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Research article

Hardy–Littlewood maximal operators and Hausdorff operators on p-adic block spaces with variable exponents

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Abstract: In this paper, we established some sufficient conditions for the boundedness of the Hardy–Littlewood maximal operators and the Hausdorff operators on p-adic Herz spaces and p-adic local block spaces with variable exponents. In particular, the boundedness of the p-adic maximal commutator operators, the p-adic Hardy–Littlewood average operators, and the p-adic Hardy-Hilbert operators on such spaces was also discussed.

Keywords: Hardy–Littlewood maximal operator; maximal commutator operator; Hardy-type operators; Hausdorff operator; Herz space; block space; variable exponent; p-adic analysis
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1. Introduction

The p-adic theory of functions plays an essential role in p-adic probability, p-adic quantum mechanics, p-adic partial differential equations, and p-adic harmonic analysis (see [2, 9, 18, 21, 23, 35, 36]). Recently, the theory of p-adic operators has garnered attention within the mathematics community. Regarding the p-adic fields, Volosivets [37] introduced the Hausdorff operator as follows:

$$\mathcal{H}_{\psi,A}(f)(x) = \int_{\mathbb{Q}_p^n} \psi(t) f(A(t)x)dt, \quad x \in \mathbb{Q}_p^n,\ \psi \in L^1_{\rm loc}(\mathbb{Q}_p^n), \tag{1.1}$$

where $A(t)$ is an $n \times n$ invertible matrix for almost everywhere t in the support of ψ. The author studied conditions that imply the boundedness of the operator $\mathcal{H}_{\psi,A}$ on the p-adic Hardy space $H^1(\mathbb{Q}_p^n)$ and the p-adic BMO space $BMO(\mathbb{Q}_p^n)$. Also, they obtained relations among the operator $\mathcal{H}_{\psi,A}$ and p-adic Fourier transform. Let us consider that $\varphi \in \mathcal{M}(\mathbb{Z}_p^*, \mathbb{C})$, $\psi(t) = \varphi(t_1)\chi_{(\mathbb{Z}_p^*)^n}(t)$, and $A(t) = t_1 I_n$, for $t = (t_1, t_2, ..., t_n)$. Then, the operator $\mathcal{H}_{\psi,A}$ reduces to the p-adic Hardy–Littlewood average operator

$\mathcal{H}_\varphi^{p,n}$ [30],

$$\mathcal{H}_\varphi^{p,n}(f)(x) = \int_{\mathbb{Z}_p^*} \varphi(t) f(tx) dt, \quad x \in \mathbb{Q}_p^n.$$

This relationship highlights how Hausdorff operators encompass existing p-adic operators like the Hardy–Littlewood average operator, potentially offering a broader framework for analyzing their properties. In 2021, Dung and Duong [10] established some sufficient conditions for the boundedness of the operators $\mathcal{H}_{\psi,A}$ on weighted Triebel–Lizorkin spaces, two weighted Morrey spaces, and Morrey–Herz spaces. By observing a special case matrix A(t) = diag[s(t), ..., s(t)], the sharp bounds of the operators $\mathcal{H}_{\psi,A}$ are given on Morrey spaces and Morrey–Herz spaces with power weights. As some applications, the authors achieved several new p-adic Hardy–Hilbert type inequalities.

It is well known that the boundedness of some operators plays a crucial role in the regularity properties of the solution of some equations. Another role in the boundedness of solutions can be found in the works [3, 26, 40–42]. On p-adic fields, the solution of some pseudo-differential equations is strongly related to the operator $\mathcal{H}_\varphi^{p,1}$. For instance, Kochubei [22] investigated the following Cauchy problem:

$$\begin{cases} D^\alpha y(|x|_p) = g(|x|_p), & x \in \mathbb{Q}_p, \\ y(0) = 0, \end{cases} \tag{1.2}$$

where D^α, $\alpha > 0$, is the Vladimirov operator introduced in [36] and g is a continuous function, such that

$$\sum_{i=1}^{\infty} |g(p^i)| < \infty, \text{ if } \alpha \neq 1, \text{ or } \sum_{i=1}^{\infty} i|g(p^i)| < \infty, \text{ if } \alpha = 1.$$

The solution y of Eq (1.2) is given by

$$\begin{aligned} y(x) = I^\alpha(g)(x) &= \mathcal{A}_p \int\limits_{|u|_p \le |x|_p} \left(|x-u|_p^{\alpha-1} - |u|_p^{\alpha-1}\right) g(u) du \\ &= |x|_p^\alpha \left(\mathcal{H}_{\varphi_1}^{p,1} g(x) - \mathcal{H}_{\varphi_2}^{p,1} g(x)\right), \end{aligned}$$

where $\mathcal{A}_p = \frac{1-p^{-\alpha}}{1-p^{\alpha-1}}$, $\varphi_1(t) = \mathcal{A}_p |1-t|_p^{\alpha-1}$, and $\varphi_2(t) = \mathcal{A}_p |t|_p^{\alpha-1}$. From this, we see that the regularity of the solution of the Eq (1.2) depends on the boundedness of $\mathcal{H}_{\varphi_1}^{p,1}$ and $\mathcal{H}_{\varphi_2}^{p,1}$.

One of the most significant operators for solving several issues in the theory of singular integral operators and partial differential equations is the Hardy–Littlewood maximal operator (see [17, 32]). Moreover, the study of weighted inequalities for Hardy–Littlewood maximal operator on function spaces is an interesting problem in harmonic analysis (see [8, 15]). On the p-adic fields, the Hardy-Littlewood maximal operator M for any locally integrable f is given by

$$M(f)(x) = \sup_{\gamma \in \mathbb{Z}} \frac{1}{p^{n\gamma}} \int_{B_\gamma(x)} |f(u)| du, \; x \in \mathbb{Q}_p^n. \tag{1.3}$$

In 2009, Kim [24] proved that the operator M is of weak type (1, 1) on $L^1(\mathbb{Q}_p^n)$. Using the p-adic version of the Marcinkiewicz interpolation theorem, the operator M is a bounded operator of $L^q(\mathbb{Q}_p^n)$ into $L^q(\mathbb{Q}_p^n)$ in Theorem 1.1 [24]. On the local fields (a generalization of the p-adic fields), Chuong and Hung [7] obtained that $\|M\|_{L^q(\omega)\to L^q(\omega)}$ is finite if and only if ω is a Muckenhoupt weight. From the Fefferman–Stein inequality on p-adic Lebesgue spaces, Volosivets [38] gave sufficient conditions for the boundedness of the operator M on the generalized Morrey space $L^{q,\omega}(\mathbb{Q}_p^n)$. On the other hand, we see that $|\mathcal{H}^p(f)(\cdot)| \lesssim M(f)(\cdot)$. Here, the p-adic Hardy operator $\mathcal{H}^p$ is defined by

$$\mathcal{H}^p(f)(x) = \frac{1}{|x|_p^n}\int_{|u|_p\le|x|_p} f(u)du, \quad x \in \mathbb{Q}_p^n \setminus \{0\}.$$

Under the above inequality, the Hardy operator can characterize a broader set of function spaces compared to the Hardy–Littlewood maximal operator.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$. Then, f is said to be in $BMO(\mathbb{R}^n)$ if the seminorm given by

$$\|f\|_{\rm BMO(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f(u) - f_B|du < \infty,$$

where the supremum is taken over balls $B \subset \mathbb{R}^n$ and $f_B = \frac{1}{|B|}\int_B f(u)du$.

The maximal commutator operator was proposed by Garcia-Cuerva et al. [16]. It is defined as follows:

$$C_b(f)(x) := \sup_{r>0} \frac{1}{|B_r|}\int_{B_r(x)} |b(x) - b(u)||f(u)|du, \quad x \in \mathbb{R}^n,$$

where $b \in \mathcal{M}(\mathbb{R}^n, \mathbb{C})$. In studying commutators of singular integral operators with BMO symbols, the operator C_b plays an important role (see [16, 25, 31]). The authors [16] stated the following theorem.

Theorem 1. *Let $p \in (1, \infty)$. Then, the operator C_b is bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

For the natural extension, the p-adic maximal commutator operator is defined by

$$C_{p,b}(f)(x) := \sup_{\xi\in\mathbb{Z}} \frac{1}{p^{\xi n}}\int_{B_\xi(x)} |b(x) - b(u)||f(u)|du, \quad x \in \mathbb{Q}_p^n.$$

From Theorem 1, the study of the boundedness of the p-adic maximal commutator operators $C_{p,b}$ on general p-adic function spaces of Lebesgue spaces needs to be posed.

In 1995, Lu and Yang [27] introduced the two weighted Herz spaces $\dot{K}_q^{\alpha,\ell}(\omega_1, \omega_2)$ and $K_q^{\alpha,\ell}(\omega_1, \omega_2)$. In the case $\omega_1 \equiv, \omega_2 \equiv 1$, $\alpha = 0$, and $\ell = q$, it is obvious that $\dot{K}_q^{\alpha,\ell}(\omega_1, \omega_2) \equiv L^q(\mathbb{R}^n)$. On the local fields, the authors [28] presented the block decomposition of Herz spaces and obtained the boundedness of the sublinear operators, generated by the operator M. As the applications of the block decomposition theory, Dung et al. [11] established the boundedness of the operator M and the operator $\mathcal{H}_{\psi,A}$ on two weighted p-adic Herz spaces. In particular, the authors gave necessary and sufficient conditions for the continuity of the operator $\mathcal{H}^p_{\Psi,A}$ on two weighted Herz spaces $\dot{K}^{\alpha,\ell}_{q,\omega_1,\omega_2}(\mathbb{Q}_p^n)$ with power weights.

Additionally, the theory of function spaces with variable exponents has certain crucial applications in electronic fluid mechanics, elasticity, recovery of graphics, partial differential equations, and harmonic analysis (see [1, 4, 5, 12–14, 29, 33, 34]). By extending the results in [1] and [27], Wang [39] established the block decomposition for the Herz spaces with two variable exponents. The authors of [43] just recently introduced the local block space with variable exponent $\mathfrak{LB}_{u,p(\cdot)}(\mathbb{R}^n)$. As a natural development, we research the block decomposition for the p-adic Herz spaces with two variable exponents and the p-adic local block spaces with variable exponent.

