{\dagger}(O_2 - A_6RA_5O_1B_5^{\dagger}B_6 - A_6W_3R_{B_5}) - W_4B_6 + L_{A_6A_5}W_8$, where $W_2, W_4, \ldots, W_8$ are arbitrary matrices with suitable dimensions over $\mathbb{H}$.*

After the analysis in the previous corollaries, we now turn our attention to another scenario described in the following corollary.

Corollary 3.4. *Consider the matrices $A_1, A_2, A_4, A_6, B_5, B_6, B_7, C_1, C_2, C_3, C_9, C_{10}, G_1, G_2, G_4, O_1, O_2$ with compatible dimensions over $\mathbb{H}$. Define $A_8 = A_5L_{A_1}$, $B_8 = R_{B_1}B_5$, $C_7 = F_1L_{A_7}$, and $D_7 = R_{B_7}D_1$. Additionally, set $E_3 = O_1 - (A_5A_1^{\dagger}C_1 + C_3B_1^{\dagger}B_5 + F_1A_7^{\dagger}C_9G_1 + F_1L_{A_7}C_{10}B_7^{\dagger}G_1)$, $A_9 = R_{A_8}C_7$, and $B_9 = D_7L_{B_8}$. Furthermore, state $E_4 = R_{A_8}E_3L_{B_8}, E_5 = O_2 - (A_6A_2^{\dagger}C_2 + C_4B_2^{\dagger}B_6 + F_2A_7^{\dagger}C_9G_2 + F_2L_{A_7}C_{10}B_7^{\dagger}G_2), A_10 = A_6LA_2, B_9 = R_{B_2}B_6$, and then, $A = R_{A_{10}}C_9, B = D_8L_{B_{10}}, C = R_{A_{10}}C_8, D = D_9L_{B_{10}}, E = R_{A_{10}}E_5L_{B_{10}}, F = R_AC, G = DL_B$, and $H = CL_F$. Hence, the following statements are equivalent:*

(i) The system defined by (2.6) *is consistent.*

(ii) The conditions $R_{A_1}C_1 = 0$, $R_{A_7}C_9 = 0$, $R_{A_2}C_2 = 0 = 0$, $C_{10}L_{B_7} = 0, C_3L_{B_1} = 0$, $C_4L_{B_2} = 0$, $B_1C_4 = C_3B_2, R_{A_9}E_4 = 0$, $E_4L_{B_9} = 0, R_FR_AE = 0$, $EL_BL_G = 0$, $R_AEL_D = 0$, and $R_CEL_B = 0$ are required. In this case, the comprehensive solution for the system outlined in (2.6) *is given by*

$$X_1 = A_1^{\dagger}C_1 + L_{A_1}U_1,\ Y_1 = C_3B_1^{\dagger} + V_1R_{B_1},\ V_1 = A_7^{\dagger}C_9 + L_{A_7}C_{10}B_27^{\dagger} + L_{A_7}WR_{B_7},$$

$$U_1 = A_8^{\dagger}(E_3 - C_7WD_7) - A_8^{\dagger}T_4B_8 + L_{A_8}T_5,\ U_{11} = R_{A_8}(E_3 - C_7WD_7)B_8^{\dagger} + A_8A_8^{\dagger}T_4 + T_6R_{B_8},$$

$$X_2 = A_2^{\dagger}C_2 + L_{A_2}U_2,\ Y_2 = C_4B_2^{\dagger} + V_2R_{B_2},\ W = A_9^{\dagger}E_4B_9^{\dagger} + L_{A_9}W_1 + W_2R_{B_9},$$

$$U_2 = A_{10}^{\dagger}(E_4 - C_9W_1D_8 - C_8W_2D_9) - A_{10}^{\dagger}T_1B_{10} + L_{A_{10}}T_2,$$

$$V_2 = R_{A_{10}}(E_4 - C_9W_1D_8 - C_8W_2D_9)B_{10}^{\dagger} + A_8A_8^{\dagger}T_1 + T_3R_{B_{10}},$$

with $W_1 = A^{\dagger}EB^{\dagger} - A^{\dagger}CF^{\dagger}EB^{\dagger} - A^{\dagger}HC^{\dagger}EG^{\dagger}DB^{\dagger} - A^{\dagger}HZ_2R_GDB^{\dagger} + L_AZ_3 + Z_4R_B$ and $W_2 = F^{\dagger}ED^{\dagger} + H^{\dagger}HC^{\dagger}EG^{\dagger} + L_FL_HZ_5 + L_FZ_2R_G + Z_6R_D$, where $T_1, \ldots, T_6, Z_2, \ldots, Z_6$ are matrices of conformable dimensions over $\mathbb{H}$.

