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Research article

A two-step Ulm-Chebyshev-like Cayley transform method for inverse eigenvalue problems with multiple eigenvalues

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Abstract: Our focus in this study was on examining the convergence problem of a novel method, inspired by the Ulm-Chebyshev-like Cayley transform method, which was designed to solve the inverse eigenvalue problems (IEPs) with multiple eigenvalues. Compared with other existing methods, the proposed method has higher convergence order and/or requires less operations. Under the assumption that the relative generalized Jacobian matrices at a solution are nonsingular, the proposed method was proved to be convergent with cubic convergence. Experimental findings demonstrated the practicality and efficiency of the suggested approaches.

Keywords: inverse eigenvalue problems; multiple eigenvalues; Ulm-Chebyshev-like method; Cayley transform; cubically convergent
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1. Introduction

Let $A_0,\ A_1,\dots,A_n$ be $n+1$ real symmetric n-by-n matrices. For any $\mathbf{c}=(c_1,c_2,\dots,c_n)^T\in\mathbb{R}^n$ such that the eigenvalues $\{\lambda_i(A(\mathbf{c}))\}_{i=1}^n$ of the matrices

$$A(\mathbf{c})\equiv A_0+\sum_{i=1}^{n}c_iA_i \tag{1.1}$$

with the order $\lambda_1(A(\mathbf{c}))\le\lambda_2(\mathbf{A}(\mathbf{c}))\le\cdots\le\lambda_\mathbf{n}(\mathbf{A}(\mathbf{c}))$. In this note, the inverse eigenvalue problem (IEP) defined here is, for the given n real numbers $\{\lambda_i^*\}_{i=1}^n$ with the order $\lambda_1^*\le\lambda_2^*\le\cdots\le\lambda_n^*$, to find a vector $\mathbf{c}^*\in\mathbb{R}^n$ such that

$$\lambda_i(A(\mathbf{c}^*))=\lambda_i^*,\quad i=1,\dots,n. \tag{1.2}$$

The IEP is utilized in a wide range of fields including the inverse Toeplitz eigenvalue problem [1–3], structural dynamics [4], molecular spectroscopy [5], the pole assignment problem [6], the inverse Sturm-Liouville problem [7], and also problems in mechanics applications [8, 9], structural integrity assessments [10], geophysical studies [11], particle physics research [12], numerical analysis [13], and dynamics systems [14]. For further insights into the diverse practical uses, underlying mathematical principles, and computational techniques of general IEPs, readers may consult the comprehensive review articles [15, 16] and the relevant literature [17, 18].

The IEP (1.2) can be represented mathematically through a set of non-linear equations:

$$\mathbf{f}(\mathbf{c}) := (\lambda_1(A(\mathbf{c})) - \lambda_1^*, \lambda_2(A(\mathbf{c})) - \lambda_2^*, \ldots, \lambda_n(A(\mathbf{c})) - \lambda_n^*)^T = \mathbf{0}. \tag{1.3}$$

In situations where the given eigenvalues are distinct, i.e.,

$$\lambda_1^* < \lambda_2^* < \cdots < \lambda_n^*. \tag{1.4}$$

Newton's method can be applied to nonlinear equation (1.3) with (1.4). However, as noted in [19–21], Newton's method has the following two disadvantages: (i) It requires computing the exact eigenvectors at each iteration; (ii) It requires solving a Jacobian equation at each iteration. These two facts make it inefficient from the point of numerical computations especially when the problem size n is large. Thus, a focus was placed on avoiding on the disadvantages: (i) The Newton-like method was proposed in [22, 23] which computed the approximate eigenvectors instead of the exact eigenvectors. The quadratic convergence rate of this type of Newton-like method was re-proved in [24]. To alleviate the over-solving problem, Chen et al. proposed in [25] an inexact Newton-like method, which stopped the inner iterations before convergence. (ii) Shen et al. proposed in [26, 27] an Ulm-type method, which avoided solving approximate Jacobian equation at each outer iteration and hence could reduce the instability problem caused by the possible ill-conditioning in solving an approximate Jacobian equation.

Note that all of the methods mentioned above are quadratically convergent. In order to speed up the convergence rate of the methods, Chen et al. [28] proposed a super quadratic convergent two-step Newton-type method where the approximate Jacobian equations are solved by inexact methods. In view of this difficulty, Wen et al. proposed, in [29], a two-step inexact Newton-Chebyshev-like method with cubic root-convergence rate, in which the approximate eigenvectors were obtained by applying the one-step inverse power method and avoided solving the approximate Jacobian equations by using the Chebyshev method to approach the inverse of the Jacobian matrix. In 2022, Wei Ma designed a two-step Ulm-Chebyshev-like Cayley transform method [30] which utilized a Cayley transform to find the approximate eigenvectors. However, the convergence analysis for the above methods became ineffective in the absence of distinct eigenvalues, due to the breakdown of $f's$ differentiability and the eigenvectors' continuity for multiple eigenvalues [22]. When multiple eigenvalues are present, all of the numerical methods in the mentioned references above are quadratic convergent, which extends to the case of multiples.

In this paper, motivated by [30], we propose a two-step Ulm-Chebyshev-like Cayley transform method for solving the IEP (1.2). Further exploration involves analyzing the performance of the newly introduced two-step Ulm-Chebyshev-like method in the presence of multiple eigenvalues. Under the assumption similar to the one used in a previous study, by the Rayleigh quotient as an approximate eigenvalue of the symmetric matrix and the estimates of eigenvalues, eigenvectors, and the relative generalized Jacobian, we show that the proposed method is still cubically convergent. Numerical experiments show the efficiency of our method and comparisons with some known methods are made.

The structure of this paper is as follows. We give some notations and preliminary results of the relative generalized Jacobian and some useful lemmas in Section 2. A novel method, the two-step Ulm-Chebyshev-like Cayley transform approach, is introduced in Section 3 and our main convergence theorems are established for the new method in Section 4. Experimental results are presented in the final section.

2. Preliminaries

Let n be a positive integer. Let $\mathbb{R}^n$ represent an n-dimensional Euclidean space, S be a subset of $\mathbb{R}^n$, and clS represent the closure of S. Usually, we use $\mathbf{B}(\mathbf{x}, \delta_1)$ to represent the empty sphere of $\mathbb{R}^n$ center $\mathbf{x} \in \mathbb{R}^n$ and radius $\delta_1 > 0$. Let $\|\cdot\|$ and $\|\cdot\|_F$ represent the Euclidean vector norms or their corresponding induced matrix norms and Frobenius norms, respectively. I is the identity matrix of appropriate dimensions. Then, by (2.3.7) in [31], we have

$$\|A\| \leq \|A\|_F \leq \sqrt{n}\|A\|, \quad \text{for each} \quad A \in \mathbb{R}^n.$$

We define

$$K = 6\beta^2\|\lambda^*\|, \quad N = (n^2 - t^2) \max_{i\in[1,n-1]} \frac{1}{\lambda^*_{i+1} - \lambda^*_i}, \quad H_1 = \frac{8n^{\frac{3}{2}}\xi\beta\rho_0 \max\limits_{1\leq j\leq n} \|A_j\|}{1 - \left(\frac{\delta}{\tau}\right)^2}, \tag{2.1}$$

$$C = \max\{2 + 2\beta + 12N\beta\|\lambda^*\|,\ 2N \max_{1\leq j\leq n} \|A_j\|\},\ \rho = \max\{2\sqrt{n}(2\beta + \beta^2 + 2NK + \frac{1}{2}\beta C),\ 3\sqrt{n}C\}, \tag{2.2}$$

$$\alpha_1 = 8n^{\frac{3}{2}}\beta\rho_0 \max_{1\leq j\leq n} \|A_j\|,\ \gamma = \frac{K}{1 - H_1\delta},\ \alpha_2 = 1 + 8n^{\frac{3}{2}}\gamma K\rho_0^2, \tag{2.3}$$

$$\alpha_3 = \rho(\alpha_2 + 4n\rho_0^2), \quad \alpha_4 = \alpha_3 + \rho_0\alpha_2, \quad \alpha_6 = 1 + 2\gamma\alpha_1,$$

$$\alpha_5 = 6\sqrt{n}\beta^2\Big(\sqrt{n} \max_{1\leq j\leq n} \|A_j\| + K\sqrt{n} + \|\lambda^*\|\Big)\alpha_3^2,$$

$$\alpha_7 = 2\gamma\alpha_5 + \alpha_2\alpha_6, \quad \delta_2 = \min\Big\{\epsilon_0, \quad \frac{1}{\rho}, \quad \frac{1}{\beta}\Big\}, \tag{2.4}$$

$$\tau = \min\Big\{1, \quad \frac{1}{\alpha_7}, \quad \frac{\sqrt{n}\rho_0}{\rho_3(\alpha_7 + \alpha_3^2)}, \quad \frac{1}{(1 + 2\gamma\alpha_1)^3}, \quad \frac{2}{H_1}\Big\} \tag{2.5}$$

and

$$0 < \delta = \min\{\mu,\ \delta_0,\ \delta_2,\ \frac{\tau}{2},\ \frac{\delta_2}{2\sqrt{n}\rho_0},\ \frac{\delta_2}{\alpha_2},\ \frac{\delta_2}{\alpha_3},\ \frac{\delta_2}{\alpha_7},\ \frac{1}{\gamma\alpha_1}\}, \tag{2.6}$$

where δ_0 and ρ_0 are defined in Lemma 2.1, β and ϵ_0 are defined in Lemma 2.2, and λ^* is defined in (2.9).

2.1. Relative generalized Jacobian

A locally Lipschitz continuous function $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ is considered. The Jacobian of $\mathbf{h}$, denoted as $\mathbf{h}'$, whenever it exists, and $D_{\mathbf{h}}$ represents the set of differentiable points of $\mathbf{h}$. Moreover, the B-differential Jacobian of $\mathbf{h}$ at $\mathbf{x} \in \mathbb{R}^n$ is denoted according to [32].

$$\partial_B \mathbf{h}(\mathbf{x}) := \left\{ U \in \mathbb{R}^{m\times n} \middle| U = \lim_{\mathbf{x}_k \to \mathbf{x}} \mathbf{h}'(\mathbf{x}_k),\ \mathbf{x}_k \in D_{\mathbf{h}} \right\}.$$

Considering the composite nonsmooth function $\mathbf{h} := \varphi \circ \psi$, in which $\varphi : \mathbb{R}^t \to \mathbb{R}^m$ is nonsmooth and $\psi : \mathbb{R}^n \to \mathbb{R}^t$ is continuously differentiable, the generalized Jacobian $\partial_Q \mathbf{h}(\mathbf{x})$ [33] and relative generalized Jacobian $\partial_{Q|S} \mathbf{h}(\mathbf{x})$ [34] are respectively defined by

$$\partial_Q \mathbf{h}(\mathbf{x}) := \partial_B(\varphi(\psi(\mathbf{x})))\psi'(\mathbf{x})$$

and

$$\partial_{Q|S} \mathbf{h}(\mathbf{x}) := \{U | U \textit{ is a limit of } U_i \in \partial_Q \mathbf{h}(\mathbf{y}_i),\ \mathbf{y}_i \in S,\ \mathbf{y}_i \to \mathbf{x}\}.$$

For $\mathbf{c} \in \mathbb{R}^n$, write

$$\Lambda(\mathbf{c}) := \operatorname{diag}(\lambda_1(\mathbf{c}), ..., \lambda_n(\mathbf{c})),$$

and define

$$\mathcal{Q}(\mathbf{c}) := \{Q(\mathbf{c}) | Q(\mathbf{c})^T Q(\mathbf{c}) = I \textit{ and } Q(\mathbf{c})^T A(\mathbf{c}) Q(\mathbf{c}) = \Lambda(\mathbf{c})\}. \tag{2.7}$$

By (1.3) and the concept of a generalized Jacobian to f, we have [34]

$$\partial_Q \mathbf{f}(\mathbf{c}) = \{J | [J]_{ij} = \mathbf{q}_i(\mathbf{c})^T A_j \mathbf{q}_i(\mathbf{c}),\ \textit{where}\ [\mathbf{q}_1(\mathbf{c}), ..., \mathbf{q}_n(\mathbf{c})] \in \mathcal{Q}(\mathbf{c})\}.$$