Motivated by the above, we obtain some results as follows:

1) We establish some inequalities for the boundedness of the operator M on the p-adic nonhomogeneous Herz space $K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ and the p-adic local block space $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$.
2) The boundedness of the operator $C_{p,b}$ with a symbol belonging to $BMO^r_*(\mathbb{Q}_p^n)$ space on the p-adic nonhomogeneous Herz space $K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ is discussed. Moreover,

$$\|C_{p,b}\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)\to K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim \|b\|_{BMO^r_*(\mathbb{Q}_p^n)}.$$

3) We show some sufficient conditions for the boundedness of the operator $\mathcal{H}_{\psi,A}$ on the p-adic homogeneous Herz space $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ and the p-adic local block space $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$.
4) As consequence, we obtain the boundedness of the operators $\mathcal{H}^p$, $\mathcal{H}^{p,n}_{\varphi}$, and $\mathcal{T}^p$ on the spaces $K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$, $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$, and $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$.

The following is the structure of our paper. Section 2 is preliminaries. Our main results are given and proved in Section 3. Finally, a conclusion is stated in Section 4.

2. Some notations and definitions

On the field of rational numbers $\mathbb{Q}$ with a prime number p, we define

$$|u|_p = \begin{cases} 0, \text{ if } u = 0, \\ p^{-\gamma}, \text{ otherwise } u = p^{\gamma}\dfrac{m}{n} \text{ with } m, n \not| p, \gamma \in \mathbb{Z}. \end{cases}$$

The field $\mathbb{Q}_p$ arises as a result of the completion of the field $\mathbb{Q}$ with the norm $|\cdot|_p$. Then,
(i) $|u|_p \geq 0$, for all $u \in \mathbb{Q}_p$;
(ii) $|u|_p = 0 \Leftrightarrow u = 0$;
(iii) $|uv|_p = |u|_p|v|_p$, for all $u, v \in \mathbb{Q}_p$;
(iv) $|u+v|_p \leq \max(|u|_p, |v|_p)$, for all $u, v \in \mathbb{Q}_p$, and $|u+v|_p = \max(|u|_p, |v|_p)$ with $|u|_p \neq |v|_p$.

For $n \in \mathbb{N}^*$, the space $\mathbb{Q}_p^n$ is defined as $\{u = (u_1, ..., u_n) : u_i \in \mathbb{Q}_p, i = 1, ..., n\}$, and equipped with the norm defined by

$$|u|_p = \max_{1\leq i\leq n} |u_i|_p. \tag{2.1}$$

We set $\mathbb{Q}_p^{n*} = \mathbb{Q}_p^n \setminus \{0\}$, $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$, and $\mathbb{Z}_p^* = \{u \in \mathbb{Q}_p : 0 < |u|_p \leq 1\}$.

Let $B_k(a) = \left\{u \in \mathbb{Q}_p^n : |u - a|_p \leq p^k\right\}$, $S_k(a) = \left\{u \in \mathbb{Q}_p^n : |u - a|_p = p^k\right\}$, $B_k = B_k(0)$, and $S_k = S_k(0)$, for all $k \in \mathbb{Z}$. Moreover, let χ_k be the characteristic function of the sphere S_k.

Corollary 2. (see Corollaries 1 in [36]) *If $b \in B_k(a)$, then $B_k(a) = B_k(b)$.*

The normalization of the Haar measure on $\mathbb{Q}_p^n$ is given by

$$\int_{B_0} du = |B_0| = 1.$$

Let us denote $|U|$ as the Haar measure of a measurable subset $U \subset \mathbb{Q}_p^n$. For any $a \in \mathbb{Q}_p^n$ and $k \in \mathbb{Z}$, we have $|B_k(a)| = p^{nk}$ and $|S_k(a)| = p^{nk}(1 - p^{-n})$.

Denote by $\mathcal{M}(\mathbb{V}, \mathbb{F})$ the set of all measurable functions $f(\cdot) : \mathbb{V} \to \mathbb{F}$ and $\mathcal{D}(\mathbb{V}, \mathbb{F})$ the collection of all functions $g(\cdot) : \mathbb{V} \to \mathbb{F}$. Besides, the set $\mathbb{W}(\mathbb{Q}_p^n)$ comprises all nonnegative weighted functions defined on $\mathbb{Q}_p^n$. For any ω belonging to the set $\mathbb{W}(\mathbb{Q}_p^n)$ and any measurable set U, we define

$$\omega(U) := \int_U \omega(u)du.$$

For $r \in (0, \infty)$, the space $L^r(\mathbb{Q}_p^n) := \left\{ f \in \mathcal{M}(\mathbb{V}, \mathbb{C}) : \|f\|_{L^r(\mathbb{Q}_p^n)} < \infty \right\}$. Here,

$$\|f\|_{L^r(\mathbb{Q}_p^n)} = \Big(\int_{\mathbb{Q}_p^n} |f(u)|^r du \Big)^{1/r}.$$

Given that f belongs to the space $L^1(\mathbb{Q}_p^n)$, we have

$$\int_{\mathbb{Q}_p^n} f(u)du = \lim_{\alpha\to\infty} \int_{B_\alpha} f(u)du = \lim_{\alpha\to\infty} \sum_{-\infty<\gamma\leq\alpha} \int_{S_\gamma} f(u)du.$$

The space $L^r_{\mathrm{loc}}(\mathbb{V}) := \left\{ f \in \mathcal{M}(\mathbb{V}, \mathbb{C}) : \|f\chi_U\|_{L^r(\mathbb{Q}_p^n)} < \infty, U \subset \mathbb{V}, U \text{ compact} \right\}$.

The definitions above are referred to in [21, 36]. Next, we recall the p-adic Lebesgue with variable exponent (see [5, 6]).

We define the set $\mathcal{P}(\mathbb{Q}_p^n) := \{q \in \mathcal{M}(\mathbb{Q}_p^n, (1, \infty)) : q_-, q_+ \in (1, \infty)\}$. Simultaneously,

$$q_- = \operatorname{ess\,inf}_{u\in\mathbb{Q}_p^n} q(u) \text{ and } q_+ = \operatorname{ess\,sup}_{u\in\mathbb{Q}_p^n} q(u).$$

For $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $f \in \mathcal{M}(\mathbb{Q}_p^n, \mathbb{C})$, and $\gamma \in (0, \infty)$, we put

$$F_{q(\cdot)}(f/\gamma) = \int\limits_{\mathbb{Q}_p^n} \left(\frac{|f(u)|}{\gamma} \right)^{q(u)} du.$$

For $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, the space $L^{q(\cdot)}(\mathbb{Q}_p^n) := \left\{ f \in \mathcal{M}(\mathbb{Q}_p^n, \mathbb{C}) : F_{q(\cdot)}(f/\gamma) < \infty, \text{ for some } \gamma \in (0, \infty) \right\}$. A norm on $L^{q(\cdot)}(\mathbb{Q}_p^n)$ is given by

$$\|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf \left\{ \gamma \in (0, \infty) : F_{q(\cdot)}(f/\gamma) \leq 1 \right\}.$$

Lemma 3. *Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$ and $f \in L^{q(\cdot)}(\mathbb{Q}_p^n)$. Then, we have*

$$\begin{cases} \big\|f\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \max\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}, \text{ if } F_{q(\cdot)}(f) \leq C, \\ \big\|f\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \geq \min\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}, \text{ otherwise.} \end{cases} \tag{2.2}$$

Proof. Let us put $\tau = \max\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}$ and $\upsilon = \min\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}$. Then, we have $C \leq \tau^{q_+}$ and $C \leq \tau^{q_-}$. Consequently,

$$\max\left\{ \frac{1}{\tau^{q_+}}, \frac{1}{\tau^{q_-}} \right\} \leq \frac{1}{C}. \tag{2.3}$$

By similar arguments as above,

$$\min\left\{\frac{1}{\upsilon^{q_+}}, \frac{1}{\upsilon^{q_-}}\right\} \geq \frac{1}{C}. \tag{2.4}$$

In case $F_{q(\cdot)}(f) \leq C$, by (2.3), we have $\max\left\{\frac{1}{\tau^{q_+}}, \frac{1}{\tau^{q_-}}\right\}\cdot\int_{\mathbb{Q}_p^n} |f(u)|^{q(u)}du \leq 1$. This leads to

$$F_{q(\cdot)}\Big((f/\tau)/1\Big) = \int_{\mathbb{Q}_p^n} \left|\frac{f(u)}{\tau}\right|^{q(u)} du \leq 1.$$

Thus, by $\|f/\tau\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} = \inf\left\{\gamma \in (0,\infty) : F_{q(\cdot)}\Big((f/\tau)/\gamma\Big) \leq 1\right\}$, we see that $\|f/\tau\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq 1$. Then,

$$\big\|f\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \max\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}.$$

In case $F_{q(\cdot)}(f) \geq C$, by (2.4), we infer $\min\left\{\frac{1}{\upsilon^{q_+}}, \frac{1}{\upsilon^{q_-}}\right\}\cdot\int_{\mathbb{Q}_p^n} |f(u)|^{q(u)}du \geq 1$. Hence,

$$\int_{\mathbb{Q}_p^n} \left|\frac{f(u)}{\upsilon}\right|^{q(u)} du \geq 1.$$

Besides, for all $\gamma \in (0,\infty)$ and $F_{q(\cdot)}\Big((f/\upsilon)/\gamma\Big) \leq 1$, we have

$$\min\left\{\frac{1}{\gamma^{q_+}}, \frac{1}{\gamma^{q_-}}\right\}\cdot\int_{\mathbb{Q}_p^n} \left|\frac{f(u)}{\upsilon}\right|^{q(u)} du \leq 1.$$

Thus, $\min\left\{\frac{1}{\gamma^{q_+}}, \frac{1}{\gamma^{q_-}}\right\} \leq 1$. This gives $\gamma \in [1;\infty)$. Accordingly,

$$\begin{aligned}\|f/\upsilon\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &= \inf\left\{\gamma \in (0,\infty) : F_{q(\cdot)}\Big((f/\upsilon)/\gamma\Big) \leq 1\right\}\\ &= \inf\left\{\gamma \in [1,\infty) : F_{q(\cdot)}\Big((f/\upsilon)/\gamma\Big) \leq 1\right\}\\ &\geq 1.\end{aligned}$$

Therefore,

$$\big\|f\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \geq \min\{C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}}\}.$$

□

For $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, the function $q'(\cdot)$ is defined by

$$\frac{1}{q(\cdot)} + \frac{1}{q'(\cdot)} = 1.$$

The space $L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n) := \left\{f \in \mathcal{M}(\mathbb{Q}_p^n, \mathbb{C}) : \big\|f\big\|_{L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n)} < \infty\right\}$ with $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$ and $\omega \in \mathbb{W}(\mathbb{Q}_p^n)$. Here,

$$\big\|f\big\|_{L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n)} = \big\|f\omega\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

The space $L^{q(\cdot)}_{\omega,\text{loc}}(\mathbb{Q}_p^{n*}) := \Big\{f \in \mathcal{M}(\mathbb{Q}_p^{n*}, \mathbb{C}) : f\chi_U \in L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n), U \subset \mathbb{Q}_p^{n*}, U \text{ compact}\Big\}$.