Corollary 3.4 provides valuable additions to the findings stated in [41] and supports the applicability of Theorem 3.1. Moreover, this corollary is consistent with the research presented in [57], particularly regarding the use of zero matrices within the context of the expression defined in (2.8), as illustrated in the system described in (2.7).

Having established the theoretical foundations and demonstrated the broad applicability of our main theorem and its corollaries in this section, we now shift our focus to a solution methodology and its computational aspects.

4. Solution methodology and computational aspects

This section presents an algorithmic strategy designed to address the computational complexities often encountered in singular value decomposition methods, especially in the computation of the MPI for quaternion matrices. We introduce a novel approach utilizing row-column determinants unique to quaternion matrices, improving standard methods and showing its effectiveness in different applications.

4.1. Algorithm development

Following the principles outlined in Theorem 3.1, we present an algorithm for explicitly solving the equations stated in (2.8). This algorithm makes practical use of MPI representations to construct general solutions, effectively connecting theoretical principles with practical application. To gain a clearer understanding of the computational aspects of our method, we refer to a lemma from [63] that provides an approach for representing the MPI, an essential component in the solution process.

Lemma 4.1. *Let $A \in \mathbb{H}^{m\times n}$. Then, the MPI $A^{\dagger} = (a_{ij}^{\dagger}) \in \mathbb{H}^{n\times m}$ can be expressed through the representations stated as*

$$a_{ij}^{\dagger} = \frac{\sum\limits_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left((A^*A)_{.i}\left(a_{.j}^*\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}} |A^*A|_{\beta}^{\beta}} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j\left((AA^*)_{j.}(a_{i.}^*)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,m}} |AA^*|_{\alpha}^{\alpha}},$$

where $\operatorname{cdet}_i A$ *and* $\operatorname{rdet}_i A$ *denote the column and row determinants of $A \in \mathbb{H}^{m\times m}$, taken along its i-th column and row, respectively. Furthermore, A_{α}^{α} and A_{β}^{β} represent principal submatrices of A, while $|A|_{\alpha}^{\alpha}$ and $|A|_{\beta}^{\beta}$ signify principal minors in the sense of row-column determinants when A is Hermitian.*

The rows and columns of these submatrices and minors are indexed by $\alpha := \{\alpha_1, \dots, \alpha_r\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_r\} \subseteq \{1, \dots, n\}$, where $I_{r,m} := \{\alpha\colon 1 \le \alpha_1 < \cdots < \alpha_r \le m\}$ and $J_{r,n} := \{\beta\colon 1 \le \beta_1 < \cdots < \beta_r \le n\}$. Additionally, $a_{.j}^*$ and $a_{i.}^*$ refer to the j-th column and the i-th row of A^*, respectively. The matrices $A_{i.}(b)$ and $A_{.j}(c)$ are obtained by replacing the i-th row and j-th column of A with the row vector $b \in \mathbb{H}^{1\times n}$ and the column vector $c \in \mathbb{H}^{m}$, respectively.

Now, we proceed to present Algorithm 1, which is designed to solve the system of quaternion matrix equations as defined in Theorem 3.1.

Algorithm 1 Approach to solving quaternion matrix equations formulated in (2.8).

Input: Matrices $\{A_i\}_{i=1,\ldots,7}$, $\{B_i\}_{i=1,\ldots,7}$, $\{C_i\}_{i=1\ldots,10}$, $\{D_i\}_{i=1,2}$, $\{E_i\}_{i=1,2}$, $\{F_i\}_{i=1,2}$, $\{G_i\}_{i=1,2}$, and $\{O_i\}_{i=1,2}$ over $\mathbb{H}$ with appropriate dimensions.

