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Research article

Iterative oscillation criteria of third-order nonlinear damped neutral differential equations

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Abstract: Using comparison principles, we examine the asymptotic characteristics of a third-order nonlinear damped neutral differential equation. Our results substantially generalize numerous previously established results as well as drastically improving them. To illustrate the relevance and effectiveness of our results, we use numerical examples.

Keywords: differential equations; third-order; oscillation; asymptotic behavior; damped
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1. Introduction

A neutral delay differential equation contains the highest-order derivative of the unknown function both with and without delays. Because of this, the theory of neutral delay differential equations is more difficult to understand than the theory of non-neutral equations. There has been an increase in interest in the theory of neutral differential equations in recent years. Studying these equations is essential for both theory and applications, as neutral equations are used to explain a wide range of real-world phenomena, including the motion of radiating electrons, population growth, the spread of epidemics, networks incorporating lossless transmission lines, etc., see [2, 17, 19, 20]. Researchers have focused a great deal of attention on the oscillation problem of functional differential equations in the recent few decades; see, for example, [1–40]. For third-order delay equations, see [1–7, 12, 25, 26, 29]. For neutral equations, see [21, 22, 32–34] and [8, 15, 24, 35, 39] for the equations with damping. Using a generalized Riccati transformation and an integral averaging technique the authors [38] obtained certain necessary conditions for oscillation for the third-order nonlinear differential equation

$$[m_2(s)\{m_1(s)x'(s)\}']' + p(s)x'(s) + q(s)f(x(\rho(s))) = 0,$$

where $\rho'(s) > 0$ and $\frac{f(u)}{u} \geq k > 0$, for all $u \neq 0$. Also, [11] improves and unifies the results of [38], reducing the third-order equations to the first and second ones. In this work, we focus our attention on the oscillation of the third-order nonlinear neutral differential equation with the form

$$\left\{m_2(s)\varphi_{\eta_2}\left(\left[m_1(s)\varphi_{\eta_1}\left(z'(s)\right)\right]'\right)\right\}' + m_3(s)\varphi_{\eta_2}\left(\left[m_1(s)\varphi_{\eta_1}\left(z'(s)\right)\right]'\right) + q(s)f\left(x(\rho(s))\right) = 0, \tag{1.1}$$

where $s \geq s_0 \geq 0$, $z(s) := x(s) + p(s)x(\mu(s))$, $\varphi_\beta(u) := |u|^{\beta-1} u$, $\beta > 0$; $\eta_1, \eta_2 > 0$, and $m_i, p, q, \rho, \mu \in C([s_0, \infty), \mathbb{R})$, $i = 1, 2, 3$. It should be noted that the oscillation of many special cases of Eq (1.1) has been studied by many authors; see, for examples, [9–12, 15, 16].

In this paper, we suppose that

(i) $0 \leq p(s) < p < 1$, $q(s) \geq 0$, $m_i(s) > 0$, $i = 1, 2$ and $m_3(s) \geq 0$;
(ii) $f \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$ and $\frac{f(x)}{\varphi_\eta(x)} \geq k > 0$, for all $x \neq 0$, $\eta := \eta_1\eta_2$;
(iii) $\rho(s) \leq s$, $\mu(s) \leq s$, and $\lim_{s\to\infty} \rho(s) = \lim_{s\to\infty} \mu(s) = \infty$;
(iv) $\int_{\mathcal{S}}^{\infty} \left(\frac{1}{m_1(t)}\right)^{1/\eta_1} \mathrm{d}t = \infty$ and $\int_{\mathcal{S}}^{\infty} \left(\frac{1}{M(t)}\right)^{1/\eta_2} \mathrm{d}t = \infty$,
where $M(s) := m_2(s) \exp\left(\int_{\mathcal{S}}^{s} \frac{m_3(r)}{m_2(r)} \mathrm{d}r\right)$, $\mathcal{S} \in [s_0, \infty)$.

A function $x(s)$ is a solution of (1.1) if it satisfies Eq (1.1) for all $s \in [s_x, \infty)$ and satisfying $\sup\{|x(s)| : s \geq \mathcal{S}\} > 0$ for any $\mathcal{S} \geq s_x$ with $x(s)$, $m_1(s)\varphi_{\eta_1}(z'(s))$, and $m_2(s)\varphi_{\eta_2}\left(\left[m_1(s)\varphi_{\eta_1}(z'(s))\right]'\right)$ are continuously differentiable for all $s \in [s_x, \infty)$. The solution on $[s_x, \infty)$ with arbitrary large zeros is said to be an oscillatory solution. In this paper, we investigate the oscillatory and asymptotic behavior of Eq (1.1) by a reduction in order and comparison with the oscillation of first-order delay differential equations.

