AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 23027–23046.
DOI: 10.3934/math.20241119
Received: 25 May 2024
Revised: 09 July 2024
Accepted: 18 July 2024
Published: 26 July 2024

Research article

Classification of Möbius homogeneous curves in $\mathbb{R}^4$

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Abstract: In this paper, we investigate the Möbius geometry of curves in $\mathbb{R}^4$. First, using moving frame methods we construct a complete system of Möbius invariants for regular curves in $\mathbb{R}^4$ by the isometric invariants. Second, we completely classify the Möbius homogeneous curves in $\mathbb{R}^4$ up to a Möbius transformation of $\mathbb{R}^4$.

Keywords: Möbius transformation group; Möbius homogeneous curve; Möbius arclength; Möbius curvature
Mathematics Subject Classification: 53A04, 53A31

1. Introduction

Recently, the study of Möbius geometry of submanifolds in space forms is a well-developed field in differential geometrisch. For examples, readers can refer to [1–3] and so on. A monograph on a comprehensive introduction to Möbius geometry is [4] of Hertrich-Jeromin.

There are also many studies on the Möbius geometry of curves (one dimensional submanifolds). By using Cartan's method of moving frames, Sulanke in [5] provided the complete Möbius invariants of the generally curved curves in $\mathbb{R}^n$ and fundamental theorem of the curves, which state the existence and uniqueness (up to Möbius transformations) of a curve with given Möbius curvatures. In [6], Magliaro et al. studied some properties of immersed curves in the conformal sphere $\mathbb{Q}_n$, which is viewed as a homogeneous space under the action of the Möbius group. Many authors have studied Möbius geometry of curves and provided the complete Möbius invariant systems for curves in $\mathbb{R}^3$ by using the osculating sphere of the curves (see [7–9]).

In this paper, we investigate the Möbius geometry of curves in $\mathbb{R}^4$. Although the Möbius invariants of the curves in $\mathbb{R}^4$ were provided in [5], exist theoretically, and expression is not specifically given. In this paper, we provide an explicit expression for the Möbius invariants of the curves in $\mathbb{R}^4$ by the isometric invariants. These Möbius invariants contain the first Möbius curvature κ_c, the second Möbius curvature ν_c, and the third Möbius curvature μ_c, expressions are given in Section 3. Thus we prove the following fundamental theorem of curves in $\mathbb{R}^4$.

Theorem 1.1. *Let γ: $I \to \mathbb{R}^4$ be a smooth regular curve in $\mathbb{R}^4$. We assume that the curve γ is not contained in any three-dimensional affine subspace. Then the first Möbius curvature κ_c, the second Möbius curvature ν_c, and the third Möbius curvature μ_c determined the curve γ up to a Möbius transformation of $\mathbb{R}^4$.*

Using the Möbius invariants, we study a notable class of immersed curves: the Möbius homogeneous curves in $\mathbb{R}^4$. A curve

$$x : I \subseteq \mathbb{R} \to \mathbb{R}^n$$

is called a *Möbius homogeneous curve* if the curve is an orbit of a subgroup of the Möbius transformation group of $\mathbb{R}^n$. Such curves are the most symmetries among all curves in $\mathbb{R}^n$ (in the extrinsic sense). In [10], Sulanke classified the Möbius homogeneous curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. In this paper, our main goal is to classify the Möbius homogeneous curves in $\mathbb{R}^4$.

Standard examples of Möbius homogeneous curves in $\mathbb{R}^n$ are the images of Möbius transformations of the isometric homogeneous curves in $\mathbb{R}^n$. The *isometric homogeneous curve* in $\mathbb{R}^n$ is an orbit of a subgroup of the isometric transformation group of $\mathbb{R}^n$. The second examples of Möbius homogeneous curves in $\mathbb{R}^4$ comes from isometric homogeneous curves in 3-dimensional sphere $\mathbb{S}^3$. In this paper, we construct the third examples of Möbius homogeneous curves in $\mathbb{R}^4$ (see Section 3), which are not isometric homogeneous curve. We also indicate which subgroup's orbit these Möbius homogeneous curves belong to. The main theorem is as follows:

Theorem 1.2. *Let γ: $I \to \mathbb{R}^4$ be a Möbius homogeneous curve. Then $\gamma(I)$ is Möbius equivalent to one of the following seven classes of curves:*

(1) The straight lines in $\mathbb{R}^2 \subset \mathbb{R}^4$;

(2) The circles in $\mathbb{R}^2 \subset \mathbb{R}^4$;

(3) The cylindric spiral in $\mathbb{R}^3 \subset \mathbb{R}^4$ given by Example 2.1;

(4) The ring-curve in $\mathbb{R}^3 \subset \mathbb{R}^4$ given by Example 2.2;

(5) The logarithmic spirals in $\mathbb{R}^2 \subset \mathbb{R}^4$ given by Example 2.3;

(6) The space cylinder spiral in $\mathbb{R}^3 \subset \mathbb{R}^4$ given by Example 2.4;

(7) The torus spirals in $\mathbb{R}^4$ given by Example 2.5.

We organize the paper as follows. In Section 2, we review the basic theory and facts about the Möbius transformation group of $\mathbb{R}^n$ and some examples of Möbius homogeneous curves. In Section 3, we give the Möbius invariants of curves in $\mathbb{R}^4$. In Section 4, we prove our main Theorem 1.2.

2. Examples of Möbius homogeneous curves

In this section, we review some facts about the Möbius transformation group. For the details, we refer to [11], or [12]. And we present and construct some examples of Möbius homogeneous curves.

A diffeomorphism ϕ: $\mathbb{S}^n \to \mathbb{S}^n$ is said to be a *Möbius transformation*, if ϕ takes the set of round $(n-1)$-spheres into the set of round $(n-1)$-spheres. All Möbius transformations form a transformation group, which is called the *Möbius transformation group* of $\mathbb{S}^n$ and denoted by $\mathbb{M}(\mathbb{S}^n)$. It is well-known that, for $n \geq 2$, the Möbius transformation group $\mathbb{M}(\mathbb{S}^n)$ of $\mathbb{S}^n$ coincides with the *conformal transformation group* of $\mathbb{S}^n$, denoted by $\mathbb{C}(\mathbb{S}^n)$, i.e.,

$$\mathbb{M}(\mathbb{S}^n) = \mathbb{C}(\mathbb{S}^n).$$

Since $\mathbb{R}^n \cup \{\infty\}$ is conformal to $\mathbb{S}^n$, We also call

$$\mathbb{M}(\mathbb{S}^n) = \mathbb{M}(\mathbb{R}^n \cup \{\infty\})$$

the Möbius transformation group of $\mathbb{R}^n$. Clearly, we have $\mathbb{C}(\mathbb{R}^n) \subseteq \mathbb{M}(\mathbb{S}^n)$.

