Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 23047–23059.
DOI:10.3934/math.20241120
Received: 16 June 2024
Revised: 22 July 2024
Accepted: 24 July 2024
Published: 29 July 2024

Research article

On Hadamard 2-(51, 25, 12) and 2-(59, 29, 14) designs

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Abstract: In this paper, we classified Hadamard 2-(51, 25, 12) designs having a non-abelian automorphism group of order 10, i.e., $\mathrm{Frob}_{10} \cong Z_5 : Z_2$, where a subgroup isomorphic to Z_2 fixed setwise all the Z_5-orbits of the sets of points and blocks. Furthermore, we classified 2-(59, 29, 14) designs having a non-abelian automorphism group of order 14, i.e., $\mathrm{Frob}_{14} \cong Z_7 : Z_2$, where a subgroup isomorphic to Z_2 fixed setwise all the Z_7-orbits of the sets of points and blocks. We also showed that there was no Hadamard 2-(59, 29, 14) design with automorphism group $\mathrm{Frob}_{21} \cong Z_7 : Z_3$, where a subgroup of order 3 fixed setwise all the Z_7-orbits of points and blocks. Additionally, we used Hadamard 2-(59, 29, 14) designs obtained to construct ternary linear codes and linear codes over the finite field of order 5. We constructed ternary self-dual codes and self-dual codes over the finite field of order 5 from the corresponding Hadamard matrices of order 60.

Keywords: Hadamard 2-design; automorphism group; Frobenius group
Mathematics Subject Classification: 05B05, 05B20, 94B05

1. Introduction

It is well-known that a Hadamard matrix of order n can exist only if $n = 1, 2$ or $n \equiv 0 \mod 4$. The Hadamard conjecture states that these necessary conditions are also sufficient. Since the discovery of a Hadamard matrix of order 428, the smallest open case is $n = 668$ (see [22]). In this paper, Hadamard matrices of order 52 and 60 are obtained. In the literature, there are many studies concerning this order and we mention few of them. In [23], the authors were interested in a construction of Hadamard matrices of order 52, and in [24], the authors established the new lower bound for Hadamard matrices of order 60. In [6], the authors obtained many new Hadamard matrices of order 60 via the genetic algorithm. In [14], the author constructed many new inequivalent Hadamard matrices of order 60.

The existence of Hadamard 2-(51, 25, 12) designs and Hadamard 2-(59, 29, 14) designs is known (see [7]), but the classification of designs with these parameters is not. In [1], the authors classify designs with parameters 2-(59, 29, 14) having an automorphism group of order 29 and one fixed point and show that there are exactly 531 such designs. The classification of designs with particular parameters is one of the most important topics in design theory. The Hadamard designs with parameters 2-$(51, 25, 12)$ and 2-$(59, 29, 14)$ assuming the action of the Frobenius groups of order 10 and 14 have not been studied before. This result gives a contribution to the studies of designs from group theory point of view. However, according to our best knowledge, in [7] there is information that the number of known Hadamard 2-$(51, 25, 12)$ is ≥ 1. We increase the number by constructing 17 new Hadamard 2-$(51, 25, 12)$ designs. Moreover, in this paper, we constructed 32 new Hadamard 2-$(59, 29, 14)$ designs, non-isomorphic to the ones obtained in [1].

In this paper, we show that, up to isomorphism, there are exactly 17 Hadamard 2-$(51, 25, 12)$ designs having a non-abelian automorphism group of order 10, i.e., $\mathrm{Frob}_{10} \cong Z_5 : Z_2$, where a subgroup isomorphic to Z_2 fixes setwise all the Z_5-orbits of the sets of points and blocks. Moreover, we show that there are exactly 32 pairwise non-isomorphic Hadamard 2-$(59, 29, 14)$ designs having a non-abelian automorphism group of order 14, i.e., $\mathrm{Frob}_{14} \cong Z_7 : Z_2$, where an involution fixes setwise all the Z_7-orbits of the sets of points and blocks. Further, we show that there is no Hadamard 2-$(59, 29, 14)$ design with automorphism group $\mathrm{Frob}_{21} \cong Z_7 : Z_3$, where a subgroup isomorphic to Z_3 fixes setwise all the Z_7-orbits of points and blocks. From the Hadamard designs obtained, we construct 9 pairwise nonequivalent Hadamard matrices of order 52 and 8 pairwise nonequivalent Hadamard matrices of order 60. From the Hadamard matrices of order 60, we construct ternary self-dual codes and self-dual codes over the finite field of order 5.

The paper is organized as follows. In Section 2, we give basic definitions and properties of designs, Hadamard matrices, and linear codes, and explain the method of constructing designs used in this paper. In Section 3, we give constructions of Hadamard 2-$(51, 25, 12)$ designs, while in Section 4 we give constructions of Hadamard 2-$(59, 29, 14)$ designs. In Section 5, we study codes from constructed Hadamard 2-$(59, 29, 14)$ designs and the corresponding Hadamard matrices.

Computer calculations in this paper were done by programs written for the system for computational discrete algebra GAP [15], and Magma [4]. The programs are tested on the orbit matrices and designs obtained in [5] and [9] by repeating the calculations given in these references. Each calculation is repeated at least twice to test the correct work of the software.