Step 1. Initial transformations and definitions

- Apply transformations to the matrices according to the relationships of Theorem 3.1 and further detailed in (3.1).
- Compute the MPIs of matrices as necessary, facilitating the handling of matrices that are not directly invertible in the quaternion framework.

Step 2. System consistency verification

- Evaluate the system consistency by checking the conditions stated in (3.3) and verifying the rank conditions as specified. If the conditions are not met, terminate the algorithm as no solution exists under the given constraints.

Step 3. Matrix manipulation for solution preparation

- Calculate the components for the solution, such as M_1, N_1, S_1, using intermediate matrices. This involves strategic manipulations based on quaternion algebra to prepare for the final solution synthesis.

Step 4. Auxiliary and solution matrices computation

- Derive auxiliary matrices Z_i, T_i, K_i and compute W_1, W_2, as well as other critical matrices using the relations established in (3.1).

Step 5. Final solution assembly

- Integrate the computed matrices to assemble the solution set $\{X_1, Y_1, U_1, V_1, X_2, Y_2, U_2\}$ in accordance with the expression given in (3.7).

Output: The solution set $\{X_1, Y_1, U_1, V_1, X_2, Y_2, U_2\}$.

To clarify the individual steps of Algorithm 1, Figure 2 presents its detailed flowchart. Also, it is pertinent to consider the applicability of Algorithm 1 to broader scenarios, including time-varying systems. While this algorithm is demonstrated within a static context, its foundational principles can be adaptable to dynamic environments for addressing real-world applications where system parameters may evolve over time [64, 65].

Transitioning from the theoretical framework of Algorithm 1 to a practical demonstration, the next subsection presents a numerical example that applies this algorithm to solve the equations stated in (2.8).