Let $\mathbb{R}^{n+1}$ denote the $(n+1)$-dimensional Euclidean space, and a dot "·" represents its inner product. The n-dimensional sphere is

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} | x \cdot x = 1\}.$$

The hypersphere $S_p(\rho)$ in $\mathbb{S}^n$ with center $p \in \mathbb{S}^n$ and radius ρ, is given by

$$S_p(\rho) = \{y \in \mathbb{S}^n | p \cdot y = \cos\rho\}, \quad 0 < \rho < \pi.$$

Let

$$\mathbb{D}^{n+1} = \{x \in \mathbb{R}^{n+1} | x \cdot x \leq 1\}.$$

Taking $o \in \mathbb{R}^{n+1}$ such that $o \notin \mathbb{D}^{n+1}$, a line l that passes through the point o intersects the sphere $\mathbb{S}^n$ in two points p, q. Now we define the Möbius inversion Υ_o for the point

$$o \notin \mathbb{D}^{n+1} \subset \mathbb{R}^{n+1}$$

as follows:

$$\Upsilon_o : \mathbb{S}^n \to \mathbb{S}^n, \quad \Upsilon_o(p) = q.$$

Clearly, $\Upsilon_o \in \mathbb{M}(\mathbb{S}^n)$. When the point o is at infinity, the Möbius inversion is indeed a reflection $\Upsilon_o \in O(n+1)$,which is an isometric transformation of $\mathbb{S}^n$. The following proposition is well-known, and readers can refer to [11] or [12] .

Proposition 2.1. *[11, 12] The Möbius transformation group $\mathbb{M}(\mathbb{S}^n)$ is generated by Möbius inversions Υ_o.*

Let $\mathbb{R}^{n+2}_1$ be the Lorentz space, i.e., $\mathbb{R}^{n+2}$ with the scalar product $\langle , \rangle$ defined by

$$\langle x, y\rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_{n+1} y_{n+1}$$

for

$$x = (x_0, x_1, \cdots, x_{n+1}), \quad y = (y_0, y_1, \cdots, y_{n+1}) \in \mathbb{R}^{n+2}.$$

Let $GL(\mathbb{R}^{n+2})$ be the set of invertible $(n+2) \times (n+2)$ matrix, then the Lorentz orthogonal group O(n+1,1) is defined by

$$\mathrm{O(n+1,1)} = \{T \in GL(\mathbb{R}^{n+2}) \mid T I_1 T^t = I_1\},$$

where T^t denotes the transpose of the matrix T and

$$I_1 = \begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix},$$

and I is the $(n+1)\times(n+1)$ unit matrix.

The positive light cone is

$$C_+^{n+1} = \{y = (y_0, \vec{y}_1) \in \mathbb{R}\times\mathbb{R}^{n+1} = \mathbb{R}_1^{n+2} \mid \langle y, y\rangle = 0,\ y_0 > 0\},$$

and $O^+(n+1,1)$ is the subgroup of O(n+1,1) defined by

$$O^+(n+1,1) = \{T \in \mathrm{O(n+1,1)} \mid T(C_+^{n+1}) = C_+^{n+1}\}.$$

Proposition 2.2. *[12] Let*

$$T = \begin{pmatrix} w & u \\ v & Q \end{pmatrix} \in \mathrm{O(n+1,1)},$$

where Q is an $(n+1)\times(n+1)$ matrix. Then $T \in O^+(n+1,1)$ if and only if $w > 0$.

It is well-known that the subgroup $O^+(n+1,1)$ is isomorphic to the Möbius transformation group $\mathbb{M}(\mathbb{S}^n)$. In fact, for any

$$T = \begin{pmatrix} w & u \\ v & Q \end{pmatrix} \in O^+(n+1,1),$$

we can define the Möbius transformation

$$\Psi(T) : \mathbb{S}^n \mapsto \mathbb{S}^n$$

by

$$\Psi(T)(x) = \frac{Qx^t + v}{ux^t + w}, \quad x = (x_1, \cdots, x_{n+1}) \in \mathbb{S}^n.$$

Then the map

$$\Psi : O^+(n+1,1) \mapsto \mathbb{M}(\mathbb{S}^n)$$

is a group isomorphism.

Let $A \in O(n+1)$ be an isometric transformation of $\mathbb{S}^n$, then $A \in \mathbb{M}(\mathbb{S}^n)$ and

$$\Psi^{-1}(A) = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in O^+(n+1,1).$$

Thus,

$$\Psi^{-1}(O(n+1)) \subset O^+(n+1,1)$$

is a subgroup.

The n-dimensional sphere $\mathbb{S}^n$ is diffeomorphic to the projective light cone PC^n,

$$PC^n = \{[z] \in PR^{n+2} \mid z \in C_+^{n+1}\}.$$

The diffeomorphism Φ: $\mathbb{S}^n \to PC^n$ is given by

$$\Phi(x) = [y] = [(1, x)].$$

The group $O^+(n+1,1)$ acts on PC^n by

$$T[p] = [Tp],\ T \in O^+(n+1,1),\ \ [p] \in PC^n.$$

By the conformal diffeomorphic σ, the Möbius transformation group of the n-dimensional Euclidean space $\mathbb{R}^n$ is

$$\mathbb{M}(\mathbb{R}^n) = \{\phi \in \mathbb{M}(\mathbb{S}^n)|\ \phi((-1, \vec{0})) = (-1, \vec{0})\}.$$

By the conformal diffeomorphic σ, we have the diffeomorphism

$$\Phi \circ \sigma : \mathbb{R}^n \cup \{\infty\} \to PC^n$$

given by

$$\Phi \circ \sigma(p) = [(p \cdot p + 1, p \cdot p - 1, 2p)], \quad \Phi \circ \sigma(\infty) = [(1, -1, 0, \cdots, 0)].$$

Let $S_{(p,r)}$ be a hypersphere in $\mathbb{R}^n$ with center p and radius r, and

$$\delta_{(p,r)} = \frac{1}{2r}(p \cdot p - r^2 + 1, p \cdot p - r^2 - 1, 2p) \tag{2.1}$$

a point in $\mathbb{R}^{n+2}_1$ such that

$$\langle \delta_{(p,r)}, \delta_{(p,r)} \rangle = 1.$$

We call $\delta_{(p,r)}$ the sphere coordinates corresponding to the hypersphere $S_{(p,r)}$.

Let $P_{(p,N)}$ be the hyperplane in $\mathbb{R}^n$ passing p with normal vector N, and

$$\pi_{(p,N)} = (p \cdot N, p \cdot N, N) \tag{2.2}$$

a point in $\mathbb{R}^{n+2}_1$ such that

$$\langle \pi_{(p,N)}, \pi_{(p,N)} \rangle = 1.$$

We call $\pi_{(p,N)}$ the sphere coordinates corresponding to the hyperplane $P_{(p,N)}$.

By direct computation, we have the results.

Proposition 2.3. *Let S_1, S_2 be two spheres or hyperplanes and δ_1, δ_2 its sphere coordinates, then*

(1) S_1 and S_2 are internally tangent if and only if $\langle \delta_1, \delta_2 \rangle = 1$.

(2) S_1 and S_2 are externally tangent if and only if $\langle \delta_1, \delta_2 \rangle = -1$.

(3) S_1 and S_2 are perpendicular if and only if $\langle \delta_1, \delta_2 \rangle = 0$.

