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Research article

Asymptotic behavior and blow-up of solutions for a nonlocal parabolic equation with a special diffusion process

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Abstract: This paper considers a class of higher-order nonlocal parabolic equations with special coefficient. We apply the Faedo-Galerkin approximation method and cut-off technique to obtain the local solvability. Furthermore, based on the framework of the modified potential well, we get the global existence, asymptotic behavior, and blow-up of the weak solutions by the Hardy-Sobolev inequality when the initial energy is subcritical $J(u_0) < d$. In the critical case of $J(u_0) = d$, the above results have also been obtained. Finally, we utilize some new processing methods to gain the blow-up criterion in finite time with supercritical initial energy $J(u_0) > 0$.

Keywords: logarithmic nonlocal source; special diffusion process; global existence; asymptotic behavior; blow-up
Mathematics Subject Classification: 35A01, 35B40, 35B44, 35K35

1. Introduction

In this paper, we consider the initial boundary value problem

$$\begin{cases} \dfrac{u_t}{|x|^s} + \Delta^2 u - \Delta u_t = |u|^{r-2} u \ln |u| - \dfrac{|x|^{-s}}{\int_\Omega |x|^{-s} dx} \displaystyle\int_\Omega |u|^{r-2} u \ln |u| \, dx, & x \in \Omega, t > 0; \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, t > 0; \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain and the boundary $\partial\Omega$ is smooth. The initial data $0 \neq u_0(x) \in W_* = \left\{ u \in H_0^2(\Omega) : \int_\Omega |x|^{-s} u(x,t) \, dx = 0 \right\}$. Moreover, $0 \leq s \leq 2$, and the parameter r satisfies

$$2 < r < \frac{8}{N} + 2. \tag{1.2}$$

Due to $\lim_{u\to 0}|u|^{r-2}u\ln|u| = 0$, then when $u = 0$, we let $|u|^{r-2}u\ln|u| = 0$.

It is widely known that the nonlocal parabolic equations are used to simulate some phenomena in biological populations, and the mass of system is often known or conserved, see [1–5] etc. Many authors have discussed the following general parabolic equations with nonlocal source terms

$$u_t - \Delta u = f(u) - \frac{1}{\Omega}\int_\Omega f(u)\,dx, \tag{1.3}$$

which satisfies the integral condition $\int_\Omega u_0(x)\,dx = 0$ and $u_0(x) \neq 0$. For the study of the properties of solutions when $f(u) = u$, readers can refer to references [6–9], where the local existence, the asymptotic behavior of the global weak solutions, and the bounds of blow-up time were established. In [10], the authors considered problem (1.3) with logarithmic nonlinearity source $f(u) = u\ln|u|$. They obtained the results of blow-up of weak solutions under some conditions by solving differential inequalities.

When Δu is extended to $\Delta_r u$ or a more higher-order operator, such problems have also been widely studied [11–13]. For example, Qu and Zhou [14] had researched the thin film equations with nonlocal sources in the following form

$$u_t + \Delta^2 u = |u|^{r-1}u - \frac{1}{\Omega}\int_\Omega |u|^{r-1}u dx. \tag{1.4}$$

They obtained the global existence of the sign-changing solutions by the potential well method. Moreover, they studied the decay estimate of the non-extinction weak solutions and established the extinction result. Subsequently, Zhou and Xu improved the research results of problem (1.4) in [15, 16], where the authors established the upper bound of blow-up time when $J(u_0) > 0$. Logarithmic nonlinearity has been studied for a long time because it naturally applies in different fields of physics. Toualbia et al. [17] studied a class of nonlocal parabolic equations

$$u_t - div\left(|\nabla u|^{r-2}\nabla u\right) = |u|^{r-2}u\ln|u| - \frac{1}{\Omega}\int_\Omega |u|^{r-2}u\ln|u|\,dx, \tag{1.5}$$

and the nonextinction, asymptotic behavior, and blow-up properties under appropriate conditions were researched.

According to the law of conservation, many reaction diffusion processes can be expressed by the equation $u_t - \nabla\cdot(D\nabla u) = f(x,t,u,\nabla u)$, where the function D is the diffusion coefficient. Tan [18] studied this kind of equation earlier when the source $f(u) = u^q$ and diffusion coefficient $D = |x|^2$. More research on the equations of the special diffusion process can be found in references [19–22].

In recent research works, the following nonlocal parabolic equation was studied:

$$\frac{u_t}{|x|^s} - \Delta u = |u|^{r-1}u - \frac{|x|^{-s}}{\int_\Omega |x|^{-s}dx}\int_\Omega |u|^{r-1}udx. \tag{1.6}$$

For the case of $s = 0$, Gao and Han [23] established a result of blow-up with positive initial energy provided that $1 < r \le \frac{N+2}{N-2}$. Khelghati and Baghaei [24] improved the blow-up conclusion for all $r > 1$. If $s \ge 0$, it is necessary to use the Hardy-Sobolev inequality to prove the main results, which is effective for $N > 2$. Feng and Zhou [25] considered Problem (1.6) when $0 \le s < \frac{N(r-1)}{r+1}$ with $1 < r < \frac{N+2}{N-2}$, and they gave the result of blow-up when $J(u_0) < d$. Moreover, they also researched the vacuum isolating phenomenon. Subsequently, Wu and Yang considered the above singular equation for which the source term is the logarithmic source $u \ln |u|$ in reference [26]. They combined the method of Faedo-Galerkin with the technique of cut-off to prove the local existence. Additionally, they obtained the decay estimate by using the Hardy-Sobolev inequality. As a result, it was also established that weak solutions blow-up in infinite time.

Considering the above works, this work is the first paper to consider the global solvability and blow-up properties of problem (1.1). We should not only overcome the difficulty of a singular potential, but also deal with a logarithmic nonlocal source. This work is extremely meaningful.

In Section 2 of this paper, the basic lemmas and definitions are introduced. Further, the potential well and its properties are also described. In Section 3, we prove the existence and uniqueness of the local solutions. In Section 4, we establish the global existence and discuss the asymptotic properties and finite blow-up of weak solutions when $J(u_0) < d$. In Section 5, we extend the above conclusions to the case of $J(u_0) = d$ in parallel. Lastly, we give the result of finite time blow-up when $J(u_0) > 0$.

2. Preliminaries

First, we introduce some symbols, lemmas, and basic definitions. For convenience, we use $\|\cdot\|_r$ for the $L_r(\Omega)$ norm, $1 \le r \le \infty$, and $\|\Delta u\|_2$ and $\|u\|_{H_0^2(\Omega)}$ are equivalent.

For $u \in W_*$, define the potential energy functional as

$$J(u) = \frac{1}{r^2} \|u\|_r^r - \frac{1}{r} \int_\Omega |u|^r \ln |u| \, dx + \frac{1}{2} \|\Delta u\|_2^2 \,, \tag{2.1}$$

$$I(u) = - \int_\Omega |u|^r \ln |u| \, dx + \|\Delta u\|_2^2 \,. \tag{2.2}$$

By a direct computation,

$$J(u) = (\frac{1}{2} - \frac{1}{r}) \|\Delta u\|_2^2 + \frac{1}{r^2} \|u\|_r^r + \frac{1}{r} I(u). \tag{2.3}$$

We also define the Nehari manifold $\mathcal{N} = \{u \in W_* \backslash \{0\} \,,\ I(u) = 0\}$, and the depth of well $d = \inf_{u \in \mathcal{N}} J(u)$. According to $I(u)$, we define the potential well sets

$$W = \{u \in W_*,\ I(u) > 0\} \cup \{0\} \,,$$

$$V = \{u \in W_*,\ I(u) < 0\} \,.$$

Next, we expand the aforementioned single potential well to the family of potential wells. For $\forall \delta > 0$, we define the modified functional and the corresponding sets respectively as

$$I_\delta(u) = - \int_\Omega |u|^r \ln |u| dx + \delta \|\Delta u\|_2^2 \,, \tag{2.4}$$

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u), \tag{2.5}$$

where $\mathcal{N}_\delta = \{u \in W_* \backslash \{0\} \,,\ I_\delta(u) = 0\}$, and

$$W_\delta = \{u \in W_* : I_\delta(u) > 0\} \cup \{0\} \,,$$

$$V_\delta = \{u \in W_* : I_\delta(u) < 0\}.$$

Before giving some important lemmas of this paper, we will show that the term $\int_\Omega |x|^{-s}dx$ is meaningful.

Remark 2.1. *Letting $R = \sup_{x\in\Omega} |x|$, $N > 2$, then we can get*

$$\begin{aligned} 0 < \int_\Omega |x|^{-s}dx &\le \int_{B(0,R)} |x|^{-s}dx \\ &= \int_0^R \left[\int_{\partial B(0,r)} |x|^{-s}dS\,(x)\right]dr \\ &= \omega_N \int_0^R r^{-s}r^{N-1}dr \\ &= \frac{\omega_N}{N-s}R^{N-s} < \infty, \end{aligned}$$

where $\omega_N = \frac{N\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$, which shows that $\int_\Omega |x|^{-s}dx$ is meaningful.

Now we introduce some important basic inequalities.

Lemma 2.1. *[27] Suppose that μ is a positive number. Then, the following inequalities hold:*

$$x^r \ln x \le (e\mu)^{-1}x^{r+\mu}, \quad x \ge 1,$$

and

$$|x^r \ln s| \le (er)^{-1}, \quad 0 < x < 1.$$

Lemma 2.2. *[27] For any $u \in H_0^2(\Omega)$, we have*

$$\|u\|_{r+\mu}^{r+\mu} \le C_G \|u\|_2^{(1-\theta)(r+\mu)} \|\Delta u\|_2^{(r+\mu)\theta},$$

where $C_G > 0$ depends on Ω, N, and p, $\theta = \frac{N(r+\mu-2)}{4(r+\mu)} \in (0,1)$, $0 < \mu < \frac{8}{N} + 2 - r$.

