AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 22910–22926.
DOI: 10.3934/math.20241114
Received: 18 June 2024
Revised: 18 July 2024
Accepted: 19 July 2024
Published: 25 July 2024

Research article

Generalized exponential stability of stochastic Hopfield neural networks with variable coefficients and infinite delay

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Abstract: This paper centers on stochastic Hopfield neural networks with variable coefficients and infinite delay. First, we propose an integral inequality that improves and extends some existing works. Second, by employing some inequalities and stochastic analysis techniques, some sufficient conditions for ensuring pth moment generalized exponential stability are established. Our results do not necessitate the construction of a complex Lyapunov function or rely on the assumption of bounded variable coefficients, and our results expand some existing works. At last, to illustrate the efficacy of our result, we present several simulation examples.

Keywords: generalized exponential stability; stochastic Hopfield neural networks; variable coefficients; infinite delay; integral inequalities
Mathematics Subject Classification: 34K50, 90B15, 93D20

1. Introduction

Hopfield neural networks have recently sparked significant interest, due to their versatile applications in various domains including associative memory [1], image restoration [2], and pattern recognition [3]. In neural networks, time delays often arise due to the restricted switching speed of amplifiers [4]. Additionally, when examining long-term dynamic behavior, nonautonomous characteristics become apparent, with system coefficients evolving over time [5]. Moreover, in biological nervous systems, synaptic transmission introduces stochastic perturbations, adding an element of randomness [6]. As we know that time delays, nonautonomous behavior, and stochastic perturbations can induce oscillations and instability in neural networks. Hence, it becomes imperative to investigate the stability of stochastic delay Hopfield neural networks (SDHNNs) with variable coefficients.

The Lyapunov technique stands out as a powerful approach for examining the stability of SDHNNs.

Wang et al. [7, 8] and Chen et al. [9] employed the Lyapunov-Krasovskii functional to investigate the (global) asymptotic stability of SHNNs characterized by constant coefficients and bounded delay, respectively. Zhou and Wan [10] and Hu et al. [11] utilized the Lyapunov technique and some analysis techniques to investigate the stability of SHNNs with constant coefficients and bounded delay, respectively. Liu and Deng [12] used the vector Lyapunov function to investigate the stability of SHNNs with bounded variable coefficients and bounded delay. It is important to note that establishing a suitable Lyapunov function or functional can pose significant challenges, especially when dealing with infinite delay nonautonomous stochastic systems.

Meanwhile, the fixed point technique presents itself as another potent tool for stability analysis, offering the advantage of not necessitating the construction of a Lyapunov function or functional. Luo used this technique to consider the stability of several stochastic delay systems in earlier research [13–15]. More recently, Chen et al. [16] and Song et al. [17] explored the stability of SDNNs characterized by constant coefficients and bounded variable coefficients using the fixed point technique, yielding intriguing results. However, the fixed point technique has a limitation in the stability analysis of stochastic systems, stemming from the inappropriate application of the Hölder inequality.

Furthermore, integral or differential inequalities are also powerful techniques for stability analysis. Hou et al. [18], and Zhao and Wu [19] used the differential inequalities to consider stability of NNs, Wan and Sun [20], Sun and Cao [21], as well as Li and Deng [22] harnessed variation parameters and integral inequalities to explore the exponential stability of various SDHNNs with constant coefficients. In a similar vein, Ruan et al. [23] and Zhang et al. [24] utilized integral and differential inequalities to probe the pth moment exponential stability of SDHNNs characterized by bounded variable coefficients.

It is worth highlighting that the literature mentioned previously exclusively focused on investigating the exponential stability of SDHNNs, without addressing other decay modes. Generalized exponential stability was introduced in [25] for cellular neural networks without stochastic perturbations, and is a more general concept of stability which contains the usual exponential stability, polynomial stability, and logarithmical stability. It provides some new insights into the stability of dynamic systems. Motivated by the above discussion, we are prompted to explore the pth moment generalized exponential stability of SHNNs characterized by variable coefficients and infinite delay.

$$\begin{cases} dz_i(t) = \Big[- c_i(t)z_i(t) + \sum\limits_{j=1}^{n} a_{ij}(t)f_j(z_j(t)) + \sum\limits_{j=1}^{n} b_{ij}(t)g_j(z_j(t-\delta_{ij}(t)))\Big]dt \\ \qquad\qquad + \sum\limits_{j=1}^{n} \sigma_{ij}(t, z_j(t), z_j(t-\delta_{ij}(t)))dw_j(t), \qquad\qquad t \geq t_0, \\ z_i(t) = \phi_i(t), \quad t \leq t_0, \quad i = 1, 2, ..., n. \end{cases} \tag{1.1}$$

It is important to note that the models presented in [20,21,25–28] are specific instances of system (1.1). System (1.1) incorporates several complex factors, including unbounded time-varying coefficients and infinite delay functions. As a result, discussing the stability and its decay rate for (1.1) becomes more complicated and challenging.

The contributions of this paper can be summarized as follows: (i) A new concept of stability is utilized for SDHNNs, namely the generalized exponential stability in pth moment. (ii) We establish a set of multidimensional integral inequalities that encompass unbounded variable coefficients and infinite delay, which extends the works in [23]. (iii) Leveraging these derived inequalities, we delve into the pth moment generalized exponential stability of SDHNNs with variable coefficients, and the work in [10, 11, 20, 21, 26, 27] are improved and extended.

The structure of the paper is as follows: Section 2 covers preliminaries and provides a model description. In Section 3, we present the primary inequalities along with their corresponding proofs. Section 4 is dedicated to the application of these derived inequalities in assessing the pth moment generalized exponential stability of SDHNNs with variable coefficients. In Section 5, we present three simulation examples that effectively illustrate the practical applicability of the main results. Finally, Section 6 concludes our paper.

2. Preliminaries and model description

Let $\mathbb{N}_n = \{1, 2, ..., n\}$. $|\cdot|$ is the norm of $\mathbb{R}^n$. For any sets A and B, $A - B := \{x|x \in A,\ x \notin B\}$. For two matrixes $C, D \in \mathbb{R}^{n\times m}$, $C \geq D$, $C \leq D$, and $C < D$ mean that every pair of corresponding parameters of C and D satisfy inequalities $\geq$, $\leq$, and $<$, respectively. E^T and E^{-1} represent the transpose and inverse of the matrix E, respectively. The space of bounded continuous $\mathbb{R}^n$-valued functions is denoted by $\mathcal{BC} := \mathcal{BC}((-\infty, t_0]; \mathbb{R}^n)$, for $\varphi \in \mathcal{BC}$, and its norm is given by

$$\|\varphi\|_\infty = \sup_{\theta\in(-\infty,t_0]} |\varphi(\theta)| < \infty.$$

$(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq t_0}, \mathbb{P})$ stands for the complete probability space with a right continuous normal filtration $\{\mathscr{F}_t\}_{t\geq t_0}$ and $\mathscr{F}_{t_0}$ contains all $\mathbb{P}$-null sets. For $p > 0$, let $\mathcal{L}^p_{\mathscr{F}_t}((-\infty, t_0]; \mathbb{R}^n) := \mathcal{L}^p_{\mathscr{F}_t}$ be the space of $\{\mathscr{F}_t\}$-measurable stochastic processes $\phi = \{\phi(\theta) : \theta \in (-\infty, t_0]\}$ which take value in $\mathcal{BC}$ satisfying

$$\|\phi\|^p_{\mathcal{L}^p} = \sup_{\theta\in(-\infty,t_0]} \mathbb{E}|\phi(\theta)|^p < \infty,$$

where $\mathbb{E}$ represents the expectation operator.