The designs and the orbit matrices constructed in this paper can be found at

`https://math.uniri.hr/~ddumicic/results/Des51-59_Had52-60.html`

2. Preliminaries

We assume that the reader is familiar with the basic facts of design theory and coding theory. We refer the reader to [3, 7] for the relevant reading in design theory and to [19] for the relevant reading in coding theory.

A 2-(v, k, λ) design is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = v$ and $1 < k < v - 1$,
2. every element (block) of $\mathcal{B}$ is incident with exactly k elements (points) of $\mathcal{P}$,
3. every two distinct points in $\mathcal{P}$ are together incident with exactly λ blocks of $\mathcal{B}$.

In a 2-(v, k, λ) design, every point is incident with exactly $r = \dfrac{\lambda(v-1)}{k-1}$ blocks, and r is called the replication number of a design. A 2-design is also called a block design. The number of blocks in a block design is denoted by b. If $v = b$, a design is called *symmetric*. In a symmetric design, each two blocks meet in exactly λ points. In case of symmetric designs, the notation (v, k, λ) design is often used. An isomorphism from one block design to another is a bijective mapping of points to points and blocks to blocks, which preserves incidence. An isomorphism from a block design $\mathcal{D}$ onto itself is called an automorphism of $\mathcal{D}$. The set of all automorphisms forms a group called the full automorphism group of $\mathcal{D}$ and is denoted by $Aut(\mathcal{D})$. If $\mathcal{D}$ is a block design, the incidence structure $\mathcal{D}'$ having as points the blocks of $\mathcal{D}$, and having as blocks the points of $\mathcal{D}$, where a point and a block are incident in $\mathcal{D}'$ if, and only if, the corresponding block and a point of $\mathcal{D}$ are incident, is a block design called the *dual* of $\mathcal{D}$. The design is called self-dual if the design and its dual are isomorphic.

If $\mathcal{D}$ is a symmetric design and $G \leq Aut(\mathcal{D})$, then the action of G on $\mathcal{D}$ induces the same number of orbits of the set of points and blocks, and G fixes the same number of points and blocks. Furthermore, each automorphism $g \in G$ has the same cyclic structure considered as a permutation on points or on blocks. We say that a group G acts *semi-standardly* on $\mathcal{D}$ if the point orbit lengths distribution is equal to the block orbit lengths distribution, and G acts *standardly* on $\mathcal{D}$ if every point orbit has length 1 or $|G|$. For further reading on symmetric designs and their automorphism groups, we refer the reader to [25, 26]. We state the following propositions that will be used later, and they can be found in [25].

Proposition 2.1. *Let $\alpha \neq 1$ be an automorphism of a symmetric 2-(v, k, λ) design which fixes F_α points. Then*

$$F_\alpha \leq v - 2n \qquad \text{and} \qquad F_\alpha \leq \frac{\lambda}{k - \sqrt{n}} v,$$

where $n = k - \lambda$. Moreover, if equality holds in either inequality, α must be an involution and every non-fixed block contains exactly λ fixed points.

Proposition 2.2. *Let $\mathcal{D}$ be a symmetric 2-(v, k, λ) design and $\alpha \in Aut(\mathcal{D})$. If α is an involution fixing $F_\alpha \neq 0$ points, then*

$$F_\alpha \geq \begin{cases} 1 + \frac{k}{\lambda}, & \text{for } k, \lambda \text{ even}, \\ 1 + \frac{k-1}{\lambda}, & \text{otherwise}. \end{cases}$$

A *Hadamard matrix* of order n is a $n \times n$ $(-1, 1)$-matrix H such that $HH^\top = nI_n$. Two Hadamard matrices are said to be equivalent if one can be transformed into the other by a series of row and column permutations and negations. A Hadamard matrix is *normalized* if all entries in the first row and the first column are 1. Hence, every Hadamard matrix is equivalent to a normalized Hadamard matrix.

Hadamard matrices are closely related to designs. Hadamard matrices of order $4t$ can be used to obtain symmetric 2-designs with parameters $(4t - 1, 2t - 1, t - 1)$ or $(4t - 1, 2t, t)$, which are called *Hadamard 2-designs* (see [8]). The construction is reversible, i.e., from symmetric 2-designs with these parameters, one can construct Hadamard matrices.

A q-ary *linear code* C of dimension k for a prime power q is a k-dimensional subspace of a vector space $\mathbb{F}_q^n$. Elements of C are called codewords. If $q = 2$, a code C is called *binary*, and for $q = 3$, a code is called *ternary*. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n) \in \mathbb{F}_q^n$. The *Hamming distance* between words x and y is the number $d(x, y) = |\{i : x_i \neq y_i\}|$. The *minimum distance* of the code C is defined by $d = \min\{d(x, y) : x, y \in C,\ x \neq y\}$. The *weight* of a codeword x is $w(x) = d(x, 0) = |\{i : x_i \neq 0\}|$. A code for which all codewords have weight divisible by four is called *doubly-even*, and *singly even* if all weights are even and there is at least one codeword x with $w(x) \equiv 2 \mod 4$. For a linear code,

$d = \min\{w(x) : x \in C, x \neq 0\}$. A q-ary linear code of length n, dimension k, and distance d is called a $[n, k, d]_q$ code. We may use the notation $[n, k]$ if the parameters d and q are unspecified. The *dual* code $C^\perp$ is the orthogonal complement under the standard inner product, i.e., $\langle \cdot, \cdot \rangle$, i.e. $C^\perp = \{v \in \mathbb{F}_q^n : \langle v, c \rangle = 0 \text{ for all } c \in C\}$. A code C is *self-orthogonal* if $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. A ternary self-dual code of length n exists if, and only if, n is divisible by four. We say that a code is *optimal* if it has the largest minimum weight among all codes of that length and dimension.