In particular, if $J(\mathbf{c})$ is a singleton, we write $\partial_Q \mathbf{f}(\mathbf{c}) = \{J(\mathbf{c})\}$. Let

$$S := \{\mathbf{c} \in \mathbb{R}^n | A(\mathbf{c}) \textit{ has distinct eigenvalues}\}.$$

Then, let $\mathbf{c} \in S$ and f be continuously differentiable at $\mathbf{c}$. Moreover,

$$\partial_Q \mathbf{f}(\mathbf{c}) = \{J(\mathbf{c})\} = \{\mathbf{f}'(\mathbf{c})\}.$$

Thus, we get the following relative generalized Jacobian [34]:

$$\partial_{Q|S} \mathbf{f}(\mathbf{c}) = \{J | J = \lim_{k \to +\infty} J(\mathbf{y}^k) \textit{ with } \{\mathbf{y}^k\} \subset S \textit{ and } \mathbf{y}^k \to \mathbf{c}\}.$$

2.2. Preliminary results

Throughout this paper, let the given eigenvalues $\{\lambda_i^*\}_{i=1}^n$ satisfy $\lambda_1^* \le \lambda_2^* \le \cdots \le \lambda_n^*$. For simplicity, without loss of generality, we assume that

$$\lambda_1^* = \lambda_2^* = \cdots = \lambda_t^* < \lambda_{t+1}^* < \cdots\cdots < \lambda_n^*, \tag{2.8}$$

where $1 \le t \le n$. Write

$$\Lambda^* = \operatorname{diag}(\lambda_1^*, \lambda_2^*, ..., \lambda_n^*) \quad \text{and} \quad \lambda^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_n^*)^T. \tag{2.9}$$

Then a solution of the IEP (1.2) can be written by $\mathbf{c}^*$ and $Q(\mathbf{c}^*)$ as

$$Q(\mathbf{c}^*)^T A(\mathbf{c}^*) Q(\mathbf{c}^*) = \Lambda^*, \tag{2.10}$$

where $Q(\mathbf{c}^*)$ is an orthogonal matrix. Recall that $Q(\mathbf{c})$ can be defined by (2.7). Let $Q(\mathbf{c}) \in \mathcal{Q}(\mathbf{c})$ and write $Q(\mathbf{c})$=$[Q^{(1)}(\mathbf{c}), Q^{(2)}(\mathbf{c})]$ in which $Q^{(1)}(\mathbf{c}) \in \mathbb{R}^{n\times t}$ and $Q^{(2)}(\mathbf{c}) \in \mathbb{R}^{n\times(n-t)}$. Let $\mathbf{c}^*$ be the solution of the IEPs with (2.8). Define

$$\Pi = Q^{(1)}(\mathbf{c}^*) Q^{(1)}(\mathbf{c}^*)^T.$$

Clearly, Π is the eigenprojection of $A(\mathbf{c}^*)$ for λ_1^* in (2.8). Given an orthogonal matrix $P = [P^{(1)}, P^{(2)}]$, where $P^{(1)} \in \mathbb{R}^{n\times t}$ and $P^{(2)} \in \mathbb{R}^{n\times(n-t)}$, we obtain the QR factorization of $\Pi P^{(1)}$ by

$$\Pi P^{(1)} = \tilde{Q}^{(1)}(\mathbf{c}^*) R, \tag{2.11}$$

where R is a $t \times t$ nonsingular upper triangular matrix and $\tilde{Q}^{(1)}(\mathbf{c}^*)$ is an $n \times t$ matrix whose columns are orthonormal. Let

$$\tilde{Q}(\mathbf{c}^*) := [\tilde{Q}^{(1)}(\mathbf{c}^*), Q^{(2)}(\mathbf{c}^*)]. \tag{2.12}$$

Obviously, $\tilde{Q}(\mathbf{c}^*) \in \mathcal{Q}(\mathbf{c}^*)$. Moreover, we define the error matrix

$$E := [P^{(1)} - \Pi P^{(1)}, P^{(2)} - Q^{(2)}(\mathbf{c}^*)].$$

Now, we state the following two lemmas, which are usefull for our proof.

Lemma 2.1. *[35, 36] Let $\mathbf{c}^* \in \mathbb{R}^n$ and the eigenvalues of the matrix $A(\mathbf{c}^*)$ satisfy (2.8). Then, there exist two positive numbers δ_0 and ρ_0 such that, for each $\mathbf{c} \in \mathbf{B}(\mathbf{c}^*, \delta_0)$ and $[Q^{(1)}(\mathbf{c}),\ Q^{(2)}(\mathbf{c})] \in \mathcal{Q}(\mathbf{c})$, we get*

$$\|A(\mathbf{c}) - A(\mathbf{c}^*)\| \leqslant \max_{1\le j\le n} \|A_j\| \|\mathbf{c} - \mathbf{c}^*\|,$$

$$\|\Lambda(\mathbf{c}) - \Lambda^*\| \leqslant \rho_0 \|\mathbf{c} - \mathbf{c}^*\|,$$

$$\|Q^{(2)}(\mathbf{c}) - Q^{(2)}(\mathbf{c}^*)\| \leqslant \rho_0 \|\mathbf{c} - \mathbf{c}^*\|$$

and

$$\|(I - \Pi) Q^{(1)}(\mathbf{c})\| \leqslant \rho_0 \|\mathbf{c} - \mathbf{c}^*\|.$$

Lemma 2.2. *There exist two positive numbers ϵ_0 and β such that, for any orthogonal matrix $P = [P^{(1)}, P^{(2)}]$, if $\|E\| = \|P^{(1)} - \Pi P^{(1)},\ P^{(2)} - Q^{(2)}(\mathbf{c}^*)\| \le \epsilon_0$ and the skew-symmetric matrix X defined by $e^X = P^T \tilde{Q}(\mathbf{c}^*)$, then we get*

$$\|X\|_F \leqslant \beta\, \|E\| \quad \text{and} \quad \|X^{(11)}\|_F \leqslant \beta\, \|E\|^2, \tag{2.13}$$

in which $X^{(11)}$ is the t-by-t leading block of X. Moreover, if $\|X\|_F < 1$, then

$$\left\| \sum_{l=2}^{\infty} \frac{X^{l-2}}{l!} \right\|_F \le 1,\ \left\| \sum_{l=2}^{\infty} \frac{(-X)^{l-2}}{l!} \right\|_F \le 1,\ \left\| \sum_{l=1}^{\infty} \frac{(-X)^{l-1}}{l!} \right\|_F \le 2,\ \text{and} \left\| \sum_{l=0}^{\infty} \frac{(-X)^{l}}{l!} \right\|_F \le 3. \tag{2.14}$$

Proof. (2.13) can be found in [22, 35, 36]. Noting that

$$\sum_{l=2}^{\infty} \frac{1}{l!} \le \sum_{l=2}^{\infty} \frac{1}{l(l-1)} = 1.$$

If $\|X\|_F \leq 1$, we get

$$\left\|\sum_{l=2}^{\infty} \frac{X^{l-2}}{l!}\right\|_F \leq 1 \quad \text{and} \quad \left\|\sum_{l=2}^{\infty} \frac{(-X)^{l-2}}{l!}\right\|_F \leq 1,$$

which implies that

$$\left\|\sum_{l=1}^{\infty} \frac{(-X)^{l-1}}{l!}\right\|_F \leq 2 \quad \text{and} \quad \left\|\sum_{l=0}^{\infty} \frac{(-X)^{l}}{l!}\right\|_F \leq 3.$$

3. The two-step Ulm-Chebyshev-like Cayley transform method

We first recall the given eigenvalues in (2.8). Suppose that P_k is the current estimate of $Q(\mathbf{c}^*)$ and Y_k is a skew-symmetric matrix, i.e., $Y_k^T = -Y_k$. Let us write $Q(\mathbf{c}^*) = P_k e^{Y_k}$. Then, by using the Taylor series of the exponential function, we can express (2.10) as

$$P_k^T A(\mathbf{c}^*) P_k = e^{Y_k} \Lambda^* e^{-Y_k} = (I + Y_k + \frac{1}{2}Y_k^2 + \cdots)\Lambda^*(1 - Y_k + \frac{1}{2}Y_k^2 + \cdots).$$

The vector $\mathbf{c}^k$ is updated as $\mathbf{c}^{k+1}$ by neglecting the second–order term of the above equality in Y_k as

$$P_k^T A(\mathbf{c}^{k+1}) P_k = \Lambda^* + Y_k \Lambda^* - \Lambda^* Y_k. \tag{3.1}$$

We obtain $\mathbf{c}^{k+1}$ by equating the diagonal elements in (3.1) as

$$J_k \mathbf{c}^{k+1} = \lambda^* - \mathbf{b}^k,$$

in which, J_k and $\mathbf{b}^k$ are defined by

$$[J_k]_{ij} = (\mathbf{p}_i^k)^T A_j \mathbf{p}_i^k, \quad 1 \leq i, j \leq n \quad \text{and} \quad [\mathbf{b}]_i^k = (\mathbf{p}_i^k)^T A_0 \mathbf{p}_i^k, \quad 1 \leq i \leq n.$$

On the other hand, equating the off-diagonal in (3.1),

$$(\mathbf{p}_i^k)^T A(\mathbf{c}^{k+1}) \mathbf{p}_j^k = [Y_k]_{ij}\left(\lambda_j^* - \lambda_i^*\right), \text{ for each } i, j \in [1, n] \text{ and } i \neq j, \tag{3.2}$$

and, assuming that the given eigenvalues are defined in (2.8), we obtain the skew-symmetric matrix Y_k as

$$[Y_k]_{ij} = \begin{cases} 0, & \text{for each } 1 \leq i, j \leq t, \text{ or } i = j; \\ \frac{(\mathbf{p}_i^k)^T A(\mathbf{c}^{k+1}) \mathbf{p}_j^k}{\lambda_j^* - \lambda_i^*}, & \text{for each } t+1 \leq i \leq n \text{ or } t+1 \leq j \leq n \text{ and } i \neq j. \end{cases} \tag{3.3}$$

Furthermore, by using the Cayley transform, we calculate the matrix P_{k+1} as

$$P_{k+1} = P_k \left(I + \frac{1}{2}Y_k\right)\left(I - \frac{1}{2}Y_k\right)^{-1}. \tag{3.4}$$

Finally, by (3.3), (3.4), and the two-step Ulm-Chebyshev iterative procedure [30], we can propose the following two-step Ulm-Chebyshev-like Cayley transform method for solving the IEP with multiple eigenvalues.

Algorithm I: The two-step Ulm-Chebyshev-like Cayley transform method

Step 1. Given $\mathbf{c}^0 \in \mathbb{R}^n$, calculate the orthogonal eigenvectors $\{\mathbf{q}_i(\mathbf{c}^0)\}_{i=1}^n$ of $A(\mathbf{c}^0)$. Let

$$P_0 = [\mathbf{p}_1^0, \mathbf{p}_2^0, \ldots, \mathbf{p}_n^0] = [\mathbf{q}_1(\mathbf{c}^0), \mathbf{q}_2(\mathbf{c}^0), \ldots, \mathbf{q}_n(\mathbf{c}^0)],$$

and $J_0 = J(\mathbf{c}^0)$ and the vector $\mathbf{b}^0$ are defined as follows:

$$\begin{aligned} &[J_0]_{ij} = (\mathbf{p}_i^0)^T A_j \mathbf{p}_i^0, \quad 1 \le i, j \le n, \\ &[\mathbf{b}]_i^0 = (\mathbf{p}_i^0)^T A_0 \mathbf{p}_i^0, \quad 1 \le i \le n. \end{aligned} \tag{3.5}$$

Let $B_0 \in \mathbb{R}^{n\times n}$ satisfy

$$\|I - B_0 J(\mathbf{c}^0)\| \le \mu,$$

where μ is a positive constant.

Step 2. For $k = 0$, until convergence, do:

(a) Calculate $\mathbf{y}^k$ by

$$\mathbf{y}^k = \mathbf{c}^k - B_k(J_k\mathbf{c}^k + \mathbf{b}^k - \lambda^*). \tag{3.6}$$

(b) Form the skew-symmetric matrix Y_k:

$$[Y_k]_{ij} = \begin{cases} 0, & \text{for } 1 \le i, j \le t, \text{ or } i = j; \\ \frac{(\mathbf{p}_i^k)^T A(\mathbf{y}^k)\mathbf{p}_j^k}{\lambda_j^* - \lambda_i^*}, & \text{for } t+1 \le i \le n \text{ or } t+1 \le j \le n \text{ and } i \ne j, \end{cases} \tag{3.7}$$

where the matrix $A(\mathbf{y}^k)$ is defined by (1.1).