Let γ: $I \to \mathbb{R}^n$ be a parameter representation of a curve in $\mathbb{R}^n$, and s its arclength. By the diffeomorphism $\Phi \circ \sigma$, we have the curve in PC^n,

$$X_\gamma = \Phi \circ \sigma \circ \gamma : I \to PC^n, \quad X_\gamma(s) = \Phi \circ \sigma \circ \gamma(s).$$

Thus we have the following well-known results:

Theorem 2.1. *[5] Two curves $\gamma(s), \tilde{\gamma}(s)$ in $\mathbb{R}^n$ are Möbius equivalent if and only if there exists $T \in$ $\mathrm{O}^+(n+1, 1)$ such that*

$$\tilde{X}_{\tilde{\gamma}} = T \circ X_\gamma.$$

Let G be a subgroup of $\mathbb{M}(\mathbb{S}^n)$. For any point $p \in \mathbb{S}^n$, the orbit of G through p is

$$G \cdot p = \{\phi(p)|\ \phi \in G\}.$$

Definition 2.1. *A regular curve γ: $I \to \mathbb{R}^n$ is called a Möbius homogeneous curve in $\mathbb{R}^n$ if there exists a subgroup $G \subset \mathbb{M}(\mathbb{R}^n)$ such that the orbit*

$$\gamma(I) = G \cdot p, \quad p \in \gamma(I).$$

It is easy to construct a Möbius homogeneous curve from a subgroup of $O^+(n+1, 1)$, whose action on $\mathbb{R}^{n+2}_1$ is linear. By Theorem 2.1, we have the following proposition:

Proposition 2.4. *A regular curve γ: $I \to \mathbb{R}^n$ is a Möbius homogeneous curvae in $\mathbb{R}^n$ if and only if there exists a subgroup $G \subset O^+(n+1, 1)$ such that the orbit*

$$G \cdot p = X_\gamma(I), \quad p \in \mathbb{C}^{n+1}_+,$$

that is, X_γ is a homogeneous curve in $\mathbb{R}^{n+2}_1$.

Standard examples of Möbius homogeneous curves in $\mathbb{R}^n$ are the images of Möbius transformations of the isometric homogeneous curves in $\mathbb{R}^n$. Clearly, the circles and the straight lines in $\mathbb{R}^n$ are isometric homogeneous curves. Next we present two other class of curves, which are all isometric homogeneous curves.

Example 2.1. *Let $a > 0, b > 0$. The cylindric spiral in $\mathbb{R}^3$ is defined by,*

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \;\; \gamma(t) = (a\cos t, a\sin t, bt).$$

Then a cylindrical spiral is an isometric homogeneous curve in $\mathbb{R}^3 \subset \mathbb{R}^4$.

The second example of Möbius homogeneous curves in $\mathbb{R}^4$ comes from isometric homogeneous curves in the 3-dimensional sphere $\mathbb{S}^3$. To clearly express this type of curve, we need the stereographic σ. The stereographic σ is a conformal diffeomorphism and is defined by

$$\sigma : \mathbb{R}^n \to \mathbb{S}^n \backslash \{(-1, \bar{0})\},$$

which are defined as follows:

$$\sigma(u) = (\frac{1 - |u|^2}{1 + |u|^2}, \frac{2u}{1 + |u|^2}), \quad u \in \mathbb{R}^n.$$

Using σ, we can regard a curve in $\mathbb{S}^n$ as a curve in $\mathbb{R}^n$. Due to conformal invariance, the theory of Möbius geometry of curves is essentially the same whether it is considered in $\mathbb{R}^n$ or $\mathbb{S}^n$.

Example 2.2. *Let $a > 1, 0 < b < 1$, and*

$$c = \frac{b\sqrt{a^2 - 1}}{\sqrt{a^2 - b^2}},$$

the ring-curve is defined in $\mathbb{S}^3$ by

$$\gamma : \mathbb{R} \to \mathbb{S}^3, \;\; \gamma(t) = (c\cos at, c\sin at, \; \sqrt{1 - c^2}\cos bt, \; \sqrt{1 - c^2}\sin bt).$$

Then the ring-curve is an isometric homogeneous curve in $\mathbb{S}^3$, and the image of σ^{-1} of the ring-curve γ is a Möbius homogeneous curve in $\mathbb{R}^3 \subset \mathbb{R}^4$, which call also the ring-curve in $\mathbb{R}^3$.

Next we construct some Möbius homogeneous curves, which are not isometric homogeneous curve. And we use it as an orbital of a subgroup of $\mathbb{M}(\mathbb{S}^n)$ to prove that it is a Möbius homogeneous curve.

Example 2.3. *Let $a > 0, b > 0$. The logarithmic spiral is defined in $\mathbb{R}^2$ by*

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \quad \gamma(t) = e^{at}(\cos bt, \sin bt).$$

Then logarithmic spiral is a Möbius homogeneous curve in $\mathbb{R}^2 \subset \mathbb{R}^4$.

Proposition 2.5. *The logarithmic spiral given by Example 2.3 is a Möbius homogeneous curve in $\mathbb{R}^2$.*

Proof. Up to a similarity, we consider the logarithmic spiral

$$\gamma(t) = e^{at}(\cos t, \sin t).$$

Thus we have the curve X_γ in PC^2,

$$X_\gamma = [(\cosh at, \sinh at, \cos t, \sin t)].$$

Let

$$G_2 = \begin{pmatrix} \cosh at & \sinh at & 0 & 0 \\ \sinh at & \cosh at & 0 & 0 \\ 0 & 0 & \cos t & \sin t \\ 0 & 0 & -\sin t & \cos t \end{pmatrix}.$$

Then G_2 is a subgroup of $\mathrm{O}^+(3, 1)$, and the logarithmic spiral X_γ is the orbit of G_2 acting on the point

$$p = (1, 0, 1, 0) \in C^3_+.$$

Thus the logarithmic spiral is a Möbius homogeneous curve in $\mathbb{R}^2$. □

Example 2.4. *Let $a > 0, b > 1$, and*

$$c = \frac{a}{b}\sqrt{\frac{b^2 - 1}{a^2 + 1}},$$

the space cylinder spiral is defined in $\mathbb{R}^3$ by

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = e^{at}(c\cos bt, c\sin bt, \sqrt{1 - c^2}).$$

Then space cylinder spiral is a Möbius homogeneous curve in $\mathbb{R}^3 \subset \mathbb{R}^4$.

Proposition 2.6. *The space cylinder spiral given by Example 2.4 is a Möbius homogeneous curve in $\mathbb{R}^3$.*

Proof. The space cylinder spiral

$$\gamma(t) = e^{at}(c\cos bt, c\sin bt, \sqrt{1 - c^2})$$

corresponds to the curve X_γ in PC^3,

$$X_\gamma = [(\cosh at, \sinh at, c\cos bt, c\sin bt, \sqrt{1 - c^2})].$$

Let

$$G_3 = \begin{Bmatrix} \cosh at & \sinh at & 0 & 0 & 0 \\ \sinh at & \cosh at & 0 & 0 & 0 \\ 0 & 0 & \cos bt & \sin bt & 0 \\ 0 & 0 & -\sin bt & \cos bt & 0 \\ 0 & 0 & 0 & 0 & 1 \end{Bmatrix}.$$

Then G_3 is a subgroup of $O^+(4,1)$, and the space cylinder spiral $X_\gamma(t)$ is the orbit of G_3 acting on the point

$$p = (1, 0, c, 0, \sqrt{1-c^2}) \in C_+^4.$$

Thus the space cylinder spiral is a Möbius homogeneous curve in $\mathbb{R}^3$. □

Example 2.5. *Let $a > b > 0, c > 0$. The torus spiral is defined in $\mathbb{R}^4$ by,*

$$\gamma : \mathbb{R} \to \mathbb{R}^4, \quad \gamma(t) = e^t(\cos at, \sin at, c\cos bt, c\sin bt).$$

Then the torus spiral is a Möbius homogeneous curve in $\mathbb{R}^4$.