P. Dembowski introduced the notion of a tactical decomposition of an incidence structure and showed that the orbits of an automorphism group acting on an incidence structure $\mathcal{I}$ induce a tactical decomposition of $\mathcal{I}$ (see [12]). Tactical decompositions induced by the action of an automorphism group leads us to orbit matrices of incidence structures, which have been successfully used for a construction of block designs for more than 40 years; see for example, in [9–11, 20, 21]. The designs constructed in this paper are obtained using orbit matrices of symmetric designs. The first step of the method is the construction of orbit matrices for the assumed action of an automorphism group on a design, and the second step is the construction of incidence matrices of designs from the obtained orbit matrices. This second step is often called the *indexing* of orbit matrices. We briefly describe the construction of designs, but more details about this construction can be found in [20].

Let $\mathcal{D}$ be a 2-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$. We denote G-orbits of points and blocks by $\mathcal{P}_1, \ldots, \mathcal{P}_m$ and $\mathcal{B}_1, \ldots, \mathcal{B}_n$, respectively, and put $|\mathcal{P}_i| = \omega_i$, $|\mathcal{B}_j| = \Omega_j$, $1 \leq i \leq m$, $1 \leq j \leq n$. It holds that $\sum_{i=1}^m \omega_i = v$, and also $\sum_{j=1}^n \Omega_j = b$.

Further, by t_{ij} we denote the number of blocks in $\mathcal{B}_j$ incident with the representative of the point orbit $\mathcal{P}_i$, i.e., $t_{ij} = |\{B \in \mathcal{B}_j \mid (P, B) \in \mathcal{I},\ P \in \mathcal{P}_i\}|$. Analogously, b_{ij} is the number of points in $\mathcal{P}_i$ incident with the representative of the block orbit $\mathcal{B}_j$, i.e., $b_{ij} = |P \in \mathcal{P}_i \mid (P, B) \in \mathcal{I},\ B \in \mathcal{B}_j|$. The numbers t_{ij} and b_{ij} do not depend on the choice of the representatives of point and block orbits, respectively.

The following conditions for t_{ij} hold (see [10] and [20]):

$$0 \leq t_{ij} \leq \Omega_j, \quad 1 \leq i \leq m, \ 1 \leq j \leq n, \tag{2.1}$$

$$\sum_{j=1}^{n} t_{ij} = r, \quad 1 \leq i \leq m, \tag{2.2}$$

$$\sum_{i=1}^{m} \frac{\omega_i}{\Omega_j} t_{ij} = k, \quad 1 \leq j \leq n, \tag{2.3}$$

$$\sum_{j=1}^{n} \frac{\omega_s}{\Omega_j} t_{sj} t_{s'j} = \lambda \omega_s + \delta_{ss'} \cdot (r - \lambda), \quad 1 \leq s, s' \leq m. \tag{2.4}$$

A $(m \times n)$ matrix $T = [t_{ij}]$ with entries satisfying conditions (2.1) − (2.4) is called a *point orbit matrix* of a 2-(v, k, λ) design with orbit lengths distributions $(\omega_1, \ldots, \omega_m)$ and $(\Omega_1, \ldots, \Omega_n)$. A $(l \times n)$ matrix $[t_{ij}]$, for $l < m$, with entries satisfying conditions (2.1), (2.2), (2.4), and the condition

$$\sum_{i=1}^{l} \frac{\omega_i}{\Omega_j} t_{ij} \leq k, \quad 1 \leq j \leq n,$$

is called a *partial point orbit matrix* of a 2-(v, k, λ) design with orbit lengths distributions $(\omega_1, \ldots, \omega_m)$ and $(\Omega_1, \ldots, \Omega_n)$.

If $\mathcal{D}$ is a symmetric 2-(v, k, λ) design, then also the following equation holds:

$$\sum_{i=1}^{m} \frac{\omega_i}{\Omega_s} t_{is} t_{it} = \lambda \Omega_t + \delta_{st}(r - \lambda), \quad 1 \le s, t \le n. \tag{2.5}$$

The set of indices of blocks in a G-orbit $\mathcal{B}_j$ indicating which blocks of $\mathcal{B}_j$ are incident with the representative of a G-orbit of points $\mathcal{P}_i$ is called the *index set* for an entry t_{ij} in a point orbit matrix, for $i \in \{1, \ldots m\}$ and $j \in \{1, \ldots, n\}$.

In the construction of (partial) orbit matrices, it is very useful to reduce the number of those from which isomorphic designs are constructed. This is achieved through appropriate permutations of rows and columns, as stated in the following proposition (see [10, Theorem 4]).