2. Main results

Throughout this paper, we define

$$\mathcal{L}_1(z(s)) := \varphi_{\eta_1}\left(z'(s)\right), \ \mathcal{L}_2(z(s)) := \varphi_{\eta_2}\left((m_1(s)\mathcal{L}_1(z(s)))'\right).$$

Also, the sequences $\{P_n(s)\}_{n=1}^{\infty}$ and $\{Q_n(s,t)\}_{n=1}^{\infty}$ are defined as follows:

$$P_n(s) = \int_{\mathcal{S}}^{s} \left[\frac{1}{m_1(u)} \int_{\mathcal{S}}^{u} \left(\frac{1}{M(t)} \exp\left(\int_t^u \overline{q}(w) P_{n-1}^{\eta}(\rho(w))\, \mathrm{d}w\right)\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1} \mathrm{d}u, \tag{2.1}$$

for $\mathcal{S} \in [s_0, \infty)$ and $s \in [\mathcal{S}, \infty)$, with

$$P_0(s) = 0 \quad \text{and} \quad \overline{q}(s) := kq(s)\left(1 - p\left(\rho(s)\right)\right)^{\eta} \exp\left(\int_{\mathcal{S}}^{s} \frac{m_3(r)}{m_2(r)} \mathrm{d}r\right),$$

and

$$Q_n(s,t) := \int_t^s \left[\frac{1}{m_1(v)} \int_v^s \left(\frac{1}{M(u)} \exp\left(\int_u^s \bar{q}_*(w) Q_{n-1}^{\eta}(w,\rho(w))\, \mathrm{d}w\right)\right)^{1/\eta_2} \mathrm{d}u\right]^{1/\eta_1} \mathrm{d}v,$$

for $s \in [t,\infty) \subseteq [\mathcal{S},\infty)$, with

$$Q_0(s,t) = 0 \text{ and } \bar{q}_*(s) := kN^{\eta} q(s) \exp\left(\int_{\mathcal{S}}^s \frac{m_3(r)}{m_2(r)}\, \mathrm{d}r\right),$$

for some $N > 0$ and $\mathcal{S} \in [s_0,\infty)$.

The subsequent lemmas will be introduced and utilized in the main result.

Lemma 2.1. *Assume that x is an eventually positive solution of Eq* (1.1). *Then there exists $\mathcal{S} \geq s_0$ such that either*

(I) $\mathcal{L}_1(z(s)) > 0$, $\mathcal{L}_2(z(s)) > 0$,

or

(II) $\mathcal{L}_1(z(s)) < 0$, $\mathcal{L}_2(z(s)) > 0$,

for all $s \geq \mathcal{S}$.

Proof. Since x is a positive solution of Eq (1.1) on $[s_1,\infty)$, $s_1 \geq s_0$ such that $x(\rho(s)) > 0$ and $x(\mu(s)) > 0$ for $s \geq s_1$. From Eq (1.1), we have for all $s \geq s_1$,

$$(m_2(s)\mathcal{L}_2(z(s)))' + m_3(s)\mathcal{L}_2(z(s)) \leq 0,$$

which implies that

$$(M(s)\mathcal{L}_2(z(s)))' \leq 0,$$

where $M(s) = m_2(s)\exp\left(\int_{s_1}^s \frac{m_3(r)}{m_2(r)}\, \mathrm{d}r\right)$. That demonstrates that $\mathcal{L}_1(z(s))$ and $\mathcal{L}_2(z(s))$ are of one sign eventually. We claim that

$$\mathcal{L}_2(z(s)) > 0 \quad \text{eventually.}$$

If not, consider the following two cases:

Case 1. There exists $s_2 \geq s_1$, sufficiently large, such that

$$\mathcal{L}_1(z(s)) > 0 \quad \text{and} \quad \mathcal{L}_2(z(s)) < 0 \quad \text{for } s \geq s_2.$$

Since $(M(s)\mathcal{L}_2(z(s)))' \leq 0$, then there exists a negative constant $\mathcal{M}$ such that

$$M(s)\varphi_{\eta_2}((m_1(s)\mathcal{L}_1(z(s)))') \leq \mathcal{M} \quad \text{for } s \geq s_2.$$

It follows that

$$(m_1(s)\mathcal{L}_1(z(s)))' \leq \varphi_{\eta_2}^{-1}(\mathcal{M})^{1/\eta_2}\left(\frac{1}{M(s)}\right)^{1/\eta_2} \quad \text{for } s \geq s_2.$$

Integrating from s_2 to s, we obtain

$$m_1(s)\mathcal{L}_1(z(s)) \leq m_1(s_2)\mathcal{L}_1(z(s_2)) + \varphi_{\eta_2}^{-1}(\mathcal{M})^{1/\eta_2}\int_{s_2}^s \left(\frac{1}{M(t)}\right)^{1/\eta_2} \mathrm{d}t.$$