Proposition 2.7. *The torus spiral given by Example 2.5 is a Möbius homogeneous curve in $\mathbb{R}^4$.*

Proof. The torus spiral

$$\gamma(t) = e^t(\sqrt{1-c^2}\cos at, \sqrt{1-c^2}\sin at, c\cos bt, c\sin bt)$$

corresponds to the curve X_γ in PC^4,

$$X_\gamma = [(\cosh t, \sinh t, \sqrt{1-c^2}\cos at, \sqrt{1-c^2}\sin at, c\cos bt, c\sin bt)].$$

Let

$$G_4 = \begin{pmatrix} \cosh t & \sinh t & 0 & 0 & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos at & \sin at & 0 & 0 \\ 0 & 0 & -\sin at & \cos at & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos bt & \sin bt \\ 0 & 0 & 0 & 0 & -\sin bt & \cos bt \end{pmatrix},$$

then G_4 is a subgroup of $O^+(5,1)$. The torus spiral $X_\gamma(t)$ is the orbit of the subgroup $O^+(5,1)$ acting on the point

$$p = (1, 0, \sqrt{1-c^2}, 0, c, 0) \in C_+^5.$$

Thus, the torus spiral in $\mathbb{R}^4$ is Möbius homogeneous. □

3. Möbius invariants of curves in $\mathbb{R}^4$

In this section, we use the osculating sphere of the curve to construct the Möbius invariants of the curve in $\mathbb{R}^n$ for $n = 2, 3, 4$. Specifically, we provide clear expressions for these Möbius invariants by the isometric invariants: curvatures.

Let $\gamma(s) \in \mathbb{R}^n$ be a parameter representation of a curve in the Euclidean space $\mathbb{R}^n$, and s its arclength. If $\gamma'(s) \neq 0$, then the curve γ is called a regular curve.

Definition 3.1. *Let $\gamma_1(s), \gamma_2(s)$ be two parameter curves, s its arclength, and*

$$f(s) = |\gamma_1(s) - \gamma_2(s)|^2.$$

If

$$f(s_0) = 0,\ f'(s_0) = 0, \cdots, f^{(n)}(s_0) = 0,$$

then two curves $\gamma_1(s), \gamma_2(s)$ are called having n-order contact at the point

$$p = \gamma_1(s_0) = \gamma_2(s_0).$$

The osculating sphere of the curve γ at an point p is the round $(n-1)$-sphere with n-order contact with γ at the point p.

An important property of an osculating sphere is that it is Möbius invariant.

Theorem 3.1. *Let $\gamma \in \mathbb{R}^n$ be a smooth curve, and $\phi \in \mathbb{M}(\mathbb{R}^n)$. For a point $p \in \gamma$, if S is the osculating sphere of γ at p, then $\phi(S)$ is the osculating sphere of $\phi(\gamma)$ at $\phi(p)$.*

Proof. Since a Möbius transformation ϕ is a diffeomorphism, so it maintain the order of contact between curves. On the other hand, a Möbius transformation ϕ takes the set of round $(n-1)$-spheres into the set of round $(n-1)$-spheres, thus $\phi(S)$ is the osculating sphere of $\phi(\gamma)$ at $\phi(p)$. □

3.1. Möbius invariants of curves in $\mathbb{R}^2$

Although Möbius invariants of curves in $\mathbb{R}^2$ are well-known (see [5]), for completeness, we provide the deduction process of the Möbius invariants in this section.

Let $\gamma(s)$ be a smooth regular curve in $\mathbb{R}^2$ with s arc length parameter, then there exists the Frenet frame $\{\alpha(s), \beta(s)\}$ along the curve $\gamma(s)$, and the Frenet formula is as follows:

$$\gamma'(s) = \alpha(s),\quad \alpha'(s) = \kappa(s)\beta(s),\quad \beta'(s) = -\kappa(s)\alpha(s). \tag{3.1}$$

Next, we consider the curve with $\kappa(s) > 0$. It is clear from the definition of osculating sphere that the radius $R(s)$ and the center $a(s)$ of the osculating sphere of $\gamma(s)$ are respectively given by

$$R(s) = \frac{1}{\kappa(s)},\quad a(s) = \gamma(s) + R(s)\beta(s).$$

Thus, the sphere coordinates of the osculating sphere in $\mathbb{R}^4_1$ are by

$$y(s) = \frac{1}{2R(s)}\Big(2a(s)\cdot\gamma(s) - |\gamma(s)|^2 + 1, 2a(s)\cdot\gamma(s) - |\gamma(s)|^2 - 1, 2a(s)\Big).$$

Thus,

$$\langle y(s), y(s)\rangle = 1,$$

and by (3.1) we have

$$\begin{aligned}
y'(s) &= \frac{\kappa'(s)}{2}\Big(|\gamma(s)|^2 + 1, |\gamma(s)|^2 - 1, 2\gamma(s)\Big),\\
y''(s) &= \frac{\kappa''(s)}{2}\Big(|\gamma(s)|^2 + 1, |\gamma(s)|^2 - 1, 2\gamma(s)\Big) + \kappa'(s)\Big(\gamma(s)\cdot\alpha(s), \gamma(s)\cdot\alpha(s), \alpha(s)\Big).
\end{aligned}$$

By direct calculation,

$$\langle y'(s), y'(s)\rangle = 0, \ \ \langle y''(s), y''(s)\rangle = \kappa'(s)^2.$$

When $\kappa'(s) \neq 0$, we can choose a parameter t such that

$$\langle y''(t), y''(t)\rangle = 1.$$

Next, we assume that $\kappa'(s) > 0$, then the relation $t = t(s)$ between the parameter t and the arclength s is as follows:

$$\frac{dt(s)}{ds} = t'(s) = \sqrt{\kappa'(s)}. \tag{3.2}$$

The parameter t is called the Möbius arclength parameter, which is invariant under $\mathbb{M}(\mathbb{R}^2)$ by Theorem 3.1. Thus, $y(t)$ is a Möbius invariant vector field in $\mathbb{R}^4_1$ along the curve γ.

Remark 3.1. *The Möbius arclength parameter can be defined when $\kappa'(s) \neq 0$. If*

$$\kappa'(s) \equiv 0,$$

then the curve is a circle or straight line. In fact, all circles and straight lines are Möbius equivalent.