Proposition 2.3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a 2-(v, k, λ) design, $G \le Aut(\mathcal{D})$, and let the $(m \times n)$ matrix T be a point orbit matrix of the design $\mathcal{D}$ with respect to the group G. Then, let $g = (\alpha, \beta) \in S = S_m \times S_n$ with the following properties:*

1. *if $\alpha(s) = t$, then the stabilizer G_{P_s} is conjugate to G_{P_t}, where $P_s, P_t \in \mathcal{P}$, $\mathcal{P}_s = P_s G$, and $\mathcal{P}_t = P_t G$,*
2. *if $\beta(i) = j$, then G_{x_i} is conjugate to G_{x_j}, where $x_i, x_j \in \mathcal{B}$, $\mathcal{B}_i = x_i G$, $\mathcal{B}_j = x_j G$.*

Then, there exists a permutation $g^* \in C_{S(\mathcal{P}) \times S(\mathcal{B})}(G)$, such that

$$\alpha(s) = t \text{ if, and only if, } g^*(\mathcal{P}_s) = \mathcal{P}_t, \text{ and}$$

$$\beta(i) = j \text{ if, and only if, } g^*(\mathcal{B}_i) = \mathcal{B}_j.$$

Two orbit matrices T and T' are *isomorphic* if there is a permutation $g = (\alpha, \beta) \in S_m \times S_n$ from T onto $T' = Tg$ that satisfies the conditions from Proposition 2.3. Isomorphic orbit matrices correspond to isomorphic designs. Hence, while constructing designs from orbit matrices, it is sufficient to construct designs from one orbit matrix from each isomorphism class of orbit matrices. So, during the construction of (partial) orbit matrices, for the isomorph rejection we used the isomorphisms of (partial) orbit matrices. For more details on this, we refer to [5] and [10].

3. Hadamard 2-$(51, 25, 12)$ designs with automorphism group $Frob_{10}$

In this section, we construct Hadamard 2-$(51, 25, 12)$ designs having an automorphism group $Frob_{10} \cong D_{10}$. To analyze the action of a non-abelian group $Frob_{10} \cong Z_5 : Z_2$ on a 2-$(51, 25, 12)$ design, first we examine the action of its subgroup Z_5 on that design and obtain the result given in the following proposition.

Proposition 3.1. *If an automorphism ρ of order 5 acts on a 2-$(51, 25, 12)$ design, then it fixes exactly one point.*

Proof. If a 2-$(51, 25, 12)$ design $\mathcal{D}$ admits an action of an automorphism ρ of order 5, then the number of points fixed by ρ is $F_\rho \in \{1, 6, 11, 16, 21\}$, since ρ acts on the set of points (and blocks) of $\mathcal{D}$ in orbits of length 1 or 5. By Proposition 2.1, it holds that $F_\rho \le 25$.

If $F_\rho = 6$, then the $\langle\rho\rangle$-orbit lengths distribution is $(\omega_1, \ldots, \omega_{15}) = (\Omega_1, \ldots, \Omega_{15}) = (1, 1, 1, 1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5, 5)$. In this case, each point fixed by ρ is incident with 0 or 5 fixed blocks, since $r = 25$. Let P be a fixed point which is not incident with a fixed block. Then, the point P together with each of the other 5 fixed points is incident with 5, 10, 15, 20, or 25 non-fixed blocks, which is a contradiction, since $\lambda = 12$. Hence, each fixed point is incident with 5 fixed blocks. Furthermore, any pair of fixed points are contained in 4 or 5 common fixed blocks, and in $5y$, $y \in \{0, 1, \ldots, 4\}$, common non-fixed blocks. It yields to a contradiction since $\lambda = 12$.

It is easy to check that for $F_\rho \in \{11, 16, 21\}$, there are no candidates for a row in a point orbit matrix corresponding to an $\langle\rho\rangle$-orbit with non-fixed points that satisfies the properties (2.1), (2.2), and (2.4), for $s = s'$. For example, if $F_\rho = 21$, then $\Omega_1 = \Omega_2 = \ldots = \Omega_{21} = 1$ and $\Omega_{22} = \Omega_{23} = \ldots = \Omega_{27} = 5$. For $s \in \{22, 23, \ldots, 27\}$, the equation

$$5(t_{s1}^2 + t_{s2}^2 + \ldots + t_{s21}^2) + t_{s22}^2 + t_{s23}^2 + \ldots + t_{s27}^2 = 73$$

does not have a solution for $t_{s1}, t_{s2}, \ldots, t_{s21} \in \{0, 1\}$ and $t_{s22}, t_{s23}, \ldots, t_{s27} \in \{0, 1, 2, 3, 4, 5\}$, under assumption that $\sum_{j=1}^{27} t_{sj} = 25$.

Therefore, it holds that $F_\rho = 1$. □

Hence, an automorphism ρ of order 5 acts standardly on a 2-$(51, 25, 12)$ design with the orbit lengths distribution $(\omega_1, \ldots, \omega_{11}) = (1, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$. It is easy to see that there is only one candidate for the first row in a point orbit matrix that represents the point fixed by ρ, which is $[0, 5, 5, 5, 5, 5, 0, 0, 0, 0, 0]$ (up to permutations of columns that correspond to the block orbits of the same length). Further, all possible candidates for rows from second to eleventh in a point orbit matrix (up to permutations of columns that correspond to the block orbits of the same length) that correspond to $\langle\rho\rangle$-orbits with non-fixed points are listed in Table 1. They satisfy the properties (2.1), (2.2), and (2.4), for $s = s'$. Note that these candidates for rows are called types in [5].

Table 1. Candidates for rows in a point orbit matrix.