The set $\mathbf{C}_0^{\log}(\mathbb{Q}_p^n) := \Big\{\alpha \in \mathcal{D}(\mathbb{Q}_p^n, \mathbb{R}) : |\alpha(u) - \alpha(0)| \lesssim \dfrac{1}{\log\left(e + |u|_p^{-1}\right)},\ \forall u \in \mathbb{Q}_p^n\Big\}$.

The set $\mathbf{C}_\infty^{\log}(\mathbb{Q}_p^n) := \Big\{\alpha \in \mathcal{D}(\mathbb{Q}_p^n, \mathbb{R}) : |\alpha(u) - \alpha_\infty| \lesssim \dfrac{1}{\log(e + |u|_p)}, \forall u \in \mathbb{Q}_p^n\Big\}$ with $\lim\limits_{|u|_p\to\infty} \alpha(u) = \alpha_\infty \in \mathbb{R}$.
The set $\mathbf{C}_{\text{loc}}^{\log}(\mathbb{Q}_p^n) := \Big\{\alpha \in \mathcal{D}(\mathbb{Q}_p^n, \mathbb{R}) : |\alpha(u) - \alpha(v)| \lesssim \frac{-1}{\log(|u-v|_p)}, \forall u, v : |u - v|_p \le \frac{1}{p}\Big\}$. Then, we denote $LH(\mathbb{Q}_p^n) := \mathbf{C}_{\text{loc}}^{\log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{\log}(\mathbb{Q}_p^n)$.

Let us give the p-adic weighted Herz spaces with variable exponents (see [20]).

Definition 1. *Let $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathcal{D}(\mathbb{Q}_p^n, \mathbb{R})$, $\ell \in (0, \infty)$, $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $\omega \in \mathbb{W}(\mathbb{Q}_p^n)$. The p-adic homogeneous Herz space $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ is defined by*

$$\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n) = \Big\{f \in L^{q(\cdot)}_{\omega,\text{loc}}(\mathbb{Q}_p^{n*}) : \|f\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} < \infty\Big\}$$

with $\|f\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} = \left(\sum\limits_{k=-\infty}^{\infty} \|p^{\alpha(\cdot)/n} f\chi_k\|^{\ell}_{L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n)}\right)^{1/\ell}$.

Definition 2. *Let $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathcal{D}(\mathbb{Q}_p^n, \mathbb{R})$, $\ell \in (0, \infty)$, $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $\omega \in \mathbb{W}(\mathbb{Q}_p^n)$. The p-adic nonhomogeneous Herz space $K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ is defined by*

$$K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n) = \Big\{f \in L^{q(\cdot)}_{\omega,\text{loc}}(\mathbb{Q}_p^{n*}) : \|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} < \infty\Big\}$$

with $\|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} = \left(\sum\limits_{k=1}^{\infty} \|p^{\alpha(\cdot)/n} f\chi_k\|^{p}_{L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n)} + \|p^{\alpha(\cdot)/n} f\chi_{B_0}\|^{p}_{L^{q(\cdot)}_{\omega}(\mathbb{Q}_p^n)}\right)^{1/\ell}$.

Definition 3. ([27, 39]) *Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\alpha(0), \alpha_\infty \in (0, \infty)$, $\omega \in \mathbb{W}(\mathbb{Q}_p^n)$, and $b(\cdot) \in \mathcal{M}(\mathbb{Q}_p^n, \mathbb{C})$. We say that $b(\cdot)$ is a central $(\alpha(\cdot), q(\cdot), \omega)$-block if there exists $k \in \mathbb{Z}$ such that*

(i) $\text{supp}(b) \subset B_k$,

(ii) $\|b\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \omega(B_k)^{-\alpha_k/n}$ *with* $\alpha_k = \begin{cases} \alpha(0), \text{ if } k < 0, \\ \alpha_\infty, \text{ otherwise.} \end{cases}$

Definition 4. ([27, 39]) *Let $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n), \alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\alpha_\infty \in (0, \infty)$, $\omega \in \mathbb{W}(\mathbb{Q}_p^n)$, and $b(\cdot) \in \mathcal{M}(\mathbb{Q}_p^n, \mathbb{C})$. We say that $b(\cdot)$ is a central $(\alpha(\cdot), q(\cdot), \omega)$-block of restricted type if there exists $k \in \mathbb{N}$ satisfying*

(i) $\text{supp}(b) \subset B_k$,
(ii) $\|b\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \omega(B_k)^{-\alpha_\infty/n}$.

From the results in [27] and [39], we develop the following theorems.

Theorem 4. *Let $\ell \in (0, 1]$, $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$, $\alpha_\infty \in (0, \infty)$, and $\omega(x) = |x|_p^\beta$ with $\beta \in (-n, \infty)$. We see that $f \in K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ if and only if*

$$f = \sum_{k=0}^{\infty} \lambda_k b_k,$$

where $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$, and each b_k is a central $(\alpha(\cdot), q(\cdot), \omega)$- block of restricted type with the support in B_k. Moreover,

$$\|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \approx inf \left\{ \sum_{k=0}^{\infty} |\lambda_k|^\ell \right\}^{1/\ell},$$

where the infimum is taken over all decomposition of f as above.

Theorem 5. *Let* $\ell \in (0, 1]$, $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}^{log}_\infty(\mathbb{Q}_p^n) \cap \mathbf{C}^{log}_0(\mathbb{Q}_p^n)$, *and* $\omega(x) = |x|_p^\beta$ *with* $\beta \in (-n, \infty)$. *We see that* $f \in \dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ *if and only if*

$$f = \sum_{k\in\mathbb{Z}} \lambda_k b_k,$$

where $\sum_{k\in\mathbb{Z}} |\lambda_k|^p < \infty$, and each b_k is a central $(\alpha(\cdot), q(\cdot), \omega)$- block with the support in B_k. Moreover,

$$\|f\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \approx inf \left\{ \sum_{k\in\mathbb{Z}} |\lambda_k|^\ell \right\}^{1/\ell},$$

where the infimum is taken over all decomposition of f as above.

Following [43], we present the definition of p-adic local block space with variable exponent $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$.

Definition 5. *Let* $q \in \mathcal{M}(\mathbb{Q}_p^n, (0, \infty))$, $u \in \mathcal{M}((0, \infty), (0, \infty))$, *and* $b \in \mathcal{M}(\mathbb{Q}_p^n, \mathbb{C})$. *We say that* $b(\cdot)$ *is a local* $(u, L^{q(\cdot)})$*-block if it is supported in* $B_k, k \in \mathbb{N}$, *and*

$$\|b\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \frac{1}{u(p^k)}.$$

The space $LB_{u,q(\cdot)}(\mathbb{Q}_p^n) := \left\{ \sum_{k=0}^{\infty} \lambda_k b_k : \sum_{k=0}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is a local } (u, L^{q(\cdot)})\text{-block} \right\}$. A norm of this space is given by

$$\|f\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} = \inf \left\{ \sum_{k=0}^{\infty} |\lambda_k| : f = \sum_{k=0}^{\infty} \lambda_k b_k \text{ a.e.} \right\}.$$

Definition 6. *Let* $q \in \mathcal{M}(\mathbb{Q}_p^n, (0, \infty))$ *and* $u \in \mathcal{M}((0, \infty), (0, \infty))$. *If there exists a constant* $C > 0$ *such that*

$$C \leq u(p^k), \text{ for all } k \in \mathbb{N}, \tag{2.5}$$

$$\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq Cu(p^k), \text{ for all } k \in \mathbb{Z} \setminus \mathbb{N}, \tag{2.6}$$

$$\sum_{j=0}^{\infty} \frac{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_{k+j+1}}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} u(p^{k+j+1}) \le Cu(p^k), \text{ for all } k \in \mathbb{Z}. \tag{2.7}$$

Then, we say that $u \in LW_{q(\cdot)}(\mathbb{Q}_p^n)$.

By similar arguments as in the proof of [43, Theorem 4], we have the following result.

Theorem 6. *Let $q \in \mathcal{M}(\mathbb{Q}_p^n, (0,\infty))$ and $u \in LW_{q'(\cdot)}(\mathbb{Q}_p^n)$. Then, we have $LB_{u,q(\cdot)}(\mathbb{Q}_p^n) \subset L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$. Moreover, $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$ is a Banach space.*

The set $\mathfrak{MB}(\mathbb{Q}_p^n) := \left\{q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n) : M \text{ is bounded on } L^{q(\cdot)}(\mathbb{Q}_p^n)\right\}$. By using Lemmas 1 and 2 in paper [19], we obtain an important lemma below.

Lemma 7. *Assume that $q(\cdot) \in \mathfrak{MB}(\mathbb{Q}_p^n)$.*

(i) If B is a ball in $\mathbb{Q}_p^n$, then we have

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \lesssim \frac{|B|}{|S|}, \text{ for all measurable subsets } S \subset B.$$

(ii) For all $a \in \mathbb{Q}_p^n$ and $k \in \mathbb{Z}$, we have

$$\|\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \|\chi_{B_k(a)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} \simeq p^{kn}.$$

Proof. Let us take a ball $B = B_{k_0}(x_0)$ and a measurable subset $S \subset B$. For all $x \in B$,

$$M(\chi_S)(x) \ge \frac{1}{p^{k_0 n}} \int_{B_{k_0}(x_0)} |\chi_S(y)| dy = \frac{1}{|B|} \int_B |\chi_S(y)| dy = \frac{|S|}{|B|}.$$

Thus,

$$B \subset \{x \in \mathbb{Q}_p^n : M(\chi_S)(x) > \lambda/2\}, \text{ for all } \lambda \in (0, |S|/|B|).$$

By $q(\cdot) \in \mathfrak{MB}(\mathbb{Q}_p^n)$, we have that the operator M is of weak type $(q(\cdot), q(\cdot))$. Clearly, for all $f \in L^{q(\cdot)}(\mathbb{Q}_p^n)$ and $\lambda \in (0,\infty)$,

$$\lambda \|\chi_{\{x \in \mathbb{Q}_p^n : M(f)(x) > \lambda\}}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

Consequently, for all $\lambda \in (0, |S|/|B|)$,

$$\frac{\lambda}{2} \|\chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \le \frac{\lambda}{2} \|\chi_{\{x \in \mathbb{Q}_p^n : M(\chi_S)(x) > \lambda/2\}}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|\chi_S\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

Hence, for all $\lambda \in (0, |S|/|B|)$,

$$\frac{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_S\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \lesssim \lambda^{-1}.$$

We choose $\lambda = \frac{|S|}{2|B|}$. Then, the proof of the section (i) of Lemma 7 is finished.