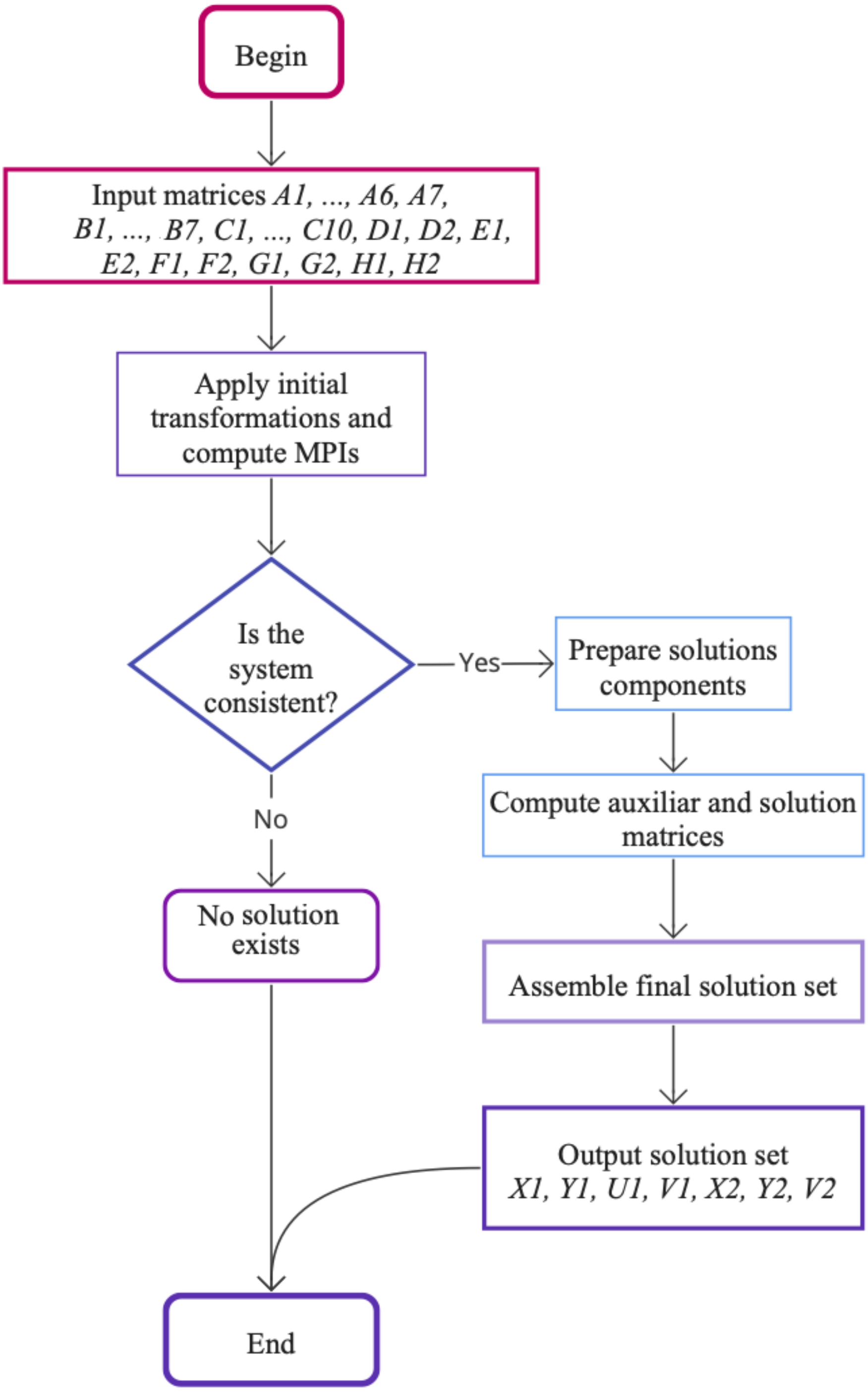

Figure 2. Detailed flowchart illustrating the steps of Algorithm 1.

4.2. Illustrative example

To demonstrate the practical application and effectiveness of Algorithm 1, we present the following numerical example. This example aims to show how our algorithm can be applied to solve quaternion matrix equations with specific sets of matrices.

Input matrices: We begin by providing the input matrices as

$$A_1 = \begin{bmatrix} 1+i & j+k \\ j-k & i-1 \\ 1-i & j-k \end{bmatrix}, A_2 = \begin{bmatrix} i & j & k \\ j & -i & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 2-i & j-k \\ 2j+k & -1-j \end{bmatrix}, A_4 = \begin{bmatrix} j & 3i \\ k & -3 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 2k & 2 \\ -2i & 2j \\ -2i & -2 \end{bmatrix}, A_6 = \begin{bmatrix} 3i+3k & 3j+3k & 3+3k \\ 3k-3 & 3k-3j & 3i-3j \end{bmatrix}, A_7 = \begin{bmatrix} i & j \\ k & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1-i+k & 1-j+k & i+j-1 \\ i+j+k & 1+j+i & -1-j-k \end{bmatrix}, B_2 = \begin{bmatrix} 1 & j \\ k & -i \\ i & k \end{bmatrix}, B_3 = \begin{bmatrix} i+k & 1-j \\ -1-j & i-k \\ i-k & 1+j \end{bmatrix}, B_4 = \begin{bmatrix} 1-i & i-j & j-k \\ k-j & i+j & 1-i \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 2+2i-2j & 2k-2i-2j & 2+2j+2k \\ 2i+2j+2k & 2i-2j-2 & 2k-2i-2 \end{bmatrix}, B_6 = \begin{bmatrix} 3+3j & 3j-3k \\ 3i+3j & -3-3k \\ 3i+3k & 3j-3 \end{bmatrix}, B_7 = \begin{bmatrix} k & i & j \\ -1 & j & -i \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -16 & -16i & -16j \\ -16j & 16k & 16 \\ 16i & -16 & 16k \end{bmatrix}, C_2 = \begin{bmatrix} -3 & -3i \\ -3k & -3j \end{bmatrix}, C_3 = \begin{bmatrix} 1-2j-k & -1-2j-k & j-i+2k \\ 1+2i+k & 1+2i-k & -2-i-j \\ -2+i-j & -2+i+j & 1-2i-k \end{bmatrix},$$

$$C_4 = \begin{bmatrix} i+j-k & -1+i+k \\ 1-i+j & -1+j-k \end{bmatrix}, C_5 = \begin{bmatrix} 3i & -3 & 3k \\ -3k & -3j & 3i \end{bmatrix}, C_6 = \begin{bmatrix} k-i & k-i \\ k-i & -1-j \end{bmatrix}, C_7 = \begin{bmatrix} j+2k-3i & -2+i+3j \\ 3-2j+k & -1-2i+3k \end{bmatrix},$$