1) [1, 5, 3, 3, 2, 2, 2, 2, 2, 2, 1]	7) [0, 5, 3, 3, 3, 2, 2, 2, 2, 2, 1]
2) [1, 4, 4, 3, 3, 2, 2, 2, 2, 1, 1]	8) [0, 4, 4, 4, 2, 2, 2, 2, 2, 2, 1]
3) [1, 4, 3, 3, 3, 3, 3, 2, 1, 1, 1]	9) [0, 4, 4, 3, 3, 3, 2, 2, 2, 1, 1]
4) [1, 4, 3, 3, 3, 3, 2, 2, 2, 2, 0]	10) [0, 4, 3, 3, 3, 3, 3, 3, 1, 1, 1]
5) [1, 3, 3, 3, 3, 3, 3, 3, 2, 1, 0]	11) [0, 4, 3, 3, 3, 3, 3, 2, 2, 2, 0]
6) [0, 5, 4, 2, 2, 2, 2, 2, 2, 2, 2]	12) [0, 3, 3, 3, 3, 3, 3, 3, 3, 1, 0]

After determining the candidates for rows of a point orbit matrix, we proceed with the construction of point orbit matrices following the algorithm given in [5].

Using our program written for GAP, we constructed 528 pairwise non-isomorphic point orbit matrices for the action of an automorphism of order 5 on a 2-$(51, 25, 12)$ design $\mathcal{D}$. Here, for isomorph rejection we used all permutations of rows and columns of (partial) point orbit matrices that correspond to the orbits of the same length, i.e., that satisfy the conditions from Proposition 2.3. We checked whether the constructed point orbit matrices are mutually non-isomorphic by using the GAP function `TransformingPermutations`. This is summarized in the following theorem.

Theorem 3.2. *Up to isomorphism, there are exactly* 528 *point orbit matrices for an action of an automorphism of order* 5 *on a* 2-$(51, 25, 12)$ *design.*

We assume that the group $G = \left\langle \rho, \alpha \mid \rho^5 = \alpha^2 = 1,\ \alpha\rho\alpha = \rho^{-1} \right\rangle \cong Frob_{10}$ acts on a Hadamard 2-(51, 25, 12) design in such a way that the involution α fixes the $\langle\rho\rangle$-orbits of points and blocks setwise. To proceed with the construction of the designs admitting such action of the group $G \leq Aut(\mathcal{D})$, while indexing the obtained orbit matrices, we may assume that its subgroup $\langle\alpha\rangle < G$ stabilizes the representatives of all $\langle\rho\rangle$-orbits of points and blocks. Since we assumed that α stabilizes the representatives of all orbits for the action of an automorphism ρ of order 5 on a 2-(51, 25, 12) design (i.e., all $\langle\rho\rangle$-orbits are also G-orbits), involution α acts on the set of indices of a $\langle\rho\rangle$-orbit of non-fixed points (and non-fixed blocks) in one of the following ways:

$$\alpha = \begin{cases} (0)(1,4)(2,3) \\ (0)(1)(2)(3)(4). \end{cases} \tag{3.1}$$

If $\alpha = (0)(1,4)(2,3)$ is the action of α on the index set of a point $\langle\rho\rangle$-orbit $\{P_0, P_1, \ldots, P_4\}$ and a block $\langle\rho\rangle$-orbit $\{B_0, B_1, \ldots, B_4\}$, then the possibilities for the first rows of the corresponding part of an incidence matrix of $\mathcal{D}$ are as given in Table 2. The other four can be obtained by their cyclic permutations.

Table 2. Indexing of t_{ij} for the group G acting on $\mathcal{D}$, $\alpha = (0)(14)(23)$.

t_{ij}	Possibilities for indexing the first rows	t_{ij}	Possibilities for indexing the first rows
0	[0,0,0,0,0]	3	[1,1,0,0,1], [1,0,1,1,0]
1	[1,0,0,0,0]	4	[0,1,1,1,1]
2	[0,1,0,0,1], [0,0,1,1,0]	5	[1,1,1,1,1]

A group acts transitively on each of its orbit, and it is known that a transitive action of a group on a set is permutation isomorphic to the action of the group on the cosets of the stabilizer of an element of the set; for example, see [13]. Let ρ and α be elements of orders 5 and 2, respectively, in D_{10}. The group D_{10} has two conjugacy classes of nontrivial subgroups, which are of orders 2 and 5. Thus, there are three ways in which D_{10} acts as a nontrivial permutation group: the regular action on 10 points, the standard action on 5 points, and an action on 2 points where elements of order 2 act nontrivially and the cyclic subgroup of order 5 is in the kernel. Hence, the following proposition holds.

Proposition 3.3. *Let $\mathcal{D}$ be a* 2-(51, 25, 12) *design having a non-abelian automorphism group $G = \left\langle \rho, \alpha \mid \rho^5 = \alpha^2 = 1,\ \alpha\rho\alpha = \rho^{-1} \right\rangle$ of order* 10. *If G acts on the set of points of $\mathcal{D}$ with the orbit lengths distribution* (1, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5) *(i.e., all $\langle\rho\rangle$-orbits are also G-orbits), then α fixes exactly* 11 *points.*

The next step in the construction of designs is the indexing of the obtained orbit matrices, i.e., the construction of 2-(51, 25, 12) designs for the described action of G. The indexing of the point orbit matrices constructed for the action of an automorphism of order 5 was carried out quickly, by using Table 2 and checking if any pair of points is incident $\lambda = 12$ common blocks. Finally, after eliminating isomorphic copies in the set of 480 constructed designs (using GAP function `IsIsomorphicBlockDesign`), we complete the classification and summarize the above results in the following theorem.