Lemma 2.3. *[28] Let $\mathbb{R}^N = \mathbb{R}^l \times \mathbb{R}^{N-l}$, $2 \le l \le N$, and $x = (x', y) \in \mathbb{R}^l \times \mathbb{R}^{N-l}$. If $1 < r < N$, $0 \le s \le r$, and $s < l$, $z\,(s, N, r) = \frac{r(N-s)}{N-r}$, then we can find a constant $H > 0$ related to s, N, r, and l that satisfies*

$$\int_{R^N} |u|^m |x'|^{-s}dx \le H\left(\int_{R^N} |\nabla u|^r dx\right)^{\frac{N-s}{N-r}}, \quad \forall u \in W_0^{1,r}(\Omega).$$

Remark 2.2. (i) *When $z = r = s$, this inequality is the classical Hardy inequality.*

(ii) *Setting $z = 2$ in Lemma 2.3, we have $r = \frac{2N}{N-s+2}$, and $H_0^1(\Omega) \hookrightarrow W_0^{1,\frac{2N}{N-s+2}}(\Omega)$, and then there exists constants $C_H > 0$ and $\tilde{C} > 0$ satisfying $\|\nabla u\|_2^2 \le \tilde{C}\|\Delta u\|_2^2$ such that Lemma 2.3 becomes*

$$\int_\Omega |u\,(x)|^2 |x|^{-s}dx \le H\|\nabla u\|^2_{\frac{2N}{N-s+2}} \le C_H \|\nabla u\|_2^2 \le C_H\tilde{C}\|\Delta u\|_2^2.$$

Remark 2.3. *The domain Ω being bounded with $\Omega \subset \mathbb{R}^N$, leads to*

$$\min\{L^{-s}, n\} \|u\|_2^2 \leq \int_\Omega \rho_n |u|^2 dx \leq C_H \tilde{C} \|\Delta u\|_2^2, \quad \forall n \in N^+,$$

where L is a normal number and large enough to satisfy $|x| \leq L$.

Lemma 2.4. *Letting $u \in H_0^2(\Omega) \setminus \{0\}$ and $\lambda > 0$, we have*

(i) $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$, $\lim_{\lambda \to 0^+} J(\lambda u) = 0$.

(ii) *$J(\lambda u)$ is increasing on $(0, \lambda^*)$, decreasing on $(\lambda^*, +\infty)$.*

(iii) *$I(\lambda u) > 0$ for $\lambda \in (0, \lambda^*)$, $I(\lambda u) < 0$ for $\lambda \in (\lambda^*, +\infty)$, and $I(\lambda^* u) = 0$.*

Proof. By (2.1), we can get

$$J(\lambda u) = \frac{\lambda^2}{2} \|\Delta u\|_2^2 + \frac{\lambda^r}{r^2} \|u\|_r^r - \frac{\lambda^r}{r} \ln \lambda \|u\|_r^r - \frac{\lambda^r}{r} \int_\Omega |u|^r \ln |u| \, dx,$$

and then clearly (i) holds. For the derivation of the above formula, we can obtain

$$\frac{d}{d\lambda} J(\lambda u) = \lambda \left(\|\Delta u\|_2^2 - \lambda^{r-2} \ln \lambda \|u\|_r^r - \lambda^{r-2} \int_\Omega |u|^r \ln |u| \, dx \right).$$

Letting $f(\lambda u) = \lambda^{-1} \frac{d}{d\lambda} J(\lambda u)$, we have

$$\frac{d}{d\lambda} f(\lambda u) = -\lambda^{r-3} \left[(r-2) \int_\Omega |u|^r \ln |u| \, dx + (r-2) \ln \lambda \|u\|_r^r + \|u\|_r^r \right].$$

Hence, by taking

$$\lambda_1 = \exp \left[\frac{(2-r) \int_\Omega |u|^r \ln |u| \, dx - \|u\|_r^r}{(r-2) \|u\|_r^r} \right] > 0,$$

such that $\frac{d}{d\lambda} f(\lambda u) > 0$ on $(0, \lambda_1)$, and $\frac{d}{d\lambda} f(\lambda u) < 0$ on $(\lambda_1, +\infty)$. As $\lim_{\lambda \to 0^+} f(\lambda u) \geq 0$ and $\lim_{\lambda \to +\infty} f(\lambda u) = -\infty$, we can find a unique $\lambda^* > 0$ that satisfies $f(\lambda^* u) = 0$, $f(\lambda u) < 0$ on $(\lambda^*, +\infty)$, and $f(\lambda u) > 0$ on $(0, \lambda^*)$. Thus, $\frac{d}{d\lambda} J(\lambda u)$ is negative on $(\lambda^*, +\infty)$ and $\frac{d}{d\lambda} J(\lambda u)$ is positive on $(0, \lambda^*)$. Hence, (ii) holds. From $I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u)$, we can then obtain the conclusions of (iii). □

Lemma 2.5. *Let $u \in H_0^2(\Omega)$ and r satisfy (1.2). Then, for any α with $0 < \alpha < \frac{8}{N} + 2 - r$, the following statements hold:*

(i) *$I_\delta(u) > 0$ when $0 < \|\Delta u\|_2 < \phi_\alpha(\delta)$;*

(ii) *$\|\Delta u\|_2 > \phi_\alpha(\delta)$ when $I_\delta(u) \leq 0$, where $\phi_\alpha(\delta) = \left(\frac{\delta \alpha}{B_\alpha^{\alpha+r}} \right)^{\frac{1}{r+\alpha-2}}$, and B_α is the optimal embedding constant of $H_0^2(\Omega) \hookrightarrow L^{r+\alpha}(\Omega)$.*

Proof. By the definition of $I_\delta(u)$, the Sobolev inequality, and

$$\ln |u(x)| < \frac{|u(x)|^\alpha}{\alpha}, \ \forall \alpha > 0,$$

we can get

$$
\begin{aligned}
I_\delta(u) &= -\int_\Omega |u|^r \ln|u|\, dx + \delta \|\Delta u\|_2^2 \\
&> -\frac{1}{\alpha}\int_\Omega |u|^{r+\alpha} dx + \delta \|\Delta u\|_2^2 \\
&> \left(\delta - \frac{B_\alpha^{\alpha+r}}{\alpha} \|\Delta u\|_2^{r+\alpha-2}\right)\|\Delta u\|_2^2 .
\end{aligned}
$$

We can derive (i) and (ii) by a direct calculation. □

Lemma 2.6. *Assume that $u \in H_0^2(\Omega)$, and r satisfies* (1.2). *Let*

$$
g(\delta) = \sup_{\alpha\in\left(0,\frac{8}{N}+2-r\right]} \phi_\alpha(\delta) \ \text{ and } h(\delta) = \sup_{\alpha\in\left(0,\frac{8}{N}+2-r\right]} \psi_\alpha(\delta),
$$

where $\psi_\alpha(\delta) = \left(\frac{\delta\alpha}{B_r^{\alpha+r}}\right)^{\frac{1}{r+\alpha-2}} |\Omega|^{\frac{\alpha}{r(r+\alpha-2)}}$, and B_r is the optimal embedding constant of $H_0^2(\Omega) \hookrightarrow L^r(\Omega)$. Then $g(\delta)$ exists and satisfies $0 < g(\delta) \le h(\delta) < +\infty$.

Proof. From Lemma 2.5 and $g(\delta)$, we can deduce that $g(\delta) > 0$ if $g(\delta)$ exists. By Hölder's inequality, we have

$$
\int_\Omega |u|^r dx \le |\Omega|^{\frac{\alpha}{r+\alpha}} \left(\int_\Omega |u|^{r+\alpha} dx\right)^{\frac{r}{r+\alpha}} .
$$

Combining the embeddings $H_0^2(\Omega) \hookrightarrow L^{r+\alpha}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^r(\Omega)$, we obtain

$$
\frac{1}{B_\alpha} = \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_2}{\|u\|_{r+\alpha}} \le |\Omega|^{\frac{\alpha}{r(r+\alpha)}} \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_2}{\|u\|_r} = \frac{1}{B_r}|\Omega|^{\frac{\alpha}{r(r+\alpha)}} .
$$

Hence,

$$
\phi_\alpha(\delta) = \left(\frac{\delta\alpha}{B_\alpha^{\alpha+r}}\right)^{\frac{1}{r+\alpha-2}} \le \psi_\alpha(\delta) .
$$

So, we have $g(\delta) \le h(\delta)$. Further, due to the continuity of $\psi_\alpha(\delta)$ on $\left[0, \frac{8}{N}+2-r\right]$, we can get $g(\delta)$ is meaningful and

$$
g(\delta) = \sup_{\alpha\in\left(0,\frac{8}{N}+2-r\right]} \psi_\alpha(\delta) \le \max_{\alpha\in\left[0,\frac{8}{N}+2-r\right]} \psi_\alpha(\delta) < +\infty.
$$

After the above discussion, we have $0 < g(\delta) \le h(\delta) < +\infty$. The proof is completed. □

Corollary 2.1. *Let $u \in H_0^2(\Omega)$ and r satisfy* (1.2). *Then, we have $I_\delta(u) > 0$ when $0 < \|\Delta u\|_2 < g(\delta)$ and $\|\Delta u\|_2 \ge g(\delta)$ when $I_\delta(u) \le 0$.*

Lemma 2.7. *Assume that $u \in \mathcal{N}_\delta$. Then:*
(i) $d(\delta) \ge \frac{r-2\delta}{2r} g^2(\delta)$, $0 < \delta < \frac{r}{2}$;
(ii) $\lim_{\delta\to+\infty} d(\delta) = -\infty$, $\lim_{\delta\to 0^+} d(\delta) > 0$;
(iii) $d(\delta)$ is monotonically decreasing on $1 < \delta < \frac{p}{2}$, and monotonically increasing on $0 < \delta \le 1$.

Proof. (*i*) With regard to $\forall u \in \mathcal{N}_\delta$, we get $\|\Delta u\|_2 \geq g(\delta)$ by Corollary 2.1. Therefore, from (2.5) and

$$J(u) = \left(\frac{1}{2} - \frac{\delta}{r}\right)\|\Delta u\|_2^2 + \frac{1}{r^2}\|u\|_r^r, \tag{2.6}$$

we have $d(\delta) \geq \frac{r-2\delta}{2r} g^2(\delta)$ directly.

(*ii*) It follows from (2.6) that $\lim\limits_{\delta\to+\infty} J(u) = -\infty$. Then, we can infer that $\lim\limits_{\delta\to+\infty} d(\delta) = -\infty$ by (2.5). Furthermore, from the conclusion of (*i*), we can get $\lim\limits_{\delta\to 0^+} d(\delta) > 0$ directly.