Letting $s \to \infty$ and using (iv), then $\mathcal{L}_1(z(s)) \to -\infty$, which contradicts that $\mathcal{L}_1(z(s)) > 0$.
Case 2. There exists $s_2 \geq s_1$, sufficiently large, such that

$$\mathcal{L}_1(z(s)) < 0 \quad \text{and} \quad \mathcal{L}_2(z(s)) < 0 \quad \text{for } s \geq s_2,$$

which implies that $(m_1(s)\mathcal{L}_1(z(s)))' < 0$ and therefore,

$$m_1(s)\mathcal{L}_1(z(s)) \leq m_1(s_2)\mathcal{L}_1(z(s_2)) = \bar{k} < 0.$$

Dividing by $m_1(s)$ and integrating from s_2 to s, we obtain

$$z(s) \leq z(s_2) + \varphi_{\eta_1}^{-1}(\bar{k}) \int_{s_2}^{s} \left(\frac{1}{m_1(t)}\right)^{1/\eta_1} \mathrm{d}t.$$

Letting $s \to \infty$, then (iv) yields $z(s) \to -\infty$, which contradicts the fact that $z(s) > 0$. This completes the proof. □

Lemma 2.2. *Assume that x is a positive solution of Eq* (1.1) *and the corresponding function z satisfies (I) of Lemma 2.1. Then*

$$(M(s)\mathcal{L}_2(z(s)))' + \overline{q}(s)z^{\eta}(\rho(s)) \leq 0. \tag{2.2}$$

Proof. Since x is a positive solution of Eq (1.1) on $[s_1, \infty)$, then there exists $s_2 \geq s_1$ such that the corresponding function z satisfies (I) of Lemma 2.1 on $[s_1, \infty)$. It is easy to see that Eq (1.1) can be written in the form

$$(m_2(s)\mathcal{L}_2(z(s)))' + m_3(s)\mathcal{L}_2(z(s)) + q(s)f(x(\rho(s))) = 0,$$

for all $s \geq s_1$. Then

$$(M(s)\mathcal{L}_2(z(s)))' + q(s)\exp\left(\int_{s_1}^{s} \frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right) f(x(\rho(s))) = 0.$$

Therefore,

$$(M(s)\mathcal{L}_2(z(s)))' + kq(s)\exp\left(\int_{s_1}^{s} \frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right) x^{\eta}(\rho(s)) \leq 0. \tag{2.3}$$

Also, we have

$$x(s) = z(s) - p(s)x(\mu(s)) \geq z(s) - p(s)z(\mu(s)).$$

Since $z' > 0$, we get

$$x(s) \geq (1 - p(s))z(s). \tag{2.4}$$

Substituting (2.4) into (2.3), we have

$$(M(s)\mathcal{L}_2(z(s)))' + \overline{q}(t)\, z^{\eta}(\rho(s)) \leq 0.$$

This completes the proof. □

Lemma 2.3. *If x is an eventually positive solution of Eq* (1.1) *and the corresponding function z satisfies Case (I) of Lemma 2.1, then for $n \in \mathbb{N}$,*

$$z(s) \geq P_n(s)\,(M(s)\mathcal{L}_2(z(s)))^{1/\eta}. \tag{2.5}$$

Proof. Since x is a positive solution of Eq (1.1) on $[s_1, \infty)$, then there exists $s_2 \geq s_1$ such that the corresponding function z satisfies (I) of Lemma 2.1 on $[s_1, \infty)$. Then

$$\begin{aligned} m_1(s)\,\mathcal{L}_1(z(s)) &= \int_{s_1}^{s} (m_1(t)\,\mathcal{L}_1(z(t)))'\,\mathrm{d}t + m_1(s_1)\,\mathcal{L}_1(z(s_1)) \\ &\geq \int_{s_1}^{s} \left(\frac{1}{M(t)}\right)^{1/\eta_2} (M(t)\,\mathcal{L}_2(z(t)))^{1/\eta_2}\,\mathrm{d}t \\ &\geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta_2} \int_{s_1}^{s} \left(\frac{1}{M(t)}\right)^{1/\eta_2} \mathrm{d}t. \end{aligned} \tag{2.6}$$

Then,

$$z'(s) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \left[\frac{1}{m_1(s)} \int_{s_1}^{s} \left(\frac{1}{M(t)}\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1}.$$

Integrating the above inequality from s_1 to $s \in [s_1, \infty)$, we obtain

$$\begin{aligned} z(s) &\geq \int_{s_1}^{s} \left\{ (M(u)\,\mathcal{L}_2(z(u)))^{1/\eta} \left[\frac{1}{m_1(u)} \int_{s_1}^{u} \left(\frac{1}{M(t)}\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1} \right\} \mathrm{d}u \\ &\geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \int_{s_1}^{s} \left\{ \left[\frac{1}{m_1(u)} \int_{s_1}^{u} \left(\frac{1}{M(t)}\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1} \right\} \mathrm{d}u \\ &= (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} P_1(s). \end{aligned}$$

This shows that (2.5) holds for $n = 1$. Consequently,

$$z(\rho(s)) \geq (M(\rho(s))\,\mathcal{L}_2(z(\rho(s))))^{1/\eta} P_1(\rho(s)). \tag{2.7}$$