Next, we assume that the curve $\gamma(t)$ satisfies $\kappa'(t) \neq 0$ with the Möbius arclength parameter t. We define

$$\begin{aligned} T_1(t) &= y(t), \\ T_2(t) &= y'(t), \\ T_3(t) &= y''(t), \\ T_4(t) &= -y'''(t) - \kappa_c y'(t), \end{aligned} \tag{3.3}$$

where

$$\kappa_c = \frac{1}{2}\langle y'''(t), y'''(t)\rangle.$$

We call κ_c the Möbius curvature of γ. Clearly, the Möbius curvature κ_c is a Möbius invariants.

By direct computation, we can obtain Table 1.

Table 1. The frame-1.

$\langle , \rangle$	$T_1(t)$	$T_2(t)$	$T_3(t)$	$T_4(t)$
$T_1(t)$	1	0	0	0
$T_2(t)$	0	0	0	1
$T_3(t)$	0	0	1	0
$T_4(t)$	0	1	0	0

Thus, $\{T_1(t), T_2(t), T_3(t), T_4(t)\}$ is a moving frame in $\mathbb{R}^4_1$ along $\gamma(s)$. By direct computation, we get the Frenet formula corresponding to the moving frame as follows:

$$\begin{pmatrix} T_1'(t) \\ T_2'(t) \\ T_3'(t) \\ T_4'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\kappa_c & 0 & -1 \\ -1 & 0 & \kappa_c & 0 \end{pmatrix} \begin{pmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \\ T_4(t) \end{pmatrix}. \tag{3.4}$$

According to the well-known moving frame theory, we have the following results:

Theorem 3.2. *Let $\gamma_1(t), \gamma_2(t)$ be two parameter curves in $\mathbb{R}^2$, and t its Möbius arclength, If their Möbius curvature*

$$\kappa_{c1}(t) = \kappa_{c2}(t),$$

then $\gamma_1(t), \gamma_2(t)$ are Möbius equivalent.

Thus, the Möbius arclength and the Möbius curvature of a regular curve in $\mathbb{R}^2$ are complete Möbius invariants of the curve. By isometric invariant: curvature $\kappa(s)$ and arclength parameter s, the Möbius curvature can be given by

$$\kappa_c(s) = \frac{1}{2\kappa'(s)}\Big[\frac{1}{4}(\ln|\kappa'(s)|)'^2_s - (\ln|\kappa'(s)|)''_s + \kappa^2(s)\Big]. \tag{3.5}$$

3.2. Möbius invariants of curves in $\mathbb{R}^3$

Although Möbius invariants of curves in $\mathbb{R}^3$ are well-known (see [5, 8]), for completeness, we provide the deduction process of the Möbius invariants in this section.

Let $\gamma(s)$ be a smooth regular curve in $\mathbb{R}^3$ with an s arc length parameter. Then there exists the Frenet frame $\{\alpha(s), N(s), \beta(s)\}$ along the curve $\gamma(s)$, and the Frenet formula is as follows:

$$\begin{aligned} \gamma'(s) &= \alpha(s),\\ \alpha'(s) &= \kappa(s)N(s),\\ N'(s) &= -\kappa(s)\alpha(s) + \tau(s)\beta(s),\\ \beta'(s) &= -\tau(s)N(s), \end{aligned} \tag{3.6}$$

where $\kappa(s), \tau(s)$ are the curvature and torsion of $\gamma(s)$, respectively.

Next, we assume that $\kappa(s) > 0, \tau(s) \neq 0$, where $\tau(s) \neq 0$ means that the curve cannot be contained in any two-dimensional affine subspace, and let

$$R(s) = \frac{1}{\kappa(s)}.$$

It follows from the definition of an osculating sphere that the radius $r(s)$ and the center $a(s)$ of the osculating sphere of $\gamma(s)$ are respectively given by

$$r(s) = \sqrt{R(s)^2 + \Big(\frac{R'(s)}{\tau(s)}\Big)^2}, \quad a(s) = \gamma(s) + R(s)N(s) + \frac{R'(s)}{\tau(s)}\beta(s).$$

Thus, the sphere coordinates of the osculating sphere $\mathbb{R}^5_1$ are by

$$y(s) = \frac{1}{2r(s)}\Big(2a(s)\cdot\gamma(s) - |\gamma(s)|^2 + 1, 2a(s)\cdot\gamma(s) - |\gamma(s)|^2 - 1, 2a(s)\Big).$$

Thus,

$$\langle y(s), y(s)\rangle = 1,$$

and by (3.6) we have

$$y'(s) = \frac{1}{2r(s)}\Big(2A(s)\gamma(s)\cdot\beta(s), 2A(s)\gamma(s)\cdot\beta(s), 2A(s)\beta(s)\Big) + \Big(\frac{1}{2r(s)}\Big)' 2r(s)y(s),$$

where

$$A(s) = R(s)\tau(s) + \Big(\frac{R'(s)}{\tau(s)}\Big)'.$$

By direct calculation,

$$\langle y'(s), y'(s)\rangle = \frac{R(s)^2 A(s)^2}{r(s)^4}.$$

If

$$\frac{R(s)^2 A(s)^2}{r(s)^4} = 0,$$

then $\tau(s) = 0$ or $A(s) = 0$.

When $\tau(s) = 0$ for every $s \in [s_1, s_2]$, then $\gamma(s)$ is a plane curve in this segment.

When $A(s) = 0$ for every $s \in [s_1, s_2]$, then $\gamma(s)$ lays on a 2-dimensional sphere in this segment, and up to a Möbius transformation, it will turn a plane curve.

Now we assume that $\tau(s) \neq 0, A(s) \neq 0$ along the curve $\gamma(s)$. We can choose a parameter t such that

$$\langle y'(t), y'(t)\rangle = 1.$$

In fact, the relation $t = t(s)$ between the parameter t and the arclength s is as follows:

$$\frac{dt(s)}{ds} = t'(s) = \frac{R(s)A(s)}{r(s)^2}. \tag{3.7}$$

The parameter t is called the Möbius arclength parameter of the curve γ, which is invariant under the Möbius transformation of $\mathbb{R}^3$ by Theorem 3.1. Thus, $y(t)$ is a Möbius invariant vector field in $\mathbb{R}^5_1$ along the curve γ.

Next, we construct a moving frame in $\mathbb{R}^5_1$ along the curve $\gamma(t)$. Let $\gamma(t)$ be the regular curve, where t is the Möbius arclength parameter. We define

$$\begin{aligned}
T_1(t) &= y(t),\\
T_2(t) &= y'(t),\\
T_3(t) &= y''(t) + y(t),\\
T_4(t) &= \frac{1}{\kappa_c} T_3'(t),\\
T_5(t) &= \frac{-1}{\kappa_c} T_4'(t) + \frac{\tau_c}{\kappa_c} T_3(t),
\end{aligned} \tag{3.8}$$

where

$$\kappa_c(t) = \sqrt{\langle T_3'(t), T_3'(t)\rangle}, \quad \tau_c(t) = \frac{-\langle T_4'(t), T_4'(t)\rangle}{2\kappa_c(t)}. \tag{3.9}$$

By direct computation, we can obtain Table 2.

Table 2. The frame-2.