(c) Calculate $P(\mathbf{y}^k) = [\mathbf{p}_1(\mathbf{y}^k), \mathbf{p}_2(\mathbf{y}^k), \ldots, \mathbf{p}_n(\mathbf{y}^k)]^T = [\mathbf{v}_1^k, \mathbf{v}_2^k, \ldots, \mathbf{v}_n^k]^T$ by solving

$$(I + \frac{1}{2}Y_k)\mathbf{v}_j^k = \mathbf{h}_j^k, \quad \text{for } 1 \le j \le n, \tag{3.8}$$

where $\mathbf{h}_j^k$ is the jth column of $H_k = (I - \frac{1}{2}Y_k)P_k^T$.

(d) Calculate the approximate eigenvalues of $A(\mathbf{y}^k)$ via

$$\hat{\lambda}_i(\mathbf{y}^k) = (\mathbf{p}_i(\mathbf{y}^k))^T A(\mathbf{y}^k)\mathbf{p}_i(\mathbf{y}^k), \quad \text{for } 1 \le i \le n.$$

(e) Calculate $\mathbf{c}^{k+1}$ by

$$\mathbf{c}^{k+1} = \mathbf{y}^k - B_k(\hat{\lambda}(\mathbf{y}^k) - \lambda^*). \tag{3.9}$$

(f) Form the skew-symmetric matrix $\hat{Y}_k$:

$$[\hat{Y}_k]_{ij} = \begin{cases} 0, & \text{for } 1 \le i, j \le t, \text{ or } i = j; \\ \frac{(\mathbf{p}_i(\mathbf{y}^k))^T A(\mathbf{c}^{k+1})(\mathbf{p}_j(\mathbf{y}^k))}{\lambda_j^* - \lambda_i^*}, & \text{for } t+1 \le i \le n \text{ or } t+1 \le j \le n \text{ and } i \ne j, \end{cases}$$

where the matrix $A(\mathbf{c}^{k+1})$ is defined by (1.1).

(g) Calculate $P^{k+1} = [\mathbf{p}_1^{k+1}, \mathbf{p}_2^{k+1}, \ldots, \mathbf{p}_n^{k+1}]^T = [\hat{\mathbf{v}}_1^k, \hat{\mathbf{v}}_2^k, \ldots, \hat{\mathbf{v}}_n^k]^T$ by solving

$$(I + \frac{1}{2}\hat{Y}_k)\hat{\mathbf{v}}_j^k = \hat{\mathbf{h}}_j^k, \quad \text{for } 1 \le j \le n, \tag{3.10}$$

where $\hat{\mathbf{h}}_j^k$ is the jth column of $\hat{H}_k = (I - \frac{1}{2}\hat{Y}_k)(P(\mathbf{y}^k))^T$.

(h) Form the matrix J_{k+1} and the vector $\mathbf{b}^{k+1}$:

$$[J_{k+1}]_{ij} = (\mathbf{p}_i^{k+1})^T A_j \mathbf{p}_i^{k+1}, 1 \leq i, j \leq n,$$

$$[\mathbf{b}]_i^{k+1} = (\mathbf{p}_i^{k+1})^T A_0 \mathbf{p}_i^{k+1}, \;\; 1 \leq i \leq n.$$

(i) Calculate the Chebyshev matrices B_{k+1} by

$$B_{k+1} = B_k + B_k(2I - J(\mathbf{c}^{k+1})B_k)(I - J(\mathbf{c}^{k+1})B_k).$$

Remark 3.1. *For $k = 0, 1, 2, \ldots$, from* (c) *and* (g) *in Step* 2 *in Algorithm* I*, we have*

$$P(\mathbf{y}^k) = P_k(I + \frac{1}{2}Y_k)(I - \frac{1}{2}Y_k)^{-1} \tag{3.11}$$

and

$$P^{k+1} = P(\mathbf{y}^k)(I + \frac{1}{2}\hat{Y}_k)(I - \frac{1}{2}\hat{Y}_k)^{-1}. \tag{3.12}$$

Remark 3.2. *Without the distinction of the given eigenvalues, the convergence analysis of the two-step Ulm-Chebyshev-like method in* [30] *cannot work properly due to the differentiability of f and the discontinuity of the eigenvectors corresponding to multiple eigenvalues* [22]*. Based on the relative generalized Jacobian of eigenvalue function* [32]*, we propose the improved method for solving the IEP (1.2) with multiple eigenvalues. Clearly, in the case when $t = 1$, the method presented below is reduced to the two-step Ulm-Chebyshev-like method proposed in* [30] *for the distinct case.*

4. Convergence analysis

In this section, we shall analyze the convergence of Algorithm I. To ensure the cubical convergence, it is assumed that all $J \in \partial_Q \mathbf{f}(\mathbf{c}^*)$ are nonsingular for robustness. Yet, a suitable choice of eigenvectors can render J nonsingular in a general manner. Therefore, we assume that all $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ are nonsingular.

Let $\mathbf{c}^k$, $\mathbf{y}^k$, Y_k, $\hat{Y}_k$, P_k, $P(\mathbf{y}^k)$, J_k and B_k be generated by Algorithm I with initial point $\mathbf{c}^0$. For $k = 0, 1, 2, \ldots$, let

$$E_k := [P_k^{(1)} - \Pi P_k^{(1)} \quad P_k^{(2)} - Q^{(2)}(\mathbf{c}^*)] \tag{4.1}$$

and

$$E(\mathbf{y}^k) := [P(\mathbf{y}^k)^{(1)} - \Pi P(\mathbf{y}^k)^{(1)} \quad P(\mathbf{y}^k)^{(2)} - Q^{(2)}(\mathbf{c}^*)]. \tag{4.2}$$

Then, we can get the following lemmas.

Lemma 4.1. *Let the given eigenvalues $\{\lambda_i^*\}_{i=1}^n$ be defined as in* (2.8)*. Then, in Algorithm I, there exists a number $0 < \delta_2 \leq 1$ such that for $k = 0, 1, 2, \ldots$, if $\|\mathbf{y}^k - \mathbf{c}^*\| \leq \delta_2$, $\|\mathbf{c}^{k+1} - \mathbf{c}^*\| \leq \delta_2$, $\|E_k\| \leq \delta_2$ and $\|E(\mathbf{y}^k)\| \leq \delta_2$, then*

$$\|\Lambda^* + X_k\Lambda^* - \Lambda^* X_k - P_k^T A(\mathbf{c}^*) P_k\| \leq K\|E_k\|^2, \tag{4.3}$$

$$\|P(\mathbf{y}^k) - P_k\| \leq \rho(\|\mathbf{y}^k - \mathbf{c}^*\| + \|E_k\|), \tag{4.4}$$

$$\|E(\mathbf{y}^k)\| \leq \rho(\|\mathbf{y}^k - \mathbf{c}^*\| + \|E_k\|^2), \tag{4.5}$$

$$\|\Lambda^* + Y_k\Lambda^* - \Lambda^* Y_k - P(\mathbf{y}^k)^T A\,(\mathbf{c}^*)\, P(\mathbf{y}^k)\| < K\|E(\mathbf{y}^k)\|^2, \tag{4.6}$$

$$\|P_{k+1} - P(\mathbf{y}^k)\| \le \rho(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|E(\mathbf{y}^k)\|) \tag{4.7}$$

and

$$\|E_{k+1}\| \le \rho(\|\mathbf{c}^{k+1} - \mathbf{c}^*\| + \|E(\mathbf{y}^k)\|^2), \tag{4.8}$$

where K and ρ are defined by (2.1) *and* (2.2)*, respectively.*

Proof. Let $e^{X_k} := P_k^T \tilde{Q}(\mathbf{c}^*)$, where X_k is the skew-symmetric matrix and $\tilde{Q}(\mathbf{c}^*)$ is defined by (2.11) and (2.12) with $P = P_k$. By $\|E_k\| \le \delta_2 \le \epsilon_0$ and Lemma 2.2, we have

$$\|X_k\|_F \leqslant \beta\|E_k\| \quad \textit{and} \quad \|X_k^{(11)}\|_F \leqslant \beta\|E_k\|^2, \tag{4.9}$$

where β is a positive number and $X_k^{(11)}$ is the t-by-t leading block of X_k. On the other hand, by $\tilde{Q}(\mathbf{c}^*) \in \mathbf{Q}(\mathbf{c}^*)$, we derive

$$e^{X_k}\Lambda_* e^{-X_k} = P_k^T A(\mathbf{c}^*) P_k. \tag{4.10}$$

Thus, by the fact of $e^{X_k} = \sum_{l=0}^{\infty}(\frac{X_k^l}{l!})$, we can express (4.10) as

$$\Lambda^* + X_k\Lambda^* - \Lambda^* X_k = P_k^T A(\mathbf{c}^*) P_k + R(X_k), \tag{4.11}$$

where

$$R(X_k) = -X_k^2 \sum_{l=2}^{\infty}(\frac{X_k^{l-2}}{l!})\Lambda^*(\sum_{l=0}^{\infty}\frac{(-X_k)^l}{l!}) + \Lambda^* X_k^2 \sum_{l=2}^{\infty}\frac{(-X_k)^{l-2}}{l!} - X_k\Lambda^* X_k \sum_{l=1}^{\infty}\frac{(-X_k)^{l-1}}{l!}.$$

By (2.1) and Lemma 2.2, we get

$$\|R(X_k)\|_F \leqslant 6\,\|X_k\|_F^2 \cdot \|\Lambda^*\|_F = 6\,\|X_k\|_F^2 \cdot \|\lambda^*\| \leqslant 6\beta^2\,\|\lambda^*\| \cdot \|E_k\|^2 \le K\|E_k\|^2. \tag{4.12}$$

Thus, (4.3) is seen to hold by (4.11) and (4.12).

In order to prove (4.4) and (4.5), assume that $\|\mathbf{y}^k - \mathbf{c}^*\| \leqslant \delta_2$. We note by (4.11) that

$$[X_k]_{ij} = \frac{1}{\lambda_j^* - \lambda_i^*}(\mathbf{p}_i^k)^T A(\mathbf{c}^*)\mathbf{p}_j^k + [R(X_k)]_{ij},$$

where $t+1 \le i \le n,\ 1 \le j \le n,\ i > j$. Combining this with (3.7), we have

$$[X_k]_{ij} - [Y_k]_{ij} = \frac{1}{\lambda_j^* - \lambda_i^*}(\mathbf{p}_i^k)^T[A(\mathbf{c}^*) - A(\mathbf{y}^k)]\mathbf{p}_j^k + [R(X_k)]_{ij},$$

in which $t+1 \le i \le n,\ 1 \le j \le n,\ i > j$. By (4.12), Lemma 2.1, and the fact that $\{\mathbf{p}_i^k\}_{i=1}^n$ are orthogonal, we get

$$\max_{t+1\le i\le n,\ 1\le j\le n, i\ne j}\left|[X_k]_{ij} - [Y_k]_{ij}\right| \leqslant \max_{1\le i\le n-1}\frac{1}{\lambda_{i+1}^* - \lambda_i^*} \times (\max_{1\le j\le n}\|A_j\| \cdot \|\mathbf{y}^k - \mathbf{c}^*\| + K\|E_k\|^2).$$

In addition, by the fact that $[Y_k]_{ij} = 0$, for each $i, j \in [1, t]$, we have

$$\|X_k - Y_k\| \leqslant \|X_k - Y_k\|_F \leqslant \|X_k^{(11)}\|_F + \left(n^2 - t^2\right)\max_{i\in[t+1,n], j\in[1,n], i\ne j}\left|[X_k]_{ij} - [Y_k]_{ij}\right|.$$

Then, it follows from (2.1) and (4.9) that

$$\|X_k - Y_k\| \leqslant \beta\|E_k\|^2 + N(\max_{1\le j\le n}\|A_j\|\cdot\|\mathbf{y}^k - \mathbf{c}^*\| + K\|E_k\|^2), \tag{4.13}$$

and so

$$\|Y_k\| \leqslant \beta\|E_k\| + \beta\|E_k\|^2 + N(\max_{1\le j\le n}\|A_j\|\cdot\|\mathbf{y}^k - \mathbf{c}^*\| + K\|E_k\|^2).$$