From (2.2) and (2.7), we obtain

$$(M(s)\,\mathcal{L}_2(z(s)))' + \overline{q}(s) P_1^{\eta}(\rho(s)) M(\rho(s))\,\mathcal{L}_2(z(\rho(s))) \leq 0.$$

Using the nonicreasing nature of $M(s)\,\mathcal{L}_2(z(s))$ and $\rho(s) \leq s$, we obtain

$$(M(s)\,\mathcal{L}_2(z(s)))' + \overline{q}(s) P_1^{\eta}(\rho(s)) M(s)\,\mathcal{L}_2(z(s)) \leq 0.$$

Integrating the above inequality from t to $s \in [t, \infty)$ implies that

$$M(t)\,\mathcal{L}_2(z(t)) \geq M(s)\,\mathcal{L}_2(z(s)) \exp\left(\int_t^s \overline{q}(w) P_1^{\eta}(\rho(w))\,\mathrm{d}w\right). \tag{2.8}$$

Using (2.8) in (2.6), we obtain

$$m_1(s)\,\mathcal{L}_1(z(s)) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta_2} \int_{s_1}^{s} \left(\frac{1}{M(t)} \exp\left(\int_t^s \overline{q}(w) P_1^{\eta}(\rho(w))\,\mathrm{d}w\right)\right)^{1/\eta_2} \mathrm{d}t.$$

It follows that

$$z'(s) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \left[\frac{1}{m_1(s)} \int_{s_1}^{s} \left(\frac{1}{M(t)} \exp\left(\int_t^s \overline{q}(w) P_1^{\eta}(\rho(w))\,\mathrm{d}w\right)\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1}.$$

Again, integrating from s_1 to s, we obtain

$$\begin{aligned} z(s) &\geq \int_{s_1}^{s} \left(M(u)\mathcal{L}_2(z(u))\right)^{1/\eta} \left[\frac{1}{m_1(u)} \int_{s_1}^{u} \left(\frac{1}{M(t)} \exp\left(\int_t^u \overline{q}(w) P_1^{\eta}(\rho(w))\, \mathrm{d}w\right)\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1} \mathrm{d}u \\ &\geq \left(M(s)\mathcal{L}_2(z(s))\right)^{1/\eta} \int_{s_1}^{s} \left[\frac{1}{m_1(u)} \int_{s_1}^{u} \left(\frac{1}{M(t)} \exp\left(\int_t^u \overline{q}(w) P_1^{\eta}(\rho(w))\, \mathrm{d}w\right)\right)^{1/\eta_2} \mathrm{d}t\right]^{1/\eta_1} \mathrm{d}u \\ &= \left(M(s)\mathcal{L}_2(z(s))\right)^{1/\eta} P_2(s). \end{aligned}$$

This shows that (2.5) holds for $n = 2$. If this process is repeated n times, we obtain (2.5). □

The asymptotic behavior of all solutions to Eq (1.1) is discussed in the results that follow.

Theorem 2.1. *Let $n \in \mathbb{N}$. Assume that the first-order delay differential equation*

$$w'(s) + \overline{q}(s) P_n^{\eta}(\rho(s))\, w(\rho(s)) = 0 \tag{2.9}$$

is oscillatory. If $x(s)$ is a solution of Eq (1.1)*, then $x(s)$ is either oscillatory or bounded.*

Proof. Assume that $x(s)$ is a nonoscillatory solution of Eq (1.1). Without loss of generality, let $x(s) > 0$, $x(\rho(s)) > 0$, and $x(\mu(s)) > 0$ on $[s_1, \infty)$, $s_1 \geq s_0$. It follows from Lemma 2.1 that there exists $s_2 \geq s_1$ such that either (I) or (II) holds on $[s_2, \infty)$. Assume (I) is valid. From (2.5), we have

$$z(\rho(s)) \geq \left(M(\rho(s))\mathcal{L}_2(z(\rho(s)))\right)^{1/\eta} P_n(\rho(s)). \tag{2.10}$$

Combining (2.2) and (2.10), we obtain

$$w'(s) + \overline{q}(s) P_n^{\eta}(\rho(s)) w(\rho(s)) \leq 0,$$

where $w(s) := M(s)\mathcal{L}_2(z(s))$. Due to [37, Theorem 1], the associated delay differential equation also has a positive solution. This is a contradiction. Now, to complete the proof, we consider (II) valid. Since $z(s) > 0$, and $z'(s) < 0$ then $z(s)$ is bounded, and therefore $x(s)$ is bounded. The proof is complete. □

Theorem 2.2. *Let $n \in \mathbb{N}$. Assume that the first-order delay differential equation* (2.9) *is oscillatory and*

$$\int^{\infty} \left[\frac{1}{m_1(v)} \int_v^{\infty} \left(\frac{1}{m_2(u)} \int_u^{\infty} q(t) \exp\left(\int_u^t \frac{m_3(r)}{m_2(r)}\, \mathrm{d}r\right) \mathrm{d}t\right)^{1/\eta_2} \mathrm{d}u\right]^{1/\eta_1} \mathrm{d}v = \infty. \tag{2.11}$$