Theorem 3.4. *Up to isomorphism, there are exactly* 17 *Hadamard* 2-(51, 25, 12) *designs having an automorphism group $G = \left\langle \rho, \alpha \mid \rho^5 = \alpha^2 = 1,\ \alpha\rho\alpha = \rho^{-1} \right\rangle$, where an involution fixes setwise all the*

$\langle\rho\rangle$-orbits of the sets of points and blocks. The G-orbit lengths distribution on the set of points (and blocks) of those designs is $(1, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$*. Among the* 17 *constructed designs* $\mathcal{D}_1, \ldots, \mathcal{D}_{17}$ *there are* 9 *self-dual and* 4 *pairs of dual designs. Their full automorphism groups are given in Table 3. Up to equivalence, these* 17 *designs yield* 9 *Hadamard matrices of order* 52*.*

Table 3. The full automorphism groups of designs $\mathcal{D}_1, \ldots, \mathcal{D}_{17}$.

$\lvert Aut(\mathcal{D})\rvert$	Description	Design
600	$E_{25} : (Z_4 \times S_3)$	$\mathcal{D}_1$
40	$Z_2 \times (Z_5 : Z_4)$	$\mathcal{D}_2, \mathcal{D}_8, \mathcal{D}_9, \mathcal{D}_{10}, \mathcal{D}_{16}, \mathcal{D}_{17}$
100	$E_{25} : Z_4$	$\mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_{11}, \mathcal{D}_{12}$
20	$Z_5 : Z_4$	$\mathcal{D}_5, \mathcal{D}_7, \mathcal{D}_{14}, \mathcal{D}_{15}$
10	D_{10}	$\mathcal{D}_6, \mathcal{D}_{13}$

4. Hadamard 2-$(59, 29, 14)$ designs with automorphism group $Frob_{14}$ and $Frob_{21}$

Similarly as in Section 3, we examine actions of a non-abelian group $Frob_{14} \cong Z_7 : Z_2$ and $Frob_{21} \cong Z_7 : Z_3$ on a 2-$(59, 29, 14)$ design. First, we take a look at the action of an automorphism of order 7 on a 2-$(59, 29, 14)$ design.

Proposition 4.1. *If an automorphism σ of order* 7 *acts on a* 2-$(59, 29, 14)$ *design, then it fixes exactly three points.*

Proof. If σ is an automorphism of order 7 acting on a 2-$(51, 25, 12)$ design $\mathcal{D}'$, then the number of points fixed by σ is $F_\sigma \in \{3, 10, 17, 24\}$, since σ acts on the set of points (and blocks) of $\mathcal{D}'$ in orbits of length 1 or 7. By Proposition 2.1, it holds that $F_\sigma \leq 29$. However, it is easy to check that for $F_\sigma \in \{10, 17, 24\}$, there are no candidates for a row in a point orbit matrix corresponding to an $\langle\sigma\rangle$-orbit of non-fixed points that satisfies the properties (2.1), (2.2), and (2.4), for $s = s'$. Therefore, it holds that $F_\sigma = 3$. □

Hence, by Proposition 4.1, an automorphism σ of order 7 acts on a 2-$(59, 29, 14)$ design with the orbit lengths distribution $(\omega_1, \ldots, \omega_{11}) = (1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7)$. It is easy to see that there is only one candidate for the rows in a point orbit matrix that represent the points fixed by σ, which is $[1, 0, 0, 7, 7, 7, 7, 0, 0, 0, 0]$ (up to permutations of columns that correspond to the block orbits of the same length). Also, all possible candidates for the rows in a point orbit matrix that correspond to $\langle\sigma\rangle$-orbits of non-fixed points (up to permutations of columns that correspond to the block orbits of the same length) are listed in Table 4. They satisfy the properties (2.1), (2.2), and (2.4), for $s = s'$.

Table 4. Candidates for rows in a point orbit matrix.

1) [1, 1, 1, 5, 4, 4, 3, 3, 3, 2, 2]	9) [1, 0, 0, 5, 5, 4, 3, 3, 3, 3, 2]
2) [1, 1, 1, 4, 4, 4, 4, 4, 2, 2, 2]	10) [1, 0, 0, 5, 4, 4, 4, 4, 3, 2, 2]
3) [1, 1, 1, 4, 4, 4, 4, 3, 3, 3, 1]	11) [1, 0, 0, 4, 4, 4, 4, 4, 4, 3, 1]
4) [1, 1, 0, 6, 3, 3, 3, 3, 3, 3, 3]	12) [0, 0, 0, 6, 4, 4, 3, 3, 3, 3, 3]
5) [1, 1, 0, 5, 5, 3, 3, 3, 3, 3, 2]	13) [0, 0, 0, 5, 5, 4, 4, 3, 3, 3, 2]
6) [1, 1, 0, 5, 4, 4, 4, 3, 3, 2, 2]	14) [0, 0, 0, 5, 4, 4, 4, 4, 4, 2, 2]
7) [1, 1, 0, 4, 4, 4, 4, 4, 3, 3, 1]	15) [0, 0, 0, 4, 4, 4, 4, 4, 4, 4, 1]
8) [1, 0, 0, 6, 4, 3, 3, 3, 3, 3, 3]	

By using our program written for GAP, we constructed 18 pairwise non-isomorphic point orbit matrices for the action of an automorphism of order 7 on a 2-(59, 29, 14) design $\mathcal{D}'$. In this case as well, for isomorph rejection we used all permutations of rows and columns of (partial) point orbit matrices that correspond to the orbits of the same length, i.e., that satisfy the conditions from Proposition 2.3. We summarize this in the following theorem.