Now, we prove the section (ii) of this lemma. First, by $q(\cdot) \in \mathfrak{MB}(\mathbb{Q}_p^n)$, we obtain the following inequality.

$$\||f|_B \chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|f \chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}, \text{ for all balls } B \text{ and } f \in L^{q(\cdot)}(\mathbb{Q}_p^n). \tag{2.8}$$

Here, $|f|_B = \dfrac{1}{|B|}\int_B |f(y)|dy$.

Indeed, we take a ball $B = B_{k_0}(x_0)$ arbitrarily. For all $x \in B$, by Corollary 2, we get $B = B_{k_0}(x)$. Hence, for all $x \in B$,

$$|f|_B = \frac{1}{|B|}\int_{B_{k_0}(x)} |f(y)\chi_B(y)|dy = \frac{1}{p^{k_0 n}}\int_{B_{k_0}(x)} |f(y)\chi_B(y)|dy \leq M(f\chi_B)(x).$$

This gives

$$|f|_B\chi_B(x) \leq M(f\chi_B)(x), \text{ for all } x \in \mathbb{Q}_p^n.$$

As a consequence, by $q(\cdot) \in \mathfrak{MB}(\mathbb{Q}_p^n)$,

$$\||f|_B\chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \|M(f\chi_B)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|f\chi_B\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.$$

Therefore, the inequality (2.8) is valid. Next, for all $B_k(a)$, by the inequality (2.8),

$$\begin{aligned}
&\frac{1}{|B_k(a)|}\|\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k(a)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\
&\leq \frac{1}{|B_k(a)|}\|\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.\sup\Big\{\int_{\mathbb{Q}_p^n} |f(x)\chi_{B_k(a)}|dx : \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq 1\Big\}\\
&= \|\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}.\sup\Big\{|f|_{B_k(a)} : \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq 1\Big\}\\
&= \sup\Big\{\||f|_{B_k(a)}\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} : \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq 1\Big\}\\
&\lesssim \sup\Big\{\|f\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} : \|f\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq 1\Big\}\\
&\lesssim 1.
\end{aligned}$$

From this, we estimate

$$\|\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k(a)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)} \lesssim p^{kn}. \tag{2.9}$$

Besides, by using the Hölder inequality,

$$p^{kn} = \|\chi_{B_k(a)}\|_{L^1(\mathbb{Q}_p^n)} \lesssim \|\chi_{B_k(a)}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k(a)}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}. \tag{2.10}$$

Certainly, by (2.9) and (2.10), we finish the proof of the section (ii) of this lemma. □

From the inequality (18) in the paper [20, Theorem 5], we finish the proof of the following lemma.

Lemma 8. *If $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n) \cap LH(\mathbb{Q}_p^n)$, then,*

$$\|\chi_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim p^{kn/q_\infty}, \textit{ for any } k \in \mathbb{N}.$$

Proof. For any $k \in \mathbb{N}$ and $x \in S_k$, by $q(\cdot) \in LH(\mathbb{Q}_p^n)$,

$$k|q(x) - q_\infty| = \log_p|x|_p.|q(x) - q_\infty| \lesssim \frac{\log_p|x|_p}{\log(e + |x|_p)} \lesssim 1.$$

Hence,

$$k(q_\infty - q(x)) \le C,$$

where the positive constant C is independent of $k \in \mathbb{N}$ and $x \in S_k$. In addition, by using the hypothesis $q(\cdot) \in \mathcal{P}(\mathbb{Q}_p^n) \cap LH(\mathbb{Q}_p^n)$, it is clearly understood that $q_\infty \in (1, \infty)$. Accordingly,

$$p^{-Cn/q_\infty} p^{kn} \le p^{knq(x)/q_\infty}, \text{ for any } k \in \mathbb{N} \text{ and } x \in S_k. \tag{2.11}$$

For any $k \in \mathbb{N}$, by (2.11), we estimate

$$\begin{aligned} F_q(\chi_k / p^{kn/q_\infty}) &= \int_{\mathbb{Q}_p^n} \left(\frac{\chi_k(x)}{p^{kn/q_\infty}} \right)^{q(x)} dx = \int_{\mathbb{Q}_p^n} \frac{\chi_k(x)}{p^{knq(x)/q_\infty}} dx = \int_{S_k} \frac{1}{p^{knq(x)/q_\infty}} dx \\ &= \frac{1}{p^{-Cn/q_\infty} p^{kn}} \int_{S_k} 1 dx \lesssim 1. \end{aligned}$$

Based on the above inequality and Lemma 3, one has

$$\|\chi_k / p^{kn/q_\infty}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim 1, \text{ for any } k \in \mathbb{N}.$$

Hence, the proof of Lemma 8 is completed. □

Definition 7. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$, and $r \in (1, \infty)$, and set*

$$\|f\|_{BMO_*^r(\mathbb{Q}_p^n)} := \sup_{B \subset Q} \Big(\frac{1}{|Q|} \int_Q |f(u) - f_B|^r du \Big)^{1/r},$$

where the supremum is taken over all balls Q and B with $B \subset Q \subset \mathbb{Q}_p^n$. The space $BMO_*^r(\mathbb{Q}_p^n)$ is defined by

$$BMO_*^r(\mathbb{Q}_p^n) := \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{BMO_*^r(\mathbb{Q}_p^n)} < \infty \Big\}.$$

3. The main results

In 2023, the authors of [11] studied the boundedness of Hardy–Littlewood maximal functions on p-adic Herz spaces through the block decomposition. Continuing to use the block decomposition technique for p-adic Herz spaces with variable exponents, we obtain the following theorem. Moreover, we hope that the following theorem will provide readers with ideas for the proof in the p-adic field without using the knowledge that weighted Herz spaces are interpolating spaces between Lebesgue weight spaces.

Theorem 9. *Let $\ell \in (0, 1]$, $q(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n) \cap LH(\mathbb{Q}_p^n)$, $q_\infty = q_+$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n)$ such that $\alpha_\infty \in (0, n - n/q_+)$. Assume that $\omega(x) = |x|_p^\beta$ with $\beta \in (-n, \infty)$. Then, we have that M is bounded on $K_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)$. Moreover,*

$$\|M\|_{K_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n) \to K_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)} \lesssim C_{1,\ell}.$$

Here, $C_{1,\ell} := \Big(\sum\limits_{\zeta=0}^{\infty} |K_\zeta|^\ell \Big)^{1/\ell}$ with $K_\zeta = \begin{cases} 1, \text{ if } \zeta = 0, \\ p^{\zeta(n/q_+ + \alpha_\infty)}(p^\zeta - 1)^{-n}, \text{ if } \zeta \in \mathbb{Z}^+. \end{cases}$

Proof. For any $f \in K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$, by Theorem 4,

$$f = \sum_{k=0}^{\infty} \lambda_k . b_k,$$

with $\Big(\sum_{k=0}^{\infty} |\lambda_k|^\ell\Big)^{1/\ell} \lesssim \|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)}$. Here, for each $k \in \mathbb{N}$, b_k is a central $(\alpha(\cdot), q(\cdot), \omega)$-block of restricted type where

$$\mathrm{supp}(b_k) \subset B_k \text{ and } \|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \omega(B_k)^{-\alpha_\infty/n}.$$

For any $k \in \mathbb{N}$, $M(b_k)$ is composed as follows:

$$M(b_k) = \chi_{B_k}.M(b_k) + \sum_{\zeta=1}^{\infty} \chi_{k+\zeta}.M(b_k) := \sum_{\zeta=0}^{\infty} N_{b_k,\zeta}. \tag{3.1}$$

Then,

$$\mathrm{supp}(N_{b_k,\zeta}) \subset B_{k+\zeta}. \tag{3.2}$$

By $q(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n)$, one has

$$\begin{aligned} \|N_{b_k,0}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\le \|M(b_k)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \\ &\lesssim \omega(B_k)^{-\alpha_\infty/n} := K_0 \omega(B_k)^{-\alpha_\infty/n}. \end{aligned} \tag{3.3}$$

For any $\zeta \in \mathbb{Z}^+$, by considering $x \in S_{k+\zeta}$, and $r \in \mathbb{Z}$ with $p^k(p^\zeta - 1) > p^r$,

$$B_r(x) \cap B_k = \emptyset.$$

Thus, by $\mathrm{supp}(b_k) \subset B_k$,

$$\begin{aligned} N_{b_k,\zeta}(x) = \chi_{k+\zeta}(x) M(b_k)(x) &\simeq \chi_{k+\zeta}(x) \sup_{r\in\mathbb{Z}} \frac{1}{p^{rn}} \int_{B_r(x)\cap B_k} |b_k(t)| dt \\ &= \chi_{k+\zeta}(x) \sup_{r\in\mathbb{Z}:\, p^k(p^\zeta-1)\le p^r} \frac{1}{p^{rn}} \int_{B_r(x)\cap B_k} |b_k(t)| dt \\ &\le \chi_{k+\zeta}(x) \frac{1}{p^{kn}(p^\zeta - 1)^n} \int_{B_k} |b_k(t)| dt. \end{aligned} \tag{3.4}$$

Consequently, by the Hölder inequality, Lemma 7 and $\|b_k\|_{L^{q(\cdot)}(B_k)} \lesssim \omega(B_k)^{-\alpha_\infty/n}$,

$$\begin{aligned} \|N_{b_k,\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|\chi_{k+\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \frac{1}{p^{kn}(p^\zeta - 1)^n} \|b_k\|_{L^{q(\cdot)}(B_k)} \|1\|_{L^{q'(\cdot)}(B_k)} \\ &\lesssim \frac{\|\chi_{k+\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{p^{kn}(p^\zeta - 1)^n} \omega(B_k)^{-\alpha_\infty/n} \frac{p^{kn}}{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \\ &\lesssim \frac{1}{(p^\zeta - 1)^n} \frac{\|\chi_{k+\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \Big(\frac{\omega(B_{k+\zeta})}{\omega(B_k)}\Big)^{\alpha_\infty/n} \omega(B_{k+\zeta})^{-\alpha_\infty/n}. \end{aligned}$$

Note that, by Lemma 3, Lemma 8 and $q_\infty = q_+$,

$$\frac{\|\chi_{k+\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \lesssim \frac{p^{(k+\zeta)n/q_\infty}}{p^{kn/q_+}} = p^{\zeta n/q_+}.$$