In system (1.1), $z_i(t)$ represents the ith neural state at time t; $c_i(t)$ is the self-feedback connection weight at time t; $a_{ij}(t)$ and $b_{ij}(t)$ denote the connection weight at time t of the jth unit on the ith unit; f_j and g_j represent the activation functions; $\sigma_{ij}(t, z_j(t), z_j(t - \delta_{ij}(t)))$ stands for the stochastic effect, and $\delta_{ij}(t) \geq 0$ denotes the delay function. Moreover, $\{\omega_j(t)\}_{j\in\mathbb{N}_n}$ is a set of Wiener processes mutually independent on the space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$; $z_i(t, \phi)$ $(i \in \mathbb{N}_n)$ represents the solution of (1.1) with an initial condition $\phi = (\phi_1, \phi_2, ..., \phi_n) \in \mathcal{L}^p_{\mathscr{F}_t}$, sometimes written as $z_i(t)$ for short. Now, we introduce the definition of generalized exponentially stable in pth $(p \geq 2)$ moment.

Definition 2.1. *System (1.1) is pth ($p \geq 2$) moment generalized exponentially stable, if for any $\phi \in \mathcal{L}^p_{\mathscr{F}_t}$, there are $\kappa > 0$ and $c(u) \geq 0$ such that $\lim_{t\to+\infty} \int_{t_0}^t c(u)du \to +\infty$ and*

$$\mathbb{E}|z_i(t, \phi)|^p \leq \kappa \max_{j\in\mathbb{N}_n}\{\|\phi_j\|^p_{\mathcal{L}^p}\}e^{-\int_{t_0}^t c(u)du}, \quad i \in \mathbb{N}_n, \quad t \geq t_0,$$

where $-\int_{t_0}^t c(u)du$ is the general decay rate.

Remark 2.1. *Lu et al. [25] proposed the generalized exponential stability for neural networks without stochastic perturbations, we extend it to the SDHNNs.*

Remark 2.2. *We replace $\int_{t_0}^t c(u)du$ by $\lambda(t-t_0)$, $\lambda \ln(t-t_0+1)$, and $\lambda \ln(\ln(t-t_0+e))$ ($\lambda > 0$), respectively. Then (1.1) is exponentially, polynomially, and logarithmically stable in pth moment, respectively.*

Lemma 2.1. *[29] For a square matrix $\Lambda \geq 0$, if $\rho(\Lambda) < 1$, then $(I - \Lambda)^{-1} \geq 0$, where $\rho(\cdot)$ is the spectral radius, and I and 0 are the identity and zero matrices, respectively.*

3. Main inequalities

Consider the following inequalities

$$\begin{cases} y_i(t) \le \psi_i(0)e^{-\int_{t_0}^{t}\gamma_i(u)du} + \sum\limits_{j=1}^{n}\alpha_{ij}\int_{t_0}^{t} e^{-\int_s^t \gamma_i(u)du}\gamma_i(s)\sup\limits_{s-\eta_{ij}(s)\le v\le s} y_j(v)ds, \quad t \ge t_0, \\ y_i(t) = \psi_i(t) \in \mathcal{BC}, \quad t \in (-\infty, t_0], \quad i \in \mathbb{N}_n, \end{cases} \tag{3.1}$$

where $y_i(t)$, $\gamma_i(t)$, and $\eta_{ij}(t)$ are non-negative functions and $\alpha_{ij} \ge 0$, $i, j \in \mathbb{N}_n$.

Lemma 3.1. *Regrading system (3.1), let the following hypotheses hold:*

(**H.1**) *For $i, j \in \mathbb{N}_n$, there exist $\gamma(t)$ and $\gamma_i > 0$ such that*

$$0 \le \gamma_i\gamma(t) \le \gamma_i(t) \text{ for } t \ge t_0, \quad \lim_{t\to+\infty}\int_{t_0}^{t}\gamma(u)du \to +\infty, \quad \sup_{t\ge t_0}\left\{\int_{t-\eta_{ij}(t)}^{t}\gamma^*(u)du\right\} := \eta_{ij} < +\infty,$$

where $\gamma^*(t) = \gamma(t)$, for $t \ge t_0$, and $\gamma^*(t) = 0$, for $t < t_0$.

(**H.2**) $\rho(\alpha) < 1$, *where $\alpha = (\alpha_{ij})_{n\times n}$.*

Then, there is a $\kappa > 0$ such that

$$y_i(t) \le \kappa \max_{j\in\mathbb{N}_n}\{\|\psi_j\|_\infty\}e^{-\lambda\int_{t_0}^{t}\gamma(u)du}, \quad i \in \mathbb{N}_n, \quad t \ge t_0.$$

Proof. For $t \ge t_0$, multiply $e^{\lambda\int_{t_0}^{t}\gamma(u)du}$ on both sides of (3.1), and one has

$$\begin{aligned} e^{\lambda\int_{t_0}^{t}\gamma(u)du}y_i(t) &\le \psi_i(t_0)e^{\lambda\int_{t_0}^{t}\gamma(u)du}e^{-\int_{t_0}^{t}\gamma_i(u)du} + \sum_{j=1}^{n} e^{\lambda\int_{t_0}^{t}\gamma(s)ds}\alpha_{ij}\int_{t_0}^{t} e^{-\int_s^t\gamma_i(u)du}\gamma_i(s)\sup_{s-\eta_{ij}(s)\le v\le s} y_j(v)ds \\ &:= I_{i1}(t) + I_{i2}(t), \quad i \in \mathbb{N}_n, \end{aligned} \tag{3.2}$$

where $\lambda \in (0, \min\limits_{i\in\mathbb{N}_n}\{\gamma_i\})$ is a sufficiently small constant which will be explained later. Define

$$H_i(t) := \sup_{\xi\le t}\left\{e^{\lambda\int_{t_0}^{\xi}\gamma^*(u)du}y_i(\xi)\right\},$$

$i \in \mathbb{N}_n$ and $t \ge t_0$. Obviously,

$$I_{i1}(t) = \psi_i(t_0)e^{\lambda\int_{t_0}^{t}\gamma(u)du}e^{-\int_{t_0}^{t}\gamma_i(u)du} \le e^{(\lambda-\gamma_i)\int_{t_0}^{t}\gamma(u)du}\psi_i(t_0) \le \psi_i(t_0), \quad i \in \mathbb{N}_n, \quad t \ge t_0. \tag{3.3}$$

Further, it follows from (**H.1**) that

$$\begin{aligned} I_{i2}(t) &\le \sum_{j=1}^{n}\alpha_{ij}\int_{t_0}^{t} e^{-\int_s^t\gamma_i(u)du}\gamma_i(s)e^{\lambda\int_{s-\eta_{ij}(s)}^{s}\gamma^*(u)du}\sup_{s-\eta_{ij}(s)\le v\le s}\{y_j(v)\}e^{\lambda\int_{t_0}^{s-\eta_{ij}(s)}\gamma^*(u)du}e^{\lambda\int_s^t\gamma(u)du}ds \\ &\le \sum_{j=1}^{n}\alpha_{ij}e^{\lambda\eta_{ij}}\int_{t_0}^{t} e^{-\int_s^t(\gamma_i(u)-\lambda\gamma(u))du}\gamma_i(s)\sup_{s-\eta_{ij}(s)\le v\le s}\{y_j(v)e^{\lambda\int_{t_0}^{v}\gamma^*(u)du}\}ds \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n} \alpha_{ij} e^{\lambda\eta_{ij}} H_j(t) \int_{t_0}^{t} e^{-\int_s^t (\gamma_i(u)-\lambda\gamma(u))du} \gamma_i(s) ds \\
&\le \sum_{j=1}^{n} \alpha_{ij} e^{\lambda\eta_{ij}} H_j(t) \int_{t_0}^{t} e^{-\int_s^t (\gamma_i(u)-\frac{\lambda}{\gamma_i}\gamma_i(u))du} \gamma_i(s) ds \\
&\le \frac{\gamma_i}{\gamma_i-\lambda} \sum_{j=1}^{n} \alpha_{ij} e^{\lambda\eta_{ij}} H_j(t), \quad i \in \mathbb{N}_n, \quad t \ge t_0. \qquad (3.4)
\end{aligned}
$$

By (3.2)–(3.4), we have

$$
e^{\lambda \int_{t_0}^{t} \gamma(s)ds} y_i(t) \le \psi_i(t_0) + \frac{\gamma_i}{\gamma_i-\lambda} \sum_{j=1}^{n} \alpha_{ij} e^{\lambda\eta_{ij}} H_j(t), \quad i \in \mathbb{N}_n, \quad t \ge t_0.
$$