Thus, thanks to the fact that $\beta\|E_k\| \le \beta\delta_2 \le 1$ and (2.2), one has

$$\begin{aligned}\|Y_k\| \leqslant\ & N\max_{1\le j\le n}\|A_j\|\cdot\|\mathbf{y}^k - \mathbf{c}^*\| + (1+\beta+6N\beta\|\lambda^*\|)\,\|E_k\| \\ \leqslant\ & \tfrac{C}{2}(\|\mathbf{y}^k - \mathbf{c}^*\| + \|E_k\|) \leqslant \tfrac{\rho}{2}(\|\mathbf{y}^k - \mathbf{c}^*\| + \|E_k\|). \end{aligned}\tag{4.14}$$

Since $\|E_k\| \leqslant \delta_2$ and $\|\mathrm{y}^k - \mathrm{c}^*\| \leqslant \delta_2$, it follows from (2.4) and (4.14) that $\|Y_k\| \leqslant 1$. Consequently,

$$\left\|(I - \frac{1}{2}Y_k)^{-1}\right\| \leqslant \frac{1}{1-\frac{1}{2}\|Y_k\|} \leqslant 2. \tag{4.15}$$

Therefore, in the following, we estimate $\|P(\mathrm{y}^k) - P_k\|$ and $\|E(\mathrm{y}^k)\|$. Indeed, by (3.11),

$$P(\mathrm{y}^k) - P_k = P_k[(I+\frac{1}{2}Y_k) - (I-\frac{1}{2}Y_k)](I-\frac{1}{2}Y_k)^{-1} = P_kY_k(I-\frac{1}{2}Y_k)^{-1}.$$

This together with (4.14), (4.15), as well as the orthogonality of P_k indicate that (4.4) holds.

As for (4.5), we note by (3.11) and X_k that

$$\begin{aligned} P(\mathrm{y}^k) - \tilde{Q}(\mathbf{c}^*) &= P_k[(I+\frac{1}{2}Y_k)(I-\frac{1}{2}Y_k)^{-1} - e^{X_k}] \\ &= P_k[(I+\frac{1}{2}Y_k) - e^{X_k}(I-\frac{1}{2}Y_k)](I-\frac{1}{2}Y_k)^{-1}. \end{aligned}$$

Combining this with $e^{X_k} = \sum_{l=0}^{\infty}(\frac{X_k^l}{l!})$, we get

$$\begin{aligned} P(\mathrm{y}^k) - \tilde{Q}(\mathbf{c}^*) &= P_k[Y_k - X_k + \frac{1}{2}X_kY_k - (X_k^2\sum_{m=2}^{\infty}\frac{X_k^{m-2}}{m!})(I-\frac{1}{2}Y_k)](I-\frac{1}{2}Y_k)^{-1} \\ &= P_k(Y_k - X_k + \frac{1}{2}X_kY_k)(I-\frac{1}{2}Y_k)^{-1} - P_kX_k^2\sum_{m=2}^{\infty}\frac{X_k^{m-2}}{m!}. \end{aligned}$$

Since P_k is orthogonal, note by (2.14) and (4.15) that

$$\|P(\mathrm{y}^k) - \tilde{Q}(\mathbf{c}^*)\| \leqslant 2\,\|\,Y_k - X_k\,\| + \|\,X_k\,\|\cdot\|\,Y_k\,\| + \|\,X_k\,\|^2\,.$$

Thus, we derive by using (2.13), (4.13), and (4.14),

$$\begin{aligned} \|P(\mathrm{y}^k) - \tilde{Q}(\mathbf{c}^*)\| \ \le\ & 2\beta\,\|E_k\|^2 + 2N(\max_{1\le j\le n}\|A_j\|\cdot\|\mathbf{y}^k - \mathbf{c}^*\| + K\|E_k\|^2) \\ & +\frac{1}{2}\beta C\,\|E_k\|\cdot(\|\mathbf{y}^k - \mathbf{c}^*\| + \|E_k\|) + \beta^2\,\|E_k\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq (2\beta+\beta^2+2NK+\frac{1}{2}\beta C)\|E_k\|^2+(2N\max_{1\leq j\leq n}\|A_j\|+\frac{1}{2}C)\|\mathbf{y}^k-\mathbf{c}^*\| \\
&\leq \max_{1\leq j\leq n}\|A_j\|\|\mathbf{y}^k-\mathbf{c}^*\|+K\|E(\mathbf{y}^k)\|^2, \quad 1\leq i\leq n, \qquad (4.16)
\end{aligned}
$$

where the second inequality holds because of the fact that $\beta\|E_k\|\leqslant 1$ while the last inequality holds because of the definition of C. Write $P(\mathrm{y}^k)=[P(\mathrm{y}^k)^{(1)}\ P(\mathrm{y}^k)^{(2)}]$, where $P(\mathrm{y}^k)^{(1)}\in\mathbb{R}^{n\times t}$ and $P(\mathrm{y}^k)^{(2)}\in\mathbb{R}^{n\times(n-t)}$. Since $(I-\Pi)\tilde{Q}^{(1)}(\mathbf{c}^*)=\mathbf{0}$, where $\mathbf{0}$ is a zero matrix, we have

$$
\begin{aligned}
\left\|(I-\Pi)P(\mathrm{y}^k)^{(1)}\right\| &= \left\|(I-\Pi)(P(\mathrm{y}^k)^{(1)}-\tilde{Q}^{(1)}(\mathbf{c}^*)+\tilde{Q}^{(1)}(\mathbf{c}^*))\right\| \\
&= \left\|(I-\Pi)(P(\mathrm{y}^k)^{(1)}-\tilde{Q}^{(1)}(\mathbf{c}^*))\right\| \\
&\leqslant \left\|P(\mathrm{y}^k)-\tilde{Q}(\mathbf{c}^*)\right\|
\end{aligned}
$$

and

$$
\|P(\mathrm{y}^k)^{(2)}-\tilde{Q}^{(2)}(\mathbf{c}^*)\|\leq\|P(\mathrm{y}^k)-\tilde{Q}(\mathbf{c}^*)\|.
$$

Hence, by (4.2) and (4.16), we obtain

$$
\begin{aligned}
\|E(\mathrm{y}^k)\| &\leqslant \|(I-\Pi)P(\mathrm{y}^k)^{(1)}\|+\|P(\mathrm{y}^k)^{(2)}-\tilde{Q}^{(2)}(\mathbf{c}^*)\|\leqslant 2\sqrt{n}\ \|\ P(\mathrm{y}^k)-\tilde{Q}(\mathbf{c}^*)\ \| \\
&\leqslant 2\ \sqrt{n}\Big[\ (2\beta+\beta^2+2NK+\frac{1}{2}\beta C)\ \|E_k\|^2+\frac{3}{2}C\ \|\ \mathbf{y}^k-\mathbf{c}^*\|\Big]. \qquad (4.17)
\end{aligned}
$$

Therefore, (4.5) is proved by (2.2) and (4.17). We defined $e^{Y_k}:=P(\mathbf{y}^k)^T\tilde{Q}(\mathbf{c}^*)$, where Y_k is the skew-symmetric matrix. Similarly, (4.6)–(4.8) also hold. □

Lemma 4.2. *Let ρ_0 and δ_0 be defined in Lemma 2.1. If $\|\mathbf{y}_k-\mathbf{c}^*\|\leq\delta_0$ and $\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)=(\hat{\lambda}_1(\mathbf{y}^k),\hat{\lambda}_2(\mathbf{y}^k),\ldots,\hat{\lambda}_n(\mathbf{y}^k))^T$, in which $\hat{\lambda}_i(\mathbf{y}^k)=(\mathbf{p}_i(\mathbf{y}^k))^TA(\mathbf{y}^k)\mathbf{p}_i(\mathbf{y}^k)$, $1\leq i\leq n$, then*

$$
\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\|\leq\sqrt{n}\max_{1\leq j\leq n}\|A_j\|\|\mathbf{y}^k-\mathbf{c}^*\|+K\sqrt{n}\|E(\mathbf{y}^k)\|^2+\|\boldsymbol{\lambda}^*\|. \qquad (4.18)
$$

Proof. From the diagonal elements of $\Lambda^*+Y_k\Lambda^*-\Lambda^*Y_k-P(\mathbf{y}^k)^TA\,(\mathbf{c}^*)\,P(\mathbf{y}^k)$, we obtain from (4.6) that

$$
|(\mathbf{p}_i(\mathbf{y}^k))^TA(\mathbf{c}^*)\mathbf{p}_i(\mathbf{y}^k)-\lambda_i^*|\leq K\|E(\mathbf{y}^k)\|^2, \quad \text{for } 1\leq i\leq n,
$$

which together with Lemma 3.1 gives

$$
\begin{aligned}
|(\mathbf{p}_i(\mathbf{y}^k))^TA(\mathbf{y}^k)\mathbf{p}_i(\mathbf{y}^k)-\lambda_i^*| &= |(\mathbf{p}_i(\mathbf{y}^k))^T(A(\mathbf{y}^k)-A(\mathbf{c}^*))\mathbf{p}_i(\mathbf{y}^k)+(\mathbf{p}_i(\mathbf{y}^k))^TA(\mathbf{c}^*)\mathbf{p}_i(\mathbf{y}^k)-\lambda_i^*| \\
&\leq |(\mathbf{p}_i(\mathbf{y}^k))^T(A(\mathbf{y}^k)-A(\mathbf{c}^*))\mathbf{p}_i(\mathbf{y}^k)|+|(\mathbf{p}_i(\mathbf{y}^k))^TA(\mathbf{c}^*)\mathbf{p}_i(\mathbf{y}^k)-\lambda_i^*| \\
&\leq \max_{1\leq j\leq n}\|A_j\|\|\mathbf{y}^k-\mathbf{c}^*\|+K\|E(\mathbf{y}^k)\|^2, \quad 1\leq i\leq n.
\end{aligned}
$$

Therefore,

$$
\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)-\boldsymbol{\lambda}^*\|\leq\sqrt{n}\max_{1\leq j\leq n}\|A_j\|\|\mathbf{y}^k-\mathbf{c}^*\|+K\sqrt{n}\|E(\mathbf{y}^k)\|^2,
$$

which together with the fact that $\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\|\leq\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)-\boldsymbol{\lambda}^*\|+\|\boldsymbol{\lambda}^*\|$, we can get (4.18). □

Lemma 4.3. *Let the Jacobian matrix of* $\tilde{J}(\mathbf{c}^*)$ *and the vector* $\tilde{\mathbf{b}}$ *be defined as follows:*

$$[\tilde{J}(\mathbf{c}^*)]_{ij} = \tilde{\mathbf{q}}_i(\mathbf{c}^*)^T A_j \tilde{\mathbf{q}}_i(\mathbf{c}^*), \quad 1 \le i, j \le n \quad \text{and} \quad [\tilde{\mathbf{b}}]_i = (\tilde{\mathbf{q}}_i(\mathbf{c}^*))^T A_0 \tilde{\mathbf{q}}_i(\mathbf{c}^*), \quad 1 \le i \le n.$$

Then, we have

$$\|\tilde{J}(\mathbf{c}^*)\mathbf{y}^k + \tilde{\mathbf{b}} - \hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\| \le 6\sqrt{n}\beta^2 \|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\| \, \|E(\mathbf{y}^k)\|^2. \tag{4.19}$$