On the other hand, by $\omega(x) = |x|_p^\beta$,

$$\Big(\frac{\omega(B_{k+\zeta})}{\omega(B_k)}\Big)^{\alpha_\infty/n} = p^{\zeta\alpha_\infty}.$$

From these, for any $k \in \mathbb{N}$ and $\zeta \in \mathbb{Z}^+$,

$$\|N_{b_k,\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \frac{p^{\zeta(n/q_+ + \alpha_\infty)}}{(p^\zeta - 1)^n}\omega(B_{k+\zeta})^{-\alpha_\infty/n} := K_\zeta \omega(B_{k+\zeta})^{-\alpha_\infty/n}. \tag{3.5}$$

By defining $\overline{N}_{b_k,\zeta} = N_{b_k,\zeta}/K_\zeta$, for any $\zeta \in \mathbb{N}$. Combining this with (3.1),

$$M(b_k) = \sum_{\zeta=0}^{\infty} K_\zeta.\overline{N}_{b_k,\zeta}.$$

Moreover, by (3.2)–(3.5),

$$\|\overline{N}_{b_k,\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \omega(B_{k+\zeta})^{-\alpha_\infty/n} \quad \text{and} \quad \operatorname{supp}(\overline{N}_{b_k,\zeta}) \subset B_{k+\zeta}.$$

Namely, for any $\zeta \in \mathbb{N}, \overline{N}_{b_k,\zeta}$ is a central $(\alpha(\cdot), q(\cdot), \omega)$-block of restricted type. Hence, in view of Theorem 4,

$$\|M(b_k)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim C_{1,\ell}. \tag{3.6}$$

Case $\ell = 1$. By the condition $\alpha_\infty + n/q_+ - n < 0$,

$$\lim_{\zeta\to\infty} \frac{p^{(\zeta+1)(n/q_+ + \alpha_\infty)}(p^\zeta - 1)^n}{p^{\zeta(n/q_+ + \alpha_\infty)}(p^{\zeta+1} - 1)^n} = p^{\alpha_\infty + n/q_+ - n} < 1.$$

Thus, by the D'Alembert criterion for convergence of series,

$$C_{1,1} = \sum_{\zeta=0}^{\infty} |K_\zeta| < \infty.$$

Case $\ell \in (0,1)$. By letting ν such that $\nu > (1-\ell)/\ell$, using the Hölder inequality and the D'Alembert criterion for convergence of series,

$$C_{1,\ell} = \Big(\sum_{\zeta=0}^{\infty} |K_\zeta|^\ell\Big)^{1/\ell} \lesssim \sum_{\zeta=1}^{\infty} \zeta^\nu.K_\zeta + 1 < \infty.$$

Consequently, by the inequality (3.6),

$$\|M(b_k)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim C_{1,\ell} < \infty, \text{ for all } k \in \mathbb{N}.$$

This leads to

$$\begin{aligned}\|M(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} &\leq \Big\| \sum_{k=0}^{\infty} |\lambda_k| M(b_k) \Big\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \\ &\leq \Big(\sum_{k=0}^{\infty} |\lambda_k|^{\ell} \|M(b_k)\|^{\ell}_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \Big)^{1/\ell} \\ &\lesssim C_{1,\ell} \|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)},\end{aligned}$$

which ends the proof of this theorem. □

According to the ideas from the proof of [43, Theorem 5], the following theorem gives some sufficient conditions for the boundedness of Hardy–Littlewood maximal functions on p-adic local block spaces with variable exponent. The following theorem and Theorem 9 provide some evaluations for the regularity of the solution of some p-adic equations.

Theorem 10. *Let $q \in \mathcal{M}(\mathbb{Q}_p^n, (1,\infty))$ and $u \in \mathcal{M}((0,\infty),(0,\infty))$. If $q \in \mathfrak{B}(\mathbb{Q}_p^n)$ and $u \in LW_{q'(\cdot)}(\mathbb{Q}_p^n)$, then M is bounded on $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$. Moreover,*

$$\|M\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n) \to LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim C_{2,u}.$$

Here, $C_{2,u} := \sum\limits_{\zeta=0}^{\infty} K^*_{\zeta}$ with $K^*_0 = 1$, and $K^*_{\zeta} = \dfrac{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_{k+\zeta}}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}} . \dfrac{u(p^{k+\zeta})}{u(p^k)}$, for all $\zeta \in \mathbb{Z}^+$.

Proof. Let us give $f \in LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$, by the definition of the local block space with variable exponent,

$$f = \sum_{k=0}^{\infty} \lambda_k . b_k,$$

with $\sum\limits_{k=0}^{\infty} |\lambda_k| \lesssim \|f\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}$ and for each $k \in \mathbb{N}$, b_k is a local $(u, L^{q(\cdot)})$-block such that

$$\operatorname{supp}(b_k) \subset B_k \text{ and } \|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \frac{1}{u(p^k)}.$$

By composing as (3.1),

$$M(b_k)(\cdot) = \sum_{\zeta=0}^{\infty} N_{b_k,\zeta}(\cdot).$$

Here, $N_{b_k,0} = \chi_{B_k} M(b_k)$, and $N_{b_k,\zeta} = \chi_{k+\zeta} M(b_k)$, for all $\zeta \in \mathbb{Z}^+$.
As $q(\cdot) \in \mathfrak{B}(\mathbb{Q}_p^n)$,

$$\|N_{b_k,0}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \frac{1}{u(p^k)} := K^*_0 \frac{1}{u(p^k)}. \tag{3.7}$$

For any $\zeta \in \mathbb{Z}^+$, by the inequality (3.4), the Hölder inequality and Lemma 7 (ii),

$$\begin{aligned}\|N_{b_k,\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \|\chi_{k+\zeta}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\frac{1}{p^{kn}(p^{\zeta}-1)^n}\|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \|\chi_{B_{k+\zeta}}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\frac{1}{p^{kn}}\|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \frac{\|\chi_{B_k}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_{k+\zeta}}\|_{L^{q'(\cdot)}(\mathbb{Q}_p^n)}}\cdot\frac{u(p^{k+\zeta})}{u(p^k)}\cdot\frac{1}{u(p^{k+\zeta})}\\ &:= K_{\zeta}^{*}\frac{1}{u(p^{k+\zeta})}.\end{aligned} \tag{3.8}$$

Note that, $u \in LW_{q'(\cdot)}(\mathbb{Q}_p^n)$, it is clear to see that $C_{2,u} < \infty$. Next, let us set as follows:

$$N_{b_k,\zeta}^{*} = \begin{cases}\frac{N_{b_k,\zeta}}{K_{\zeta}^{*}}, \text{ if } K_{\zeta}^{*} \neq 0,\\ 0, \qquad \text{otherwise}.\end{cases}$$

This gives

$$M(b_k)(\cdot) = \sum_{\zeta=0}^{\infty} N_{b_k,\zeta}(\cdot) = \sum_{\zeta=0}^{\infty} K_{\zeta}^{*}N_{b_k,\zeta}^{*}(\cdot).$$

On the other hand, for any $\zeta \in \mathbb{N}$, by (3.7), (3.8) and $\operatorname{supp}(N_{b_k,\zeta}^{*}) \subset B_{k+\zeta}$, we have that $N_{b_k,\zeta}^{*}$ is a local $(u, L^{q(\cdot)})$-block. Thus,

$$\|M(b_k)\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim C_{2,u}, \text{ for all } k \in \mathbb{N}.$$

This leads to

$$\begin{aligned}\|M(f)\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} &\leq \Big\|\sum_{k=0}^{\infty}|\lambda_k|M(b_k)\Big\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}\\ &\leq \sum_{k=0}^{\infty}|\lambda_k|\Big\|M(b_k)\Big\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim C_{2,u}\|f\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

Hence, the proof of this theorem is concluded. □

As a consequence, we immediately have the following two results.

Corollary 11. *If the assumptions of Theorem 9 hold, then $\mathcal{H}^p$ is bounded on $K_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)$. Moreover,*

$$\|\mathcal{H}^p\|_{K_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)\to K_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)} \lesssim C_{1,\ell},$$

Here, $C_{1,\ell}$ is defined in Theorem 9.

Corollary 12. *Let the assumptions of Theorem 10 hold. Then, we have that $\mathcal{H}^p$ is bounded on $LB_{u,q(\cdot)(\mathbb{Q}_p^n)}$. Moreover,*

$$\|\mathcal{H}^p\|_{LB_{u,q(\cdot)(\mathbb{Q}_p^n)}\to LB_{u,q(\cdot)(\mathbb{Q}_p^n)}} \lesssim C_{2,u}.$$

Here, $C_{2,u}$ is defined in Theorem 10.

Theorem 13. *Let $\delta \in (0,1)$ and $r \in (\max(\frac{\sigma}{1-\sigma}, 1), \infty)$. If $b \in BMO_*^r(\mathbb{Q}_p^n)$, then we have*

$$M_\delta(C_{p,b}(f))(\cdot) \le \|b\|_{BMO_*^r(\mathbb{Q}_p^n)}.\Big(M_{r'}(f)(\cdot) + M^2(f)(\cdot)\Big), \textit{for all } f \in L^1_{loc}(\mathbb{R}^n).$$

Here, $M_\delta g(x) := \Big[M(|g|^\delta)(x)\Big]^{1/\delta}$.