By the definition of $H_i(t)$, we get

$$
H_i(t) \le \psi_i(t_0) + \frac{\gamma_i}{\gamma_i-\lambda} \sum_{j=1}^{n} \alpha_{ij} e^{\lambda\eta_{ij}} H_j(t), \quad i \in \mathbb{N}_n, \quad t \ge t_0,
$$

i.e.,

$$
H(t) \le \psi(t_0) + \frac{\Gamma\alpha e^{\lambda\eta} H(t)}{\Gamma - \lambda I}, \quad t \ge t_0, \qquad (3.5)
$$

where $H(t) = (H_1(t), ..., H_n(t))^T$, $\psi(t_0) = (\psi_1(t_0), ..., \psi_n(t_0))^T$, $\Gamma = diag(\gamma_1, ..., \gamma_n)$, and $\alpha e^{\lambda\eta} = (\alpha_{ij} e^{\lambda\eta_{ij}})_{n\times n}$. Since $\rho(\alpha) < 1$ and $\alpha \ge 0$, then there is a small enough $\lambda > 0$ such that

$$
\rho\Big(\frac{\Gamma\alpha e^{\lambda\eta}}{\Gamma-\lambda I}\Big) < 1 \text{ and } \frac{\Gamma\alpha e^{\lambda\eta}}{\Gamma-\lambda I} \ge 0.
$$

From Lemma 2.1, we get

$$
\Big(I - \frac{\Gamma\alpha e^{\lambda\eta}}{\Gamma-\lambda I}\Big)^{-1} \ge 0.
$$

Denote

$$
N(\lambda) = \Big(I - \frac{\Gamma\alpha e^{\lambda\eta}}{\Gamma-\lambda I}\Big)^{-1} = (N_{ij}(\lambda))_{n\times n}.
$$

From (3.5), we have

$$
H(t) \le N(\lambda)\psi(t_0), \quad t \ge t_0.
$$

Therefore, for $i \in \mathbb{N}_n$, we get

$$
y_i(t) \le \sum_{i=1}^{n} N_{ij}(\lambda)\psi_i(t_0) e^{-\lambda \int_{t_0}^{t} \gamma(u)du} \le \sum_{j=1}^{n} N_{ij}(\lambda) \|\psi_j\|_\infty e^{-\lambda \int_{t_0}^{t} \gamma(u)du}, \quad t \ge t_0,
$$

and then there exists a $\kappa > 0$ such that

$$
y_i(t) \le \kappa \max_{i\in\mathbb{N}_n} \{\|\psi_i\|_\infty\} e^{-\lambda \int_{t_0}^{t} \gamma(u)du}, \quad i \in \mathbb{N}_n, \quad t \ge t_0.
$$

This completes the proof. □

Consider the following differential inequalities

$$\begin{cases} D^+y_i(t) \le -\gamma_i(t)y_i(t) + \sum\limits_{j=1}^{n} \alpha_{ij}\gamma_i(t) \sup\limits_{t-\eta_{ij}(t)\le s\le t} y_j(s), \quad t \ge t_0, \\ y_i(t) = \psi_i(t) \in \mathcal{BC}, \quad t \in (-\infty, t_0], \quad i \in \mathbb{N}_n, \end{cases} \tag{3.6}$$

where D^+ is the Dini-derivative, $y_i(t)$, $\gamma_i(t)$, and $\eta_{ij}(t)$ are non-negative functions, and $\alpha_{ij} \ge 0$, $i, j \in \mathbb{N}_n$.

Lemma 3.2. *For system (3.6), under hypotheses* (**H**.**1**) *and* (**H**.**2**)*, there are* $\kappa > 0$ *and* $\lambda > 0$ *such that*

$$y_i(t) \le \kappa \max_{i\in\mathbb{N}_n}\{\|\psi_i\|_\infty\} e^{-\lambda\int_{t_0}^{t}\gamma(u)du}, \quad i \in \mathbb{N}_n, \quad t \ge t_0.$$

Proof. For $t > t_0$, multiply $e^{\int_{t_0}^{t}\gamma_i(u)du}$ ($i \in \mathbb{N}_n$) on both sides of (3.6) and perform the integration from t_0 to t. We have

$$y_i(t) \le \psi_i(0)e^{-\int_{t_0}^{t}\gamma_i(u)du} + \sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_{s}^{t}\gamma_i(u)du}\alpha_{ij}\gamma_i(s) \sup_{s-\eta_{ij}(s)\le v\le s} y_j(v)ds, \quad i \in \mathbb{N}_n.$$

The proof is deduced from Lemma 3.1. □

Remark 3.1. *For a given matrix* $M = (m_{ij})_{n\times n}$*, we have* $\rho(M) \le \|M\|$*, where* $\|\cdot\|$ *is an arbitrary norm, and then we can obtain some conditions for generalized exponential stability. In addition, for any nonsingular matrix* S*, define the responding norm by* $\|M\|_S = \|S^{-1}MS\|$*. Let* $S = diag\{\xi_1, \xi_2, ..., \xi_n\}$*, then for the row, column, and the Frobenius norm, the following conditions imply* $\|M\|_S < 1$*:*

(1) $\sum\limits_{j=1}^{n}\left(\frac{\xi_i}{\xi_j}|m_{ij}|\right) < 1$ *for* $i \in \mathbb{N}_n$*;*

(2) $\sum\limits_{i=1}^{n}\left(\frac{\xi_i}{\xi_j}|m_{ij}|\right) < 1$ *for* $i \in \mathbb{N}_n$*;*

(3) $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\left(\frac{\xi_i}{\xi_j}|m_{ij}|\right)^2 < 1.$

Remark 3.2. *Ruan et al. [23] investigated the special case of inequalities (3.6), i.e.,* $\gamma_i(t) = \gamma_i$ *and* $\eta_{ij}(t) = \eta_{ij}$*. They obtained that system (3.6) is exponentially stable provided*

$$\gamma_i > \sum_{j=1}^{n}\alpha_{ij}, \quad i \in \mathbb{N}_n. \tag{3.7}$$

From Remark 3.1, we know condition $\rho(\alpha) < 1$ ($\alpha = (\frac{\alpha_{ij}}{\gamma_i})_{n\times n}$) *is weaker than (3.7). Moreover, we discuss the generalized exponential stability which contains the normal exponential stability. This means that our result improves and extends the result in [23].*

4. Main result

This section considers the pth moment generalized exponential stability of (1.1) by applying Lemma 3.1. To obtain the pth moment generalized exponential stability, we need the following conditions.

(**C**.**1**) For $i, j \in \mathbb{N}_n$, there are $c(t)$ and $c_i > 0$ such that

$$0 \le c_i c(t) \le c_i(t) \text{ for } t \ge t_0, \quad \lim_{t\to+\infty}\int_{t_0}^{t} c(s)ds \to +\infty, \quad \sup_{t\ge t_0}\left\{\int_{t-\delta_{ij}(t)}^{t} c^*(s)ds\right\} := \delta_{ij} < +\infty,$$

where $c^*(t) = c(t)$, for $t \ge t_0$, and $c(t) = 0$, for $t < t_0$.