Proof. Let $e^{Y_k} := P(\mathbf{y}^k)^T \tilde{Q}(\mathbf{c}^*)$ where Y_k is the skew-symmetric matrix and $\tilde{Q}(\mathbf{c}^*)$ is defined by (2.11) and (2.12) with $P = P(\mathbf{y}^k)$. By $\|E(\mathbf{y}^k)\| \le \delta_2 \le \epsilon_0$ and Lemma 2.2, we get

$$\|Y_k\|_F \leqslant \beta \|E(\mathbf{y}^k)\|,$$

where β is a positive number. By $\hat{\Lambda}(\mathbf{y}^k) = P(\mathbf{y}^k)^T A(\mathbf{y}^k) P(\mathbf{y}^k)$ and $e^{-Y_k}\hat{\Lambda}(\mathbf{y}^k)e^{Y_k} = \tilde{Q}(\mathbf{c}^*)^T A(\mathbf{y}^k)\tilde{Q}(\mathbf{c}^*)$, we have

$$\hat{\Lambda}(\mathbf{y}^k) - Y_k\hat{\Lambda}(\mathbf{y}^k) + \hat{\Lambda}(\mathbf{y}^k)Y_k = \tilde{Q}(\mathbf{c}^*)^T A(\mathbf{y}^k)\tilde{Q}(\mathbf{c}^*) + R(Y_k),$$

where

$$R(Y_k) = (Y_k)^2 \sum_{l=2}^{\infty} (\frac{(-Y_k)^{l-2}}{l!})\hat{\Lambda}(\mathbf{y}^k)\Big(\sum_{l=0}^{\infty} \frac{(Y_k)^l}{l!}\Big) - \hat{\Lambda}(\mathbf{y}^k)(Y_k)^2 \sum_{l=2}^{\infty} \frac{(Y_k)^{l-2}}{l!} + Y_k\hat{\Lambda}(\mathbf{y}^k)Y_k \sum_{l=1}^{\infty} \frac{(Y_k)^{l-1}}{l!}.$$

By Lemma 2.2, we get

$$\|R(Y_k)\|_F \leqslant 6\,\|Y_k\|_F^2\,\|\hat{\Lambda}(\mathbf{y}^k)\|_F = 6\,\|Y_k\|_F^2\,\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\| \leqslant 6\beta^2\,\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\|\,\|E(\mathbf{y}^k)\|^2.$$

Thus,

$$\|\tilde{Q}(\mathbf{c}^*)^T A(\mathbf{y}^k)\tilde{Q}(\mathbf{c}^*) + Y_k\hat{\Lambda}(\mathbf{y}^k) - \hat{\Lambda}(\mathbf{y}^k)Y_k - \hat{\Lambda}(\mathbf{y}^k)\| \le 6\beta^2\,\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\|\,\|E(\mathbf{y}^k)\|^2. \tag{4.20}$$

By the diagonal entries of $\tilde{Q}(\mathbf{c}^*)^T A(\mathbf{y}^k)\tilde{Q}(\mathbf{c}^*) + Y_k\hat{\Lambda}(\mathbf{y}^k) - \hat{\Lambda}(\mathbf{y}^k)Y_k - \hat{\Lambda}(\mathbf{y}^k)$ and (4.20), we get

$$|(\tilde{\mathbf{q}}_i(\mathbf{c}^*))^T A(\mathbf{y}^k)\tilde{\mathbf{q}}_i(\mathbf{c}^*) - \hat{\lambda}_i(\mathbf{y}^k)| < 6\beta^2\,\|\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)\|\,\|E(\mathbf{y}^k)\|^2, \quad \text{for } 1 \le i \le n.$$

Therefore, by the definitions of $\tilde{J}(\mathbf{c}^*)$, $\hat{\boldsymbol{\lambda}}(\mathbf{y}^k)$, $\tilde{\mathbf{b}}$ and $A(\mathbf{y}^k)$, we can get (4.19). □

Lemma 4.4. *[35] Let* J_0 *be defined in* (3.5)*. Suppose that* J_0 *is invertible. Let* $k \ge 1$ *such that*

$$2n\|J_0^{-1}\| \max_{1\le j\le n} \|A_j\|\|P_k - P_0\| < 1. \tag{4.21}$$

Then, the matrix J_k *is nonsingular and*

$$\|J_k^{-1}\| \le \frac{\|J_0^{-1}\|}{1 - 2n\|J_0^{-1}\| \max_{1\le j\le n} \|A_j\|\|P_k - P_0\|}.$$

Lemma 4.5. *Let the vector* $\mathbf{c}^* \in clS$ *and the given eigenvalues of the matrix* $A(\mathbf{c}^*)$ *satisfy* (2.8)*. Let all* $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ *be nonsingular. If for* $\mathbf{c}^0 \in \mathbf{B}(\mathbf{c}^*, \delta) \bigcap S$ *where* $\delta > 0$ *and* $k = 0, 1, 2, \ldots$*, then there exist three numbers* $0 < \tau \le 1$, $0 < \delta < \tau$ *and* $0 \le \mu \le \delta$*, that if* $\|\mathbf{c}^0 - \mathbf{c}^*\| \le \delta$*, the conditions*

$$\|E_k\| \le 2\sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^k}, \tag{4.22}$$

$$\|\mathbf{c}^k - \mathbf{c}^*\| \leq \tau\Big(\frac{\delta}{\tau}\Big)^{3^k}, \tag{4.23}$$

and

$$\|I - B_k J_k\| \leq \tau\Big(\frac{\delta}{\tau}\Big)^{3^k} \tag{4.24}$$

imply

$$\|J_k^{-1}\| \leq \gamma, \quad \|B_k\| \leq 2\gamma, \quad \text{and} \quad \|\mathbf{c}^{k+1} - \mathbf{c}^*\| \leq \tau\Big(\frac{\delta}{\tau}\Big)^{3^{k+1}}. \tag{4.25}$$

Proof. Since all $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ are nonsingular, and from Theorem 3.2 in [34], for each $\mathbf{c}^0 \in \mathbf{B}(\mathbf{c}^*, \delta_0) \bigcap S$, we have

$$\sup_{J\in\partial_{Q|S}\mathbf{f}(\mathbf{c}^0)} \|J^{-1}\| \leq \xi,$$

where $\xi > 0$ and $\delta_0 > 0$. From (2.3), (2.5), and (2.6), we know that

$$\tau \leq 1. \tag{4.26}$$

By (2.6) and (4.22), we get

$$\|E_k\| \leq 2\sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} \leq 2\sqrt{n}\rho_0\delta \leq \delta_2,$$

and then, from Lemma 2.2, we obtain

$$\|X_k\|_F \leqslant \beta\|E_k\|, \tag{4.27}$$

where $\beta > 0$. By the definitions of X_k and the Taylor series of the exponential function e^{X_k}, we have

$$P_k - \tilde{Q}(\mathbf{c}^*) = P_k(I - e^{X_k}) = P_k(-X_k)\sum_{l=1}^{\infty}\frac{(X)^{l-1}}{l!}.$$

Since P_k is orthogonal, note by (2.14) and (4.27) that

$$\|P_k - \tilde{Q}(\mathbf{c}^*)\| \leq 2\|X_k\| \leq 2\|X_k\|_F \leq 2\beta\|E_k\|. \tag{4.28}$$

Similarly, we also have

$$\|P_{k-1} - \tilde{Q}(\mathbf{c}^*)\| \leq 2\|X_{k-1}\| \leq 2\|X_{k-1}\|_F \leq 2\beta\|E_{k-1}\|.$$

Thus, by (2.5), (4.22), and (4.28), we have

$$\begin{aligned}
\|P_k - P_{k-1}\| &\leq \|P_k - \tilde{Q}(\mathbf{c}^*)\| + \|P_{k-1} - \tilde{Q}(\mathbf{c}^*)\| \\
&\leq 2\beta\|E_k\| + 2\beta\|E_{k-1}\| \\
&\leq 2\beta\Big(2\sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} + 2\sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^{k-1}}\Big) \\
&\leq 4\sqrt{n}\beta\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^{k-1}}.
\end{aligned} \tag{4.29}$$

Therefore, we further have

$$\|P_m - P_0\| \leq \sum_{k=1}^{m}\|P_k - P_{k-1}\| \leq 4\sqrt{n}\beta\rho_0\tau\Big[\Big(\frac{\delta}{\tau}\Big) + \Big(\frac{\delta}{\tau}\Big)^3 + \Big(\frac{\delta}{\tau}\Big)^{3^2} + \cdots + \Big(\frac{\delta}{\tau}\Big)^{3^{m-1}}\Big]. \tag{4.30}$$

Since $3^n \geq 2n+1$ for each $n \geq 0$, we obtain from (4.30) that

$$\begin{aligned} \|P_m - P_0\| &\leq 4\sqrt{n}\beta\rho_0\tau\Big[\Big(\frac{\delta}{\tau}\Big) + \Big(\frac{\delta}{\tau}\Big)^3 + \Big(\frac{\delta}{\tau}\Big)^5 + \cdots + \Big(\frac{\delta}{\tau}\Big)^{2m-1}\Big] \\ &\leq 4\sqrt{n}\beta\rho_0\delta\frac{\Big[1-\Big(\frac{\delta}{\tau}\Big)^{2m}\Big]}{1-\Big(\frac{\delta}{\tau}\Big)^2}, \end{aligned}$$

which together with (2.1) and (2.6), we obtain

$$2n\xi \max_{1\leq j\leq n}\|A_j\|\|P_m - P_0\| \leq \frac{8n^{\frac{3}{2}}\xi\beta\rho_0\delta \max\limits_{1\leq j\leq n}\|A_j\|}{1-\Big(\frac{\delta}{\tau}\Big)^2} = H_1\delta < \frac{1}{2}H_1\tau < 1.$$

Consequently, using Lemma 4.4, we can derive that J_k is nonsingular and moreover

$$\|J_m^{-1}\| \leq \frac{\xi}{1-2n\xi\max\limits_{1\leq j\leq n}\|A_j\|\|P_m - P_0\|} \leq \frac{\xi}{1-H_1\delta} = \gamma. \tag{4.31}$$

Furthermore, by (2.5), (2.6), and (4.24), we have

$$\|B_k\| \leq \|B_kJ_k\|\|J_k^{-1}\| \leq (I + \|I - B_kJ_k\|)\|J_k^{-1}\| \leq (1+\tau\Big(\frac{\delta}{\tau}\Big)^{3^k})\gamma \leq (1+\tau)\gamma \leq 2\gamma. \tag{4.32}$$

On the other hand, considering the diagonal elements of $\Lambda^* + X_k\Lambda^* - \Lambda^*X_k - P_k^TA(\mathbf{c}^*)P_k$, we obtain from (4.3) that

$$|(\mathbf{p}_i^k)^TA(\mathbf{c}^*)\mathbf{p}_i^k - \lambda_i^*| \leq K\|E_k\|^2, \text{ for } 1 \leq i \leq n.$$

Therefore, by the definitions of $\boldsymbol{\lambda}^*$, J_k, $\mathbf{b}^k$ and $A(\mathbf{c}^*)$, we have

$$\|J_k\mathbf{c}^* - \boldsymbol{\lambda}^* + \mathbf{b}^k\| \leq \sqrt{n}K\|E_k\|^2. \tag{4.33}$$

From (3.6), we get

$$\mathbf{y}^k - \mathbf{c}^* = B_k(\boldsymbol{\lambda}^* - \mathbf{b}^k - J_k\mathbf{c}^*) + (I - B_kJ_k)(\mathbf{c}^k - \mathbf{c}^*).$$

It follows that

$$\|\mathbf{y}^k - \mathbf{c}^*\| \leq \|B_k\|\|J_k\mathbf{c}^* - \boldsymbol{\lambda}^* + \mathbf{b}^k\| + \|I - B_kJ_k\|\|\mathbf{c}^k - \mathbf{c}^*\|,$$

which together with (4.22)–(4.24), (4.32), and (4.33) gives

$$\begin{aligned} \|\mathbf{y}^k - \mathbf{c}^*\| &\leq 2\gamma\sqrt{n}K\|E_k\|^2 + \tau\Big(\frac{\delta}{\tau}\Big)^{3^k}\cdot\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} \\ &\leq (1+8\gamma Kn^{\frac{3}{2}}\rho_0^2)\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k} := \alpha_2\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k}. \end{aligned} \tag{4.34}$$

By (2.6), (4.22), and (4.34), we have

$$\|\mathbf{y}^k - \mathbf{c}^*\| \leq \alpha_2\delta \leq \delta_2 \leq 1 \tag{4.35}$$

and

$$\|E_k\| \le 2\sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} \le 2\sqrt{n}\rho_0\delta \le \delta_2 \le 1, \tag{4.36}$$

which together with (4.22) and (4.28), we obtain

$$\|\mathbf{p}_i^k - \tilde{\mathbf{q}}_i(\mathbf{c}^*)\| \le \|P_k - \tilde{Q}(\mathbf{c}^*)\| \le 4\sqrt{n}\beta\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^k}, \quad 1 \le i \le n.$$