(*iii*) If we shall prove the monotonicity of $d(\delta)$, we just need to prove the following conclusions: For arbitrary $0 < \delta' < \delta'' < 1$ or $1 < \delta'' < \delta' < \frac{r}{2}$, as well as $u \in \mathcal{N}_{\delta''}$, we can find $v \in \mathcal{N}_{\delta'}$ and $\varepsilon(\delta', \delta'') > 0$ which satisfy $J(u) - J(v) > \varepsilon(\delta', \delta'')$. Next, we define $\lambda(\delta) > 0$ satisfying

$$\delta\|\Delta u\|_2^2 = \lambda^{r-2}\left(\|u\|_r^r \ln\lambda + \int_\Omega |u|^r \ln|u|\,dx\right),$$

such that $I_\delta(\lambda(\delta)u) = 0$. Specifically, it follows from $\lambda(\delta'') = 1$ that $I''_\delta(\lambda(\delta)u) = 0$ for any $u \in \mathcal{N}_{\delta''}$. In addition, we take $\phi(\lambda) = J(\lambda u)$. Then, combining with (2.4) we have

$$\frac{d}{d\lambda}\phi(\lambda) = \lambda(1-\delta)\|\Delta u\|_2^2.$$

Let $v = \lambda(\delta')u$. Then $v \in \mathcal{N}_{\delta'}$. If $\forall 0 < \delta' < \delta'' < 1$, from Corollary 2.1 we have

$$\begin{aligned} J(u) - J(v) &= \phi(1) - \phi(\lambda(\delta')) \\ &= \int_{\lambda(\delta')}^1 \lambda(1-\delta)\|\Delta u\|_2^2\,d\lambda \\ &> \lambda(\delta')(1-\lambda(\delta'))(1-\delta'')g^2(\delta') \\ &= \varepsilon(\delta', \delta'') > 0. \end{aligned}$$

Similarly, if $\forall 1 < \delta'' < \delta' < \frac{p}{2}$, the above results can also be obtained. Therefore, we have (*iii*). □

Lemma 2.8. *Let δ_1, δ_2 be two solutions of the equation $d(\delta) = J(u)$, where $0 < J(u) < d$. So, the sign of $I_\delta(u)$ remains unchanged for $\delta_1 < \delta < \delta_2$.*

Proof. $J(u) > 0$ means that $\|\Delta u\|_2 \neq 0$. On the contrary, if the sign of $I_\delta(u)$ is changed for $\delta_1 < \delta < \delta_2$, we can find a $\bar{\delta} \in (\delta_1, \delta_2)$ which satisfies $I_{\bar{\delta}}(u) = 0$. Therefore, we can get $J(u) \geq d(\bar{\delta})$ by (2.5). From Lemma 2.7 (*iii*), we know that $J(u) = d(\delta_1) = d(\delta_2) < d(\bar{\delta})$, which contradicts the fact $J(u) \geq d(\bar{\delta})$. □

Lemma 2.9. *Let $u \in H_0^2(\Omega)$ and r satisfy (1.2). If $I(u) < 0$, we can find a $\lambda^* \in (0,1)$ that makes $I(\lambda^* u) = 0$.*

Proof. Let

$$\varphi(\lambda) = \lambda^{r-2}\left(-\|u\|_r^r \ln\lambda + \int_\Omega |u|^r \ln|u|\,dx\right),$$

so we can know that

$$I(\lambda u) = \lambda^2\left[\|\Delta u\|_2^2 - \varphi(\lambda)\right].$$

Combining $I(u) < 0$, Corollary 2.1, and the fact of $r > 2$, we can conclude that

$$\begin{cases}\varphi(\lambda) - \|\Delta u\|_2^2 < 0, \, as \ \lambda \to 0^+, \\ \varphi(\lambda) - \|\Delta u\|_2^2 > 0, \, as \ \lambda = 1.\end{cases}$$

So, there exists $\lambda^* \in (0,1)$ that satisfies $\varphi(\lambda_*) = \|\Delta u\|_2^2$, namely $I(\lambda^* u) = 0$. □

Lemma 2.10. *[29] Suppose $\Xi(t)$ is a second-order differentiable normal function satisfying*

$$\Xi''(t)\Xi(t) - (1+\alpha)(\Xi'(t))^2 \geq 0,$$

where $\alpha > 0$. If $\Xi(0) > 0$ and $\Xi'(0) > 0$, then when

$$t \to t_* \leq t^* = \frac{\Xi(0)}{\alpha \Xi'(0)},$$

we have $\Xi(t) \to \infty$.

Definition 2.1. *$u(x,t)$ is called a weak solution of problem* (1.1) *on $\Omega \times [0,T)$, if the initial data $u(x,0) = u_0(x) \in W_*$, $u \in L^\infty(0,T;W_*)$, $u_t \in L^2\left(0,T;H_0^1(\Omega)\right)$ with $\int_0^t \int_\Omega \frac{u_\tau^2}{|x|^s} dx d\tau < \infty$, and $u(x,t)$ satisfies*

$$\left(\frac{u_t}{|x|^s}, \omega\right) + (\Delta u, \Delta \omega) + (\nabla u_t, \nabla \omega) = \left\langle |u|^{r-2} u \ln |u|, \omega \right\rangle - \left\langle \frac{|x|^{-s}}{\int_\Omega |x|^{-s} dx} \int_\Omega |u|^{r-2} u \ln |u| \, dx, \omega \right\rangle,$$

for any $t \in [0,T)$ and $\omega \in H_0^2(\Omega)$, where $(\cdot,\cdot)$ represents the inner product in $L^2(\Omega)$ and $\langle \cdot,\cdot \rangle$ stands for the dual product between $H^{-2}(\Omega)$ and $H^2(\Omega)$.

Definition 2.2. *Assume that $u(x,t)$ is a weak solution of* problem (1.1)*, and suppose the maximal existence time T_{max} is finite, satisfying*

$$\lim_{t \to T_{\max}^-} \int_0^t \left(\|\nabla u(\tau)\|_2^2 + \left\| |x|^{-\frac{s}{2}} u(\tau) \right\|_2^2 \right) d\tau = +\infty.$$

Then $u(x,t)$ blows up in finite time.

3. Local existence

Theorem 3.1. *Let $u_0 \in W_*$ and r satisfy* (1.2)*. We can find a $T > 0$ such that* problem (1.1) *possesses a unique weak solution $u(x,t) \in L^\infty(0,T;W_*)$, $u_t \in L^2\left(0,T;H_0^1(\Omega)\right)$ with $\int_0^t \int_\Omega \frac{u_\tau^2}{|x|^s} dx d\tau < \infty$ and satisfies $u(x,0) = u_0(x)$. Moreover, for $0 \leq t \leq T$, the following energy equality holds:*

$$J(u(t)) + \int_0^t \left(\|\nabla + u_\tau(\tau)\|_2^2 + \left\| |x|^{-s} u_\tau(\tau) \right\|_2^2 \right) d\tau = J(u_0). \tag{3.1}$$

Proof. We provide the following cut-off function to solve the singularity:

$$\rho_n(x) = \min\left\{ |x|^{-s}, n \right\}, \ n \in N^+.$$

Step 1. Local existence

For $n \in N^+$, we denote the solutions relevant to ρ_n of problem (1.1) as u_n, and let

$$0 \neq u_{n0}(x) \in \tilde{W}_* = \left\{ u_n \in H_0^2(\Omega) : \int_\Omega \rho_n(x) u_n(x,t)\, dx = 0 \right\}.$$

Let $W_m = Span\{e_1, \cdots, e_m\}$, where $\left\{e_j\right\}_{j=1}^{\infty}$ is a system of basis of $H_0^2(\Omega)$ and normalized orthogonal in $L^2(\Omega)$. On the basis of their multiplicity of $-\Delta e_j = \lambda_j e_j$, we define the related eigenvalues repeated by λ_j. We will establish the approximate solutions

$$u_n^m(x,t) = \sum_{j=1}^{m} h_{nj}^m(t)\, e_j(x),$$

satisfying the problem

$$\begin{aligned}\left(\rho_n(x) u_{nt}^m, e_j\right) + \left(\Delta u_n^m, \Delta e_j\right) + \left(\nabla u_{nt}^m, \nabla e_j\right) =& \left\langle \left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right|, e_j \right\rangle \\ &- \left\langle \frac{\rho_n(x)}{\int_\Omega \rho_n(x) dx} \int_\Omega \left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right| dx, e_j \right\rangle,\end{aligned} \tag{3.2}$$

and

$$u_n^m(x,0) = \sum_{j=1}^{k} h_{nj}^m e_j(x) = u_{n0}^m \to u_0(x) \;\; in\ W_* \tag{3.3}$$

as $m \to \infty,\ n \to \infty$. Let

$$\begin{aligned}F_j =& -\lambda_j{}^2 h_{nj}^m(t) + \int_\Omega \left|\sum_{j=1}^m h_{nj}^m(t) e_j(x)\right|^{r-2} \sum_{j=1}^m h_{nj}^m(t) e_j(x) \ln\left|\sum_{j=1}^m h_{nj}^m(t) e_j(x)\right| e_j dx \\ &- \frac{\int_\Omega \rho_n(x) e_j dx}{\int_\Omega \rho_n(x) dx} \int_\Omega \left|\sum_{j=1}^m h_{nj}^m(t) e_j(x)\right|^{r-2} \sum_{j=1}^m h_{nj}^m(t) e_j(x) \ln\left|\sum_{j=1}^m h_{nj}^m(t) e_j(x)\right| dx.\end{aligned}$$

Hence, $\left\{h_{nj}^m\right\}_{j=1}^m$ satisfies the following Cauchy problem

$$\begin{cases} \sum\limits_{j=1}^m \left[\int_\Omega \rho_n(x) e_j(x) e_j(x) dx\right] \dot{h}_{nj}^m(t) - \lambda_j \dot{h}_{nj}^m(t) = F_j\left(t, h_{n1}^m(t), h_{n2}^m(t), \cdots, h_{nm}^m(t)\right), \\ h_{nj}^m(0) = \int_\Omega u_{n0} w_j dx, \end{cases}$$

for $j = 1, \cdots, m$; this is an ordinary differential equation with h_{nj}^m. According to Peano's Theorem, the above problem has a local solution $h_{nj}^m \in C^1[0, T_m]$.