$\langle , \rangle$	$T_1(t)$	$T_2(t)$	$T_3(t)$	$T_4(t)$	$T_5(t)$
$T_1(t)$	1	0	0	0	0
$T_2(t)$	0	1	0	0	0
$T_3(t)$	0	0	0	0	1
$T_4(t)$	0	0	0	1	0
$T_5(t)$	0	0	1	0	0

Thus, $\{T_1(t), T_2(t), T_3(t), T_4(t), T_5(t)\}$ is a moving frame in $\mathbb{R}^5_1$ along $\gamma(t)$. By direct computation, we get the Frenet formula corresponding to the moving frame as follows:

$$\begin{pmatrix} T_1'(t) \\ T_2'(t) \\ T_3'(t) \\ T_4'(t) \\ T_5'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \kappa_c & 0 \\ 0 & 0 & \tau_c & 0 & -\kappa_c \\ 0 & -1 & 0 & -\tau_c & 0 \end{pmatrix} \begin{pmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \\ T_4(t) \\ T_5(t) \end{pmatrix}. \tag{3.10}$$

Since the moving frame $\{T_1(t), T_2(t), T_3(t), T_4(t), T_5(t)\}$ is invariant under the Möbius transformation of $\mathbb{R}^3$, from the Frenet formula (3.10) and the well-known moving frame theory, we have the results.

Theorem 3.3. *In (3.9), κ_c and τ_c are two Möbius invariants of the curve γ, which are called the Möbius curvature and Möbius torsion of the curve γ, respectively.*

Theorem 3.4. *Let $\gamma_1(t), \gamma_2(t)$ be two parameter curves in $\mathbb{R}^3$, and t its Möbius arclength, If their Möbius curvature $\kappa_{c1}(t) = \kappa_{c2}(t)$ and the Möbius torsion*

$$\tau_{c1}(t) = \tau_{c2}(t),$$

then $\gamma_1(t), \gamma_2(t)$ are Möbius equivalent.

Thus, the Möbius arclength, the Möbius curvature, and the Möbius torsion of a regular curve in $\mathbb{R}^3$ are complete Möbius invariants of the curve. By isometric invariant: curvatures $\kappa(s), \tau(s)$ and arclength parameter s, the Möbius curvature κ_c and the Möbius torsion τ_c can be given by, respectively,

$$\begin{aligned} \kappa_c(s) &= \frac{\tau(s) r(s)^5}{R(s)^4 A(s)^2}, \\ \tau_c(s) &= \frac{-R(s)^2}{2\tau(s) r(s)} \Big[\Big(\ln \Big(\frac{\tau(s) r(s)^3}{R(s)^3 A(s)} \Big) \Big)'^2_s + \frac{1}{R(s)^2} - 2 \Big(\ln \Big(\frac{\tau(s) r(s)^3}{R(s)^3 A(s)} \Big) \Big)''_s \Big]. \end{aligned} \tag{3.11}$$

In Eq (3.11), we can assume that $\tau(s) > 0$, since $\tau(s) \neq 0$ along the curve γ.

3.3. Möbius invariants of curves in $\mathbb{R}^4$

Let $\gamma(s)$ be a smooth regular curve in $\mathbb{R}^4$ with s arc length parameter. We assume that the curve is not contained in any three-dimensional affine subspace. Let $\alpha(s), N(s), \beta(s), \delta(s)$ be the Frenet frame, then the Frenent formula is given by

$$\begin{pmatrix} \alpha'(s) \\ N'(s) \\ \beta'(s) \\ \delta'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & 0 & 0 \\ -\kappa_1(s) & 0 & \kappa_2(s) & 0 \\ 0 & -\kappa_2(s) & 0 & \kappa_3(s) \\ 0 & 0 & -\kappa_3(s) & 0 \end{pmatrix} \begin{pmatrix} \alpha(s) \\ N(s) \\ \beta(s) \\ \delta(s) \end{pmatrix}. \tag{3.12}$$

The functions $\kappa_1(s), \kappa_2(s), \kappa_3(s)$ are respectively the first, second, third curvatures of the curve $\gamma(s)$. Since the curve $\gamma(s)$ is not contained in any three-dimensional affine subspace, we can assume that $\kappa_1(s) > 0, \kappa_2(s) > 0, \kappa_3(s) > 0$.

It follows from the definition of an osculating sphere that the radius $r(s)$ and the center $a(s)$ of the osculating sphere $\mathbb{R}^4$ of $\gamma(s)$ are respectively given by

$$r(s) = \sqrt{R(s)^2 + \Big(\frac{R'(s)}{\kappa_2(s)}\Big)^2 + A_1(s)^2},$$
$$a(s) = \gamma(s) + R(s)N(s) + \frac{R'(s)}{\kappa_2(s)}\beta(s) + A_1(s)\delta(s),$$

where

$$R(s) = \frac{1}{\kappa_1(s)}, \quad A_1(s) = \frac{R''(s)}{\kappa_2(s)\kappa_3(s)} + \frac{R(s)\kappa_2(s)}{\kappa_3(s)} - \frac{R'(s)\kappa_2'(s)}{\kappa_2(s)^2\kappa_3(s)}.$$

Thus, the sphere coordinates of the osculating sphere $\mathbb{R}^6_1$ are by

$$y(s) = \frac{1}{2r(s)}\Big(2a(s)\cdot\gamma(s) - |\gamma(s)|^2 + 1, 2a(s)\cdot\gamma(s) - |\gamma(s)|^2 - 1, 2a(s)\Big),$$

Thus,

$$\langle y(s), y(s)\rangle = 1,$$

and by (3.12) we have

$$y'(s) = \frac{1}{r(s)}\Big(A(s)\gamma(s)\cdot\delta(s), A(s)\gamma(s)\cdot\delta(s), A(s)\delta(s)\Big) - \frac{A(s)A_1(s)}{r(s)^2}y(s),$$

where

$$A(s) = \frac{R'(s)\kappa_3(s)}{\kappa_2(s)} + A_1'(s), \quad B(s) = \sqrt{R(s)^2 + \Big(\frac{R'(s)}{\kappa_2(s)}\Big)^2}.$$

By direct calculation, we have the following equation,

$$\langle y'(s), y'(s)\rangle = \frac{A(s)^2B(s)^2}{r(s)^4}.$$

When $A(s) = 0$ for every $s \in [s_1, s_2]$, then $\gamma(s)$ lays on a 3-dimensional sphere in this segment, up to a Möbius transformation, $\gamma(s)$ is contained in some three-dimensional affine subspace. Thus, we can assume that $A(s) > 0$ along the curve γ. We can choose a parameter t such that

$$\langle y'(t), y'(t)\rangle = 1.$$

In fact, the relation $t = t(s)$ between the parameter t and the arclength s is as follows:

$$\frac{dt(s)}{ds} = t'(s) = \frac{A(s)B(s)}{r(s)^2}. \tag{3.13}$$

The parameter t is called the Möbius arclength parameter, which is invariant under the Möbius transformation group of $\mathbb{R}^4$ by Theorem 3.1. Thus $y(t)$ is a Möbius invariant vector field in $\mathbb{R}^6_1$ along the curve $\gamma(s)$.