If $x(s)$ is a solution of Eq (1.1)*, then $x(s)$ is either oscillatory or tends to zero eventually.*

Proof. Assume that $x(s)$ is a nonoscillatory solution of Eq (1.1). Without loss of generality, let $x(s) > 0$, $x(\rho(s)) > 0$, and $x(\mu(s)) > 0$ on $[s_1, \infty)$, $s_1 \geq s_0$. It follows from Lemma 2.1 that there exists $s_2 \geq s_1$ such that either (I) or (II) holds on $[s_2, \infty)$. The proof of Case (I) is identical to the proof of Theorem 2.1, Case (I), and so it has been omitted. Assume (II) is valid. It is obvious that Eq (1.1) can be written as

$$\left(M(s)\mathcal{L}_2(z(s))\right)' + kq(s) \exp\left(\int_{s_1}^{s} \frac{m_3(r)}{m_2(r)}\, \mathrm{d}r\right) x^{\eta}(\rho(s)) \leq 0. \tag{2.12}$$

Since $z(s) > 0$ and $z'(s) < 0$, there exists a constant $l \geq 0$ such that $\lim_{t\to\infty} z(s) = l$. We claim $l = 0$. If not, then for sufficiently small $\epsilon > 0$, there exists $s_3 \geq s_2$ such that $l - p(l+\epsilon) > 0$ and $l < z(s) < l + \epsilon$ for all $s > s_3$. Then

$$x(s) = z(s) - p(s)x(\mu(s)) \geq z(s) - pz(\mu(s)) \geq l - p(l+\epsilon) \geq N\,(l+\epsilon) > N\,z(s), \tag{2.13}$$

$N := \frac{l-p(l+\epsilon)}{l+\epsilon} > 0$. From (2.12) and (2.13), we obtain

$$(M\,(s)\,\mathcal{L}_2(z(s)))' + kN^{\eta}q(s)\exp\left(\int_{s_1}^{s}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)z^{\eta}\,(\rho\,(s)) \leq 0. \tag{2.14}$$

Then

$$(M\,(s)\,\mathcal{L}_2(z(s)))' + Kq(s)\exp\left(\int_{s_1}^{s}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right) \leq 0,$$

where $K := kN^{\eta}l^{\eta} > 0$. Integrating the above inequality from $s \in [s_3, \infty)$ to ∞, we obtain

$$\begin{aligned} M\,(s)\,\mathcal{L}_2(z(s)) &\geq K\int_{s}^{\infty} q(t)\exp\left(\int_{s_1}^{t}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\mathrm{d}t \\ &= K\exp\left(\int_{s_1}^{s}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\int_{s}^{\infty} q(t)\exp\left(\int_{s}^{t}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\mathrm{d}t. \end{aligned}$$

It follows that

$$(m_1\,(s)\,\mathcal{L}_1(z(s)))' \geq K^{1/\eta_2}\left(\frac{1}{m_2(s)}\int_{s}^{\infty} q(t)\exp\left(\int_{s}^{t}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\mathrm{d}t\right)^{1/\eta_2},$$

Integrating the above inequality from s to ∞, we obtain

$$-z'(s) \geq K^{1/\eta}\left[\frac{1}{m_1(s)}\int_{s}^{\infty}\left(\frac{1}{m_2(u)}\int_{u}^{\infty} q(t)\exp\left(\int_{u}^{t}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\mathrm{d}t\right)^{1/\eta_2}\mathrm{d}u\right]^{1/\eta_1}.$$

Again, integrating the above inequality from s_2 to ∞, we obtain

$$z(s_2) \geq K^{1/\eta}\int_{s_2}^{\infty}\left[\frac{1}{m_1(v)}\int_{v}^{\infty}\left(\frac{1}{m_2(u)}\int_{u}^{\infty} q(t)\exp\left(\int_{u}^{t}\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\mathrm{d}t\right)^{1/\eta_2}\mathrm{d}u\right]^{1/\eta_1}\mathrm{d}v,$$

which is a contradiction to (2.11), then $\lim_{s\to\infty} z(s) = 0$. Since $0 < x(s) \leq z(s)$, then $\lim_{s\to\infty} x(s) = 0$. The proof is complete. □

Lemma 2.4. *If x is an eventually positive solution of Eq* (1.1) *and the corresponding function z satisfies Case (II) of Lemma 2.1, then for $n \in \mathbb{N}$ and $s \in [t, \infty)$*,

$$z(t) \geq (M\,(s)\,\mathcal{L}_2(z(s)))^{1/\eta}\;Q_n(s,t). \tag{2.15}$$

Proof. Let x be a positive solution of Eq (1.1) such that the Case (II) of Lemma 2.1 is satisfied on $[s_1, \infty)$, for some $s_1 \geq s_0$. Then, for $s \geq v \geq s_1$,

$$-m_1\,(v)\,\mathcal{L}_1(z(v)) = \int_{v}^{s}(m_1\,(u)\,\mathcal{L}_1(z(u)))'\;\mathrm{d}u - m_1\,(s)\,\mathcal{L}_1(z(s))$$

$$\geq \int_v^s \left(\frac{1}{M(u)}\right)^{1/\eta_2} (M(u)\,\mathcal{L}_2(z(u)))^{1/\eta_2}\,\mathrm{d}u \tag{2.16}$$
$$\geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta_2} \int_v^s \left(\frac{1}{M(u)}\right)^{1/\eta_2}\,\mathrm{d}u.$$