Multiplying both sides of (3.2) by $h_{nj}^m(t)$, and summing over j from 1 to m, we have

$$(\rho_n(x) u_{nt}^m, u_n^m) + (\Delta u_n^m, \Delta u_n^m) + (\nabla u_{nt}^m, \nabla u_n^m) = \left\langle \left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right|, u_n^m \right\rangle.$$

Integrating the above equation from $(0, t)$, we have

$$\frac{1}{2}\left\||\rho_n(x)|^{\frac{1}{2}}u_n^m(t)\right\|_2^2+\int_0^t\left\|\Delta u_n^m(\tau)\right\|_2^2d\tau+\frac{1}{2}\left\|\nabla u_n^m(t)\right\|_2^2$$
$$=\frac{1}{2}\left\||\rho_n(x)|^{\frac{1}{2}}u_n^m(0)\right\|_2^2+\frac{1}{2}\left\|\nabla u_n^m(0)\right\|_2^2+\int_0^t\int_\Omega\left|u_n^m(\tau)\right|^r\ln\left|u_n^m(\tau)\right|dxd\tau.$$

So, we have

$$S_n^m(t)=S_n^m(0)+\int_0^t\int_\Omega\left|u_n^m\right|^r\ln\left|u_n^m\right|dxd\tau,\quad 0\le t\le T,\tag{3.4}$$

where

$$S_n^m(t)=\frac{1}{2}\left\||\rho_n(x)|^{\frac{1}{2}}u_n^m(t)\right\|_2^2+\frac{1}{2}\left\|\nabla u_n^m(t)\right\|_2^2+\int_0^t\left\|\Delta u_n^m(\tau)\right\|_2^2d\tau.\tag{3.5}$$

For the term $\int_0^t\int_\Omega\left|u_n^m\right|^r\ln\left|u_n^m\right|dxd\tau$ in (3.4), let $\Omega_1=\{x\in\Omega|\,|u_n(x)|<1\}$, $\Omega_2=\{x\in\Omega|\,|u_n(x)|\ge 1\}$. By virtue of Lemma 2.1, we get

$$\begin{aligned}\int_\Omega\left|u_n^m\right|^r\ln\left|u_n^m\right|dx&=\int_{\Omega_1}\left|u_n^m\right|^r\ln\left|u_n^m\right|dx+\int_{\Omega_2}\left|u_n^m\right|^r\ln\left|u_n^m\right|dx\\&\le(e\mu)^{-1}\int_{\Omega_2}\left|u_n^m\right|^{r+\mu}dx\\&\le(e\mu)^{-1}\left\|u_n^m\right\|_{r+\mu}^{r+\mu}.\end{aligned}\tag{3.6}$$

We choose $0<\mu<\frac{8}{N}+2-r$, and then combining (3.6) and Lemma 2.2, we apply Young's inequality with ε and the embedding theorem, giving

$$\begin{aligned}\int_\Omega\left|u_n^m\right|^r\ln\left|u_n^m\right|dx&\le(e\mu)^{-1}\left\|u_n^m\right\|_{r+\mu}^{r+\mu}\\&\le(e\mu)^{-1}C_G\left\|\Delta u_n^m\right\|_2^{\theta(r+\mu)}\left\|u_n^m\right\|_2^{(1-\theta)(r+\mu)}\\&\le(e\mu)^{-1}C_G\varepsilon\left\|\Delta u_n^m\right\|_2^2+(e\mu)^{-1}C_GC(\varepsilon)B_1\left\|\nabla u_n^m\right\|_2^{\frac{2(1-\theta)(r+\mu)}{2-\theta(r+\mu)}},\end{aligned}\tag{3.7}$$

where $\varepsilon\in(0,1)$. Substituting (3.7) into (3.4), we get

$$S_n^m(t)\le C_1+C_2\int_0^t\left[S_n^m(\tau)\right]^a d\tau,\tag{3.8}$$

where $a=\frac{4r+4\mu-Nr-N\mu+2N}{8-N(r+\mu-2)}>1$, and the positive constants $C_1=\frac{S_n^m(0)}{1-(e\mu)^{-1}C_G\varepsilon}$ and $C_2=\frac{(e\mu)^{-1}C_GC(\varepsilon)2^a}{1-(e\mu)^{-1}C_G\varepsilon}$, which are independent of n and m. Then, the following inequality can be obtained by direct calculation:

$$S_n^m(t)\le C_3,\tag{3.9}$$

where C_3 is a normal constant and only depends on T.

Next, multiplying both sides of Eq (3.2) by $\dot{h}_{nj}^m(t)$, summing over j from 1 to m, and then integrating on $(0, t)$, due to the continuity of $J(u)$ in $H_0^2(\Omega)$ as well as (3.3), we obtain

$$\int_0^t\left(\left\||\rho_n(x)|^{\frac{1}{2}}u_{n\tau}^m\right\|_2^2+\|\nabla u_{n\tau}\|_2^2\right)d\tau+J\left(u_n^m(t)\right)=J\left(u_{n0}^m\right)\le C,\tag{3.10}$$

where C is a normal constant that depends of n and m.

Combining (3.5), (3.7), (3.9), (3.10), and Remark 2.3, we obtain

$$
\begin{aligned}
&\int_0^t \left(\left\||\rho_n(x)|^{\frac{1}{2}} u_{n\tau}^m\right\|_2^2 + \|\nabla u_{n\tau}\|_2^2\right) d\tau + \frac{1}{2}\left\|\Delta u_n^m\right\|_2^2 + \frac{1}{r^2}\left\|u_n^m\right\|_r^r \\
&= J\left(u_{n0}^m\right) + \frac{1}{r}\int_\Omega \left|u_n^m\right|^r \ln\left|u_n^m\right| dx \\
&\leq C + (re\mu)^{-1} C_G \varepsilon \left\|\Delta u_n^m\right\|_2^2 + (re\mu)^{-1} C_G C(\varepsilon) B_1 \left\|\nabla u_n^m\right\|_2^{2a} \\
&\leq C + (re\mu)^{-1} C_G \varepsilon \left\|\Delta u_n^m\right\|_2^2 + (re\mu)^{-1} C_G C(\varepsilon) 2^\alpha B_1 (C_3)^a,
\end{aligned}
$$

meaning that

$$
\begin{aligned}
&\int_0^t \left(\left\||\rho_n(x)|^{\frac{1}{2}} u_{n\tau}^m\right\|_2^2 + \left\|\nabla u_{n\tau}^m\right\|_2^2\right) d\tau + \left(\frac{1}{2} - \frac{C_G \varepsilon}{re\mu}\right)\left\|\Delta u_n^m\right\|_2^2 + \frac{1}{r^2}\left\|u_n^m\right\|_r^r \\
&\leq C + (re\mu)^{-1} C_G C(\varepsilon) 2^a B_1 (C_3)^a.
\end{aligned} \tag{3.11}
$$

From (3.11), it can be inferred that

$$\left\|u_n^m(t)\right\|_{L^\infty\left(0,T;H_0^2(\Omega)\right)} \leq C_T, \tag{3.12}$$

$$\left\|u_n^m(t)\right\|_{L^\infty(0,T;L^p(\Omega))} \leq C_T, \tag{3.13}$$

$$\left\|u_{nt}^m(t)\right\|_{L^2\left(0,T;H_0^1(\Omega)\right)} \leq C_T, \tag{3.14}$$

$$\left\||\rho_n(x)|^{\frac{1}{2}} u_{nt}^m\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \leq C_T, \tag{3.15}$$

where $C_T > 0$ only depends on T. By (3.12), (3.14), and the Aubin-Lions-Simon lemma [30], we have

$$u_n^m \to u \ \ in \ \ C(0,T;L^2(\Omega)), \tag{3.16}$$

as m *and* $n \to +\infty$. Thus, we have $u_n^m(x,0) \to u(x,0)$ in $L^2(\Omega)$. From (3.16), we can get $u_n^m \to u$, and $\left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right| \to |u|^{r-2} u \ln|u|$ a.e. in $\Omega \times (0,T)$.

Next, we can choose $\mu = \frac{2N-Nr+4r}{2N} > 0$ in Lemma 2.3. Then, from Lemma 2.3 and (3.11), we obtain

$$
\begin{aligned}
\int_\Omega \left|\left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right|\right|^{\frac{2N}{N+4}} dx &= \int_{\Omega_1} \left|\left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right|\right|^{\frac{2N}{N+4}} dx + \int_{\Omega_2} \left|\left|u_n^m\right|^{r-2} u_n^m \ln\left|u_n^m\right|\right|^{\frac{2N}{N+4}} dx \\
&\leq (e\mu)^{-\frac{2N}{N+4}} \int_{\Omega_1} \left|u_n^m\right|^{\frac{2N(r-1+\mu)}{N+4}} dx + [e(r-1)]^{-\frac{2N}{N+4}} |\Omega| \\
&\leq (e\mu)^{-\frac{2N}{N+4}} \left\|u_n^m\right\|_r^r + [e(r-1)]^{-\frac{2N}{N+4}} |\Omega| < C_T.
\end{aligned}
$$

Combining Hölder's inequality and the above inequality, we have

$$\begin{aligned}\left\|\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\right\|_{H^{-2}(\Omega)} &= \sup_{\varphi\in H_0^2(\Omega)}\frac{\int_\Omega \left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\varphi dx}{\|\varphi\|_{H_0^2(\Omega)}}\\ &\le \frac{\left(\int_\Omega \left|\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\right|^{\frac{2N}{N+4}}dx\right)^{\frac{N+4}{2N}}\left(\int_\Omega|\varphi|^{\frac{2N}{N-4}}dx\right)^{\frac{N-4}{2N}}}{\|\varphi\|_{H_0^2(\Omega)}}\\ &\le B_2\left(\int_\Omega \left|\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\right|^{\frac{2N}{N+4}}dx\right)^{\frac{N+4}{2N}} < C_T,\end{aligned}$$

where $B_2 > 0$ satisfies the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{\frac{2N}{N-4}}(\Omega)$. Hence, it follows that

$$\left\|\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\right\|_{L^\infty(0,T;H^{-2}(\Omega))} \le C_T. \tag{3.17}$$

So, we can know that

$$\begin{aligned}&\int_\Omega\left(\frac{\rho_n(x)}{\int_\Omega\rho_n(x)\,dx}\int_\Omega\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|dx\right)^{\frac{2N}{N+4}}dx\\ \le\ &\frac{\left(\min\{L^{-s},n\}\right)^{\frac{2N}{N+4}}|\Omega|}{\left(\min\{L^{-s},n\}|\Omega|\right)^{\frac{2N}{N+4}}}|\Omega|^{\frac{N-4}{2N}}\left\|\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\right\|_{\frac{2N}{N+4}}\\ =\ &|\Omega|^{-\frac{(N-4)^2}{2N(N+4)}}\left\|\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right|\right\|_{\frac{2N}{N+4}} < C_T.\end{aligned}$$

The above estimations allow us to get a subsequence of $\{u_n^m\}_{m,n=1}^\infty$ satisfying

$$u_n^m \to u \ \ in\ \ L^\infty(0,T;H_0^2(\Omega))\ weakly\ \ star, \tag{3.18}$$

$$u_{nt}^m \to u_t \ \ in\ \ L^2(0,T;H_0^1(\Omega))\ \ weakly, \tag{3.19}$$

$$|\rho_n(x)|^{\frac{1}{2}}u_{nt}^m \to |x|^{-\frac{s}{2}}u_t \ \ in\ \ L^2(0,T;L^2(\Omega))\ \ weakly, \tag{3.20}$$

$$\left|u_n^m\right|^{r-2}u_n^m\ln\left|u_n^m\right| \to |u|^{r-2}u\ln|u| \ \ in\ \ L^\infty(0,T;H^{-2}(\Omega))\ \ weakly\ \ star. \tag{3.21}$$