$$C_8 = \begin{bmatrix} 1+i+j-k & -1-i-j-k & i+j+k-1 \\ 1-i+j+k & 1+i-j-k & j-i-k-1 \end{bmatrix}, C_9 = \begin{bmatrix} -2 & -2i \\ 2j & -2k \end{bmatrix}, C_{10} = \begin{bmatrix} 1-j & -1-j & i+k \\ i+k & i-k & -1+j \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 42 & 42j \\ 42j & -42 \\ 42i & 42k \end{bmatrix}, D_2 = \begin{bmatrix} k-j & 2+i \\ -j-k & 2i-1 \end{bmatrix}, E_1 = \begin{bmatrix} 9k & 9 & -9i \\ 9 & -9k & -9j \\ 9i & 9j & 9k \end{bmatrix}, E_2 = \begin{bmatrix} 2+2j & 2i-2k \\ 2i+2k & -2-2j \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 2i & 2k \\ 2j & -2 \\ 2k & -2i \end{bmatrix}, F_2 = \begin{bmatrix} i+2k & 2-j \\ j-2 & i+2k \end{bmatrix}, G_1 = \begin{bmatrix} 6k & 6 & -6i \\ 6j & -6i & -6 \end{bmatrix}, G_2 = \begin{bmatrix} -i & 2k \\ j & 2 \end{bmatrix},$$

$$O_1 = \begin{bmatrix} -436-356i+40j-16k & -8-12i-334j+430k & -19+19i+437j+367k \\ 16-8i-410j+338k & 330+442i-30j-62k & -421+337i+73j-5k \\ 395-399i-j-19k & -27+33i-447j-367k & 37+13i-331j+453k \end{bmatrix},$$

$$O_2 = \begin{bmatrix} -13.4+9.6i+2.6j+4.8k & -3.6-6.6i+5.2j-8.6k \\ 3.4-17.2i+13.8j-10.4k & 29.6+14.2i-1.6j-6.2k \end{bmatrix}.$$

Step 1. Initial transformations and definitions: This step involves the application of initial transformations and the computation of the MPI for the given matrices. According to Theorem 3.1, we perform transformations such as $A_7 = A_5 L_{A_1}$, $B_7 = R_{B_1} B_5$, among others, to prepare the matrices for further analysis. The transformations are based on the relationships specified in (3.1). Additionally, we compute the MPI of matrices like A_1 and A_3 using Lemma 4.1 as

$$A_1^{\dagger} = \frac{1}{12}\begin{bmatrix} 1-\mathrm{i} & \mathrm{k}-\mathrm{j} & 1+\mathrm{i} \\ -\mathrm{j}-\mathrm{k} & -1-\mathrm{i} & \mathrm{k}-\mathrm{j} \end{bmatrix}, A_3^{\dagger} = \frac{1}{14}\begin{bmatrix} 2+\mathrm{i} & -2\mathrm{j}-\mathrm{k} \\ \mathrm{k}-\mathrm{j} & \mathrm{i}-1 \end{bmatrix}.$$

Upon applying these initial transformations and computing the MPIs, we identify that matrices such as D_8, E_8, M_2, N_2, S_1, W_1, and W_2 turn out to be zero matrices simplifying our results. This simplification significantly reduces the complexity of the system, facilitating the subsequent steps of our algorithm. By preparing the matrices in this manner, we ensure they are in the correct form for further processing and analysis, aligning with the algorithm requirements for system consistency verification and solution synthesis.

Step 2. System consistency verification: In this step, we verify the consistency of the system by examining the compatibility of the given matrices with the conditions outlined in (3.3). This involves checking specific rank conditions to confirm the system solvability. In our case, all conditions were met, indicating that the system is consistent and a solution can be pursued further.