Then

$$-z'(v) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \left[\frac{1}{m_1(v)} \int_v^s \left(\frac{1}{M(u)}\right)^{1/\eta_2}\,\mathrm{d}u\right]^{1/\eta_1}.$$

Integrating the above inequality from t to $s \in [t, \infty)$ with respect to v, we obtain

$$z(t) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \int_t^s \left[\frac{1}{m_1(v)} \int_v^s \left(\frac{1}{M(u)}\right)^{1/\eta_2}\,\mathrm{d}u\right]^{1/\eta_1}\,\mathrm{d}v$$
$$\geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta}\, Q_1(s,t).$$

This shows that (2.15) holds for $n = 1$. Consequently,

$$z(\rho(s)) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta}\, Q_1(s,\rho(s)). \tag{2.17}$$

From (2.14) and (2.17), we obtain

$$(M(s)\,\mathcal{L}_2(z(s)))' + \overline{q}_*(s) Q_1^{\eta}(s,\rho(s)) M(s)\,\mathcal{L}_2(z(s)) \leq 0, \tag{2.18}$$

where $\overline{q}_*(s) = kN^{\eta} q(s) \exp\left(\int_{s_1}^{s} \frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)$. Integrating the latter inequality from u to $s \in [u, \infty)$ gives

$$M(u)\,\mathcal{L}_2(z(u) \geq M(s)\,\mathcal{L}_2(z(s)) \exp\left(\int_u^s \bar{q}_*(w) Q_1^{\eta}(w,\rho(w))\,\mathrm{d}w\right). \tag{2.19}$$

From (2.16) and (2.19), we obtain

$$-m_1(v)\mathcal{L}_1(z(v)) \geq \int_v^s \left(\frac{1}{M(u)}\right)^{1/\eta_2} (M(u)\,\mathcal{L}_2(z(u)))^{1/\eta_2}\,\mathrm{d}u$$
$$\geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta_2} \int_v^s \left(\frac{1}{M(u)} \exp\left(\int_u^s \bar{q}_*(w) Q_1^{\eta}(w,\rho(w))\,\mathrm{d}w\right)\right)^{1/\eta_2}\,\mathrm{d}u.$$

It follows that

$$-z'(v) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \left[\frac{1}{m_1(v)} \int_v^s \left(\frac{1}{M(u)} \exp\left(\int_u^s \bar{q}_*(w) Q_1^{\eta}(w,\rho(w))\,\mathrm{d}w\right)\right)^{1/\eta_2}\,\mathrm{d}u\right]^{1/\eta_1}.$$

Therefore,

$$z(s) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta} \int_t^s \left[\frac{1}{m_1(v)} \int_v^s \left(\frac{1}{M(u)} \exp\left(\int_u^s \bar{q}_*(w) Q_1^{\eta}(w,\rho(w))\,\mathrm{d}w\right)\right)^{1/\eta_2}\,\mathrm{d}u\right]^{1/\eta_1}\,\mathrm{d}v.$$

Then,

$$z(s) \geq (M(s)\,\mathcal{L}_2(z(s)))^{1/\eta}\, Q_2(s,t).$$

This shows that (2.15) holds for $n = 2$. To obtain (2.15) for arbitrary $n \in \mathbb{N}$, this procedure can be done n times. □

Theorem 2.3. *Let $\rho(s)$ be nondecreasing on $[s_0, \infty)$. Suppose there exists $n \in \mathbb{N}$ such that one of the following first-order delay differential equations* (2.9) *is oscillatory and*

$$\limsup_{s\to\infty} \int_{\rho(s)}^{s} \bar{q}_*(t) Q_n^{\eta}(\rho(s), \rho(t))\,\mathrm{d}t > 1. \tag{2.20}$$

Then Eq (1.1) *is oscillatory.*

Proof. Assume that $x(s)$ is a nonoscillatory solution of Eq (1.1). Without loss of generality, let $x(s) > 0$, $x(\rho(s)) > 0$, and $x(\mu(s)) > 0$ on $[s_1, \infty)$, $s_1 \geq s_0$. It follows from Lemma 2.1 that there exists $s_2 \geq s_1$ such that either (I) or (II) holds on $[s_1, \infty)$. The proof of Case (I) is identical to the proof of Theorem 2.1, Case (I), and so it has been omitted. Assume (II) is valid. As in the proof of Theorem 2.2, Case (II), we have

$$-\left(M(s)\,\mathcal{L}_2(z(s))\right)' \geq kN^{\eta} q(s) \exp\left(\int_{s_1}^{s} \frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right) z^{\eta}(\rho(s)) = \bar{q}_*(s) z^{\eta}(\rho(s)).$$