(**C**.**2**) The mappings f_j and g_j satisfy $f_j(0) = g_j(0) = 0$ and the Lipchitz condition with Lipchitz constants $F_j > 0$ and $G_j > 0$ such that

$$|f_j(v_1) - f_j(v_2)| \le F_j|v_1 - v_2|, \quad |g_j(v_1) - g_j(v_2)| \le G_j|v_1 - v_2|, \quad j \in \mathbb{N}_n, \quad \forall v_1, v_2 \in \mathbb{R}.$$

(**C**.**3**) The mapping σ_{ij} satisfies $\sigma_{ij}(t, 0, 0) \equiv 0$ and $\forall u_1, u_2, v_1, v_2 \in \mathbb{R}$, and there are $\mu_{ij}(t) \ge 0$ and $\nu_{ij}(t) \ge 0$ such that

$$\left|\sigma_{ij}(t, u_1, v_1) - \sigma_{ij}(t, u_2, v_2)\right|^2 \le \mu_{ij}(t)\left|u_1 - u_2\right|^2 + \nu_{ij}(t)\left|v_1 - v_2\right|^2, \quad i, j \in \mathbb{N}_n, \quad t \ge t_0.$$

(**C**.**4**) For $i, j \in \mathbb{N}_n$,

$$\sup_{\{t|t\ge t_0\}-\{t|c_i(t)=|a_{ij}(t)|F_j+|b_{ij}(t)|G_j=0\}}\left\{\frac{|a_{ij}(t)|F_j + |b_{ij}(t)|G_j}{c_i(t)}\right\} := \rho_{ij}^{(1)},$$

$$\sup_{\{t|t\ge t_0\}-\{t|c_i(t)=\mu_{ij}(t)+\nu_{ij}(t)\}}\left\{\frac{\mu_{ij}(t) + \nu_{ij}(t)}{c_i(t)}\right\} := \rho_{ij}^{(2)}.$$

(**C**.**5**)

$$\rho\left(M + \frac{\Omega^{(1)}}{p} + \frac{(p-1)\Omega^{(2)}}{p}\right) < 1,$$

where $M = diag(m_1, m_2, ..., m_n)$, $m_i = \frac{(p-1)\sum_{j=1}^{n}\rho_{ij}^{(1)}}{p} + \frac{(p-1)(p-2)\sum_{j=1}^{n}\rho_{ij}^{(2)}}{2p}$, $\Omega^{(k)} = (\rho_{ij}^{(k)})_{n\times n}$, $k \in \mathbb{N}_2$, and $p \ge 2$.

Conditions (**C**.**1**)–(**C**.**4**) guarantee the existence and uniqueness of (1.1) [30].

Theorem 4.1. *Under conditions* (**C**.**1**)–(**C**.**5**)*, system (1.1) is pth moment generalized exponentially stable with decay rate* $-\lambda\int_{t_0}^{t} c(s)ds$*,* $\lambda > 0$*.*

Proof. By the Itô formula, one can obtain

$$\begin{aligned} dz_i^p(t) = \Big[& - pc_i(t)z_i^p(t) + \sum_{j=1}^{n} pa_{ij}(t)f_j(z_j(t))z_i^{p-1}(t) + \sum_{j=1}^{n} pb_{ij}(t)g_j(z_j(t-\delta_{ij}(t)))z_i^{p-1}(t) \\ & + \sum_{j=1}^{n}\frac{p(p-1)}{2}|\sigma_{ij}(t, z_j(t), z_j(t-\delta_{ij}(t)))|^2 z_i^{p-2}(t)\Big]dt \\ & + \sum_{j=1}^{n} p\sigma_{ij}(t, z_j(t), z_j(t-\delta_{ij}(t)))z_i^{p-1}(t)dw_j(t), \quad i \in \mathbb{N}_n, \quad t \ge t_0. \end{aligned}$$

So we get

$$
\begin{aligned}
z_i^p(t) = \phi_i^p(t_0) + \int_{t_0}^{t} \Big[& - pc_i(s) z_i^p(s) + \sum_{j=1}^{n} pa_{ij}(s) f_j(z_j(s)) z_i^{p-1}(s) \\
& + \sum_{j=1}^{n} pb_{ij}(s) g_j(z_j(s - \delta_{ij}(s))) z_i^{p-1}(s) \\
& + \sum_{j=1}^{n} \frac{p(p-1)}{2} |\sigma_{ij}(s, z_j(s), z_j(s - \delta_{ij}(s)))|^2 z_i^{p-2}(s) \Big] ds \\
& + \sum_{j=1}^{n} \int_{t_0}^{t} p\sigma_{ij}(s, z_j(s), z_j(s - \delta_{ij}(s))) z_i^{p-1}(s) dw_j(s), \quad i \in \mathbb{N}_n, \quad t \geq t_0.
\end{aligned}
$$

Since $\mathbb{E}\Big[\int_{t_0}^{t} p\sigma_{ij}(s, z_j(s), z_j(s - \delta_{ij}(s))) z_i^{p-1}(s) dw_j(s) \Big] = 0$ for $i \in \mathbb{N}_n$ and $t \geq t_0$, we have

$$
\begin{aligned}
\mathbb{E}[z_i^p(t)] = \mathbb{E}[\phi_i^p(t_0)] + \int_{t_0}^{t} \mathbb{E}\Big[& - pc_i(s) z_i^p(s) + \sum_{j=1}^{n} pa_{ij}(s) f_j(z_j(s)) z_i^{p-1}(s) \\
& + \sum_{j=1}^{n} pb_{ij}(s) g_j(z_j(s - \delta_{ij}(s))) z_i^{p-1}(s) \\
& + \sum_{j=1}^{n} |\sigma_{ij}(s, z_j(s), z_j(t - \delta_{ij}(s)))|^2 \frac{p(p-1)}{2} z_i^{p-2}(s) \Big] ds, \quad i \in \mathbb{N}_n, \quad t \geq t_0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
d\mathbb{E}[z_i^p(t)] = -pc_i(t) \mathbb{E}[z_i^p(t)] dt + \mathbb{E}\Big[& \sum_{j=1}^{n} pa_{ij}(t) f_j(z_j(t)) z_i^{p-1}(t) + \sum_{j=1}^{n} pb_{ij}(t) g_j(z_j(t - \delta_{ij}(t))) z_i^{p-1}(t) \\
& + \sum_{j=1}^{n} \frac{p(p-1)}{2} |\sigma_{ij}(t, z_j(t), z_j(t - \delta_{ij}(t)))|^2 z_i^{p-2}(t) \Big] dt, \quad i \in \mathbb{N}_n, \quad t \geq t_0.
\end{aligned}
$$

For $i \in \mathbb{N}_n$ and $t \geq t_0$, using the variation parameter approach, we get

$$
\begin{aligned}
\mathbb{E}[z_i^p(t)] = \mathbb{E}[\phi_i^p(t_0)] e^{-\int_{t_0}^{t} pc_i(s) ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} pc_i(u) du} \mathbb{E}\Big[& \sum_{j=1}^{n} pa_{ij}(s) f_j(z_j(s)) z_i^{p-1}(s) \\
+ \sum_{j=1}^{n} pb_{ij}(s) g_j(z_j(s - \delta_{ij}(s))) z_i^{p-1}(s) + & \sum_{j=1}^{n} \frac{p(p-1)}{2} |\sigma_{ij}(s, z_j(s), z_j(s - \delta_{ij}(s)))|^2 z_i^{p-2}(s) \Big] ds.
\end{aligned}
$$

For $i \in \mathbb{N}_n$ and $t \geq t_0$, conditions (**C.2**)–(**C.4**) and the Young inequality yield

$$
\begin{aligned}
\mathbb{E}|z_i(t)|^p \leq & \mathbb{E}|\phi_i(t_0)|^p e^{-\int_{t_0}^{t} pc_i(u) du} \\
& + \sum_{j=1}^{n} \int_{t_0}^{t} e^{-\int_{s}^{t} pc_i(u) du} p|a_{ij}(s)| F_j \mathbb{E}|z_j(s) z_i^{p-1}(s)| ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du} p|b_{ij}(s)|G_j\mathbb{E}|z_j(s-\delta_{ij}(s))z_i^{p-1}(s)|ds \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du} \frac{p(p-1)}{2}\mu_{ij}(s)\mathbb{E}|z_j^2(s)z_i^{p-2}(s)|ds \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du} \frac{p(p-1)}{2}\nu_{ij}(s)\mathbb{E}|z_j^2(s-\delta_{ij}(s))z_i^{p-2}(s)|ds \\
\leq\ &\mathbb{E}|\phi_i(t_0)|^p e^{-\int_{t_0}^t pc_i(u)du} \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du}|a_{ij}(s)|F_j\Big(\mathbb{E}|z_j(s)|^p+(p-1)\mathbb{E}|z_i(s)|^p\Big)ds \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du}|b_{ij}(s)|G_j\Big(\mathbb{E}|z_j(s-\delta_{ij}(s))|^p+(p-1)\mathbb{E}|z_i(s)|^p\Big)ds \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du}\mu_{ij}(s)\Big((p-1)\mathbb{E}|z_j(s)|^p+\frac{(p-1)(p-2)}{2}\mathbb{E}|z_i(s)|^p\Big)ds \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du}\nu_{ij}(s)\Big((p-1)\mathbb{E}|z_j(s-\delta_{ij}(s))|^p+\frac{(p-1)(p-2)}{2}\mathbb{E}|z_i(s)|^p\Big)ds \\
\leq\ &\mathbb{E}|\phi_i(t_0)|^p e^{-\int_{t_0}^t pc_i(u)du} \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du}(\rho_{ij}^{(1)}+(p-1)\rho_{ij}^{(2)})c_i(s)\sup_{s-\delta_{ij}(s)\leq v\leq s}\mathbb{E}|z_j(v)|^p ds \\
&+\sum_{j=1}^{n}\int_{t_0}^{t} e^{-\int_s^t pc_i(u)du}\Big((p-1)\rho_{ij}^{(1)}+\frac{(p-1)(p-2)}{2}\rho_{ij}^{(2)}\Big)c_i(s)\sup_{s-\delta_{ij}(s)\leq v\leq s}\mathbb{E}|z_i(v)|^p ds.
\end{aligned}
$$