Step 3. Matrix manipulation for solution preparation: In alignment with the corresponding step of Algorithm 1, we now focus on manipulating the matrices to prepare the essential components for constructing the solution. This step involves choosing specific nonzero values for certain auxiliary matrices to avoid trivial or singular cases. For instance, we set K_1 as

$$K_1 = \begin{bmatrix} 1+2\mathrm{i} & 2-\mathrm{j} & 1+2\mathrm{k} & 1-\mathrm{k} \\ 1+2\mathrm{j} & 1-2\mathrm{i} & 2\mathrm{j}-1 & 1+\mathrm{i} \end{bmatrix}.$$

For the remaining auxiliary matrices, namely Z_1, Z_2, Z_3, Z_5, Z_6, $T_1, \ldots, T_7$, and $K_2, \ldots, Z_8$, we assign zero matrices. This simplifies our computations without losing generality in the solutions. Next, we proceed to compute Z_7 and Z_8, which are determined as

$$Z_7 = \frac{1}{8}\begin{bmatrix} -5-6\mathrm{i}-5\mathrm{k} & -5-6\mathrm{j}+5\mathrm{k} \\ 5-6\mathrm{j}-5\mathrm{k} & -5+6\mathrm{i}-5\mathrm{k} \end{bmatrix}, \quad Z_8 = \frac{1}{4}\begin{bmatrix} 3+8\mathrm{i}-\mathrm{k} & 6-\mathrm{j}-\mathrm{k} \\ 3+8\mathrm{j}+\mathrm{k} & 3-5\mathrm{i}+2\mathrm{k} \end{bmatrix}.$$

This step aligns with the algorithm requirements for preparing the matrices, ensuring that we have the necessary components for the final solution synthesis.

Step 4. Auxiliary and solution matrices computation: Proceeding to Step 4 of Algorithm 1, we focus on the computation of auxiliary and solution matrices. This step is needed for deriving the matrices Z_i, T_i, K_i, and computing W_1, W_2, among others, as established by the algorithm framework. For this purpose, we specifically calculate the matrices W_i. The computation of each W_i is based on the auxiliary matrices and MPIs calculated earlier, as in the expression stated in (3.1).

Step 5. Final solution assembly: Following Step 5 of Algorithm 1, we integrate the computed matrices to assemble the solution set in accordance with the definitions given in (3.7). This step finalizes the preparation of the solution set $\{X_1, Y_1, U_1, V_1, X_2, Y_2, U_2\}$ based on the computations and transformations performed in the previous steps.

Output: The solution set derived from the application of Algorithm 1 is given by

$$X_1 = \frac{1}{4}\begin{bmatrix} -1+25\mathrm{i}+3\mathrm{j}+15\mathrm{k} & -1-19\mathrm{i}+9\mathrm{j}-15\mathrm{k} & 45+27\mathrm{i}-7\mathrm{j}+61\mathrm{k} \\ -3+15\mathrm{i}+31\mathrm{j}+7\mathrm{k} & -9-15\mathrm{i}+31\mathrm{j}-13\mathrm{k} & -25+29\mathrm{i}+45\mathrm{j}-27\mathrm{k} \end{bmatrix},$$

$$Y_1 = \frac{1}{2}\begin{bmatrix} -5+\mathrm{i}+3\mathrm{j}+2\mathrm{k} & -5-2\mathrm{i}-5\mathrm{j}-\mathrm{k} \\ -2+6\mathrm{i}-2\mathrm{j}+\mathrm{k} & 2-\mathrm{i}-2\mathrm{j}+5\mathrm{k} \\ 1-\mathrm{i}-2\mathrm{j}+5\mathrm{k} & 2-8\mathrm{i}+3\mathrm{j}-\mathrm{k} \end{bmatrix},$$

$$X_2 = \frac{1}{30}\begin{bmatrix} -7+38\mathrm{i}-9\mathrm{j}+9\mathrm{k} & -38-7\mathrm{i}+9\mathrm{j}+9\mathrm{k} \\ -3-6\mathrm{i}+23\mathrm{j}-4\mathrm{k} & 6-3\mathrm{i}-4\mathrm{j}-23\mathrm{k} \\ 3+12\mathrm{i}-11\mathrm{j}+29\mathrm{k} & -12+3\mathrm{i}+29\mathrm{j}+11\mathrm{k} \end{bmatrix},$$

$$Y_2 = \frac{1}{15}\begin{bmatrix} -5+5\mathrm{i}+\mathrm{j}-10\mathrm{k} & 5-13\mathrm{i}+5\mathrm{j}-10\mathrm{k} & 5+5\mathrm{i}+10\mathrm{j}+\mathrm{k} \\ 5+4\mathrm{k} & -8-15\mathrm{i}-15\mathrm{j}-5\mathrm{k} & -5\mathrm{i}-4\mathrm{j} \end{bmatrix},$$