Next, we construct a moving frame in $\mathbb{R}^6_1$ along the curve γ. Let $\gamma(t)$ be the regular curve, where t is the Möbius arclength parameter. We define

$$\begin{aligned}
T_1(t) &= y(t),\\
T_2(t) &= y'(t),\\
T_3(t) &= \frac{1}{\kappa_c(t)}(y''(t) + y(t)),\\
T_4(t) &= T_3'(t) + \kappa_c(t)y'(t),\\
T_5(t) &= \frac{1}{\nu_c(t)}T_4'(t),\\
T_6(t) &= \frac{-1}{\nu_c(t)}T_5'(t) + \frac{\mu_c(t)}{\nu_c(t)}T_4(t),
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\kappa_c(t) &= \sqrt{\langle T_2'(t) + T_1(t), T_2'(t) + T_1(t)\rangle},\\
\nu_c(t) &= \sqrt{\langle T_4'(t), T_4'(t)\rangle},\\
\mu_c(t) &= \frac{-\langle T_5'(t), T_5'(t)\rangle}{2\nu_c(t)}.
\end{aligned} \tag{3.15}$$

By direct computation, we can obtain Table 3.

Table 3. The frame-3.

$\langle , \rangle$	$T_1(t)$	$T_2(t)$	$T_3(t)$	$T_4(t)$	$T_5(t)$	$T_6(t)$
$T_1(t)$	1	0	0	0	0	0
$T_2(t)$	0	1	0	0	0	0
$T_3(t)$	0	0	1	0	0	0
$T_4(t)$	0	0	0	0	0	1
$T_5(t)$	0	0	0	0	1	0
$T_6(t)$	0	0	0	1	0	0

Thus, $\{T_1(t), T_2(t), T_3(t), T_4(t), T_5(t), T_6(t)\}$ is a moving frame in $\mathbb{R}^6_1$ along $\gamma(t)$. By direct computation, we get the Frenet formula corresponding to the moving frame as follows:

$$\begin{pmatrix} T_1'(t)\\ T_2'(t)\\ T_3'(t)\\ T_4'(t)\\ T_5'(t)\\ T_6'(t)\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & \kappa_c(t) & 0 & 0 & 0\\ 0 & -\kappa_c(t) & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & \nu_c(t) & 0\\ 0 & 0 & 0 & \mu_c(t) & 0 & -\nu_c(t)\\ 0 & 0 & -1 & 0 & -\mu_c(t) & 0\end{pmatrix}\begin{pmatrix} T_1(t)\\ T_2(t)\\ T_3(t)\\ T_4(t)\\ T_5(t)\\ T_6(t)\end{pmatrix}. \tag{3.16}$$

Since the moving frame $\{T_1(t), T_2(t), T_3(t), T_4(t), T_5(t), T_6(t)\}$ is invariant under the Möbius transformation of $\mathbb{R}^4$, from the Frenet formula (3.16) and the well-known moving frame theory, we have the results:

Theorem 3.5. *In (3.15), $\kappa_c(t), \nu_c(t), \mu_c(t)$ are three Möbius invariants, which are called the first Möbius curvature, the second Möbius curvature and the third Möbius curvature of the curve γ in $\mathbb{R}^4$, respectively.*

Theorem 3.6. *Let $\gamma_1(t), \gamma_2(t)$ be two regular curves in $\mathbb{R}^4$, and t its Möbius arclength, If their the first, the second and the third Möbius curvature*

$$\kappa_{c1}(t) = \kappa_{c2}(t), \quad \nu_{c1}(t) = \nu_{c2}(t), \quad \mu_{c1}(t) = \mu_{c2}(t),$$

then $\gamma_1(t), \gamma_2(t)$ are Möbius equivalent.

Thus, the Möbius arclength, the first, the second, and the third Möbius curvature of a regular curve in $\mathbb{R}^4$ are complete Möbius invariants of the curve. By isometric invariant: curvatures $\kappa_1(s), \kappa_2(s), \kappa_3(s)$ and arclength parameter s, the first, the second, and the third Möbius curvature can be given by, respectively,

$$\begin{aligned} \kappa_c(s) &= \frac{\kappa_3(s)r(s)^3R(s)}{A(s)B(s)^3}, \\ \nu_c(s) &= \frac{\kappa_2(s)r(s)^4}{A(s)^2B(s)R(s)^2}, \\ \mu_c(s) &= \frac{-R(s)^2}{2B(s)\kappa_2(s)}\Big[\Big(\ln\Big(\frac{A(s)R(s)^2}{\kappa_2(s)r(s)^2}\Big)\Big)_s'^2 + \frac{1}{R(s)^2} + 2\Big(\ln\Big(\frac{A(s)R(s)^2}{\kappa_2(s)r(s)^2}\Big)\Big)_s''\Big]. \end{aligned} \tag{3.17}$$

4. The proof of Theorem 1.2.

Let γ: $I \to \mathbb{R}^4$ be a Möbius homogeneous regular curve in $\mathbb{R}^4$. Since the Möbius homogeneous curve γ is an orbit of a subgroup of the Möbius transformation group of $\mathbb{R}^4$, then for any two point $s_1, s_2 \in I$, there exists a Möbius transformation $\phi \in \mathbb{M}(\mathbb{R}^4)$ such that

$$\gamma(s_1) = \phi(\gamma(s_2)), \quad \gamma(I) = \phi(\gamma(I)).$$

If there is a point $\gamma(s_0)$ such that $\gamma'(s_0) = 0$, then $\gamma'(s) \equiv 0$ for all $s \in I$, and the curve is a point. In this paper, we do not consider this trivial case. So we can assume that $\gamma'(s) \neq 0$ for all $s \in I$, that is the Möbius homogeneous curve is a regular curve.

We divide the proof into three cases:

Case 1. The curve γ is contained in some two-dimensional affine subspace.

Case 2. The curve γ is contained in some three-dimensional affine subspace, but the curve γ is not contained in any two-dimensional affine subspace.

Case 3. The curve γ is not contained in any three-dimensional affine subspace.

Now we consider Case 1. Since the curve γ is contained in some two-dimensional affine subspace, we can assume that the curve γ is a planar curve, that is, γ: $I \to \mathbb{R}^2$.

If the curvature $\kappa'(s) \equiv 0$, then the curve $\gamma(s)$ is a circle or a straight line.

If the curvature $\kappa'(s) \neq 0$ then the Möbius curvature $\kappa_c(t)$ can be defined. Since conformal curvature $\kappa_c(t)$ is a Möbius invariant, it is a constant.

By direct computation, the conformal curvature $\kappa_c(t)$ of the logarithmic spiral

$$\gamma(t) = e^{at}(\cos bt, \sin bt)$$

in $\mathbb{R}^2$ is

$$\kappa_c(t) = \frac{b^2 - a^2}{2ab}.$$

Since the constants a, b are arbitrary, by Theorem 3.2, we know that the Möbius homogeneous curve is Möbius equivalent to the logarithmic spiral. Thus, we have the following results.