Now, from (3.18)–(3.21), taking the limit in (3.2), as m *and* $n \to +\infty$, we have

$$\left(\frac{u_t}{|x|^s},\omega\right)+(\Delta u,\Delta\omega)+(\nabla u_t,\nabla\omega)=\left\langle|u|^{r-2}u\ln|u|,\omega\right\rangle-\left\langle\frac{|x|^{-s}}{\int_\Omega|x|^{-s}dx}\int_\Omega|u|^{r-2}u\ln|u|\,dx,\omega\right\rangle,$$

for $t \in [0,T]$, $\omega \in H_0^2(\Omega)$, and initial data satisfying $u(0) = u_0$. Moreover, problem (1.1) is multiplied by u_t and integrated on $\Omega \times (0,t)$ to get the following energy equality:

$$J(u(t))+\int_0^t\left(\|\nabla u_\tau\|_2^2+\left\||x|^{-\frac{s}{2}}u_\tau(\tau)\right\|_2^2\right)d\tau = J(u_0).$$

Step 2. Uniqueness

Assume that u_1 and u_2 are the two solutions to problem (1.1) satisfying $u_1(x,0) = u_2(x,0) = u_0(x) \in W_*$. Thus, we have the following two equations:

$$\left(\frac{u_{1t}}{|x|^s}, w\right) + (\Delta u_1, \Delta w) + (\nabla u_1, \nabla w) = \left\langle |u_1|^{r-2} u_1 \ln |u_1|, w\right\rangle - \left\langle \frac{|x|^{-s}}{\int_\Omega |x|^{-s} dx} \int_\Omega |u_1|^{r-2} u_1 \ln |u_1| \, dx, w\right\rangle,$$

and

$$\left(\frac{u_{2t}}{|x|^s}, w\right) + (\Delta u_2, \Delta w) + (\nabla u_2, \nabla w) = \left\langle |u_2|^{r-2} u_2 \ln |u_2|, w\right\rangle - \left\langle \frac{|x|^{-s}}{\int_\Omega |x|^{-s} dx} \int_\Omega |u_2|^{r-2} u_2 \ln |u_2| \, dx, w\right\rangle.$$

Let $v = u_1 - u_2$ and $v(0) = 0$. Then, by subtracting the above two equations, we can derive

$$\int_\Omega |x|^{-s} v_t w dx + \int_\Omega \Delta v \Delta w dx + \int_\Omega \nabla v_t \nabla w dx = \int_\Omega |u_1|^{r-2} u_1 w \ln |u_1| dx - \int_\Omega |u_2|^{r-2} u_2 w \ln |u_2| dx.$$

Let $w = v$ and integrate above equation on $[0, t]$. Then, by Remark 2.3 and a direct calculation, we have

$$\|v\|_2^2 \le 2M \int_0^t \int_\Omega \frac{f(u_1) - f(u_2)}{v} v^2 dx dt,$$

where M is a normal number and $F(x) = |x|^{r-2} x \ln |x|$. Due to $F : \mathbb{R}^+ \to \mathbb{R}^+$ being Lipschitz continuous, we can get

$$\|v\|_2^2 \le C_U \int_0^t \|v\|_2^2 \, dt.$$

It follows from the above inequality and Gronwall's inequality that $\|v\|_2^2 = 0$. Therefore, we can get $v = 0$ a.e. in $\Omega \times (0, T)$. □

4. Subcritical initial energy $J(u_0) < d$

First, we will show that W_δ is invariant.

Lemma 4.1. *Assume that $u_0 \in W_*$, r satisfies* (1.2)*, $0 < e < d$, and (δ_1, δ_2) is the maximum section including $\delta' = 1$ satisfies $d(\delta) > e$ for any $\delta \in (\delta_1, \delta_2)$. We can get all weak solutions of* problem (1.1) *where $J(u_0) = e$ and $I(u_0) > 0$ belong to W_δ, $0 \le t < T_{max}$.*

Proof. For $\delta \in (\delta_1, \delta_2)$, combining $I(u_0) > 0$ and Lemma 2.8, we have $I_\delta(u_0) > 0$, which means that $u_0 \in W_\delta$. Combining (3.1) and $J(u_0) < d(\delta)$, we obtain

$$0 < J(u(t)) + \int_0^t \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\||x|^{-s} u_\tau(\tau)\right\|_2^2\right) d\tau = J(u_0) < d(\delta). \tag{4.1}$$

Then, we will prove that $u(t) \in W_\delta$ for any $\delta \in (\delta_1, \delta_2)$. Otherwise, there exists a minimum time $t_0 \in (0, T_{max})$ that satisfies $u(t_0) \in \partial W_\delta$, i.e. $I_\delta(u(t_0)) = 0$ with $u(t_0) \neq 0$. Thus, by (2.5), we have $J(u(t_0)) \ge \inf_{u \in \mathcal{N}_\delta} J(u) = d(\delta)$. Obviously, this contradicts (4.1). Therefore, the conclusion is proved. □

Remark 4.1. *Providing the condition $J(u_0) = e$ is changed to $0 < J(u_0) \leq e$ in Lemma 4.1, then the conclusion also holds.*

Theorem 4.1. *Let $u_0 \in W_*$, $J(u_0) < d$, $I(u_0) > 0$, and r satisfy* (1.2). *Then* problem (1.1) *allows a global weak solution $u \in L^\infty(0,\infty; W_*)$, $u_t \in L^2(0,\infty; H_0^1(\Omega))$ with $\int_0^t \int_\Omega \frac{u_\tau^2}{|x|^s} dx d\tau < \infty$. Moreover, for any $0 \leq t \leq \infty$, $u(t)$ satisfies*

$$J(u(t)) + \int_0^t \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\| |x|^{-s} u_\tau(\tau) \right\|_2^2 \right) d\tau = J(u_0). \tag{4.2}$$

Proof. Since we know that the case of $I(u_0) > 0$ and $J(u_0) < 0$ is contradictive with (2.3), we just need to consider the case of $0 < J(u_0) < d$ and $I(u_0) > 0$. Then, we can get $u \in W$ by Lemma 4.1 if $\delta = 1$, which means that $I(u(t)) > 0$. Combining $J(u_0) < d$ and (3.1), we get

$$J(u(t)) + \int_0^t \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\| |x|^{-\frac{s}{2}} u_\tau(\tau) \right\|_2^2 \right) d\tau < d, \tag{4.3}$$

where $0 \leq t \leq T_{max}$. Combining (2.3) and (4.3), we can get

$$\left(\frac{1}{2} - \frac{1}{r} \right) \|\Delta u\|_2^2 + \frac{1}{r^2} \|u\|_r^r + \int_0^t \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\| |x|^{-\frac{s}{2}} u_\tau(\tau) \right\|_2^2 \right) d\tau < d. \tag{4.4}$$

This estimate enables us to take $T_{max} = +\infty$. Hence, we have proved the uniqueness, and the energy equation can also be obtained. □

Theorem 4.2. *Let $u_0 \in W_*$, r satisfy* (1.2)*, and $u(t)$ be the weak solution of* problem (1.1). *If $0 < J(u_0) < d$, and $I(u_0) > 0$, we can get the inequality*

$$\|\nabla u(t)\|_2^2 + \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \leq c_1 \exp\{-c_2 t\},$$

where $c_1 = \|\nabla u_0\|_2^2 + \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2$, $c_2 = \frac{2(1-\delta_1)}{C_H \tilde{C} + \tilde{C}}$.

Proof. Multiplying (1.1) by $u(t)$ and integrating at Ω, for any $0 \leq t \leq +\infty$ we have

$$\frac{1}{2}\frac{d}{dt} \|\nabla u(t)\|_2^2 + \frac{1}{2}\frac{d}{dt} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 + \|\Delta u(t)\|_2^2 = \int_\Omega |u(t)|^r \ln |u(t)| \, dx.$$

Combining (2.4), by a direct computation we arrive at

$$\frac{1}{2}\frac{d}{dt} \|\nabla u(t)\|_2^2 + \frac{1}{2}\frac{d}{dt} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 + (1-\delta_1) \|\Delta u(t)\|_2^2 + I_{\delta_1}(u(t)) = 0. \tag{4.5}$$

From Lemma 4.1, we obtain $u(t) \in W_\delta$, which means that $I_\delta(u(t)) > 0$, $\delta \in (\delta_1, \delta_2)$. Through the continuity of I_δ relative to δ, we have $I_{\delta_1}(u(t)) \geq 0$, and (4.5) becomes

$$\frac{1}{2}\frac{d}{dt} \|\nabla u(t)\|_2^2 + \frac{1}{2}\frac{d}{dt} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \leq -(1-\delta_1) \|\Delta u(t)\|_2^2. \tag{4.6}$$

Combining Remark 2.2 and (4.6), we have

$$\frac{1}{2}\frac{d}{dt} \|\nabla u(t)\|_2^2 + \frac{1}{2}\frac{d}{dt} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \leq -\frac{(1-\delta_1)}{C_H \tilde{C} + \tilde{C}} \left(\left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 + \|\nabla u(t)\|_2^2 \right).$$

Integrating the above inequality, we get

$$\|\nabla u(t)\|_2^2 + \left\||x|^{-\frac{s}{2}} u(t)\right\|_2^2 \le c_1 \exp\{-c_2 t\},$$

where $c_1 = \|\nabla u_0\|_2^2 + \left\||x|^{-\frac{s}{2}} u_0\right\|_2^2$, $c_2 = \frac{2(1-\delta_1)}{C_H \tilde{C} + \tilde{C}}$. □

Next, the following lemma claims that the set V_δ is invariant

Lemma 4.2. *Let $u_0 \in W_*$, r satisfy* (1.2)*, $0 < e < d$, and (δ_1, δ_2) be the maximum section including $\delta' = 1$ satisfying $d(\delta) > e$ for any $\delta \in (\delta_1, \delta_2)$. We can get all weak solutions of* problem (1.1) *where $J(u_0) = e$ and $I(u_0) < 0$ belong to V_δ, $0 \le t < T_{max}$.*