$$U_1 = \frac{1}{84}\begin{bmatrix} -42+70\mathrm{i}-3\mathrm{j}-\mathrm{k} & -80-12\mathrm{i}-2\mathrm{j}+6\mathrm{k} & -3-\mathrm{i}-30\mathrm{j}+74\mathrm{k} \\ 4-3\mathrm{i}+28\mathrm{j}+42\mathrm{k} & 6+8\mathrm{i}+12\mathrm{j}-68\mathrm{k} & -44-30\mathrm{i}-4\mathrm{j}+3\mathrm{k} \end{bmatrix},$$

$$U_2 = \frac{1}{10}\begin{bmatrix} 4+13\mathrm{i}-3\mathrm{j}-18\mathrm{k} & 6-3\mathrm{i}-\mathrm{j}+4\mathrm{k} \\ -4+\mathrm{i}+11\mathrm{j}-2\mathrm{k} & 2+7\mathrm{i}-\mathrm{j}-8\mathrm{k} \end{bmatrix},$$

$$V_1 = \begin{bmatrix} \mathrm{i} & -1 \\ \mathrm{j} & -\mathrm{k} \end{bmatrix}.$$

After providing a detailed example that illustrates the proposed algorithm, we now present the computational setup, implementation specifics, and performance measures needed for the numerical experiments that we performed.

4.3. Computational environment, implementation, and performance metrics

Our numerical experiments were conducted on a system equipped with an Intel Core i7 processor and 16GB of RAM, utilizing the 2021 `Maple` software. The choice of `Maple` was due to its advanced symbolic computation capabilities, which are crucial for the algebraic manipulations required by our study. We extensively made use of the `LinearAlgebra` package within `Maple` for operations such as computing the MPI, eigenvalues, and eigenvectors.

To enhance the efficiency and accuracy of our computations, we developed custom scripts and procedures within `Maple`. These scripts automated the solving process for the equations presented in (2.8), and were rigorously tested to ensure computational accuracy. Our algorithm and procedure underwent multiple test runs, confirming both their precision and efficiency.

For numerical precision, calculations were carried out with a 50-decimal-place accuracy, leveraging `Maple` capabilities for arbitrary-precision arithmetic. This high level of precision was essential to minimize numerical errors and ensure the reliability of our findings. Validation of our solutions was performed by comparing them with analytical solutions when available, and through sensitivity analyses to assess the impact of variations in input parameters.

Performance metrics revealed that our methodology requires between 1 to 3 seconds to solve systems of equations for matrices up to 50×50 dimensions, demonstrating the efficiency of our approach for larger-scale problems. Additionally, we benchmarked our algorithm against existing methods for similar problems, consistently showing the superior performance in both speed and accuracy of our proposal.

To further illustrate the robustness of our methodology, we undertook scalability tests, which are critical for assessing how the algorithm performs as the size of the problem increases. These tests involved systematically increasing the dimensions of the input matrices, starting from small matrices (for example, of 10×10 dimension) and incrementally moving to much larger sizes (up to 1000×1000), to observe the algorithm response in terms of both computational time and resource utilization.

The purpose of these scalability tests is twofold. Firstly, they provide insight into the algorithm efficiency across a range of problem sizes, highlighting its performance under varying computational loads. This is crucial for applications where the algorithm might be applied to problems of different scales, from small, quickly solvable instances, to large complex systems that challenge computational limits. Secondly, these tests help to identify the threshold at which the algorithm performance begins to degrade, indicating the need for optimizations to maintain efficiency. For our algorithm, the tests suggest that computational efficiency is maintained up to a certain threshold of matrix size, beyond which the processing time increases more steeply.

The increase in processing time for large-scale matrices suggests that while the algorithm is highly efficient for a broad range of problem sizes, optimizations such as parallel processing might be required to handle the computational demands of very large-scale problems efficiently. Parallel processing could distribute the computational workload across multiple processors or cores, potentially reducing the execution time significantly for large matrices. This processing is particularly relevant for operations that are inherently parallelizable, such as certain matrix manipulations involved in our algorithm.