Then, all of the hypotheses of Lemma 3.1 are satisfied. So there exists $\kappa > 0$ and $\lambda > 0$ such that

$$\mathbb{E}|z_i(t)|^p \leq \kappa \max_{j\in\mathbb{N}_n}\{||\phi_j||^p_{\mathcal{L}^p}\}e^{-\lambda\int_{t_0}^t c(u)du}, \quad i\in\mathbb{N}_n, \quad t\geq t_0.$$

This completes the proof. □

Remark 4.1. *Huang et al. [27] and Sun and Cao [21] considered the special case of (1.1), i.e.,* $a_{ij}(t)\equiv a_{ij}$, $b_{ij}(t)\equiv b_{ij}$, $c_i(t)\equiv c_i$, $\mu_{ij}(t)\equiv\mu_j$, $\nu_{ij}(t)\equiv\nu_j$, *and* $\delta_{ij}(t)\equiv\delta_j(t)$ *is a bounded delay function. [27] showed that system (1.1) is pth moment exponentially stable provided that there are positive constants* $\xi_i,...,\xi_n$ *such that* $N_1 > N_2 > 0$, *where*

$$N_1 = \min_{i\in\mathbb{N}_n}\{pc_i - \sum_{j=1}^{n}(p-1)|a_{ij}|(F_j+G_j)+\sum_{j=1}^{n}\frac{\xi_j}{\xi_i}(|a_{ji}|F_i+(p-1)\mu_i)+\sum_{j=1}^{n}\frac{(p-1)(p-2)}{2}(\mu_i+\nu_i)\}$$

and

$$N_2 = \max_{i\in\mathbb{N}_n}\Big\{\sum_{j=1}^{n}\frac{\xi_j}{\xi_i}(|b_{ji}|G_i+(p-1)\nu_i)\Big\}.$$

The above conditions imply that for each $i \in \mathbb{N}_n$,

$$pc_i - \sum_{j=1}^{n}(p-1)|a_{ij}|(F_j+G_j) + \sum_{j=1}^{n}\frac{\xi_j}{\xi_i}(|a_{ji}|F_i + |b_{ji}|G_i + (p-1)(\mu_i+\nu_j)) + \sum_{j=1}^{n}\frac{(p-1)(p-2)}{2}(\mu_i+\nu_i) > 0.$$

Then

$$0 \le \sum_{j=1}^{n}\frac{(p-1)|a_{ij}|(F_j+G_j)}{pc_i} + \sum_{j=1}^{n}\frac{\xi_j}{\xi_i}\Big(\frac{|a_{ji}|F_i + |b_{ji}|G_i + (p-1)(\mu_i+\nu_j)}{pc_i}\Big) + \sum_{j=1}^{n}\frac{(p-1)(p-2)}{2pc_i}(\mu_i+\nu_i) < 1. \tag{4.1}$$

From Remark 3.1, we know condition (4.1) implies

$$\rho\Big(M + \frac{\Omega^{(1)}}{p} + \frac{(p-1)\Omega^{(2)}}{p}\Big) < 1,$$

and this means that this paper improves and enhances the results in [27]. Similarly, our results also improve and enhance the results in [10, 11, 26]. Besides, the results in [21] required the following conditions to guarantee the pth moment exponential stability, i.e.,

$$\rho(C^{-1}(M^*M_1I + M^*M_2I + NN_1 + NN_2)) < 1,$$

where

$$C = diag(c_1, c_2, ..., c_n), \quad M^* = diag((4c_1)^{p-1}, (4c_2)^{p-1}, ..., (4c_n)^{p-1}),$$
$$N_1 = (d_{ij})_{n\times n}, \quad d_{ij} = \mu_j^{p/2}, \quad N_2 = (e_{ij})_{n\times n}, \quad e_{ij} = \nu_j^{p/2},$$
$$M_1 = diag\Big((\sum_{j=1}^{n}|a_{1j}F_j|^{\frac{p}{p-1}})^{p-1}, (\sum_{j=1}^{n}|a_{2j}F_j|^{\frac{p}{p-1}})^{p-1}, ..., (\sum_{j=1}^{n}|a_{nj}F_j|^{\frac{p}{p-1}})^{p-1}\Big),$$
$$M_2 = diag\Big((\sum_{j=1}^{n}|b_{1j}G_j|^{\frac{p}{p-1}})^{p-1}, (\sum_{j=1}^{n}|b_{2j}G_j|^{\frac{p}{p-1}})^{p-1}, ..., (\sum_{j=1}^{n}|b_{nj}G_j|^{\frac{p}{p-1}})^{p-1}\Big),$$
$$N = diag(4^{p-1}C_pn^{p-1}c_1^{1-p/2}, 4^{p-1}C_pn^{p-1}c_2^{1-p/2}, ..., 4^{p-1}C_pn^{p-1}c_n^{1-p/2})\ (C_p \ge 1).$$

From the matrix spectral analysis [29], we can get

$$\rho\Big(M + \frac{\Omega^{(1)}}{p} + \frac{(p-1)\Omega^{(2)}}{p}\Big) < \rho(C^{-1}(M^*M_1I + M^*M_2I + NN_1 + NN_2).$$

The above discussion shows that our results improve and extend the works in [21]. Similarly, our results also improve and broaden the results in [20].

Remark 4.2. *When $c_i(t) \equiv c_i$, $a_{ij}(t) \equiv a_{ij}$, $b_{ij}(t) \equiv b_{ij}$, $\delta_{ij}(t) \equiv \delta_j$, and $\sigma_{ij}(t, z_j(t), z_j(t-\delta_{ij}(t))) \equiv 0$, then (1.1) turns to be the following HNNs*

$$dz_i(t) = \Big[-c_iz_i(t) + \sum_{j=1}^{n}a_{ij}f_j(z_j(t)) + \sum_{j=1}^{n}b_{ij}g_j(z_j(t-\delta_j))\Big]dt, \quad i \in \mathbb{N}_n, \quad t \ge t_0, \tag{4.2}$$

or

$$dz(t) = \Big[- Cz(t) + Af(z(t)) + Bg(z_\delta(t)) \Big] dt, \quad t \geq t_0, \tag{4.3}$$

where $z(t) = (z_1(t), ..., z_n(t))^T$, $C = diag(c_1, ..., c_n) > 0$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $f(x(t)) = (f_1(z_1(t)), ..., f_n(z_n(t)))^T$, *and* $g(z_\delta(t)) = (g_1(z_1(t-\delta_1)), ..., g_n(z_n(t-\delta_n)))^T$. *This model was discussed in [16, 28]. For (4.3), using our approach can get the subsequent corollary.*

Corollary 4.1. *Under condition* (**C.2**), *if* $\rho(C^{-1}D) < 1$, *then (4.3) is exponentially stable, where* $D = (|a_{ij}F_j| + |b_{ij}G_j|)_{n\times n}$.