Proof. From Lemma 2.8 and $I(u_0) < 0$, we can get $I_\delta(u_0) < 0$. Combining with $0 < J(u_0) < d(\delta)$, it follows from (4.2) that

$$0 < J(u(t)) + \int_0^t \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\||x|^{-s} u_\tau(\tau)\right\|_2^2\right) d\tau = J(u_0) < d(\delta). \tag{4.7}$$

Next, we will prove that $I_\delta(u(t)) < 0$ for all $t \in [0, T_{max})$. Otherwise, due to continuity of $I_\delta(u)$, there exists a $t_1 \in (0, T_{max})$ which satisfies $u(t_1) \in \partial V_\delta$, i.e. $I_\delta(u(t_1)) = 0$ with $u(t_1) \neq 0$. According to (2.5), we have $J(u(t_1)) \ge d(\delta)$, which contradicts with (4.7). Therefore, the conclusion is proved. □

Remark 4.2. *Providing the condition $J(u_0) = e$ is changed to $0 < J(u_0) \le e$ in Lemma 4.2, then the conclusion also holds.*

Theorem 4.3. *($0 < J(u_0) < d$) Let $u_0 \in W_*$, $0 < J(u_0) < d$, $I(u_0) < 0$, and r satisfies* (1.2)*. Then $u(t)$ blows up in finite time and the upper bound for blow-up time T_{max} is*

$$T_{\max} \le \frac{p\sigma^2}{(r-2)\, p\sigma - \left(\|\nabla u_0\|_2^2 + \left\||x|^{-\frac{s}{2}} u_0\right\|_2^2\right)},$$

where p and σ are given in (4.13) *and* (4.14)*, respectively.*

Proof. First, it is clear that $J(u) < d$ from $J(u_0) < d$ and (4.7). Then, we can get $u \in V$ by Lemma 4.2 with $\delta = 1$. From Lemma 2.9, we can find a $\lambda^* < 1$ that satisfies $I(\lambda^* u) = 0$. Hence,

$$\begin{aligned} d \le J(\lambda^* u) &= (\lambda^*)^2 \left(\frac{r-2}{2r}\right) \|\Delta u\|_2^2 + \frac{(\lambda^*)^r}{r^2} \|u\|_r^r + \frac{1}{r} I(\lambda^* u) \\ &< \left(\frac{1}{2} - \frac{1}{r}\right) \|\Delta u\|_2^2 + \frac{1}{r^2} \|u\|_r^r. \end{aligned} \tag{4.8}$$

On the contrary, it is assumed that u is the global weak solution of problem (1.1). Then $T_{max} = +\infty$. For any $T \in [0, T_{max})$, we construct

$$H(t) = \int_0^t R(\tau)\, d\tau + (T - t) R(0) + \frac{p}{2}(t + \sigma)^2, \tag{4.9}$$

and $H(t) > 0$, where $R(t) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \left\||x|^{-\frac{s}{2}}u\right\|_2^2\right)$, $p > 0, \sigma > 0$. We calculate the derivative of $H(t)$ as

$$\begin{aligned} H'(t) &= R(t) - R(0) + p(t+\sigma) = \int_0^t \frac{d}{d\tau} R(\tau)\, d\tau + p(t+\sigma) \\ &= \int_0^t \left(\int_\Omega \nabla u \cdot \nabla u_\tau dx + \int_\Omega |x|^{-s} u \cdot u_\tau dx \right) d\tau + p(t+\sigma), \end{aligned} \tag{4.10}$$

$$H''(t) = R'(t) + p = -I(u) + p = -rJ(u) + \left(\frac{r}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{r}\|u\|_r^r + p. \tag{4.11}$$

So, we can get

$$\begin{aligned} &H(t) H''(t) - (1+\alpha)[H'(t)]^2 = H(t) H''(t) \\ &+ (1+\alpha) \cdot \left\{ \varrho(t) - [2H(t) - 2(T-t)R(0)] \left[\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 \right) d\tau + p \right] \right\}, \end{aligned} \tag{4.12}$$

in which we define

$$\begin{aligned} \varrho(t) = &\left[\int_0^t \left(\|\nabla u\|_2^2 + \left\||x|^{-\frac{s}{2}} u\right\|_2^2 \right) d\tau + p(t+\sigma)^2 \right] \cdot \left[\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 \right) d\tau + p \right] \\ &- \left[\int_0^t \int_\Omega (\nabla u \cdot \nabla u_\tau + |x|^{-s} u \cdot u_\tau dx) dx d\tau + p(t+\sigma) \right]^2. \end{aligned}$$

Applying Holder's inequality and the Cauchy-Schwarz inequality, we can get $\varrho(t) \geq 0$. Next, we choose $\alpha = \frac{r-2}{2} > 0$, and (4.12) becomes

$$\begin{aligned} &H(t) H''(t) - \frac{r}{2}[H'(t)]^2 \\ &\geq H(t) H''(t) - \frac{r}{2}[2H(t) - 2(T-t)R(0)] \left[\int_0^t \left(\left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 + \|\nabla u_\tau\|_2^2 \right) d\tau + p \right] \\ &\geq H(t) \left[H''(t) - r \int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 \right) d\tau - rp \right] \\ &= H(t) \left[-rJ(u) + \frac{r-2}{2}\|\Delta u\|_2^2 + \frac{1}{r}\|u\|_r^r - r \int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 \right) d\tau + (1-r)p \right] \\ &= H(t) M(t), \end{aligned}$$

we denote $\kappa(t)$ as

$$M(t) = -rJ(u) + \left(\frac{r}{2} - 1\right)\|\Delta u\|_2^2 + \frac{1}{r}\|u\|_r^r - r \int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 \right) d\tau + (1-r)p.$$

From (4.2) and (4.8), the conditions under which we can choose p is

$$p \in \left(0, \frac{r(d - J(u_0))}{r-1}\right] \tag{4.13}$$

such that

$$M(t)=\frac{r-2}{2}\|\Delta u\|_2^2+\frac{1}{r}\|u\|_r^r+(1-r)p-rJ(u_0)$$
$$\geq r(d-J(u_0))+(1-r)p\geq 0.$$

From what has been discussed above, we arrive at

$$H(t)H''(t)-(1+\alpha)[H'(t)]^2\geq 0.$$

After direct calculation, we have $H(0)>0$, $H'(0)=p\sigma>0$. Therefore, by Lemma 2.10, we can find a T_* that satisfies $0<T_*<\dfrac{2H(0)}{(r-2)H'(0)}$ where $H(t)\to\infty,\ t\to T_*$, and we can obtain that

$$T_{\max}\leq\frac{p\sigma^2}{(r-2)p\sigma-\left(\|\nabla u_0\|_2^2+\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2\right)},$$

where

$$\beta>\max\left\{0,\frac{\|\nabla u_0\|_2^2+\left\||x|^{-\frac{s}{2}}u_0\right\|_2^2}{(r-2)p}\right\}. \tag{4.14}$$

This obviously contradicts the assumption that $H(t)$ is clearly defined on $[0,T]$ for any $T>0$. Thus, the finite time blow-up result is proved. □

Theorem 4.4. *($J(u_0)<0$) Assume that $u_0\in W_*$, $I(u_0)<0$, $J(u_0)<0$, and r satisfies* (1.2). *Then $u(t)$ blows up in finite time and*

$$T_{max}\leq-\frac{\|\nabla u(0)\|_2^2+\left\||x|^{-\frac{s}{2}}u(0)\right\|_2^2}{(r^2-2r)J(u_0)}.$$

Proof. In view of (2.2), (2.3), and $R(t)$, we can get

$$\begin{aligned}\frac{d}{dt}R(t)&=\int_\Omega\nabla u\cdot\nabla u_t dx+\int_\Omega|x|^{-s}u\cdot u_t dx\\&=\int_\Omega|u|^r\ln|u|\,dx-\|\Delta u\|_2^2=-I(u)\\&=\left(\frac{r}{2}-1\right)\|\Delta u\|_2^2+\frac{1}{r}\|u\|_r^r-rJ(u).\end{aligned} \tag{4.15}$$

According to (4.2) and (4.15), we arrive at

$$\frac{d}{dt}R(t)\geq-rJ(u)\geq-rJ(u_0)>0. \tag{4.16}$$

Then, from problem (1.1), we have

$$\frac{d}{dt}J(u(t))=-\left(\|\nabla u_t\|_2^2+\left\||x|^{-\frac{s}{2}}u_t\right\|_2^2\right)\leq 0. \tag{4.17}$$

Making use of Hölder's inequality and the Cauchy-Schwarz inequality, we get

$$\begin{aligned} -R(t)\frac{d}{dt}J(u) &= \frac{1}{2}\left(\|\nabla u\|_2^2 + \left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2\right)\left(\|\nabla u_t\|_2^2 + \left\||x|^{-\frac{s}{2}}u_t(t)\right\|_2^2\right) \\ &\geq \frac{1}{2}\left(\frac{d}{dt}R(t)\right)^2 \geq -\frac{r}{2}J(u)\frac{d}{dt}R(t). \end{aligned} \tag{4.18}$$

By a simple calculation, we have

$$\frac{d}{dt}\left(J(u)(R(t))^{-\frac{r}{2}}\right) = (R(t))^{-\frac{r}{2}-1}\left(R(t)\frac{d}{dt}J(u) - \frac{r}{2}J(u)\frac{d}{dt}R(t)\right) \leq 0, \tag{4.19}$$

for all $t \in [0, t)$. According to (4.19), $J(u_0) < 0$, and $L(0) > 0$, we obtain

$$J(u)(R(t))^{-\frac{r}{2}} \leq J(u_0)(R(0))^{-\frac{r}{2}} \equiv -b < 0. \tag{4.20}$$

Combining (4.16) and (4.19), we have

$$\begin{aligned} \frac{d}{dt}(R(t))^{\frac{2-r}{2}} &= \frac{2-r}{2}(R(t))^{-\frac{r}{2}}\frac{d}{dt}R(t) \\ &\leq r\left(\frac{r-2}{2}\right)(R(t))^{-\frac{r}{2}}J(u) \\ &\leq \frac{2r-r^2}{2}b < 0. \end{aligned} \tag{4.21}$$

Integrating (4.21) over $[0, t]$ for any $t \in (0, T_{max})$, then combining this with the fact $r > 2$, we have

$$0 < (R(t))^{1-\frac{r}{2}} \leq (R(0))^{1-\frac{r}{2}} - \left(\frac{r^2-2r}{2}\right)bt, \quad t \in (0, T_{max}]. \tag{4.22}$$