This together with the orthogonality of P_k and $\tilde{Q}(\mathbf{c}^*)$ and the Cauchy-Schwarz inequality indicates that

$$\begin{aligned}
|[J_k]_{ij} - [\tilde{J}(\mathbf{c}^*)]_{ij}| &= |(\mathbf{p}_i^k)^T A_j \mathbf{p}_i^k - \tilde{\mathbf{q}}_i(\mathbf{c}^*)^T A_j \tilde{\mathbf{q}}_i(\mathbf{c}^*)| \\
&= |(\mathbf{p}_i^k - \tilde{\mathbf{q}}_i(\mathbf{c}^*))^T A_j \mathbf{p}_i^k - \tilde{\mathbf{q}}_i(\mathbf{c}^*)^T A_j(\tilde{\mathbf{q}}_i(\mathbf{c}^*) - \mathbf{p}_i^k)| \\
&\le 2\|A_j\|\|\mathbf{p}_i^k - \tilde{\mathbf{q}}_i(\mathbf{c}^*)\| \\
&\le 8\sqrt{n}\beta\rho_0\|A_j\|\tau\Big(\frac{\delta}{\tau}\Big)^{3^k}, \quad 1 \le i, j \le n.
\end{aligned}$$

Thus, we get

$$\|J_k - \tilde{J}(\mathbf{c}^*)\| \le \|J_k - \tilde{J}(\mathbf{c}^*)\|_F \le 8n^{\frac{3}{2}}\beta\rho_0 \max_{1\le j\le n}\|A_j\|\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} := \alpha_1\tau\Big(\frac{\delta}{\tau}\Big)^{3^k}. \tag{4.37}$$

By (4.24), (4.32), and (4.37), we have

$$\begin{aligned}
\|I - B_k\tilde{J}(\mathbf{c}^*)\| &\le \|I - B_kJ_k\| + \|B_k\|\|J_k - \tilde{J}(\mathbf{c}^*)\| \\
&\le \tau\Big(\frac{\delta}{\tau}\Big)^{3^k} + 2\gamma\alpha_1\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} \\
&\le (1 + 2\gamma\alpha_1)\tau\Big(\frac{\delta}{\tau}\Big)^{3^k} \\
&:= \alpha_6\tau\Big(\frac{\delta}{\tau}\Big)^{3^k}.
\end{aligned} \tag{4.38}$$

By (4.34)–(4.36) and Lemma 4.1, we obtain

$$\begin{aligned}
\|E(\mathbf{y}^k)\| &\le \rho\Big(\alpha_2\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k} + 4n\rho_0^2\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k}\Big) \\
&\le \rho(\alpha_2 + 4n\rho_0^2)\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k} \\
&:= \alpha_3\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k},
\end{aligned} \tag{4.39}$$

which together with (2.6), (4.22), and (4.34), we have

$$\|E(\mathbf{y}^k)\| \le \alpha_3\delta \le \delta_2 \le 1,$$

which together with (4.35) and Lemmas 4.2 and 4.3, we get

$$\begin{aligned}
\|\tilde{J}(\mathbf{c}^*)\mathbf{y}^k + \tilde{\mathbf{b}} - \hat{\lambda}(\mathbf{y}^k)\| &\le 6\sqrt{n}\beta^2\,\|\hat{\lambda}(\mathbf{y}^k)\|\|E(\mathbf{y}^k)\|^2 \\
&\le 6\sqrt{n}\beta^2\Big(\sqrt{n}\max_{1\le j\le n}\|A_j\|\|\mathbf{y}^k - \mathbf{c}^*\| + K\sqrt{n}\|E(\mathbf{y}^k)\|^2 + \|\boldsymbol{\lambda}^*\|\Big)\|E(\mathbf{y}^k)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad 6\sqrt{n}\beta^2\Big(\sqrt{n}\max_{1\leq j\leq n}\|A_j\| + K\sqrt{n} + \|\boldsymbol{\lambda}^*\|\Big)\alpha_3^2\tau^4\Big(\frac{\delta}{\tau}\Big)^{4\cdot 3^k}\\
&\leq \quad 6\sqrt{n}\beta^2\Big(\sqrt{n}\max_{1\leq j\leq n}\|A_j\| + K\sqrt{n} + \|\boldsymbol{\lambda}^*\|\Big)\alpha_3^2\tau^3\Big(\frac{\delta}{\tau}\Big)^{3^{k+1}}\\
&:= \quad \alpha_5\tau^3\Big(\frac{\delta}{\tau}\Big)^{3^{k+1}}. \qquad (4.40)
\end{aligned}
$$

Together with (3.9) and $\boldsymbol{\lambda}^* = \tilde{J}(\mathbf{c}^*)\mathbf{c}^* + \tilde{\mathbf{b}}$, we get

$$\mathbf{c}^{k+1} - \mathbf{c}^* = B_k(\tilde{J}(\mathbf{c}^*)\mathbf{y}^k + \mathbf{b} - \hat{\lambda}(\mathbf{y}^k)) + (I - B_k\tilde{J}(\mathbf{c}^*))(\mathbf{y}^k - \mathbf{c}^*).$$

It follows from (4.26), (4.32), (4.34), (4.38), and (4.40) that

$$
\begin{aligned}
\|\mathbf{c}^{k+1} - \mathbf{c}^*\| &\leq \|B_k\|\|\tilde{J}(\mathbf{c}^*)\mathbf{y}^k + \tilde{\mathbf{b}} - \hat{\lambda}(\mathbf{y}^k)\| + \|I - B_k\tilde{J}(\mathbf{c}^*)\|\|\mathbf{y}^k - \mathbf{c}^*\|\\
&\leq 2\gamma\alpha_5\tau^3\Big(\frac{\delta}{\tau}\Big)^{3^{k+1}} + \alpha_6\tau\Big(\frac{\delta}{\tau}\Big)^{3^k}\cdot\alpha_2\tau^2\Big(\frac{\delta}{\tau}\Big)^{2\cdot 3^k}\\
&\leq (2\gamma\alpha_5 + \alpha_2\alpha_6)\tau^2\Big(\frac{\delta}{\tau}\Big)^{3^{k+1}} = \tau\Big(\frac{\delta}{\tau}\Big)^{3^{k+1}}. \qquad (4.41)
\end{aligned}
$$

Finally, by (4.31), (4.32), and (4.41), we have (4.25). □

Next, we can analyze the convergence of Algorithm I.

Theorem 4.1. *Let the vector* $\mathbf{c}^* \in clS$ *and the given eigenvalues* $\{\lambda_i^*\}_{i=1}^n$ *satisfy* (2.8). *All* $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ *are nonsingular. Then Algorithm I is locally cubic convergent.*

Proof. Let us start by mathematical induction that (4.22)–(4.24) are true for all $k > 0$. Clearly, by assumptions $\mu \leq \delta$ and $\|\mathbf{c}^0 - \mathbf{c}^*\| \leq \delta$, (4.23) and (4.24) for $k = 0$ are trivial. From Lemma 2.1, we have

$$\|E_0\| \leq \|\|E_0\|_F \leq \|(I - \Pi)Q^{(1)}(\mathbf{c}^0)\|_F + \|Q^{(2)}(\mathbf{c}^0) - Q^{(2)}(\mathbf{c}^*)\|_F \leq 2\sqrt{n}\alpha_1\|\mathbf{c}^0 - \mathbf{c}^*\| \leq 2\sqrt{n}\alpha_1\delta,$$

and this gives that (4.22) is true for $k = 0$.

Now assume that (4.22)–(4.24) are true for all $k \leq m - 1$. Recalling that (2.5) and (2.6), we get

$$\|E_{m-1}\| \leq \sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}} \leq \sqrt{n}\rho_0\delta \leq \delta_2,$$

which together with Lemma 4.1, (2.5), (2.6), (4.34), and (4.39) with $k = m - 1$ gives

$$\|E(\mathbf{y}^{m-1})\| \leq \alpha_3\delta \leq \delta_2.$$

It follows from Lemma 4.1, (2.5), (4.39), and (4.41) with $k = m - 1$ that

$$\|\mathbf{c}^m - \mathbf{c}^*\| \leq \tau\Big(\frac{\delta}{\tau}\Big)^{3^m} \leq \delta \leq \delta_2$$

and

$$\|E_m\| \leq \rho_3\Big(\alpha_7\tau^2\Big(\frac{\delta}{\tau}\Big)^{3^m} + \alpha_3^2\tau^4\Big(\frac{\delta}{\tau}\Big)^{4\cdot 3^{m-1}}\Big) \leq \rho_3(\alpha_7 + \alpha_3^2)\tau^2\Big(\frac{\delta}{\tau}\Big)^{3^m} \leq 2\sqrt{n}\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^m} \leq 2\sqrt{n}\rho_0\delta.$$

Thus, (4.22) and (4.23) hold for $k = m$. From Lemma 4.5 with $k = m - 1$, we get $\|I - B_{m-1}J_{m-1}\| \leq \tau\left(\frac{\delta}{\tau}\right)^{3^{m-1}}$ and $\|B_{m-1}\| \leq 2\gamma$. Thanks to (4.29) (with $k = m$), one can see that

$$\|P_m - P_{m-1}\| \leq 4\sqrt{n}\beta\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}},$$

which implies that

$$\|\mathbf{p}_i^m - \mathbf{p}_i^{m-1}\| \leq 4\sqrt{n}\beta\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}}, \quad 1 \leq i \leq n.$$

Consequently,

$$\begin{aligned} |[J_m]_{ij} - [J_{m-1}]_{ij}| &= |(\mathbf{p}_i^m - \mathbf{p}_i^{m-1})^T A_j \mathbf{p}_i^m - (\mathbf{p}_i^m)^T A_j(\mathbf{p}_i^m - \mathbf{p}_i^{m-1})| \\ &\leq 2\|A_j\|\|(\mathbf{p}_i^m - \mathbf{p}_i^{m-1})\| \\ &\leq 8\|A_j\|\sqrt{n}\beta\rho_0\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}}, \ 1 \leq i, j \leq n. \end{aligned}$$

Hence,

$$\|J_m - J_{m-1}\| \leq \|J_m - J_{m-1}\|_F \leq 8n^{\frac{3}{2}}\beta\rho_0 \max_{1\leq j\leq n} \|A_j\|\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}} := \alpha_1\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}}.$$

It follows that

$$\begin{aligned} \|I - B_{m-1}J_m\| &\leq \|I - B_{m-1}J_{m-1}\| + \|B_{m-1}\|J_{m-1} - J_m\| \\ &\leq \tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}} + 2\gamma \cdot \alpha_1\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}} \leq (1 + 2\gamma\alpha_1)\tau\Big(\frac{\delta}{\tau}\Big)^{3^{m-1}}. \end{aligned} \tag{4.42}$$

Notice that $B_m = B_{m-1} + B_{m-1}(2I - J(\mathbf{c}^m)B_{m-1})(I - J(\mathbf{c}^m)B_{m-1})$, and we obtain

$$I - B_mJ_m = I - (B_{m-1} + B_{m-1}(2I - J(\mathbf{c}^m)B_{m-1})(I - J(\mathbf{c}^m)B_{m-1}))J_m = (I - B_{m-1}J_m)^3.$$

Together with (2.5), (4.26), and (4.42), one has

$$\|I - B_mJ_m\| \leq \|I - B_{m-1}J_m\|^3 \leq (1 + 2\gamma\alpha_1)^3\tau^3\Big(\frac{\delta}{\tau}\Big)^{3^m} \leq \tau\Big(\frac{\delta}{\tau}\Big)^{3^m},$$

and therefore, (4.24) is true for $k = m$ and the proof is complete. □

5. Numerical experiments

In this section, we present the computational performance of Algorithm I in addressing the IEP with several multiple eigenvalues. Algorithm I is contrasted with the two-step Ulm-Chebyshev-like Cayley transform method (TUCT method), inexact Cayley transform method (ICT method), and Ulm-like Cayley transform method (UCT method) as presented in [30, 35, 36], respectively. The tests were carried out in MATLAB 7.10 running on a PC Intel Pentium IV of 3.0 GHz CPU. We will now consider the problem of finding the eigenvalues for a matrix with a Toeplitz structure, as previously investigated in references [35, 36].