Note that Lai and Zhang [28] (Theorem 4.1) and Chen et al. [16] (Corollary 5.2) required the following conditions

$$\max_{i\in\mathbb{N}_n} \left[\frac{1}{c_i} \sum_{j=1}^{n} |a_{ij}F_j| + \frac{1}{c_i} \sum_{j=1}^{n} |b_{ij}G_j| \right] < \frac{1}{\sqrt{n}}$$

and

$$\sum_{j=1}^{n} \frac{1}{c_i} \max_{i\in\mathbb{N}_n} |a_{ij}F_j| + \sum_{j=1}^{n} \frac{1}{c_i} \max_{i\in\mathbb{N}_n} |b_{ij}G_j| < 1$$

to ensure the exponential stability, respectively. From Remark 3.1, we know that Corollary 4.1 is weaker than Theorem 4.1 in [28] and Corollary 5.2 in [16]. This improves and extends the results in [16, 28].

5. Examples

Now, we give three examples to illustrate the effectiveness of the main result.

Example 5.1. *Consider the following SDHNNs:*

$$\begin{cases} dz_i(t) = \Big[- c_i(t)z_i(t) + \sum\limits_{j=1}^{2} a_{ij}(t)f_j(z_j(t)) + \sum\limits_{j=1}^{2} b_{ij}(t)g_j(z_j(0.5t)) \Big] dt \\ \qquad\qquad + \sum\limits_{j=1}^{2} \sigma_{ij}(t, z_j(t), z_j(0.5t))dw_j(t), \quad t \geq 0, \\ z_i(0) = \phi_i(0), \quad i \in \mathbb{N}_2, \end{cases} \tag{5.1}$$

where $c_1(t) = 10(t+1)$, $c_2(t) = 20(t+2)$, $a_{11}(t) = b_{11}(t) = 0.5(t+1)$, $a_{12}(t) = b_{12}(t) = t+1$, $a_{21}(t) = b_{21}(t) = 2(t+2)$, $a_{22}(t) = b_{22}(t) = 2.5(t+2)$, $f_1(u) = f_2(u) = arctanu$, $g_1(u) = g_2(u) = 0.5(|u+1| - |u-1|)$, $\sigma_{11}(t, u, v) = \frac{\sqrt{2(t+1)}(u-v)}{2}$, $\sigma_{12}(t, u, v) = 2\sqrt{(t+1)}(u-v)$, $\sigma_{21}(t, u, v) = \sqrt{(t+1)}(u-v)$, $\sigma_{22}(t, u, v) = \frac{\sqrt{10(t+2)}(u-v)}{2}$, $\delta_{11}(t) = \delta_{21}(t) = \delta_{12}(t) = \delta_{22}(t) = 0.5t$, *and* $\phi(0) = (40, 20)$.

Choose $c(t) = \frac{1}{t+1}$, *and then* $\sup\limits_{t\geq 0} \left\{ \int_{0.5t}^{t} \frac{1}{s+1} ds \right\} = \ln 2$. *We can find* $F_1 = F_2 = G_1 = G_2 = 1$, $\rho_{11}^{(1)} = 0.1$, $\rho_{12}^{(1)} = 0.2$, $\rho_{21}^{(1)} = 0.2$, $\rho_{22}^{(1)} = 0.25$, $\rho_{11}^{(2)} = 0.2$, $\rho_{12}^{(2)} = 1.6$, $\rho_{21}^{(2)} = 0.2$, *and* $\rho_{22}^{(2)} = 0.5$. *Then*

$$\rho \begin{pmatrix} \rho_{11}^{(1)} + 0.5\rho_{12}^{(1)} + 0.5\rho_{11}^{(2)} & 0.5\rho_{12}^{(1)} + 0.5\rho_{12}^{(2)} \\ 0.5\rho_{21}^{(1)} + 0.5\rho_{21}^{(2)} & \rho_{22}^{(1)} + 0.5\rho_{21}^{(1)} + 0.5\rho_{22}^{(2)} \end{pmatrix} = \rho \begin{pmatrix} 0.3 & 0.9 \\ 0.2 & 0.6 \end{pmatrix} = 0.9 < 1.$$

Then (**C.1**)–(**C.5**) *are satisfied* ($p = 2$). *So (5.1) is generalized exponentially stable in mean square with a decay rate* $-\lambda \int_0^t \frac{1}{1+s} ds = -\lambda ln(1+t)$, $\lambda > 0$ *(see Figure 1).*

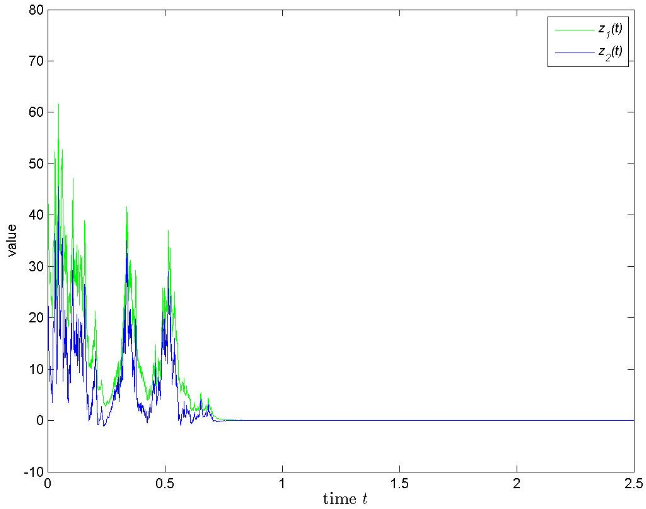

Figure 1. States $z_1(t)$ and $z_2(t)$ of Example 5.1.

Remark 5.1. *It is noteworthy that all variable coefficients and delay functions in Example 5.1 are unbounded, and then the results in [12, 23] are not applicable in this example.*

Example 5.2. *Consider the following SDHNNs:*

$$\begin{cases} dz_i(t) = \Big[-c_i(t)z_i(t) + \sum\limits_{j=1}^{2} a_{ij}(t)f_j(z_j(t)) + \sum\limits_{j=1}^{2} b_{ij}(t)g_j(z_j(t-\delta_{ij}(t)))\Big]dt \\ \qquad\qquad + \sum\limits_{j=1}^{2} \sigma_{ij}(t, z_j(t), z_j(t-\delta_{ij}(t)))dw_j(t), \qquad\qquad t \geq 0, \\ z_i(t) = \phi_i(t), \quad t \in [-\pi, 0], \quad i \in \mathbb{N}_2, \end{cases} \tag{5.2}$$

where $c_1(t) = 20(1-sint)$, $c_2(t) = 10(1-sint)$, $a_{11}(t) = b_{11}(t) = 2(1-sint)$, $a_{12}(t) = b_{12}(t) = 4(1-sint)$, $a_{21}(t) = b_{21}(t) = 0.5(1-sint)$, $a_{22}(t) = b_{22}(t) = 1.5(1-sint)$, $f_1(u) = f_2(u) = arctanu$, $g_1(u) = g_2(u) = 0.5(|u+1| - |u-1|)$, $\sigma_{11}(t,u,v) = \sqrt{2(1-sint)}(u-v)$, $\sigma_{12}(t,u,v) = \sqrt{6(1-sint)}(u-v)$, $\sigma_{21}(t,u,v) = \frac{\sqrt{(1-sint)}(u-v)}{2}$, $\sigma_{22}(t,u,v) = \frac{\sqrt{(1-sint)}(u-v)}{2}$, $\delta_{11}(t) = \delta_{21}(t) = \delta_{12}(t) = \delta_{22}(t) = \pi|\cos t|$, *and* $\phi(t) = (40, 20)$ *for* $t \in [-\pi, 0]$.