Obviously, (4.22) cannot be established for all $t > 0$. Thus, $T_{max} < +\infty$. Besides, it follows from (4.22) that

$$T_{max} \leq \frac{(R(0))^{\frac{2-r}{2}}}{\left(\frac{r^2-2r}{2}\right)b} = -\frac{\|\nabla u(0)\|_2^2 + \left\||x|^{-\frac{s}{2}}u(0)\right\|_2^2}{(r^2-2r)J(u_0)} < \infty.$$

The proof is completed. □

5. Critical initial energy $J(u_0) = d$

Lemma 5.1. *If $u_0 \in W_*$, letting u be a solution to problem (1.1) which is not a steady-state solution, we can find a $t_* \in (0, T_{max})$ that satisfies*

$$\int_0^{t_*}\left(\|\nabla u_\tau(\tau)\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau(\tau)\right\|_2^2\right)d\tau > 0.$$

Proof. Assume that $u(x, t)$ is an arbitrary solution to problem (1.1) with $J(u_0) = d$ which is not a steady-state solution. Using the reduction to absurdity, we suppose that

$$\int_0^{t_*}\left(\left\|\|\nabla u_\tau(\tau)\|_2^2 + |x|^{-\frac{s}{2}}u_\tau(\tau)\right\|_2^2\right)d\tau \equiv 0, \quad 0 \leq t < T_{max}.$$

Thus, we can conclude $u_t = 0$, which gives $u(x, t) = u_0(x)$ for $x \in \Omega$ and $t \in [0, T)$, namely $u(x, t)$ is a steady-state solution, which is a contradiction. □

Theorem 5.1. *Let $u_0 \in W_*$ and r satisfy* (1.2)*. If $J(u_0) = d$, $I(u_0) > 0$, then* problem (1.1) *allows a global weak solution $u \in L^\infty(0, \infty; W_*)$, $u_t \in L^2(0, \infty; H_0^1(\Omega))$ with $\int_0^t \int_\Omega \frac{u_\tau^2}{|x|^s} dx d\tau < \infty$.*

Proof. First, we select a sequence $\{\theta_k\}_{k=1}^\infty \subset (0, 1)$ that satisfies $\lim_{k\to\infty} \theta_k = 1$. Then, we discuss problem (1.1) with initial data $u(x, 0) = u_{0k} = \theta_k u_0(x)$.

We then claim that $I(u_{0k}) > 0$ and $J(u_{0k}) < d$. In fact, from $\theta_k \subset (0, 1)$ and $I(u_0) > 0$, we get

$$\begin{aligned} I(u_{0k}) &= \theta_k^2 \|\Delta u_0\|_2^2 - \theta_k^r \ln \theta_k \|u_0\|_r^r - \theta_k^r \int_\Omega |u_0|^r \ln |u_0| dx \\ &\geq \theta_k^2 \left(\|\Delta u_0\|_2^2 - \theta_k^{r-2} \int_\Omega |u_0|^r \ln |u_0| dx \right) \\ &\geq \theta_k^2 I(u_0) > 0. \end{aligned} \tag{5.1}$$

In addition, combining the above inequality and Lemma 2.6, we have

$$\frac{d}{d\theta_k} J(\theta_k u_0) = \frac{1}{\theta_k} \left(\theta_k^2 \|\Delta u_0\|_2^2 - \theta_k^r \ln \theta_k \|u_0\|_r^r - \theta_k^r \int_\Omega |u_0|^r \ln |u_0| dx \right) = \frac{1}{\theta_k} I(u_{0k}) > 0,$$

which implies that

$$J(u_{0k}) = J(\theta_k u_0) < J(u_0) = d. \tag{5.2}$$

Since $u_{0k} \to u_0$ as $k \to +\infty$, by (5.1) and (5.2), we will use a method similar to Theorem 4.1 in the subsequent proof. □

Theorem 5.2. *Let $u_0 \in W_*$ and $u(t)$ be the weak solution of* problem (1.1)*, and r satisfy* (1.2)*. If $J(u_0) = d$, $I(u_0) > 0$, then we can find positive constants satisfying*

$$\|\nabla u(t)\|_2^2 + \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \leq c_3 \exp\{-c_2 t\},$$

where $c_3 = \left(\|\nabla u(\bar{t_0})\|_2^2 + \left\| |x|^{-\frac{s}{2}} u(\bar{t_0}) \right\|_2^2 \right) \exp\{c_2 \bar{t_0}\}$, $c_2 = \frac{2(1-\delta_1)}{C_H \tilde{C}}$.

Proof. First, on the premise of $J(u_0) = d$, we claim that $I(u) > 0$ for $0 < t < \infty$ provided that $I(u_0) > 0$. Arguing by contradiction, we can find a $t_0 \in (0, T_{max})$ that satisfies $I(u(t_0)) = 0$, which means that $J(u(t_0) \geq d$. In view of energy equality (4.2), we get

$$J(u(t_0)) + \int_0^{t_0} \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\| |x|^{-s} u_\tau(\tau) \right\|_2^2 \right) d\tau = J(u_0) = d.$$

So, we get $\int_0^{t_0} \left(\left\| |x|^{-s} u_\tau(\tau) \right\|_2^2 + \|\nabla u_\tau(\tau)\|_2^2 \right) d\tau = 0$ for $0 \leq t \leq t_0$, which implies $\frac{du}{dt} = 0$. It follows that $u(x, t) = u_0$ for $0 \leq t \leq t_0$. Then, we can conclude that $I(u(t_0)) = I(u_0) > 0$, which contradicts $I(u(t_0)) = 0$. Thus, for $0 < t < \infty$, we can get $I(u) > 0$.

Next, combining (4.2) and Lemma 5.1, we obtain

$$J(u(t_0)) = d - \int_0^{t_0} \left(\|\nabla u_\tau(\tau)\|_2^2 + \left\| |x|^{-s} u_\tau(\tau) \right\|_2^2 \right) d\tau < d.$$

So we can select the initial time $\bar{t_0} > 0$ sufficiently small that satisfies $I(u(\bar{t_0})) > 0$ and $J(u(\bar{t_0})) < d$. Then, using similar arguments as used in Theorem 4.2, we have

$$\|\nabla u(t)\|_2^2 + \left\||x|^{-\frac{s}{2}} u(t)\right\|_2^2 \le c_0 \exp\{-c_2(t - \bar{t_0})\},$$

where $c_0 = \|\nabla u(\bar{t_0})\|_2^2 + \left\||x|^{-\frac{s}{2}} u(\bar{t_0})\right\|_2^2$, $c_2 = \frac{2(1-\delta_1)}{C_H \tilde{C}}$. Thus, by direct calculation, we can get

$$\|\nabla u(t)\|_2^2 + \left\||x|^{-\frac{s}{2}} u(t)\right\|_2^2 \le c_3 \exp\{-c_2 t\},$$

where $c_3 = c_0 \exp\{c_2 \bar{t_0}\}$. The proof is completed. □

Theorem 5.3. *Assume that $u_0 \in W_*$, r satisfies* (1.2)*, $I(u_0) < 0$, and $J(u_0) = d$. So the weak solution $u(t)$ of* problem (1.1) *blows up in finite time.*

Proof. From Lemma 5.1, we can find a sufficiently small $\bar{t_1} > 0$ that satisfies $I(u) < 0$ for $0 \le t \le \bar{t_1}$. Combining with (4.2), we obtain

$$J(u(\bar{t_1})) = d - \int_0^{\bar{t_1}} \|\nabla u_\tau(\tau)\|_2^2 + \left(\left\||x|^{-s} u_\tau(\tau)\right\|_2^2\right) d\tau < d.$$

Therefore, we let $t = \bar{t_1}$ as the new initial time, and we have $I(u(\bar{t_1})) > 0$ and $J(u(\bar{t_1})) < d$. Hence, similar to Theorem 4.3, Theorem 5.3 is proved. □

6. Blow-up for high initial energy $J(u_0) > 0$

Lemma 6.1. *Assume that $u_0 \in W_*$ satisfies*

$$0 < J(u_0) < \frac{r-2}{2p(C_H + 1)\tilde{C}} \|\nabla u_0\|_2^2, \tag{6.1}$$

where C_H and $\tilde{C}$ is defined in Remark 2.2. Then, for $u \in W_*$, we have $I(u) < 0$.

Proof. In view of (2.1), we have

$$\begin{aligned} J(u_0) &= \frac{1}{r} I(u_0) + (\frac{1}{2} - \frac{1}{r}) \|\Delta u_0\|_2^2 + \frac{1}{r^2} \|u_0\|_r^r \\ &> \frac{1}{r} I(u_0) + (\frac{1}{2} - \frac{1}{r}) \|\Delta u_0\|_2^2. \end{aligned} \tag{6.2}$$

Meanwhile, inequality (6.1) means that $J(u_0) < \frac{r-2}{2p(C_H+1)} \|\Delta u_0\|_2^2$, and combining (6.2) and the fact $C_H + 1 > 1$, we have $I(u_0) < 0$.

Next, we shall prove $I(u(t)) < 0$ for $t \in (0, T_{max})$. If it does not hold, by the continuity of $I(u)$ with respect to t, we can find the first $\tilde{t} \in (0, T_{max})$ satisfies $I(u(\tilde{t})) < 0$ for $t \in (0, \tilde{t})$ and $I(u(\tilde{t})) = 0$ with $u(\tilde{t}) \ne 0$. Then, by Corollary 2.1, we know $\|\Delta u\|_2 \ne 0$. From (4.15), we obtain

$$\frac{d}{dt}\left(\|\nabla u(t)\|_2^2 + \left\||x|^{-\frac{s}{2}} u(t)\right\|_2^2\right) = -2I(u) > 0, \quad t \in (0, \tilde{t}), \tag{6.3}$$

and, further, by Remark 2.2, we can get

$$\|\nabla u_0\|_2^2 < \|\nabla u_0\|_2^2 + \left\||x|^{-\frac{s}{2}} u_0\right\|_2^2 < \left\|\nabla u\left(\tilde{t}\right)\right\|_2^2 + \left\||x|^{-\frac{s}{2}} u(\tilde{t})\right\|_2^2 < (C_H + 1)\left\|\nabla u(\tilde{t})\right\|_2^2. \tag{6.4}$$

Now we look for the contradiction. In view of (4.17), we arrive at

$$J(u(\tilde{t})) \leq J(u_0). \tag{6.5}$$

Combining (2.3) (6.5), and $I(u(\tilde{t})) = 0$, we have

$$\begin{aligned} J(u_0) \geq J(u(\tilde{t})) &= \frac{r-2}{2r}\left\|\Delta u(\tilde{t})\right\|_2^2 + \frac{1}{r^2}\left\|u(\tilde{t})\right\|_r^r + \frac{1}{r} I(u(\tilde{t})) \\ &> \frac{r-2}{2r}\left\|\Delta u(\tilde{t})\right\|_2^2 \geq \frac{r-2}{2r\tilde{C}}\left\|\nabla u(\tilde{t})\right\|_2^2. \end{aligned}$$