Proof. Let $x \in \mathbb{Q}_p^n$, and fix a ball $B_k(x_0)$ with $k \in \mathbb{Z}$ and $x \in B_k(x_0)$. We set up $f = f_1 + f_2$, where $f_1 = f\chi_{B_k(x_0)}$ and $f_2 = f\chi_{B_k^c(x_0)}$. Thus, for any $y \in \mathbb{Q}_p^n$,

$$C_{p,b}(f)(y) \le M((b - b_{B_k(x_0)})f_1)(y) + M((b - b_{B_k(x_0)})f_2)(y) + |b(y) - b_{B_k(x_0)}|Mf(y).$$

This leads to

$$\begin{aligned}\Big(\frac{1}{p^{kn}}\int\limits_{B_k(x_0)} (C_{p,b}(f)(y))^\delta dy\Big)^{1/\delta} &\lesssim \Big(\frac{1}{p^{kn}}\int\limits_{B_k(x_0)} |M((b - b_{B_k(x_0)})f_1)(y)|^\delta dy\Big)^{1/\delta}\\ &+ \Big(\frac{1}{p^{kn}}\int\limits_{B_k(x_0)} |M((b - b_{B_k(x_0)})f_2)(y)|^\delta dy\Big)^{1/\delta}\\ &+ \Big(\frac{1}{p^{kn}}\int\limits_{B_k(x_0)} |b(y) - b_{B_k(x_0)}|^\delta (Mf(y))^\delta dy\Big)^{1/\delta}\\ &:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{3.9}$$

By Theorem 1.1 in the paper [24], we have that M is a bounded operator from $L^1(\mathbb{Q}_p^n)$ to $L^{1,\infty}(\mathbb{Q}_p^n)$. This implies that

$$\begin{aligned}\int\limits_{B_k(x_0)} |M((b - b_{B_k(x_0)})f_1)(y)|^\delta dy &= \int\limits_0^{p^{kn}} |M((b - b_{B_k(x_0)})f_1)^*(t)|^\delta dt\\ &\le \Big(\sup_{0<t\le p^{kn}} t(M_1((b - b_{B_k(x_0)})f_1))^*(t)\Big)^\delta \Big(\int\limits_0^{p^{kn}} t^{-\delta} dt\Big)\\ &\lesssim p^{kn(-\delta+1)}\|(b - b_{B_k(x_0)})f\|^\delta_{L^1(B_k(x_0))}.\end{aligned}$$

Besides, by applying Corollary 2, we obtain $B_k(x_0) = B_k(x)$. Combining this with the Hölder inequality,

$$\mathcal{I}_1 \lesssim \frac{1}{p^{kn}}\int\limits_{B_k(x_0)} |b(y) - b_{B_k(x_0)}||f(y)|dy$$

$$\leq \frac{1}{p^{kn}}\Big(\int\limits_{B_k(x_0)}|b(y)-b_{B_k(x_0)}|^r dy\Big)^{1/r}.\Big(\int\limits_{B_k(x_0)}|f(y)|^{r'}dy\Big)^{1/r'}$$
$$\leq \|b\|_{BMO^r_*(\mathbb{Q}_p^n)}.M_{r'}(f)(x). \tag{3.10}$$

Next, for any $y\in B_k(x_0)$, by using Corollary 2 again, we also have

$$B_k(x_0)=B_k(x)=B_k(y).$$

Consequently, for any $y\in B_k(x_0)$,

$$B_{\tilde a}(y)\subset B_{\tilde a}(x) \text{ with } \tilde a\geq k \text{ and } \tilde a\in\mathbb{Z}. \tag{3.11}$$

Indeed, let $z\in B_{\tilde a}(y)$,

$$|z-x|_p\leq \max\left\{|z-y|_p,|z-x|_p\right\}\leq \max\left\{p^{\tilde a},p^k\right\}\leq p^{\tilde a}.$$

This leads to $z\in B_{\tilde a}(x)$. Hence, the relation (3.11) is right. Thus,

$$\begin{aligned}\mathcal{I}_2&=\Big\{\frac{1}{p^{kn}}\int\limits_{B_k(x_0)}\Big(\sup_{\tilde a\in\mathbb{Z}}\frac{1}{p^{\tilde a n}}\int\limits_{B_{\tilde a}(y)\cap B_k^c(x_0)}|b(z)-b_{B_k(x_0)}||f(z)|dz\Big)^{\delta}dy\Big\}^{1/\delta}\\&\leq\Big\{\frac{1}{p^{kn}}\int\limits_{B_k(x_0)}\Big(\sup_{\tilde a\in\mathbb{Z},\tilde a\geq k+1}\frac{1}{p^{\tilde a n}}\int\limits_{B_{\tilde a}(y)}|b(z)-b_{B_k(x_0)}||f(z)|dz\Big)^{\delta}dy\Big\}^{1/\delta}\\&\leq\Big\{\frac{1}{p^{kn}}\int\limits_{B_k(x_0)}\Big(\sup_{\tilde a\in\mathbb{Z},\tilde a\geq k+1}\frac{1}{p^{\tilde a n}}\int\limits_{B_{\tilde a}(x)}|b(z)-b_{B_k(x)}||f(z)|dz\Big)^{\delta}dy\Big\}^{1/\delta}\\&\leq\sup_{\tilde a\in\mathbb{Z},\tilde a\geq k+1}\frac{1}{p^{\tilde a n}}\int\limits_{B_{\tilde a}(x)}|b(z)-b_{B_k(x)}||f(z)|dz.\end{aligned}$$

Moreover, for any $\tilde a\in\mathbb{Z}$ with $\tilde a\geq k+1$, by using the Hölder inequality as in (3.10),

$$\frac{1}{p^{\tilde a n}}\int\limits_{B_{\tilde a}(x)}|b(z)-b_{B_k(x)}||f(z)|dz\leq\|b\|_{BMO^r_*(\mathbb{Q}_p^n)}.M_{r'}(f)(x).$$

This implies that

$$\mathcal{I}_2\leq\|b\|_{BMO^r_*(\mathbb{Q}_p^n)}.M_{r'}(f)(x). \tag{3.12}$$

To estimate $\mathcal{I}_3$, we use the Hölder inequality with $\gamma=r/(r-\delta)\in(1,\infty)$, and $B_k(x_0)=B_k(x)$ as follows:

$$\mathcal{I}_3\leq\Big(\frac{1}{p^{kn}}\int\limits_{B_k(x_0)}|b(y)-b_{B_k(x_0)}|^{\delta\gamma'}dy\Big)^{1/(\delta\gamma')}.\Big(\frac{1}{p^{kn}}\int\limits_{B_k(x)}(Mf(y))^{\delta\gamma}dy\Big)^{1/(\delta\gamma)}$$
$$\leq\|b\|_{BMO^r_*(\mathbb{Q}_p^n)}.M_{\delta\gamma}(Mf)(x). \tag{3.13}$$

By $\delta\gamma\in(0,1)$, it is clear to see that $M_{\delta\gamma}(Mf)(x)\leq M^2f(x)$. From these,

$$\mathcal{I}_3\leq\|b\|_{BMO^r_*(\mathbb{Q}_p^n)}.M^2(f)(x).$$

Therefore, in view of (3.11) and (3.12), the proof of the theorem is finished. □

Since $C_{p,b}(f)(\cdot) \leq M_\delta(C_{p,b}(f))(\cdot)$, Theorem 13 guarantees the following result.

Corollary 14. *Let $r \in (1,\infty)$ and $b \in BMO^r_*(\mathbb{Q}^n_p)$. Then,*

$$C_{p,b}(f)(\cdot) \leq \|b\|_{BMO^r_*(\mathbb{Q}^n_p)}\Big(M_{r'}(f)(\cdot) + M^2(f)(\cdot)\Big), \text{ for all } f \in L^1_{loc}(\mathbb{Q}^n_p).$$

From [5, Proposition 2.18] and the definition of the p-adic Herz spaces with variable exponents, we immediately have the following property.

Lemma 15. *Let $\ell \in (1,\infty)$, $q(\cdot) \in \mathcal{P}(\mathbb{Q}^n_p)$, $\xi \in (1/q_-,\infty)$, and $\alpha(\cdot) \in L^\infty(\mathbb{Q}^n_p)$. Then, we have*

(i) $\||f|^\xi\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} = \|f\|^\xi_{\dot{K}^{\alpha(\cdot)/\xi,\xi\ell}_{\xi q(\cdot),\omega}(\mathbb{Q}^n_p)}$,

(ii) $\||f|^\xi\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} = \|f\|^\xi_{K^{\alpha(\cdot)/\xi,\xi\ell}_{\xi q(\cdot),\omega}(\mathbb{Q}^n_p)}$.

By using Corollary 14, Lemma 15, and Theorem 9, we achieve the following theorem. To prove the following theorem, we rely on the estimation of the p-adic maximal operator M. Furthermore, attentive readers see that we need to use the properties of the p-adic fields (such as Corollary 2) in the proof.

Theorem 16. *Let $\ell \in (0,1]$, $q(\cdot) \in \mathfrak{B}(\mathbb{Q}^n_p) \cap LH(\mathbb{Q}^n_p)$, $r' \in (1,q_-)$, $q(\cdot)/r' \in \mathfrak{B}(\mathbb{Q}^n_p)$, $q_\infty = q_+$, and $\alpha(\cdot) \in L^\infty(\mathbb{Q}^n_p) \cap \mathbf{C}^{log}_\infty(\mathbb{Q}^n_p)$ such that $\alpha_\infty \in (0, n/r' - n/q_+)$. Assume that $\omega(x) = |x|^\beta_p$ with $\beta \in (-n,\infty)$ and $b \in BMO^r_*(\mathbb{Q}^n_p)$. Then, $C_{p,b}$ is bounded on $K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)$. Moreover,*

$$\|C_{p,b}\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)\to K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} \lesssim \Big(C^2_{1,\ell} + C^{1/r'}_{1,\ell/r'}\Big)\|b\|_{BMO^r_*(\mathbb{Q}^n_p)}.$$

Here, $C_{1,\ell}$ is defined in Theorem 9.

Proof. By Corollary 14, we have

$$\|C_{p,b}(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} \leq \|b\|_{BMO^r_*(\mathbb{Q}^n_p)}\Big(\|M_{r'}(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} + \|M^2(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)}\Big). \tag{3.14}$$

Besides, by Theorem 9,

$$\|M^2(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} \lesssim C_{1,\ell}.\|M(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} \lesssim C^2_{1,\ell}.\|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)}. \tag{3.15}$$

In addition, by Lemma 15 and Theorem 9, we obtain

$$\begin{aligned}\|M_{r'}(f)\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)} &= \|M(|f|^{r'})\|^{1/r'}_{K^{\alpha(\cdot)r',\ell/r'}_{q(\cdot)/r',\omega}(\mathbb{Q}^n_p)}\\ &\lesssim C^{1/r'}_{1,\ell/r'}\||f|^{r'}\|^{1/r'}_{K^{\alpha(\cdot)r',\ell/r'}_{q(\cdot)/r',\omega}(\mathbb{Q}^n_p)}\\ &= C^{1/r'}_{1,\ell/r'}\|f\|_{K^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}^n_p)}.\end{aligned}$$

From the above estimation, by using the inequalities (3.14) and (3.15), the proof of this theorem is concluded. □

To present the next result, we set

$$\mathcal{K}_{A,n,q}(t) = \max\{\left|\det A^{-1}(t)\right|_p^{n/q_+}, \left|\det A^{-1}(t)\right|_p^{n/q_-}\},$$
$$\mathcal{P}_A(\mathbb{Q}_p^n) = \{u \in \mathcal{P}(\mathbb{Q}_p^n) : u(A^{-1}(t)\cdot) = u(\cdot), \text{ for a.e. } t \in supp(\psi)\}.$$

Very recently, the authors of [11] established the boundedness of the Hausdorff operators on p-adic Herz spaces. Expanding this work, we state the two following theorems about the boundedness of the Hausdorff operators on p-adic Herz spaces and p-adic local block spaces with variable exponent. As a consequence, the two following theorems contribute to the regularity of the solution of the Eq (1.2).