Theorem 4.2. *Up to isomorphism, there are exactly* 18 *point orbit matrices for an action of an automorphism of order* 7 *on a* 2-(59, 29, 14) *design.*

Let us assume that a non-abelian group $H = \left\langle \sigma, \alpha \mid \sigma^7 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma^{-1} \right\rangle \cong Frob_{14}$ acts on a 2-(59, 29, 14) design in such a way that an involution fixes $\langle\sigma\rangle$-orbits of points and blocks setwise. In other words, all $\langle\sigma\rangle$-orbits are also H-orbits. As in Section 3, here we assume that the subgroup $\langle\alpha\rangle \cong Z_2$ of $H \leq Aut(\mathcal{D}')$ stabilizes the representatives of all $\langle\sigma\rangle$-orbits of points and blocks. The following proposition can be proven in a similar way as Proposition 3.3.

Proposition 4.3. *Let $\mathcal{D}'$ be a* 2-(59, 29, 14) *design having a non-abelian automorphism group $H = \left\langle \sigma, \alpha \mid \sigma^7 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma^{-1} \right\rangle$ of order* 14*. If H acts on the set of points (and blocks) with the orbit lengths distribution* (1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7)*, then α fixes exactly* 11 *points.*

Therefore, α acts on the set of indices of all $\langle\sigma\rangle$-orbit of non-fixed points (and non-fixed blocks) as

$$\alpha = (0)(1, 6)(2, 5)(3, 4). \tag{4.1}$$

Further, the corresponding element t_{ij} in a point orbit matrix for the described action of H on $\mathcal{D}'$ can be indexed in a few ways, as given in Table 5. Only the first rows are listed, since the other six can be obtained by their cyclic permutations.

Table 5. Indexing of t_{ij} for the group H acting on $\mathcal{D}'$.

t_{ij}	Possibilities for indexing the first rows	t_{ij}	Possibilities for indexing the first rows
1	[1,0,0,0,0,0,0]	4	[0,1,1,0,0,1,1], [0,1,0,1,1,0,1], [0,0,1,1,1,1,0]
2	[0,1,0,0,0,0,1], [0,0,1,0,0,1,0], [0,0,0,1,1,0,0]	5	[1,1,1,0,0,1,1], [1,1,0,1,1,0,1], [1,0,1,1,1,1,0]
3	[1,1,0,0,0,0,1], [1,0,1,0,0,1,0], [1,0,0,1,1,0,0]	6	[0,1,1,1,1,1,1]

The last step is indexing of the obtained orbit matrices, i.e., the construction of 2-(59, 29, 14) designs for the assumed action of H. The indexing of the point orbit matrices constructed for the action of an automorphism of order 7 was carried out very quickly, by using Table 5 and by checking if any pair of points is incident with $\lambda = 14$ common blocks. In this way, we construct all Hadamard 2-(59, 29, 14) designs admitting the described action of the automorphism group H. After eliminating isomorphic copies in the set of 96 constructed designs, we complete the classification and summarize the above results in the following theorem.

Theorem 4.4. *Let $\mathcal{D}'$ be a* 2-(59, 29, 14) *design having a non-abelian automorphism group $H = \left\langle \sigma, \alpha \mid \sigma^7 = \alpha^2 = 1,\ \alpha\sigma\alpha = \sigma^{-1} \right\rangle$ of order* 14*. If the involution α fixes setwise all the $\langle\sigma\rangle$-orbits of points and blocks, then H acts on $\mathcal{D}'$ with the orbit lengths distribution* (1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7)*. Up to isomorphism, there are exactly* 32 *Hadamard* 2-(59, 29, 14) *designs admitting that action of H. Among them there are* 16 *pairs of dual designs and their full automorphism group is D_{28}. Up to equivalence, these* 32 *designs yield* 8 *Hadamard matrices of order* 60.

Next, we analyze the action of a non-abelian group of order 21, i.e., $H' \cong Frob_{21} \cong Z_7 : Z_3$ on a Hadamard 2-(59, 29, 14) design $\mathcal{D}'$, assuming that a subgroup of order 3 fixes setwise all the Z_7-orbits of points and blocks. The 18 pairwise non-isomorphic point orbit matrices for the action of an automorphism of order 7 on a 2-(59, 29, 14) design are used here to prove the following proposition.