The QMR method [37] was utilized to solve all linear systems in all algorithms, using the MATLAB QMR function and setting the maximum number of iterations to 1000. Notably, in the context of solving approximate Jacobian equations within the framework of the inexact Cayley transform method, an advanced approach was employed. This approach involved leveraging the preconditioned QMR method with a specific stopping tolerance, and integrating the MATLAB incomplete LU factorization as the designated preconditioner. Specifically, the drop tolerance in LUINC(A, drop-tolerance) is fixed at 0.001. Furthermore, the precision level for the linear systems involved in all computational methods is adjusted to match the machine accuracy, ensuring the attainment of the desired solutions. The termination criterion for the outer (Newton) iterations is met in every algorithm when

$$\|P_k^T A(\mathbf{c}^k) P_k - \Lambda^*\| \leq 10^{-12}.$$

Example 5.1. (See [35, 36]) Referring to the Toeplitz matrices stated in [35, 36], consider $\{A_i\}_{i=0}^n$ as

$$A_0 = O,\ \ A_1 = I,\ \ A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \cdots, A_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Hence, the matrix $A(\mathbf{c})$ can be characterized as a symmetric Toeplitz matrix where the first column is identical to the vector $\mathbf{c}$.

Here, we consider three cases: $n = 100, 200, 300$. For any n, we generate a vector $\tilde{\mathbf{c}}^*$ such that $|\lambda_{k+1}(\tilde{\mathbf{c}}^*) - \lambda_k(\tilde{\mathbf{c}}^*)| < \eta$ with $1 \leq l \leq n-1$, where

$$\eta = \begin{cases} 5 \times 10^{-5}, & n = 100; \\ 1 \times 10^{-5}, & n = 200; \\ 1 \times 10^{-6}, & n = 300. \end{cases}$$

Set

$$\lambda_i^* = \begin{cases} \lambda_k(\tilde{\mathbf{c}}^*), & i = k,\ k+1; \\ \lambda_i(\tilde{\mathbf{c}}^*), & \text{otherwise.} \end{cases}$$

Subsequently, we choose $\{\lambda_i^*\}_{i=1}^n$ as the prescribed eigenvalues. It is clear that, in this way of selecting $\{\lambda_i^*\}_{i=1}^n$, multiple eigenvalues are present. Since all of the algorithms are locally convergent, the initial guess $\mathbf{c}_0$ is formed by chopping the components of $\tilde{\mathbf{c}}^*$ to six decimal places for $n =$ 100, 200, 300. The information in Table 1 presents the average values of $\|P_k^T A(\mathbf{c}^k) P_k - \Lambda^*\|$, and " it." denotes the the averaged numbers of outer iterations. The data in Table 2 presents the average total numbers of outer iterations N_0 throughout ten test scenarios and the average total numbers of inner iterations N_i essential for solving the IEPs. A comparison of the averaged CPU time of all of the algorithms for the ten tests with different n is shown in Table 3.

Table 1. Averaged values of $\|P_k^T A(\mathbf{c}^k)P_k - \Lambda^*\|$ for the ten tests.

n	it.	ICT method $\beta = 1.5$	ICT method $\beta = 1.8$	UCT method	Algorithm I	TUCT method
100	0	$1.8973e-5$	$1.8973e-5$	$1.8973e-5$	$1.8973e-5$	$1.8973e-5$
	1	$1.0006e-6$	$9.5329e-5$	$3.4389e-5$	$7.2881e-9$	$2.3548e+3$
	2	$3.4623e-7$	$5.2316e-7$	$1.5623e-7$	$4.6864e-14$	$2.5421e+15$
	3	$1.5456e-9$	$2.5695e-9$	$9.2635e-9$		
	4	$8.6994e-12$	$9.9999e-11$	$8.2312e-12$		
	5	$9.5684e-15$	$1.0001e-14$	$6.2351e-15$		
200	0	$2.6051e-5$	$2.6051e-5$	$2.6051e-5$	$2.6051e-5$	$2.6051e-5$
	1	$1.0003e-7$	$4.1858e-6$	$3.1889e-6$	$9.9999e-8$	$4.2151e+4$
	2	$9.9998e-8$	$3.9856e-8$	$4.4668e-8$	$6.2392e-13$	$8.2357e+21$
	3	$6.1254e-9$	$2.2254e-9$	$3.5563e-9$		
	4	$1.5684e-12$	$9.5695e-11$	$1.5623e-12$		
	5	$2.3564e-14$	$3.2356e-13$	$8.8563e-14$		
300	0	$4.0904e-5$	$4.0904e-5$	$4.0904e-5$	$4.0904e-5$	$4.0904e-5$
	1	$9.1364e-7$	$4.2864e-7$	$7.2789e-7$	$1.9604e-9$	$3.2546e+3$
	2	$3.3874e-8$	$2.9856e-8$	$3.3668e-8$	$2.2275e-13$	$6.2534e+22$
	3	$4.2356e-9$	$8.5695e-9$	$4.5564e-9$		
	4	$1.0012e-12$	$9.9898e-11$	$2.5873e-11$		
	5	$6.5684e-14$	$1.2325e-13$	$8.2344e-13$		

Table 2. Averaged total numbers of outer and inner iterations for the ten tests.

n		ICT method $\beta = 1.5$	ICT method $\beta = 1.8$	Algorithm I	UCT method
100	N_0	4.9	4.9	2	4.9
	N_i	32.1	32.9	29.2	32.5
200	N_0	5.1	5.1	2	5.1
	N_i	47.2	47.8	38.9	47.3
300	N_0	5.1	5.1	2	5.1
	N_i	71.2	71.8	60.1	71.6

Table 3. Averaged CPU time in seconds of all algorithms for the ten tests.

n	50	100	150	200	250	300
UCT method	0.66	2.14	7.23	18.11	40.56	98.88
ICT method with $\beta = 1.5$	0.64	1.91	8.12	17.68	32.43	86.37
Algorithm I	0.33	1.18	5.53	12.52	28.99	55.67

From the data presented in Table 1, we see that Algorithm I requires a lower number of iterations compared to the ICT and UCT methods, and the TUCT method fails to converge. It is evident from the data presented in Table 2 that Algorithm I outperforms the inexact Cayley transform method and Ulm-like Cayley transform method. Table 3 shows that Algorithm I demonstrates a more cost-effective CPU time compared to the other approaches.

Example 5.2. [22] This is an inverse problem with multiple eigenvalues and $n = 8$. We are given that $B = I + WW^T$, where

$$W = \begin{bmatrix} 1 & -1 & -3 & -5 & -6 \\ 1 & 1 & -2 & -5 & -17 \\ 1 & -1 & -1 & 5 & 18 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & -1 \\ 2.5 & 0.2 & 0.3 & 0.5 & 0.6 \\ 2 & -0.2 & 0.3 & 0.5 & 0.8 \end{bmatrix}_{8\times 5}.$$

Define

$$A_0 = 0, \quad A_i := b_{ii}\mathbf{e}_i\mathbf{e}_i^T + \sum_{j=1}^{i-1} b_{ij}(\mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T), \quad i = 1, \ldots, 8.$$

$$\boldsymbol{\lambda}^* = (1, 1, 1, 2.120754, 9.218868, 17.28137, 35.70822, 722.6808)^T,$$

$$\mathbf{c}^* = (1, 1, 1, 1, 1, 1, 1, 1)^T.$$

We report our numerical results for different starting points: (a) $\mathbf{c}^0 = 10^{-5} \cdot (1, 1, 1, 1, 1, 1, 1, 1)^T$; (b) $\mathbf{c}^0 = (0, 0, 0, 0, 0, 0, 0, 0)^T$.

Table 4 presents the average values of $\|P_k^T A(\mathbf{c}^k)P_k - \Lambda^*\|$, and " it." denotes the the averaged numbers of outer iterations.

Table 4. Averaged values of $\|P_k^T A(\mathbf{c}^k)P_k - \Lambda^*\|$ for the ten tests.

	it.	ICT method		UCT method	Algorithm I	TUCT method
		$\beta = 1.5$	$\beta = 1.8$			
(a)	0	$2.2113e-5$	$2.2113e-5$	$2.2113e-5$	$2.2113e-5$	$2.2113e-5$
	1	$3.2154e-6$	$4.2568e-6$	$8.2135e-5$	$9.889e-14$	$4.2356e+3$
	2	$8.3549e-8$	$4.2648e-8$	$2.1350e-8$		$3.2589e+15$
	3	$3.2111e-10$	$1.2309e-10$	$8.2156e-10$		
	4	$8.2543e-13$	$5.2236e-13$	$9.9998e-14$		
(b)	0	$7.2383e+2$	$7.2383e+2$	$7.2383e+2$	$7.2383e+2$	$7.2383e+2$
	1	$5.2469e+5$	$3.2699e+6$	$5.3216e+6$	$8.3269e+18$	$3.2156e+14$
	2	$5.2148e+12$	$7.2584e+12$	$5.2318e+13$		

It can be observed from Table 4 that Algorithm I requires a lower number of iterations compared to the ICT and UCT methods and Algorithm I has global non-convergence and local cubic convergence.

To further illustrate the effectiveness of Algorithm I, we present a practical engineering application in vibrations [15, 16]. We consider the vibration of a taut string with n beads. Figure 1 shows such a model for the case where $n = 4$. Here, we assume that the n beads are placed along the string, where the ends of the string are clamped. The mass of the jth bead is denoted by m_j. The horizontal lengths between masses m_j and m_{j+1} (and between each bead at each end and the clamped support) are set to be a constant L. The horizontal tension is set to be a constant T. Then the equation of motion is governed by

$$m_j y_j''(t) = T\frac{y_{j+1} - y_j}{L} - T\frac{y_j - y_{j-1}}{L}, \quad j = 1, \ldots, n, \tag{5.1}$$

where $y_0 = y_{n+1} = 0$. That is, the ends of the string are fixed. The matrix form of (5.1) is given by

$$y''(t) = -CJy(t), \tag{5.2}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$, $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ with $c_j = \frac{T}{m_j L}$, and J is the discrete Laplacian matrix

$$J = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathcal{S}^n.$$

The general solution of (5.2) is given in terms of the eigenvalue problem

$$CJy = \lambda y,$$

where λ is the square of the natural frequency of the vibration system and the nonzero vector y accounts for the interplay between the masses. The inverse problem for the beaded string is to compute the masses $\{m_j\}_{j=1}^n$ so that the resulting system has a prescribed set of natural frequencies.

It is easy to check that the eigenvalues of J are given by

$$\lambda_j(J) = 4\left(\sin\frac{j\pi}{n+1}\right)^2, \quad j = 1, 2, \ldots, n.$$

Thus, J is symmetric and positive definite and CJ is similar to $L^T CL$, where L is the Cholesky factor of $J = LL^T$ [31]. Then, the inverse problem is converted into the form of the IEP where $A_0 = 0$ and $A_j = L^T E_j L$ with $E_j = \mathrm{diag}(e_j)$ for $j = 1, 2, \ldots, n$. The beaded string data in Examples 5.3 and 5.4 comes from the website `http://www.caam.rice.edu/~beads`.

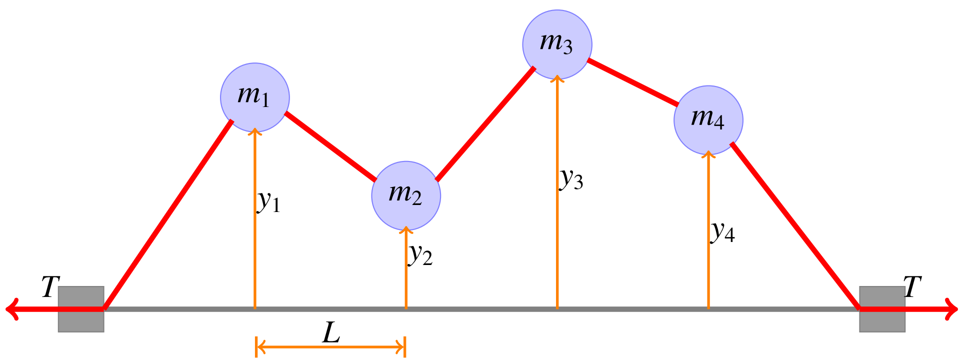

Figure 1. A string with $n = 4$ beads.