Theorem 17. *Let $q(\cdot) \in \mathcal{P}_A(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}_\infty^{log}(\mathbb{Q}_p^n) \cap \mathbf{C}_0^{log}(\mathbb{Q}_p^n)$, $\beta \in (-n, \infty)$, $\alpha(0) = \alpha_\infty \in (0, \infty)$, and $\omega(x) = |x|_p^\beta$. Assume that $\ell \in (0, 1)$, $\sigma > (1-\ell)/\ell$, and*

$$\mathcal{L}_\sigma = \int_{\mathbb{Q}_p^n} |\psi(t)|\mathcal{K}_{A,n,q}(t)\|A^{-1}(t)\|_p^{(\beta+n)\alpha_\infty/n} max\{|\log_p\|A^{-1}(t)\|_p|^\sigma, 1\}dt < \infty.$$

Then, $\mathcal{H}_{\psi,A}$ is bounded on $\dot{K}_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)$. Moreover,

$$\left\|\mathcal{H}_{\psi,A}\right\|_{\dot{K}_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n) \to \dot{K}_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)} \lesssim \mathcal{L}_{(1-\ell)/\ell}.$$

Here, $\mathcal{L}_{(1-\ell)/\ell} = \int_{\mathbb{Q}_p^n} |\psi(t)|\mathcal{K}_{A,n,q}(t)\|A^{-1}(t)\|_p^{(\beta+n)\alpha_\infty/n} max\{|\log_p\|A^{-1}(t)\|_p|^{(1-\ell)/\ell}, 1\}dt$.

Proof. For any $f \in \dot{K}_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)$, by Theorem 5,

$$f = \sum_{k=-\infty}^{\infty} \lambda_k b_k,$$

where $\Big(\sum_{k=-\infty}^{\infty} |\lambda_k|^\ell\Big)^{1/\ell} \lesssim \|f\|_{\dot{K}_{q(\cdot),\omega}^{\alpha(\cdot),\ell}(\mathbb{Q}_p^n)}$, and for $k \in \mathbb{Z}$, b_k is a central $(\alpha(\cdot), q(\cdot), \omega)$-block such that

$$\text{supp}(b_k) \subset B_k \text{ and } \|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \omega(B_k)^{-\alpha_\infty/n}.$$

Thus,

$$\left|\mathcal{H}_{\psi,A}(f)(x)\right| \leq \sum_{k=-\infty}^{\infty} |\lambda_k| \int_{\mathbb{Q}_p^n} |\psi(t)||b_k(A(t)x)|dt := \sum_{k=-\infty}^{\infty} |\lambda_k|\widetilde{\mathcal{H}}_{\psi,A}(b_k)(x).$$

Next, we compose

$$\begin{aligned}\widetilde{\mathcal{H}}_{\psi,A}(b_k)(x) &= \int_{\mathbb{Q}_p^n} |\psi(t)||b_k(A(t)x)|dt = \sum_{j=-\infty}^{\infty} \int_{\|A^{-1}(t)\|_p = p^j} |\psi(t)||b_k(A(t)x)|dt \\ &:= \sum_{j=-\infty}^{\infty} h_{\psi,A,b_k,j}(x). \end{aligned} \tag{3.16}$$

By $\mathrm{supp}(b_k) \subset B_k$, and $\|A^{-1}(y)\|_p = p^j$,

$$\mathrm{supp}(h_{\psi,A,b_k,j}) \subset B_{k+j}. \tag{3.17}$$

By the Minkowski inequality,

$$\|h_{\psi,A,b_k,j}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \int\limits_{\|A^{-1}(t)\|_p = p^j} |\psi(t)| \|b_k(A(t)\cdot)\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} dt.$$

For $\eta > 0$, we obtain

$$\begin{aligned}\int\limits_{\mathbb{Q}_p^n} \Big(\frac{|b_k(A(t)x)|}{\eta}\Big)^{q(x)} dx &= \int\limits_{\mathbb{Q}_p^n} \Big(\frac{|b_k(z)|}{\eta}\Big)^{q(A^{-1}(t)z)} \big|\det A^{-1}(t)\big|_p dz \\ &\le \int\limits_{\mathbb{Q}_p^n} \Big(\frac{\mathcal{K}_{A,n,q}(t)|b_k(z)|}{\eta}\Big)^{q(z)} dz.\end{aligned}$$

Thus, $\big\|b_k(A(t).)\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \mathcal{K}_{A,n,q}(t)\big\|b_k\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}$. This leads to

$$\|h_{\psi,A,b_k,j}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \int\limits_{\|A^{-1}(t)\|_p = p^j} |\psi(t)| \mathcal{K}_{A,n,q}(t) \big\|b_k\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} dt. \tag{3.18}$$

On the other hand, by using $\|b_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \omega(B_k)^{-\alpha_\infty/n}$,

$$\big\|b_k\big\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \Big(\frac{\omega(B_{k+j})}{\omega(B_k)}\Big)^{\alpha_\infty/n} \omega(B_{k+j})^{-\alpha_\infty/n} = p^{j(\beta+n)\alpha_\infty/n} \omega(B_{k+j})^{-\alpha_\infty/n}.$$

From these, we have

$$\begin{aligned}\|h_{\psi,A,b_k,j}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \Big(\int\limits_{\|A^{-1}(t)\|_p = p^j} |\psi(t)| \mathcal{K}_{A,n,q}(t) \|A^{-1}(t)\|_p^{(\beta+n)\alpha_\infty/n} dt\Big) \omega(B_{k+j})^{-\alpha_\infty/n} \\ &:= h_j \omega(B_{k+j})^{-\alpha_\infty/n}.\end{aligned} \tag{3.19}$$

Let us put $\tilde{h}_{\psi,A,b_k,j} = \begin{cases} \dfrac{h_{\psi,A,b_k,j}}{h_j}, & \text{if } h_j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$

Combining this with (3.16),

$$\widetilde{\mathcal{H}}_{\psi,A}(b_k) = \sum_{j=-\infty}^{\infty} h_j \tilde{h}_{\psi,A,b_k,j}.$$

Note that, by (3.17) and (3.19), $\tilde{h}_{\psi,A,b_k,j}$ is a central $(\alpha(\cdot), q(\cdot), \omega)$-block. Hence, by Theorem 5,

$$\|\widetilde{\mathcal{H}}_{\psi,A}(b_k)\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim \Big(\sum_{j=-\infty}^{\infty} |h_j|^\ell\Big)^{1/\ell}.$$

By using $\ell \in (0,1)$, $\sigma > (1-\ell)/\ell$, and the Hölder inequality,

$$\begin{aligned}\Big(\sum_{j=-\infty}^{\infty}|h_j|^{\ell}\Big)^{1/\ell} &\lesssim \sum_{j\in\mathbb{Z}\setminus\{0\}}|j|^{\sigma}h_j + h_0\\ &\lesssim \sum_{j\in\mathbb{Z}\setminus\{0\}}\int\limits_{\|A^{-1}(t)\|_p=p^j}|\psi(t)|\mathcal{K}_{A,n,q}(t)\|A^{-1}(t)\|_p^{(\beta+n)\alpha_\infty/n}|\log_p\|A^{-1}(t)\|_p|^{\sigma}dt+\\ &\quad+\int\limits_{\|A^{-1}(t)\|_p=1}|\psi(t)|\mathcal{K}_{A,n,q}(t)\|A^{-1}(t)\|_p^{(\beta+n)\alpha_\infty/n}dt\\ &\le \mathcal{L}_{\sigma}.\end{aligned}$$

According to the above estimation, for any $k \in \mathbb{Z}$,

$$\|\widetilde{\mathcal{H}}_{\psi,A}(b_k)\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim \mathcal{L}_{\sigma}.$$

This gives

$$\begin{aligned}\|\mathcal{H}_{\psi,A}(f)\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} &\le \Big\|\sum_{k=-\infty}^{\infty}|\lambda_k|\mathcal{H}_{\psi,A}(b_k)\Big\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)}\\ &\le \Big(\sum_{k=-\infty}^{\infty}|\lambda_k|^{\ell}\|\mathcal{H}_{\psi,A}(b_k)\|^{\ell}_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)}\Big)^{1/\ell}\\ &\lesssim \mathcal{L}_{\sigma}\|f\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)}.\end{aligned}$$

Combining this with the dominated convergence theorem of Lebesgue,

$$\Big\|\mathcal{H}_{\psi,A}\Big\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)\to\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim \mathcal{L}_{(1-\ell)/\ell}.$$

□

Theorem 18. *Let $q(\cdot) \in \mathcal{P}_A(\mathbb{Q}_p^n)$ and $u \in LW_{q(\cdot)}(\mathbb{Q}_p^n)$. If*

$$\mathcal{M} = \int\limits_{\mathbb{Q}_p^n}|\psi(t)|\mathcal{K}_{A,n,q}(t)\max\{1,\|A^{-1}(t)\|_p^n\}dt < \infty,$$

then, $\mathcal{H}_{\psi,A}$ is bounded on $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$. Moreover,

$$\Big\|\mathcal{H}_{\psi,A}\Big\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)\to LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \mathcal{M}.$$

Proof. Let $f \in LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$. From the definition of the local block space with variable exponent,

$$f = \sum_{k=0}^{\infty}\lambda_k c_k,$$

with $\sum_{k=0}^{\infty}|\lambda_k| \lesssim \|f\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}$, and for $k \in \mathbb{N}$, c_k is a local $(u, L^{q(\cdot)})$-block such that

$$\operatorname{supp}(c_k) \subset B_k \text{ and } \|c_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \frac{1}{u(p^k)}.$$

Based on the above arguments, one has

$$|\mathcal{H}_{\psi,A}(f)(x)| \leq \sum_{k=0}^{\infty}|\lambda_k|\Big(\sum_{j=0}^{\infty} H_{\psi,A,c_k,j}(x)\Big). \tag{3.20}$$