Proposition 4.1. *Let γ: $I \to \mathbb{R}^2$ be a Möbius homogeneous curve in $\mathbb{R}^2$, then the curve is Möbius equivalent to a circle, a straight line, or a logarithmic spiral.*

Now we consider Case 2. Since the curve γ is contained in some three-dimensional affine subspace and is not contained in any two-dimensional affine subspace. Thus, we can assume that the curve γ: $I \to \mathbb{R}^3$ and

$$\tau(s) \neq 0, \ \ A(s) \neq 0$$

along the curve $\gamma(s)$. And the Möbius curvature $\kappa_c(t)$ and the Möbius torsion $\tau_c(t)$ can be defined.

Since the curve γ is Möbius homogeneous, the Möbius curvature $\kappa_c(t)$ and the Möbius torsion $\tau_c(t)$ are constant. Thus, $\frac{\tau_c}{\kappa_c}$ is a constant.

If

$$\frac{\tau_c}{\kappa_c} > \frac{-1}{2},$$

we consider the space cylinder spiral

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \ \ \gamma(t) = e^{at}(c\cos bt, c\sin bt, \sqrt{1-c^2}), \ \ a > 0, \ b > 1, \ c = \frac{a}{b}\sqrt{\frac{b^2-1}{a^2+1}},$$

which is a Möbius homogeneous curve in $\mathbb{R}^3$.

By direct computation, we obtain the Möbius curvature $\kappa_c(t)$ and the Möbius torsion of the space cylinder spiral, respectively,

$$\kappa_c = \sqrt{(b^2-1)(a^2+1)}, \quad \tau_c = \frac{a^2 - b^2 + 1}{2\sqrt{(b^2-1)(a^2+1)}}.$$

Clearly, we have

$$\frac{\tau_c}{\kappa_c} > \frac{-1}{2}.$$

Since the constants a, b are arbitrary, thus the curve γ is Möbius equivalent to the space cylinder spiral.

If

$$\frac{\tau_c}{\kappa_c} < \frac{-1}{2},$$

we consider the ring curve

$$\gamma = \sigma^{-1} \circ \tilde{\gamma},$$

here, the curve $\tilde{\gamma}$ is a ring-curve in $\mathbb{S}^3$

$$\tilde{\gamma} : \mathbb{R} \to \mathbb{S}^3, \quad \tilde{\gamma}(t) = (c\cos at, c\sin at, \sqrt{1-c^2}\cos bt, \sqrt{1-c^2}\sin bt),$$

where $a > 1, 0 < b < 1$ and

$$c = b\sqrt{\frac{a^2 - 1}{a^2 - b^2}}.$$

In fact, the ring curve $\tilde{\gamma}$ is a homogeneous curve in $\mathbb{S}^3$.

By direct computation, we obtain the Möbius curvature $\kappa_c(t)$ and the Möbius torsion of the ring-curve $\sigma^{-1} \circ \tilde{\gamma}$, respectively,

$$\kappa_c = \sqrt{(a^2 - 1)(1 - b^2)}, \quad \tau_c = \frac{1 - a^2 - b^2}{2\sqrt{(a^2 - 1)(1 - b^2)}}.$$

Clearly, we have

$$\frac{\tau_c}{\kappa_c} > \frac{-1}{2}.$$

Since the constants a and b are arbitrary, the curve is Möbius equivalent to the ring curve $\sigma^{-1} \circ \tilde{\gamma}$.

If

$$\frac{\tau_c}{\kappa_c} = \frac{-1}{2},$$

we consider the cylindric spiral γ: $\mathbb{R} \to \mathbb{R}^3$,

$$\gamma(t) = (a\cos\sqrt{1 + a^2}t, a\sin\sqrt{1 + a^2}t, t), \quad a > 0.$$

which is an isometric homogeneous curve in $\mathbb{R}^3$.

By direct computation, we obtain the Möbius curvature $\kappa_c(t)$ and the Möbius torsion of the cylindric spiral, respectively

$$\kappa_c = a, \quad \tau_c = \frac{-a}{2}.$$

Clearly, we have

$$\frac{\tau_c}{\kappa_c} = \frac{-1}{2},$$

thus, the curve is Möbius equivalent to the cylindric spiral.

Thus, we have the following results:

Proposition 4.2. *Let γ: $I \to \mathbb{R}^3$ be a Möbius homogeneous curve in $\mathbb{R}^3$, which is not contained in any two-dimensional affine subspace, then the curve is Möbius equivalent to a space cylinder spiral, a cylinder spiral, or the ring curve.*

Finally, we consider **Case 3**. Since the curve γ is not contained in any three-dimensional affine subspace, we can assume that $\kappa_1(s) > 0, \kappa_2(s) > 0, \kappa_3(s) > 0$ and $A(s) \neq 0$ along the curve $\gamma(s)$. And the Möbius curvatures $\kappa_c(t), \nu_c(t), \mu_c(t)$ can be defined.

Since the curve γ is Möbius homogeneous, the Möbius curvatures $\kappa_c(t), \nu_c(t), \mu_c(t)$ are constant, and $\kappa_c > 0, \nu_c > 0$.

By direct computation, we obtain the Möbius curvatures $\kappa_c(t), \nu_c(t), \mu_c(t)$ of the torus spiral

$$\gamma(t) = e^t(\sqrt{1 - c^2}\cos at, \sqrt{1 - c^2}\sin at, c\cos bt, c\sin bt), \quad a > 0, \ b > 0, \ 0 < c < 1,$$

are constants, which are given by

$$\begin{aligned}\kappa_c &= \frac{(a^2-b^2)(a^2+1)(b^2+1)c\sqrt{1-c^2}}{ab\alpha\sqrt{\alpha}},\\ \nu_c &= \Big[(a^2+1)(b^2+1)\beta-\alpha^2a^2b^2\Big]^{\frac{3}{2}}\frac{1}{\alpha^3a^2b^2},\\ \mu_c &= \frac{\alpha-\beta}{2\Big[(a^2+1)(b^2+1)\beta-\alpha^2a^2b^2\Big]^{\frac{1}{2}}},\end{aligned} \tag{4.1}$$

where

$$\alpha = a^2(1-c^2)+b^2c^2+1, \quad w\,\beta = a^2(a^2+1)(1-c^2)+b^2(b^2+1)c^2.$$

Since for any constants $\kappa_c > 0, \nu_c > 0, \mu_c$, we can choose the constants $a > 0,\ b > 0,\ 0 < c < 1$ satisfying the Eq (4.1), thus the curve γ is Möbius equivalent to a torus spiral.

Proposition 4.3. *Let γ: $I \to \mathbb{R}^4$ be a Möbius homogeneous curve in $\mathbb{R}^4$, which is not contained in any three-dimensional affine subspace, then the curve is Möbius equivalent to a torus spiral.*

Combining Propositions 4.1–4.3, we finish the proof of our main Theorem 1.2.

5. Conclusions

In this paper, we construct a complete system of Möbius invariants for regular curves in $\mathbb{R}^4$ by the isometric invariants. Secondly, we completely classify the Möbius homogeneous curves in $\mathbb{R}^4$ up to a Möbius transformation of $\mathbb{R}^4$.

Author contributions

Tongzhu Li: writing, the idea of the theorem proof process; Ruiyang Lin: writing, the specific calculation. Both authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Authors are supported by the Grant No. 12071028 of NSFC.

Conflict of interest

The authors declare that they have no conflicts of interest.

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