Integrating the above inequality from $\rho(s)$ to s, we obtain

$$M(\rho(s))\,\mathcal{L}_2(z(\rho(s))) \geq \int_{\rho(s)}^{s} \bar{q}_*(t) z^{\eta}(\rho(t))\,\mathrm{d}t.$$

In view of the nondecreasing nature of ρ and (2.15), we obtain

$$M(\rho(s))\,\mathcal{L}_2(z(\rho(s))) \geq M(\rho(s))\,\mathcal{L}_2(z(\rho(s))) \int_{\rho(s)}^{s} \bar{q}_*(t) Q_n^{\eta}(\rho(s), \rho(t))\,\mathrm{d}t.$$

This is a contradiction with (2.20). The proof is complete. □

Applying the results of [27, 28, 30, 31] in Theorems 2.1–2.3, we get the asymptotic behavior of the solutions to Eq (1.1).

Corollary 2.1. *Let $\rho(s)$ be nondecreasing on $[s_0, \infty)$. Suppose there exists $n \in \mathbb{N}$ such that one of the following conditions is satisfied:*

(a) $\liminf_{s\to\infty} \int_{\rho(s)}^{s} \overline{q}(t) P_n^{\eta}(\rho(t))\,\mathrm{d}t > \dfrac{1}{\mathrm{e}}$;
(b) $\limsup_{s\to\infty} \int_{\rho(s)}^{s} \overline{q}(t) P_n^{\eta}(\rho(t))\,\mathrm{d}t > 1$;
(c) $\liminf_{s\to\infty} \int_{\rho(s)}^{s} \overline{q}(t) P_n^{\eta}(\rho(t))\,\mathrm{d}t > \alpha$ *and* $\limsup_{s\to\infty} \int_{\rho(s)}^{s} \overline{q}(t) P_n^{\eta}(\rho(t))\,\mathrm{d}t > 1 - \left(1 - \sqrt{1-\alpha}\right)^2$.

If $x(s)$ is a solution of Eq (1.1)*, then*

(I) *$x(s)$ is either oscillatory or bounded;*
(II) *$x(s)$ is either oscillatory or tends to zero eventually if* (2.11) *holds;*
(III) *$x(s)$ is oscillatory if* (2.20) *holds.*

Remark 2.1. *We note that Theorems 2.2 and 2.3 are reduced to [10, Theorems 1 and 2] when $\eta_1 = \eta_2 = 1$, $m_3(s) = 0$, $k = 1$, and $p(s) = 0$.*

3. Numerical examples

Examples are provided to demonstrate the significance of our results.

Example 3.1. *Consider the third-order nonlinear neutral differential equation with a damping term of the form*

$$\begin{aligned}&\left\{\frac{1}{s}\varphi_1\left(\left[(s-1)\varphi_3\left(\left(x(s)+\frac{1}{2}x\left(\frac{s}{2}\right)\right)'\right)\right]'\right)\right\}'+\frac{1}{s^2}\varphi_1\left(\left[(s-1)\varphi_3\left(\left(x(s)+\frac{1}{2}x\left(\frac{s}{2}\right)\right)'\right)\right]'\right)+6sx^3(s-1)\\ =\;& 0,\ s\geq 1,\end{aligned} \tag{3.1}$$

where $m_1(s)=s-1,\ m_2(s)=\frac{1}{s},\ m_3(s)=\frac{1}{s^2},\ q(s)=6s,\ \mu(s)=\frac{s}{2},\ \rho(s)=s-1,\ p(s)=\frac{1}{2},\ \eta_1=3,$ *and* $\eta_2=1$. *Using Maple software, we see that condition* (*iv*) *holds and* $\overline{q}(s)=\frac{3}{4}s^2$.

$$\begin{aligned}&\int_{s_2}^{\infty}\left[\frac{1}{m_1(v)}\int_v^{\infty}\left(\frac{1}{m_2(u)}\int_u^{\infty}q(t)\exp\left(\int_u^t\frac{m_3(r)}{m_2(r)}\,\mathrm{d}r\right)\mathrm{d}t\right)^{1/\eta_2}\mathrm{d}u\right]^{1/\eta_1}\mathrm{d}v\\ =\;&\int_1^{\infty}\left[\frac{1}{v-1}\int_v^{\infty}\left(u\int_u^{\infty}16t\exp\left(\int_u^t\frac{1}{r}\,\mathrm{d}r\right)\mathrm{d}t\right)\mathrm{d}u\right]^{\frac{1}{3}}\mathrm{d}v=\infty.\end{aligned}$$

Also, we have, for $n=1$

$$\begin{aligned}&\liminf_{s\to\infty}\int_{\rho(s)}^{s}\bar{q}(t)P_n^{\eta}(\rho(t))\,\mathrm{d}t\\ =\;&\liminf_{s\to\infty}\int_{s-1}^{s}\bar{q}(t)P_1^{3}(t-1)\,\mathrm{d}t\\ =\;&\liminf_{s\to\infty}((1/8)s^6-(1/8)(s-1)^6-(9/10)s^5+(9/10)(s-1)^5+(9/4)s^4-(9/4)(s-1)^4\\ -\;&2s^3+2(s-1)^3)>\frac{1}{\mathrm{e}}.\end{aligned}$$

Then, according to Corollary 2.1, every solution to Eq (3.1) *is either oscillatory or tends to zero as* $s\to\infty$.