Here

$$H_{\psi,A,c_k,j}(x) = \int_{V_j} |\psi(t)||c_k(A(t)x)|dt,$$

with $V_0 = \{t \in \mathbb{Q}_p^n : \|A^{-1}(t)\|_p \leq 1\}$, and $V_j = \{t \in \mathbb{Q}_p^n : \|A^{-1}(t)\|_p = p^j\}$ for any $j \in \mathbb{Z}^+$. It is clear to see that

$$\operatorname{supp}(H_{\psi,A,c_k,j}) \subset B_{k+j}. \tag{3.21}$$

By estimating as in (3.18) and using $\|c_k\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \leq \frac{1}{u(p^k)}$,

$$\|H_{\psi,A,c_k,j}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \Big(\int_{V_j} |\psi(t)|\mathcal{K}_{A,n,q}(t)\frac{u(p^{k+j})}{u(p^k)}dt\Big)\frac{1}{u(p^{k+j})}.$$

By applying $u \in LW_{q(\cdot)}(\mathbb{Q}_p^n)$ and Lemma 7 (i), for any $j \in \mathbb{N}$,

$$\frac{u(p^{k+j})}{u(p^k)} \lesssim \frac{\|\chi_{B_{k+j}}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}}{\|\chi_{B_k}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)}} \lesssim \frac{|B_{k+j}|}{|B_k|} = p^{jn}.$$

Thus, for any $j \in \mathbb{N}$,

$$\begin{aligned}\|H_{\psi,A,c_k,j}\|_{L^{q(\cdot)}(\mathbb{Q}_p^n)} &\lesssim \Big(\int_{V_j} |\psi(t)|\mathcal{K}_{A,n,q}(t)\max\{1, \|A^{-1}(t)\|_p^n\}dt\Big)\frac{1}{u(p^{k+j})}\\ &:= d_j\frac{1}{u(p^{k+j})}.\end{aligned} \tag{3.22}$$

By setting $\tilde{H}_{\psi,A,c_k,j} = \begin{cases}\dfrac{H_{\psi,A,c_k,j}}{d_j}, & \text{if } d_j \neq 0,\\ 0, & \text{otherwise.}\end{cases}$

Then,

$$\sum_{j=0}^{\infty} H_{\psi,A,c_k,j}(\cdot) = \sum_{j=0}^{\infty} d_j\tilde{H}_{\psi,A,c_k,j}(\cdot).$$

Hence, by (3.21) and (3.22), $\tilde{H}_{\psi,A,c_k,j}$ is a local $(u, L^{q(\cdot)})$-block. Consequently, we deduce

$$\Big\|\sum_{j=0}^{\infty} H_{\psi,A,c_k,j}\Big\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \sum_{j=0}^{\infty}|d_j| = \mathcal{M}, \text{ for all } k \in \mathbb{N}.$$

In view of the above inequality, by using (3.20), we obtain

$$\begin{aligned}\|\mathcal{H}_{\psi,A}(f)\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} &\le \Big\|\sum_{k=0}^{\infty}|\lambda_k|\Big(\sum_{j=0}^{\infty}H_{\psi,A,c_k,j}\Big)\Big\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}\\ &\le \sum_{k=0}^{\infty}|\lambda_k|\Big\|\Big(\sum_{j=0}^{\infty}H_{\psi,A,c_k,j}\Big)\Big\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}\\ &\lesssim \mathcal{M}\|f\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)}.\end{aligned}$$

This completes the proof of this theorem. □

For simplicity of notation in the following two results, we write

$$\mathcal{P}_*(\mathbb{Q}_p^n) := \{u \in \mathcal{P}(\mathbb{Q}_p^n) : u(t^{-1}\cdot) = u(\cdot), \text{ for a.e. } t \in supp(\varphi)\}.$$

Consequently, by Theorems 17 and 18, we establish the boundedness of Hardy–Littlewood average operators on the spaces $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$ and the spaces $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$.

Corollary 19. *Let $q(\cdot) \in \mathcal{P}_*(\mathbb{Q}_p^n)$, $\alpha(\cdot) \in L^\infty(\mathbb{Q}_p^n) \cap \mathbf{C}^{\log}_\infty(\mathbb{Q}_p^n) \cap \mathbf{C}^{\log}_0(\mathbb{Q}_p^n)$, $\beta \in (-n, \infty)$, $\alpha(0) = \alpha_\infty \in (0, \infty)$, and $\omega(x) = |x|_p^\beta$. If $\ell \in (0, 1)$, $\sigma > (1 - \ell)/\ell$, and*

$$\mathcal{N}_\sigma = \int_{\mathbb{Z}_p^*} |\varphi(t)|.|t|_p^{-n/q_- - (\beta+n)\alpha_\infty/n} \max\Big\{|\log_p |t|_p|^\sigma, 1\Big\}dt < \infty,$$

then, $\mathcal{H}^{p,n}_\varphi$ is bounded on $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)$. Moreover,

$$\|\mathcal{H}^{p,n}_\varphi\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)\to\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p^n)} \lesssim \mathcal{N}_{(1-\ell)/\ell}.$$

Here, $\mathcal{N}_{(1-\ell)/\ell} = \int_{\mathbb{Z}_p^*} |\varphi(t)|.|t|_p^{-n/q_- - (\beta+n)\alpha_\infty/n} \max\Big\{|\log_p |t|_p|^{(1-\ell)/\ell}, 1\Big\}\, dt.$

Corollary 20. *Let $q(\cdot) \in \mathcal{P}_*(\mathbb{Q}_p^n)$ and $u \in LW_{q(\cdot)}(\mathbb{Q}_p^n)$. If*

$$\mathcal{M}_1 = \int_{\mathbb{Z}_p^*} |\varphi(t)|.|t|_p^{-n/q_- - n}dt < \infty,$$

then, $\mathcal{H}^{p,n}_\varphi$ is bounded on $LB_{u,q(\cdot)}(\mathbb{Q}_p^n)$. Moreover,

$$\|\mathcal{H}^{p,n}_\varphi\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p^n)\to LB_{u,q(\cdot)}(\mathbb{Q}_p^n)} \lesssim \mathcal{M}_1.$$

Let $\mathcal{G}$ be a nonnegative function on $\mathbb{R}^2_+$. Assume that $\mathcal{G}$ is a homogeneous function of degree -1. Then, the p-adic Hardy-Hilbert type integral operator is defined by

$$\mathcal{T}^p(f)(x) = \int_{\mathbb{Q}_p^*} \mathcal{G}(|x|_p, |y|_p) f(y)dy,\ x \in \mathbb{Q}_p^*.$$

By setting $y = tx$, the p-adic Hardy-Hilbert operator is rewritten as follows:

$$\mathcal{T}^p(f)(x) = \int\limits_{\mathbb{Q}_p^*} \mathcal{G}(1, |t|_p) f(tx) dt.$$

Consequently, the p-adic Hardy-Hilbert operator $\mathcal{T}^p$ is a special case of the operator $\mathcal{H}^p_{\varphi,A}$ by choosing $n = 1, A(t) = t$, and $\psi(t) = \mathcal{G}(1, |t|_p)$. Besides, we define

$$\mathcal{H}(\mathbb{Q}_p) = \Big\{u \in \mathcal{P}(\mathbb{Q}_p) : u(t^{-1}\cdot) = u(\cdot), \text{ for a.e. } t \in supp\Big(\mathcal{G}(1, |\cdot|_p)\Big)\Big\}.$$

In view of Theorems 17 and 18, we obtain the boundedness of the Hardy-Hilbert operator $\mathcal{T}^p$ on the space $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p)$, and the space $LB_{u,q(.)}(\mathbb{Q}_p)$.

Corollary 21. *Let $q(\cdot) \in \mathcal{H}(\mathbb{Q}_p), \alpha(\cdot) \in L^\infty(\mathbb{Q}_p) \cap \mathbf{C}^{\log}_\infty(\mathbb{Q}_p) \cap \mathbf{C}^{\log}_0(\mathbb{Q}_p), \beta \in (-1, \infty), \alpha(0) = \alpha_\infty \in (0, \infty)$, and $\omega(x) = |x|_p^\beta$. If $\ell \in (0, 1), \sigma > (1 - \ell)/\ell$, and*

$$\widetilde{\mathcal{N}}_{\sigma,\mathcal{G}} = \int\limits_{\mathbb{Q}_p^*} \mathcal{G}(1, |t|_p).|t|_p^{-1/q_- -(\beta+1)\alpha_\infty} \max\Big\{|\log_p |t|_p|^\sigma, 1\Big\} dt < \infty,$$

then, $\mathcal{T}^p$ is bounded on $\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p)$. Moreover,

$$\|\mathcal{T}^p\|_{\dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p) \to \dot{K}^{\alpha(\cdot),\ell}_{q(\cdot),\omega}(\mathbb{Q}_p)} \lesssim \widetilde{\mathcal{N}}_{(1-\ell)/\ell,\mathcal{G}}.$$

Here, $\widetilde{\mathcal{N}}_{(1-\ell)/\ell,\mathcal{G}} = \int\limits_{\mathbb{Q}_p^*} \mathcal{G}(1, |t|_p).|t|_p^{-1/q_- -(\beta+1)\alpha_\infty} \max\Big\{|\log_p |t|_p|^{(1-\ell)/\ell}, 1\Big\} dt.$

Corollary 22. *Let $q(\cdot) \in \mathcal{H}(\mathbb{Q}_p)$ and $u \in LW_{q(\cdot)}(\mathbb{Q}_p)$. If*

$$\widetilde{\mathcal{M}}_{1,\mathcal{G}} = \int\limits_{\mathbb{Q}_p^*} \mathcal{G}(1, |t|_p).|t|_p^{-1/q_- -1} dt < \infty,$$

then, $\mathcal{T}^p$ is bounded on $LB_{u,q(.)}(\mathbb{Q}_p)$. Moreover,

$$\|\mathcal{T}^p\|_{LB_{u,q(\cdot)}(\mathbb{Q}_p) \to LB_{u,q(\cdot)}(\mathbb{Q}_p)} \lesssim \widetilde{\mathcal{M}}_{1,\mathcal{G}}.$$

4. Conclusions

This paper aims to investigate some inequalities for the boundedness of the Hardy–Littlewood maximal operators, the maximal commutator operators, the Hausdorff operators, the Hardy–Littlewood average operators, the Hardy–Hilbert operators on p-adic Herz spaces, and p-adic local block spaces with variable exponents. Theorems 9, 10, and 16–18 are the main and important results.

As a natural development, we discuss some future works as follows:

1) The study of sharp bounds for the boundedness of the above operators on p-adic Herz spaces and p-adic local block spaces with variable exponents is an open problem.

2) By establishing extrapolation theorems, we hope to obtain some new results for the boundedness of the above p-adic operators.

3) From Theorems 9, 10, 17, and 18 in this paper, we will research the regularity of the solution of some p-adic equations.

Author contributions

Pham Thi Kim Thuy and Kieu Huu Dung: writing-original draft, writing-review & editing. Both authors have read and agreed to the published version of this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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