Choose $c(t) = 1 - \sin t$, *and then* $\sup\limits_{t\geq 0} \int_{t-\pi|\cos t|}^{t} (1-\sin s)^* ds = \pi + 2$. *We can find* $F_1 = F_2 = G_1 = G_2 = 1$, $\rho_{11}^{(1)} = 0.2$, $\rho_{12}^{(1)} = 0.4$, $\rho_{21}^{(1)} = 0.1$, $\rho_{22}^{(1)} = 0.3$, $\rho_{11}^{(2)} = 0.4$, $\rho_{12}^{(2)} = 1.2$, $\rho_{21}^{(2)} = 0.1$, *and* $\rho_{22}^{(2)} = 0.1$. *Then*

$$\rho\begin{pmatrix} \rho_{11}^{(1)} + 0.5\rho_{12}^{(1)} + 0.5\rho_{11}^{(2)} & 0.5\rho_{12}^{(1)} + 0.5\rho_{12}^{(2)} \\ 0.5\rho_{21}^{(1)} + 0.5\rho_{21}^{(2)} & \rho_{22}^{(1)} + 0.5\rho_{21}^{(1)} + 0.5\rho_{22}^{(2)} \end{pmatrix} = \rho\begin{pmatrix} 0.6 & 0.8 \\ 0.1 & 0.4 \end{pmatrix} = 0.8 < 1.$$

Then (**C.1**)–(**C.5**) *are satisfied* ($p = 2$). *So (5.2) is generalized exponentially stable in mean square with a decay rate* $-\lambda \int_0^t (1-sins)ds = -\lambda(t - cost + 1)$, $\lambda > 0$ *(see Figure 2).*

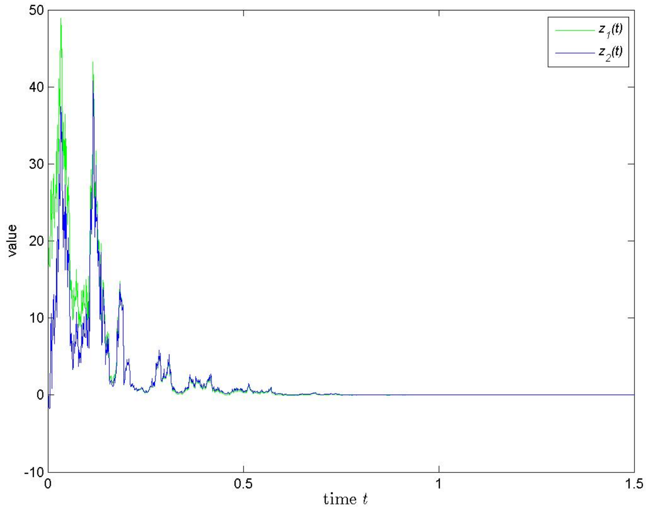

Figure 2. States $z_1(t)$ and $z_2(t)$ of Example 5.2.

Remark 5.2. *It should be pointed out that in Example 5.2 the variable coefficients* $c_i(t) = 0$ *for* $t = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{N}$. *This means that the results in [12, 23] cannot solve this case.*

To compare to some known results, we consider the following SDHNNs which are the special case of [12, 16, 20–23].

Example 5.3.

$$\begin{cases} dz_i(t) = \Big[-c_i z_i(t) + \sum\limits_{j=1}^{2} a_{ij} f_j(z_j(t)) + \sum\limits_{j=1}^{2} b_{ij} g_j(z_j(t-\delta_{ij}(t))) \Big] dt + \sigma_i(z_i(t)) dw_i(t), \quad t \geq 0, \\ z_i(t) = \phi_i(t), \quad t \in [-1, 0], \quad i \in \mathbb{N}_2, \end{cases} \tag{5.3}$$

where $c_1 = 2$, $c_2 = 4$, $a_{11} = 0.5$, $a_{12} = 1$, $b_{11} = 0.25$, $b_{12} = 0.5$, $a_{21} = \frac{1}{3}$, $a_{22} = \frac{2}{3}$, $b_{21} = \frac{1}{3}$, $b_{22} = \frac{2}{3}$, $f_1(u) = f_2(u) = arctanu$, $g_1(u) = g_2(u) = 0.5(|u+1| - |u-1|)$, $\sigma_1(u) = 0.5u$, $\sigma_2(u) = 0.5u$, $\delta_{11}(t) = \delta_{21}(t) = \delta_{12}(t) = \delta_{22}(t) = 1$, *and* $\phi(t) = (40, 20)$ *for* $t \in [-1, 0]$.

Choose $c(t) = 1$, *and then* $\sup\limits_{t \geq 0} \int_{t-1}^{t} (1)^* ds = 1$. *We can find* $F_1 = F_2 = G_1 = G_2 = 1$, $\rho_{11}^{(1)} = \frac{3}{8}$, $\rho_{12}^{(1)} = \frac{3}{4}$, $\rho_{21}^{(1)} = \frac{1}{6}$, $\rho_{22}^{(1)} = \frac{1}{3}$, $\rho_{11}^{(2)} = \frac{1}{8}$, $\rho_{12}^{(2)} = \rho_{21}^{(2)} = 0$, *and* $\rho_{22}^{(2)} = \frac{1}{16}$. *Then*

$$\rho \begin{pmatrix} \rho_{11}^{(1)} + 0.5\rho_{12}^{(1)} + 0.5\rho_{11}^{(2)} & 0.5\rho_{12}^{(1)} + 0.5\rho_{12}^{(2)} \\ 0.5\rho_{21}^{(1)} + 0.5\rho_{21}^{(2)} & \rho_{22}^{(1)} + 0.5\rho_{21}^{(1)} + 0.5\rho_{22}^{(2)} \end{pmatrix} = \rho \begin{pmatrix} \frac{7}{8} & \frac{3}{8} \\ \frac{1}{12} & \frac{43}{96} \end{pmatrix} < 1.$$

Then (**C.1**)–(**C.5**) *are satisfied* ($p = 2$). *So (5.3) is exponentially stable in mean square (see Figure 3).*

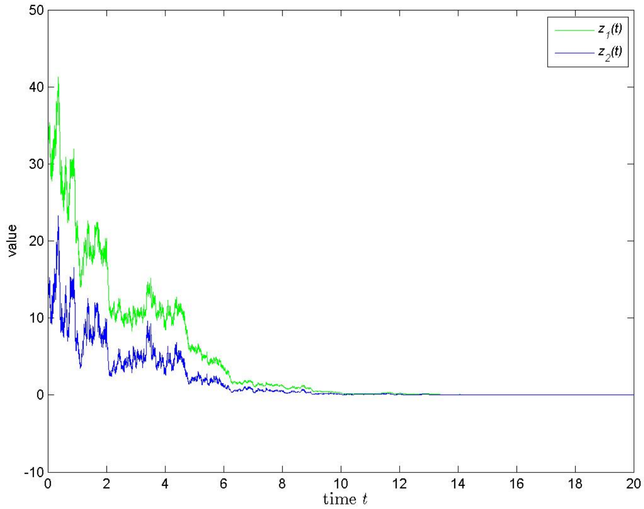

Figure 3. States $z_1(t)$ and $z_2(t)$ of Example 5.3.

Remark 5.3. *It is noteworthy that in Example 5.3,*

$$\rho\begin{pmatrix} \frac{4(a_{11}^2F_1^2+a_{12}^2F_2^2+b_{11}^2G_1^2+b_{12}^2G_2^2)}{c_1^2}+\frac{4\mu_{11}}{c_1} & 0 \\ 0 & \frac{4(a_{21}^2F_1^2+a_{22}^2F_2^2+b_{21}^2G_1^2+b_{22}^2G_2^2)}{c_2^2}+\frac{4\mu_{22}}{c_2} \end{pmatrix}=\rho\begin{pmatrix} \frac{33}{16} & 0 \\ 0 & \frac{19}{36} \end{pmatrix}=\frac{33}{16}>1,$$

which makes the result in [20–22] invalid. In addition,

$$\frac{4(a_{11}^2F_1^2+a_{12}^2F_2^2+b_{11}^2G_1^2+b_{12}^2G_2^2)}{c_1^2}+\frac{4(a_{21}^2F_1^2+a_{22}^2F_2^2+b_{21}^2G_1^2+b_{22}^2G_2^2)}{c_2^2}>1,$$

which makes the result in [16] not applicable in this example. Moreover

$$-c_1+(a_{11}F_1+a_{12}F_2+b_{11}G_1+b_{11}F_2+\frac{1}{2}\mu_{11})>0,$$

which makes the results in [12, 23] inapplicable in this example.