Example 5.3. This is an inverse problem for the beaded string with $n = 4$ beads, where

$$(m_1, m_2, m_3, m_4) = (0.030783, 0.017804, 0.017804, 0.030783)\,(kg = kilogram),$$
$$(n+1)L = 1.12395\,(meter), \quad T = 191.8199\,(Newton),$$
$$\boldsymbol{\lambda}^* = (15041.90, 42344.26, 88328.78, 15041.90)^T,$$
$$c^* = (27720.80, 47929.08, 47929.08, 27720.80)^T.$$

We report our numerical result for the starting points: $c^0 = 10^{-5} \cdot (27720.80, 47929.08, 47929.08, 27720.80)^T$.

Example 5.4. This is an inverse problem for the beaded string with $n = 6$ beads, where

$$(m_1, m_2, m_3, m_4, m_5, m_6) = (0.017804, 0.030783, 0.017804, 0.017804, 0.030783, 0.017804)(kg),$$
$$(n+1)L = 1.12395\,(meter), \quad T = 166.0370\,(Newton),$$
$$\boldsymbol{\lambda}^* = (9113.978, 30746.32, 83621.69, 148694.4, 148694.4, 193537.0)^T,$$
$$c^* = (58081.57, 33592.71, 58081.57, 58081.57, 33592.71, 58081.57)^T.$$

We report our numerical results for the starting points: $c^0 = 10^{-5} \cdot (58081.57, 33592.71, 58081.57, 58081.57, 33592.71, 58081.57)^T$.

The information in Table 5 presents the average values of $\|P_k^T A(\mathbf{c}^k) P_k - \Lambda^*\|$, and Table 6 displays the computed masses for the beaded string.

Table 5. Averaged values of $\|P_k^T A(\mathbf{c}^k) P_k - \Lambda^*\|$ in the ten tests for Examples 5.3 and 5.4.

	Example 5.3
	$\|P_k^T A(\mathbf{c}^k) P_k - \Lambda^*\|$-value (last 3 iterations)
ICT method	$(6.0235 \times 10^{-8}, 3.6564 \times 10^{-10}, 9.3356 \times 10^{-13})$
UCT method	$(1.9587 \times 10^{-8}, 8.3985 \times 10^{-10}, 2.8872 \times 10^{-14})$
Algorithm I	$(5.9254 \times 10^{-5}, 2.8123 \times 10^{-11}, 6.5865 \times 10^{-21})$
	Example 5.4
	$\|P_k^T A(\mathbf{c}^k) P_k - \Lambda^*\|$-value (last 3 iterations)
ICT method	$(1.0562 \times 10^{-7}, 5.0420 \times 10^{-9}, 7.4001 \times 10^{-13})$
UCT method	$(2.9125 \times 10^{-7}, 9.0859 \times 10^{-9}, 3.7523 \times 10^{-13})$
Algorithm I	$(1.9862 \times 10^{-5}, 2.9546 \times 10^{-11}, 5.5555 \times 10^{-21})$

Table 6. Recovered masses for Examples 5.3 and 5.4.

			Example 5.3			
	m_1	m_2	m_3	m_4		
true	0.030783	0.017804	0.017804	0.030783		
recovered	0.030783	0.017804	0.017804	0.030783		
			Example 5.4			
	m_1	m_2	m_3	m_4	m_5	m_6
true	0.017804	0.030783	0.017804	0.017804	0.030783	0.017804
recovered	0.017804	0.030783	0.017804	0.017804	0.030783	0.017804

From Table 5, we know that Algorithm I requires a lower number of iterations compared to the ICT and UCT methods, and the TUCT method fails to converge. Table 6 shows that the desired masses are recovered. All of these numerical observations agree with our prediction and further validate our theoretical results.

6. Conclusions

In this paper, we have proposed a two-step Ulm-Chebyshev-like Cayley transform method for solving the IEP (1.2) with multiple eigenvalues, which avoids solving (approximate) Jacobian equations in each outer iteration. Furthermore, the proposed algorithm is proved to have cubical convergence under the following nonsingular condition in terms of the relative generalized Jacobian evaluated at a solution $\mathbf{c}^*$: Each $J \in \partial_{Q|S}\mathbf{f}(\mathbf{c}^*)$ is nonsingular. Furthermore, this kind of method can be worked with larger problems for the Toeplitz inverse eigenvalue problem and is efficient when working with small and medium-sized problems for a more general perspective on this problem. Nevertheless, the effective methods for general larger inverse eigenvalue problems should be studied in the future.

Author contributions

Wei Ma, Zhenhao Li and Yuxin Zhang: Algorithms, Software, Numerical examples, Writing-original draft, Writing-review & editing. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. F. Diele, T. Laudadio, N. Mastronardi, On some inverse eigenvalue problems with Toeplitz-related structure, *SIAM J. Matrix Anal. Appl.*, **26** (2004), 285–294. https://doi.org/10.1137/S0895479803430680
2. J. Peinado, A. M. Vidal, A new parallel approach to the Toeplitz inverse eigenproblem using Newton-like methods, In: *Vector and parallel processing–VECPAR 2000*, Springer, 2001. https://doi.org/10.1007/3-540-44942-6_29
3. W. F. Trench, Numerical solution of the inverse eigenvalue problem for real symmetric Toeplitz matrices, *SIAM J. Sci. Comput.*, **18** (1997), 1722–1736. https://doi.org/10.1137/S1064827595280673
4. N. Wagner, Inverse eigenvalue problems in structural dynamics, *Proc. Appl. Math. Mech.*, **6** (2006), 339–340. https://doi.org/10.1002/pamm.200610151
5. P. J. Brussard, P. W. M. Glaudemans, *Shell model applications in nuclear spectroscopy*, North-Holland, 1977.
6. M. S. Ravi, J. Rosenthal, X. A. Wang, On decentralized dynamic pole placement and feedback stabilization, *IEEE Trans. Automat. Control*, **40** (1995), 1603–1614. https://doi.org/10.1109/9.412629
7. O. Hald, *On discrete and numerical Sturm-Liouville problems*, New York University, 1972.
8. G. M. L. Gladwell, Inverse problems in vibration, *Appl. Mech. Rev.*, **39** (1986), 1013–1018. https://doi.org/10.1115/1.3149517
9. G. M. L. Gladwell, Inverse problems in vibration-II, *Appl. Mech. Rev.*, **49** (1996), S25–S34. https://doi.org/10.1115/1.3101973
10. K. T. Joseph, Inverse eigenvalue problem in structural design, *AIAA J.*, **30** (1992), 2890–2896. https://doi.org/10.2514/3.11634
11. R. L. Parker, K. A. Whaler, Numerical methods for establishing solutions to the inverse problem of electromagnetic induction, *J. Geophys. Res.*, **86** (1981), 9574–9584. https://doi.org/10.1029/JB086iB10p09574
12. N. Li, A matrix inverse eigenvalue problem and its application, *Linear Algebra Appl.*, **266** (1997), 143–152. https://doi.org/10.1016/S0024-3795(96)00639-8
13. M. Müller, An inverse eigenvalue problem: computing B-stable Runge-Kutta methods having real poles, *BIT Numer. Math.*, **32** (1992), 676–688. https://doi.org/10.1007/BF01994850
14. S. Elhay, Y. M. Ram, An affine inverse eigenvalue problem, *Inverse Problems*, **18** (2002), 455. https://doi.org/10.1088/0266-5611/18/2/311
15. M. T. Chu, Inverse eigenvalue problems, *SIAM Rev.*, **40** (1998), 1–39. https://doi.org/10.1137/S0036144596303984

16. M. T. Chu, G. H. Golub, Structured inverse eigenvalue problems, *Acta Numer.*, **11** (2002), 1–71. https://doi.org/10.1017/S0962492902000016

17. M. T. Chu, G. H. Golub, *Inverse eigenvalue problems: theory, algorithms, and applications*, Oxford: Oxford University Press, 2005. https://doi.org/10.1093/acprof:oso/9780198566649.001.0001

18. S. F. Xu, *An introduction to inverse algebric eigenvalue problems*, Beijing: Peking University Press, 1998.

19. Z. J. Bai, Inexact Newton methods for inverse eigenvalue problems, *Appl. Math. Comput.*, **172** (2006), 682–689. https://doi.org/10.1016/j.amc.2004.11.023

20. V. N. Kublanovskaja, On one approach to the solution of the inverse eigenvalue problem, In: *Automatic programming and numerical methods of analysis*, Springer, 1972, 80–86. https://doi.org/10.1007/978-1-4615-8588-6_10

21. Y. Wang, W. Shen, An extended two-step method for inverse eigenvalue problems with multiple eigenvalues, *Numer. Math. Theor. Methods Appl.*, **16** (2023), 968–992.

22. S. Friedland, J. Nocedal, M. L. Overton, The formulation and analysis of numerical methods for inverse eigenvalue problems, *SIAM. J. Numer. Anal.*, **24** (1987), 634–667. https://doi.org/10.1137/0724043

23. Z. J. Bai, R. H. Chan, B. Morini, An inexact Cayley transform method for inverse eigenvalue problems, *Inverse Problems*, **20** (2004), 1675. https://doi.org/10.1088/0266-5611/20/5/022

24. R. H. Chan, S. F. Xu, H. M. Zhou, On the convergence rate of a quasi-Newton method for inverse eigenvalue problems, *SIAM J. Numer. Anal.*, **36** (1999), 436–441. https://doi.org/10.1137/S0036142997327051

25. R. H. Chan, H. L. Chung, S. F. Xu, The inexact Newton-like method for inverse eigenvalue problem, *BIT Numer. Math.*, **43** (2003), 7–20. https://doi.org/10.1023/a:1023611931016

26. W. P. Shen, C. Li, X. Q. Jin, A Ulm-like method for inverse eigenvalue problems, *Appl. Numer. Math.*, **61** (2011), 356–367. https://doi.org/10.1016/j.apnum.2010.11.001

27. W. P. Shen, C. Li, An Ulm-like Cayley transform method for inverse eigenvalue problems, *Taiwanese J. Math.*, **16** (2012), 367–386. https://doi.org/10.11650/twjm/1500406546

28. X. S. Chen, C. T. Wen, H. W. Sun, Two-step Newton-type methods for solving inverse eigenvalue problems, *Numer. Linear Algebra Appl.*, **25** (2018), e2185. https://doi.org/10.1002/nla.2185

29. C. T. Wen, X. S. Chen, H. W. Sun, A two-step inexact Newton-Chebyshev-like method for inverse eigenvalue problems, *Linear Algebra Appl.*, **585** (2020), 241–262. https://doi.org/10.1016/j.laa.2019.10.004

30. W. Ma, Two-step Ulm-Chebyshev-like Cayley transform method for inverse eigenvalue problems, *Int. J. Comput. Math.*, **99** (2022), 391–406. https://doi.org/10.1080/00207160.2021.1913728

31. G. H. Golub, C. F. Van Loan, *Matrix computations*, 3 Eds., Johns Hopkins University Press, 1996.

32. L. Q. Qi, Convergence analysis of some algorithms for solving nonsmooth equations, *Math. Oper. Res.*, **18** (1993), 227–244. https://doi.org/10.1287/moor.18.1.227

33. F. A. Potra, L. Q. Qi, D. F. Sun, Secant methods for semismooth equations, *Numer. Math.*, **80** (1998), 305–324. https://doi.org/10.1007/s002110050369

34. D. F. Sun, J. Sun, Strong semismoothness of eigenvalues of symmetric matrices and its application to inverse eigenvalue problems, *SIAM J. Numer. Anal.*, **40** (2003), 2352–2367. https://doi.org/10.1137/s0036142901393814

35. W. P. Shen, C. Li, X. Q. Jin, An inexact Cayley transform method for inverse eigenvalue problems with multiple eigenvalues, *Inverse Problems*, **31** (2015), 085007. https://doi.org/10.1088/0266-5611/31/8/085007

36. W. P. Shen, C. Li, X. Q. Jin, An Ulm-like Cayley transform method for inverse eigenvalue problems with multiple eigenvalues, *Numer. Math. Theor. Methods Appl.*, **9** (2016), 664–685. https://doi.org/10.4208/nmtma.2016.y15030

37. R. W. Freund, N. M. Nachtigal, QMR: a quasi-minimal residual method for non-Hermitian linear systems, *Numer. Math.*, **60** (1991), 315–339. https://doi.org/10.1007/BF01385726

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