Theorem 4.5. *There does not exist a Hadamard* 2-(59, 29, 14) *design* $\mathcal{D}'$ *having a non-abelian automorphism group* $H' = \left\langle \sigma, \beta \mid \sigma^7 = \beta^3 = 1,\ \beta\sigma\beta = \sigma^{-1} \right\rangle \cong Frob_{21}$ *such that the automorphism* β *fixes setwise all the* $\langle\sigma\rangle$*-orbits of points and blocks.*

Proof. By Proposition 4.1, the automorphism σ acts on $\mathcal{D}'$ with the point orbit lengths distribution $(1, 1, 1, 7, 7, 7, 7, 7, 7, 7, 7)$. Since we may assume that β stabilizes the representatives of all $\langle\sigma\rangle$-orbits (i.e. $\langle\sigma\rangle$-orbits are also H'-orbits), the action of β on the set of indices of a $\langle\sigma\rangle$-orbit of non-fixed points (and non-fixed blocks) is as follows:

$$\beta = (0)(1, 2, 4)(3, 6, 5). \tag{4.2}$$

Since $\beta = (0)(1, 2, 4)(3, 6, 5)$ is a representation of the action of β on the set of indices of a point $\langle\sigma\rangle$-orbit $\{P_0, P_1, \ldots, P_6\}$ and a block $\langle\sigma\rangle$-orbit $\{B_0, B_1, \ldots, B_6\}$, then it is clear that the corresponding element t_{ij} in a point orbit matrix for the assumed action of H' is

$$t_{ij} \in \{0, 1, 3, 4, 6, 7\}. \tag{4.3}$$

The automorphism β acts on every $\langle\sigma\rangle$-orbit of non-fixed points O_P (and non-fixed blocks) as the permutation $\beta = (0)(1, 2, 4)(3, 6, 5)$. Hence, if the i-th row in a point orbit matrix for the assumed action of H' on $\mathcal{D}'$ is the one that corresponds to the orbit O_P, then the elements t_{ij}, $\forall j \in \{4, \ldots, 11\}$ must be from the set as denoted in (4.3). However, each of the 18 constructed pairwise non-isomorphic point orbit matrices contains an element equal to 2 or 5 in each row corresponding to an orbit with non-fixed points. This gives a contradiction with (4.3). Therefore, point orbit matrices for the assumed action of the group H' on $\mathcal{D}'$ do not exist. □

5. Self-dual codes constructed from obtained Hadamard matrices

Codes obtained from Hadamard designs and Hadamard matrices gain a lot of attention among people working in coding theory. In this section, we use Hadamard 2-(59, 29, 14) designs and Hadamard matrices of order 60 obtained in this paper to construct linear codes. Recently, in [1, 2, 18], the authors were interested in self-dual ternary codes of length 60 obtained form Hadamard matrices and designs. In the aforementioned papers, the authors have studied extremal self-dual codes, while the codes that we constructed are not extremal. Further, in [27], the authors classify all optimal singly even self-dual codes with an automorphism of order 5 with 12 cycles, and in [17] the authors were interested in binary self-dual codes of length 60.

It is well-known that designs with high degree of symmetry could produce codes with low dimension. If p is a prime, p-rank of the incidence matrix of a 2-(v, k, λ) design can be smaller than $v-1$ only if p divides the order of a design, i.e., if p divides $k-\lambda$ in the case of a symmetric design (see [16]). For a symmetric 2-(59, 29, 14) design, the primes that could be of interest are 3 and 5, whereas for 2-(51, 25, 12), we obtain trivial results whenever $p \neq 13$. Ternary codes and codes over the finite field of order 5 spanned by incidence matrices of 2-(59, 29, 14) designs give raise to 8 non-equivalent ternary linear codes having the length 59 and dimension 30, i.e., linear codes with parameters [59, 30]. In the cases of ternary codes, the best known minimal weight for $[59, 30]_3$ codes is 17, while in the case of the codes over the finite field of order 5, i.e., $[59, 30]_5$, is 18.

The minimum weights of the constructed linear codes are presented in Table 6.

Table 6. Linear codes from 2-(59,29,14) designs.

$GF(q)$	$C(\mathcal{D}_i)$	Minimum weight
$q = 3$	$i = 1, 2, 3, 5$	8
$q = 3$	$i = 6, 8$	11
$q = 3$	$i = 4, 7$	12
$q = 5$	$i = 1, 2, 3, 4, 5, 8$	12
$q = 5$	$i = 6, 7$	13

Moreover, we construct ternary self-dual codes and self-dual codes over the finite field of order 5 spanned by the Hadamard matrices of order 60 obtained in this paper. Self-dual codes constructed from Hadamard matrices of order 60 with an automorphism group of order 29 were studied in [1].

Ternary codes spanned by Hadamard matrices of order 60 give raise to 8 nonequivalent ternary self-dual codes having the length 60 and dimension 30, i.e., linear codes with parameters [60, 30], and to 8 nonequivalent [60, 30] linear codes over finite field of order 5. In the cases of ternary codes, the best known minimal weight for $[60, 30]_3$ codes is 18, while in the case of the codes over the finite field of order 5, i.e., $[60, 30]_5$, is 19.

The minimum weights of constructed self-dual codes are presented in Table 7.

Table 7. Self-dual codes from Hadamard matrices of order 60.

$GF(q)$	$C(\mathcal{H}_i)$	Minimum weight
$q = 3$	$i = 4, 6, 7, 8$	9
$q = 3$	$i = 1, 2, 3, 5$	12
$q = 5$	$i = 2, 3, 4, 5, 7, 8$	12
$q = 5$	$i = 1, 6$	14

Author contributions

This is a joint collaboration with all authors contributing substantially throughout.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the reviewers for comments that improved the presentation of the paper. This work has been fully supported by Croatian Science Foundation under the project 4571 and 5713.

Conflict of interest

The authors declare no conflict of interest.

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