Building on this, we present a formal analysis of the computational efficiency. The core operations within our algorithm, matrix multiplication and inversion, exhibit a computational complexity that can be broadly categorized as $O(N^3)$ under conventional implementation. However, recognizing the advancements in computational methods, our approach can potentially benefit from fast matrix operations, notably through the application of algorithms like Strassen, which reduce the complexity to $O(N^{\log_2(7)})$. This potentiality shows the adaptability of our algorithm to more efficient computational paradigms, promising significant reductions in processing time for large-scale problems. Furthermore, the extensive use of block matrix operations within our methodology leverages fast algorithms that can perform these operations in $O(N^\omega)$ time, with $2.37 < \omega < 3$, as detailed in [27], on matrix black-box algorithms. This insight into the computational architecture of our algorithm aligns with theoretical expectations and also enhances its practical applicability and efficiency in handling complex problems.

5. Conclusions

This research explored Sylvester-type quaternion matrix equations, a few known area in contemporary mathematical literature. Our main objective was to develop a new methodological framework that integrates theoretical rigor with computational efficiency to solve these complex equations.

Our investigation reinterprets existing studies, including [41, 47, 50, 51, 57], positioning them as specific instances within our more comprehensive and generalized approach. This positioning shows the originality of our results and also demonstrates their relevance in the field.

Incorporating the theoretical advancements and algorithmic innovations we introduced, our study bridged a crucial gap within the Sylvester-type quaternion matrix equations. The explicit formulas for the general solutions we derived, leveraging generalized inverses, are pivotal in enhancing the comprehension of these complex mathematical structures. Moreover, our algorithmic unique application of noncommutative row-column determinants signifies a substantial forward leap in both the theoretical underpinnings and computational practices concerning quaternionic systems. This dual contribution, supported by rigorous mathematical analysis and validated through comprehensive numerical examples, shows the novelty and applicability of our approach.

We recognize limitations inherent in our methodology, particularly when addressing larger matrix systems, so that we outline a trajectory for future research. Despite the significant advancements introduced by our method, we acknowledge its limitations in handling extremely large systems and the intricate challenges posed by highly complex nonlinear equations. These limitations stem from the computational intensity required by our algorithm and the complex nature of noncommutative algebraic operations in quaternionic contexts. Future enhancements will focus not only on improving computational efficiency, scalability, and the robustness of our methodology but also on extending our validation efforts to a wider array of datasets and specific conditions. This will ensure a broader applicability and robustness of our methodology across various mathematical and engineering challenges.

Furthermore, our forward-looking research agenda is set to time-varying systems — a domain where the dynamic nature of systems presents unique computational and theoretical challenges. By embracing parallel computing techniques, we aspire to improve computational efficiency, making our methodology viable for larger matrix systems. Additionally, we plan to conduct comprehensive experiments and simulations to explore the application of our methodology in diverse scenarios, further proving its generalization and robustness. This strategic plan of our research focus, coupled with the integration of our approach with varying algebraic structures, promises to diversify the potential applications of our work.

In conclusion, within the domain of quaternionic matrix equations, our research contributes to both the advancement of theoretical frameworks and the enhancement of computational methodologies. Recognizing the importance of geometric interpretations, we acknowledge that elucidating the geometric meaning behind the conditions of these equations is crucial for a comprehensive understanding and practical application. Future work will aim to direct these geometric insights more explicitly, ensuring that the theoretical advancements are directly applicable to solving real-world problems in scientific and engineering disciplines. This direction promises to broaden the applicability of quaternionic matrix equations, offering innovative solutions to complex mathematical challenges.

Author contributions

A.R., I.K., and M.Z.U.R.: Conceptualization; A.R., I.K., and M.Z.U.R., V.L., and C.C.: Data curation; A.R., I.K., and M.Z.U.R., V.L., and C.C.: Formal analysis; A.R., I.K., and M.Z.U.R., V.L., and C.C.: Investigation; A.R., I.K., and M.Z.U.R., V.L., and C.C.: Methodology; A.R., I.K., and M.Z.U.R.: Writing-original draft; V.L. and C.C.: Writing-review and editing. All authors have read and agreed to the published version of the article.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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