6. Conclusions

In this paper, we have addressed the issue of pth moment generalized exponential stability concerning SHNNs characterized by variable coefficients and infinite delay. Our approach involves the utilization of various inequalities and stochastic analysis techniques. Notably, we have extended and enhanced some existing results. Lastly, we have provided three numerical examples to showcase the practical utility and effectiveness of our results.

Author contributions

Dehao Ruan: Writing and original draft. Yao Lu: Review and editing. Both of the authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by the Talent Special Project of Guangdong Polytechnic Normal University (2021SDKYA053 and 2021SDKYA068), and Guangzhou Basic and Applied Basic Research Foundation (2023A04J0031 and 2023A04J0032).

Conflict of interest

The authors declare that they have no competing interests.

References

1. T. Deb, A. K. Ghosh, A. Mukherjee, Singular value decomposition applied to associative memory of Hopfield neural network, *Mater. Today: Proc.*, **5** (2018), 2222–2228. https://doi.org/10.1016/j.matpr.2017.09.222
2. M. Muneyasu, K. Yamamoto, T. Hinamoto, Image restoration using layered neural networks and Hopfield networks, *Proceedings., International Conference on Image Processing*, **2** (1995), 33–36. https://doi.org/10.1109/ICIP.1995.537408
3. P. N. Suganthan, E. K. Teoh, D. P. Mital, Pattern recognition by homomorphic graph matching using Hopfield neural networks, *Image Vision Comput.*, **13** (2015), 45–60. https://doi.org/10.1016/0262-8856(95)91467-R
4. C. M. Marcus, R. M. Westervelt, Stability of analog neural networks with delay, *Phys. Rev. A*, **39** (1989), 347–359. https://doi.org/10.1103/physreva.39.347
5. X. Wang, M. Jiang, S. Fang, Stability analysis in Lagrange sense for a non-autonomous Cohen-Grossberg neural network with mixed delays, *Nonlinear Anal.: Theory Methods Appl.*, **70** (2009), 4294–4306. https://doi.org/10.1016/j.na.2008.09.019
6. S. Haykin, *Neural networks: a comprehensive foundation*, 3 Eds., Prentice-Hall, Inc., 2007.
7. Z. Wang, H. Shu, J. Fang, X. Liu, Robust stability for stochastic Hopfield neural networks with time delays, *Nonlinear Anal.: Real World Appl.*, **7** (2006), 1119–1128. https://doi.org/10.1016/j.nonrwa.2005.10.004
8. Z. Wang, Y. Liu, K. Fraser, X. Liu, Stochastic stability of uncertain Hopfield neural networks with discrete and distributed delays, *Phys. Lett. A*, **354** (2006), 288–297. https://doi.org/10.1016/j.physleta.2006.01.061
9. Y. Chen, Z. Wang, Y. Liu, F. E. Alsaadi, Stochastic stability for distributed delay neural networks via augmented Lyapunov-Krasovskii functionals, *Appl. Math. Comput.*, **338** (2018), 869–881. https://doi.org/10.1016/j.amc.2018.05.059

10. Q. Zhou, L. Wan, Exponential stability of stochastic delayed Hopfield neural networks, *Appl. Math. Comput.*, **199** (2008), 84–89. https://doi.org/10.1016/j.amc.2007.09.025

11. J. Hu, S. Zhong, L. Liang, Exponential stability analysis of stochastic delayed cellular neural network, *Chaos Soliton. Fract.*, **27** (2006), 1006–1010. https://doi.org/10.1016/j.chaos.2005.04.067

12. L. Liu, F. Deng, Stability analysis of time varying delayed stochastic Hopfield neural networks in numerical simulation, *Neurocomputing*, **316** (2018), 294–305. https://doi.org/10.1016/j.neucom.2018.08.004

13. J. Luo, Fixed points and stability of neutral stochastic delay differential equations, *J. Math. Anal. Appl.*, **334** (2007), 431–440. https://doi.org/10.1016/j.jmaa.2006.12.058

14. J. Luo, Fixed points and exponential stability of mild solutions of stochastic partial differential equations with delays, *J. Math. Anal. Appl.*, **342** (2008), 753–760. https://doi.org/10.1016/j.jmaa.2007.11.019

15. J. Luo, Fixed points and exponential stability for stochastic Volterra-Levin equations, *J. Comput. Appl. Math.*, **234** (2010), 934–940. https://doi.org/10.1016/j.cam.2010.02.013

16. G. Chen, D. Li, L. Shi, O. van Ganns, S. V. Lunel, Stability results for stochastic delayed recurrent neural networks with discrete and distributed delays, *J. Differ. Equations*, **264** (2018), 3864–3898. https://doi.org/10.1016/j.jde.2017.11.032

17. Q. Song, R. Zeng, Z. Zhao, Y. Liu, F. E. Alsaadi, Mean-square stability of stochastic quaternion-valued neural networks with variable coefficients and neutral delays, *Neurocomputing*, **471** (2022), 130–138. https://doi.org/10.1016/j.neucom.2021.11.033

18. X. Hou, H. Wu, J. Cao, Observer-based prescribed-time synchronization and topology identification for complex networks of piecewise-smooth systems with hybrid impulses, *Comput. Appl. Math.*, **43** (2008), 180. https://doi.org/10.1007/s40314-024-02701-x

19. Y. Zhao, H. Wu, Fixed/Prescribed stability criterions of stochastic system with time-delay, *AIMS Math.*, **9** (2024), 14425–14453. https://doi.org/10.3934/math.2024701

20. L. Wan, J. Sun, Mean square exponential stability of stochastic delayed Hopfield neural networks, *Phys. Lett. A*, **343** (2005), 306–318. https://doi.org/10.1016/j.physleta.2005.06.024

21. Y. Sun, J. Cao, *p*th moment exponential stability of stochastic recurrent neural networks with time-varying delays, *Nonlinear Anal.: Real World Appl.*, **8** (2007), 1171–1185. https://doi.org/10.1016/j.nonrwa.2006.06.009

22. X. Li, D. Deng, Mean square exponential stability of stochastic Hopfield neural networks with mixed delays, *Stat. Probab. Lett.*, **126** (2017), 88–96. https://doi.org/10.1016/j.spl.2017.02.029

23. D. Ruan, Z. Huang, X. Guo, Inequalities and stability of stochastic Hopfield neural networks with discrete and distributed delays, *Neurocomputing*, **407** (2020), 281–291. https://doi.org/10.1016/j.neucom.2020.05.005

24. F. Zhang, C. Fei, W. Fei, Stability of stochastic Hopfield neural networks driven by G-Brownian motion with time-varying and distributed delays, *Neurocomputing*, **520** (2023), 320–330. https://doi.org/10.1016/j.neucom.2022.10.065

25. B. Lu, H. Jiang, A. Abdurahman, C. Hu, Global generalized exponential stability for a class of nonautonomous cellular neural networks via generalized Halanay inequalities, *Neurocomputing*, **214** (2016), 1046–1052. https://doi.org/10.1016/j.neucom.2016.06.068
26. C. Huang, Y. He, H. Wang, Mean square exponential stability of stochastic recurrent neural networks with time-varying delays, *Comput. Math. Appl.*, **56** (2008), 1773–1778. https://doi.org/10.1016/j.camwa.2008.04.004
27. C. Huang, Y. He, L. Huang, W. Zhu, pth moment stability analysis of stochastic recurrent neural networks with time-varying delays, *Inf. Sci.*, **178** (2008), 2194–2203. https://doi.org/10.1016/j.ins.2008.01.008
28. X. Lai, Y. Zhang, Fixed point and asymptotic analysis of cellular neural networks, *J. Appl. Math.*, **2012** (2012), 701–708. https://doi.org/10.1155/2012/689845
29. R. A. Horn, C. R. Johnson, *Matrix analysis*, Cambridge University Press, 1985. https://doi.org/10.1017/CBO9780511810817
30. A. Friedman, Stochastic differential equations and applications, In: J. Cecconi, *Stochastic differential equations*, C.I.M.E. Summer Schools, Springer, **77** (1997), 75–148. https://doi.org/10.1007/978-3-642-11079-5_2

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