Example 3.2. *Consider the third-order nonlinear neutral differential equation with a damping term of the form*

$$\begin{aligned}&\left\{\frac{1}{s^2}\varphi_1\left(\left[\frac{1}{9s^2}\varphi_1\left((x(s)+\frac{1}{3}x(s-1))'\right)\right]'\right)\right\}'+\frac{2}{s^3}\varphi_1\left(\left[\frac{1}{9s^2}\varphi_1\left((x(s)+\frac{1}{3}x(s-1)'\right)\right]'\right)+\frac{14}{s^7}x(\frac{s}{2})\\ =\;& 0,\ t\geq 1,\end{aligned} \tag{3.2}$$

where $m_1(s)=\frac{1}{9s^2},\ m_2(s)=\frac{1}{s^2},\ m_3(s)=\frac{2}{s^3},\ q(s)=\frac{42}{s^7},\ \mu(s)=s-1,\ \rho(s)=\frac{s}{2},\ p(s)=\frac{1}{3},\eta_1=\eta_2=1,$ $k=\frac{1}{3}$, *and* $N=\frac{1}{2}$. *Using Maple software, we see that condition* (*iv*) *holds and* $\overline{q}(s)=\frac{28}{3s^5}, q_*(s)=\frac{7}{s^5}$. *Also, we have, for* $n=1$,

$$\liminf_{s\to\infty}\int_{\rho(s)}^{s}\overline{q}(t)P_n^{\eta}(\rho(t))\,\mathrm{d}t$$

$$
\begin{aligned}
&= \liminf_{s\to\infty} \int_{\frac{s}{2}}^{s} \overline{q}(t) P_1(\frac{t}{2})\, \mathrm{d}t \\
&= \liminf_{s\to\infty} ((21/16)\ln(2) - 7/(2s) + 105/(4s^4)) \\
&= 0.90976 > \frac{1}{\mathrm{e}},
\end{aligned}
$$

$$
\begin{aligned}
&\limsup_{s\to\infty} \int_{\rho(s)}^{s} q_*(t) Q_n^{\eta}(\rho(s),\rho(t))\, \mathrm{d}t \\
=\ &\limsup_{s\to\infty} \int_{\frac{s}{2}}^{s} q_*(t) Q_1(\frac{s}{2},\frac{t}{2})\, \mathrm{d}t \\
=\ &\limsup_{s\to\infty} \int_{\frac{s}{2}}^{s} q_*(t) \int_{t/2}^{s/2} \frac{\int_v^{s/2} (M\,(u))^{-1}\, \mathrm{d}u}{m1\,(v)}\, \mathrm{d}v\mathrm{d}t \\
=\ &\limsup_{s\to\infty} \int_{\frac{s}{2}}^{s} \frac{7}{t^5} \frac{\left(3\,s^2+6\,st+9\,t^2\right)(s-t)^2}{64} \mathrm{d}t = 0.60029 < 1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\limsup_{s\to\infty} \int_{\rho(s)}^{s} q_*(t) Q_n^{\eta}(\rho(s),\rho(t))\, \mathrm{d}t \\
=\ &\limsup_{s\to\infty} \int_{\frac{s}{2}}^{s} q_*(t) Q_2(\frac{s}{2},\frac{t}{2})\, \mathrm{d}t \\
=\ &\limsup_{s\to\infty} \int_{\frac{s}{2}}^{s} q_*(t) \int_{t/2}^{s/2} \frac{1}{m1\,(v)} \int_v^{s/2} \frac{\mathrm{e}^{\int_u^{s/2} q_*(w) Q1(w,w/2)\, \mathrm{d}w}}{M\,(u)}\, \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}s \\
=\ &1.111 > 1.
\end{aligned}
$$

Thus, (a) *is satisfied for* $n = 1$, *and* (2.20) *is satisfied for* $n = 2$. *Then, according to Corollary 2.1, every solution to Eq* (3.2) *is oscillatory.*

Author contributions

Taher S. Hassan: Supervision, Writing-review editing, Software and Investigation; Emad R. Attia: Supervision, Writing-review editing, Software and Investigation; Bassant M. ElMatary: Supervision, Writing-original draft, Writing-review editing, Software and Investigation. All authors have read and agreed to the published version of the manuscript

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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