Then, by (6.1), we have the inequality

$$\|\nabla u_0\|_2^2 > (C_H + 1)\left\|\nabla u(\tilde{t})\right\|_2^2,$$

which is contradictory to (6.4). Hence, we have $I(u) < 0$ for $u \in W_*$. □

Theorem 6.1. *Assume that $u_0 \in W_*$, $J(u_0)$ satisfies* (6.1)*, and r satisfies* (1.2)*. Then $u(t)$ blows up in finite time, and the blow-up time T_{max} satisfies*

$$T_{\max} \leq \frac{2q\kappa}{(\epsilon - 1)\left(\|\nabla u_0\|_2^2 + \left\||x|^{-\frac{s}{2}} u_0\right\|_2^2\right)^2},$$

where ϵ, q, and κ are given in (6.14)*,* (6.15)*, and* (6.20)*, respectively.*

Proof. We suppose that $u(x,t)$ is a global solution, which means that $T_{max} = +\infty$. Henceforth, we structure the following auxiliary functions

$$L(t) = \int_0^t \left(\|\nabla u(\tau)\|_2^2 + \left\||x|^{-\frac{s}{2}} u(\tau)\right\|_2^2\right) d\tau,$$

$$K(t) = (L(t))^2 + q^{-1} L'(0) L(t) + \kappa, \tag{6.6}$$

where q and κ are given later. First, from (6.3), we have

$$\begin{aligned} L''(t) &= -2I(u) = -2\left(\|\Delta u\|_2^2 - \int_\Omega |u|^r \ln|u|\, dx\right) \\ &= \frac{4}{r^2}\|u\|_r^r + \frac{2(r-2)}{r}\int_\Omega |u|^r \ln|u|\, dx - 4\left(\frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{r}\int_\Omega |u|^r \ln|u|\, dx + \frac{1}{r^2}\|u\|_r^r\right) \\ &= -4J(u) + \frac{4}{r^2}\|u\|_r^r + \frac{2(r-2)}{r}\int_\Omega |u|^r \ln|u|\, dx. \end{aligned} \tag{6.7}$$

Next, we will consider two situations.

Case 1: $J(u) \geq 0$ **for** $t \in (0, \infty)$

Let $\epsilon > 1$. Combining energy equation (4.2), (6.7), and $J(u_0) \geq 0$, we have

$$\begin{aligned} L''(t) &= \frac{4}{r^2}\|u\|_r^r + \frac{2(r-2)}{r}\int_\Omega |u|^r \ln|u|\,dx + 4(\epsilon-1)J(u) - 4\epsilon J(u) \\ &> \frac{4}{r^2}\|u\|_r^r + \frac{2(r-2)}{r}\int_\Omega |u|^r \ln|u|\,dx - 4\epsilon J(u) \\ &= 4\epsilon\int_0^t\left(\|\nabla u_\tau(\tau)\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau(\tau)\right\|_2^2\right)d\tau + \frac{4}{r^2}\|u\|_r^r + \frac{2(r-2)}{r}\int_\Omega |u|^r\ln|u|\,dx - 4\epsilon J(u_0). \end{aligned} \tag{6.8}$$

From Lemma 6.1, we have $I(u) < 0$, which means

$$\|\Delta u\|_2^2 < \int_\Omega |u|^r \ln|u|\,dx. \tag{6.9}$$

By Remark 2.2, (6.8), and (6.9), we arrive at

$$\begin{aligned} L''(t) &> -4\epsilon J(u_0) + 4\epsilon\int_0^t\left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + \frac{4}{r^2}\|u\|_r^r + \frac{2(r-2)}{r}\|\Delta u\|_2^2 \\ &\geq -4\epsilon J(u_0) + 4\epsilon\int_0^t\left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + \frac{2(r-2)}{r\tilde{C}(C_H+1)}\left(\|\nabla u\|_2^2 + \left\||x|^{-\frac{s}{2}}u\right\|_2^2\right) \\ &= -4\epsilon J(u_0) + 4\epsilon\int_0^t\left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + \frac{2(r-2)}{r\tilde{C}(C_H+1)}L'(t). \end{aligned} \tag{6.10}$$

We can also derive from (6.10) that

$$L''(t) > -4\epsilon J(u_0) + \frac{2(r-2)}{r\tilde{C}(C_H+1)}L'(t). \tag{6.11}$$

Then, by solving the above inequality, we arrive at

$$L'(t) > L'(0)\,e^{\frac{2(r-2)}{r\tilde{C}(C_H+1)}t} + \frac{2r\tilde{C}(C_H+1)}{r-2}\epsilon J(u_0)\left(1 - e^{\frac{2(r-2)}{r\tilde{C}(C_H+1)}t}\right). \tag{6.12}$$

Substituting (6.12) into (6.10), we get

$$L''(t) > 4\epsilon\int_0^t\left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + \left[\frac{2(r-2)}{r\tilde{C}(C_H+1)}L'(0) - 4\epsilon J(u_0)\right]e^{\frac{2(r-2)}{r\tilde{C}(C_H+1)}t}. \tag{6.13}$$

We can therefore choose ϵ that satisfies the condition

$$1 < \epsilon < \frac{(r-2)L'(0)}{2r\tilde{C}(C_H+1)J(u_0)}, \tag{6.14}$$

which means that we can choose such ϵ that ensures the following condition on q holds

$$0 < q < \frac{1}{2\epsilon L'(0)}\left[\frac{2(r-2)}{r\tilde{C}(C_H+1)}L'(0) - 4\epsilon J(u_0)\right]. \tag{6.15}$$

From (6.14), (6.15), and $e^{\frac{2(r-2)}{r\tilde{C}(C_H+1)}t} > 1$, (6.13) becomes

$$L''(t) > 4\epsilon\int_0^t\left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + 2q\epsilon L'(0). \tag{6.16}$$

By the definition of $K(t)$ in (6.6), we directly calculate that

$$K'(t) = \left(2L(0) + q^{-1}L'(0)\right)L'(t), \tag{6.17}$$

and

$$K''(t) = 2(L'(t))^2 + \left(2L(t) + q^{-1}L'(0)\right)L''(t). \tag{6.18}$$

Then,

$$\begin{aligned}(K'(t))^2 &= \left(4(L(t))^2 + 4q^{-1}L'(0) + q^{-2}(L'(0))^2\right)(L'(t))^2 \\ &= \left(4K(t) - 4\kappa + q^{-2}(L'(0))^2\right)(L'(t))^2 \\ &= (4K(t) - \varsigma)(L'(t))^2,\end{aligned} \tag{6.19}$$

where $\varsigma = 4c - q^{-2}(L'(0))^2 > 0$, which means that

$$\kappa > (2q)^{-2}(L'(0))^2. \tag{6.20}$$

Then, (6.19) becomes

$$4K(t)(L'(t))^2 = \varsigma(L'(t))^2 + (K'(t))^2. \tag{6.21}$$

By (6.16), (6.18), (6.19), and (6.21), we have

$$\begin{aligned}&2K(t)K''(t) - (1+\epsilon)(K'(t))^2 \\ &> 2K(t)\left[2(L'(t))^2 + \left(2L(t) + q^{-1}L'(0)\right)L''(t)\right] - (1+\epsilon)(4K(t) - \varsigma)(L'(t))^2 \\ &> 4\epsilon K(t)\left(2L(t) + q^{-1}L'(0)\right)\left(2\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + qL'(0)\right) \\ &\quad + (K'(t))^2 + \varsigma(L'(t))^2 - (1+\epsilon)(K'(t))^2.\end{aligned} \tag{6.22}$$

On the other hand, $L'(t)$ can be written as

$$L'(t) = L'(0) + 2\int_0^t \int_\Omega (\nabla u \nabla u_\tau + |x|^{-s}u \cdot u_\tau)dxd\tau. \tag{6.23}$$

Combining the Cauchy-Schwarz inequality and Hölder's inequality, we have

$$\begin{aligned}(L'(t))^2 &= (L'(0))^2 + \left[2\int_0^t \int_\Omega (\nabla u \nabla u_\tau + |x|^{-s}u \cdot u_\tau)dxd\tau\right]^2 \\ &\quad + 4L'(0)\int_0^t \int_\Omega (\nabla u \nabla u_\tau + |x|^{-s}u \cdot u_\tau)dxd\tau \\ &\le (L'(0))^2 + 4L(t)\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau \\ &\quad + 4L'(0)(L(t))^{\frac{1}{2}}\left[\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau\right]^{\frac{1}{2}} \\ &\le (L'(0))^2 + 4L(t)\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau + 2qL'(0)L(t) \\ &\quad + 2q^{-1}L'(0)\int_0^t \left(\|\nabla u_\tau\|_2^2 + \left\||x|^{-\frac{s}{2}}u_\tau\right\|_2^2\right)d\tau \\ &:= A(t).\end{aligned} \tag{6.24}$$

Combining (6.21), (6.22), and (6.24), we can infer that

$$\begin{aligned}
&2K(t)K''(t)-(1+\epsilon)(K'(t))^2\\
&>4\epsilon K(t)A(t)+(K'(t))^2+\varsigma(L'(t))^2-(1+\epsilon)(K'(t))^2\\
&\geq 4\epsilon K(t)(L'(t))^2+(4K(t)-\varsigma)(L'(t))^2+\varsigma(L'(t))^2-(1+\epsilon)(K'(t))^2\\
&=4\epsilon K(t)(L'(t))^2+4K(t)(L'(t))^2-(1+\epsilon)(K'(t))^2\\
&=(1+\epsilon)\varsigma(L'(t))^2>0,
\end{aligned}$$

which says that, for $t\in[0,\infty)$, there is

$$K(t)K''(t)-(1+\alpha)(K'(t))^2>0,$$

and here we choose $\alpha=\dfrac{\epsilon-1}{2}$. Besides, through simple calculation, we have

$$K'(0)=q^{-1}(L'(0))^2>0 \ \ \textit{and} \ \ K(0)=\kappa>0.$$

Therefore, by Lemma 2.10, there is a T_2 with $0<T_2\leq\dfrac{2K(0)}{(\epsilon-1)K'(0)}$ which satisfies $K(t)\to\infty,\ t\to T_2$, and we can obtain that

$$T_{\max}\leq\frac{2q\kappa}{(\epsilon-1)(L'(0))^2}.$$

Case 2: $J(u(\tilde{t}))<0$ **for some** $\tilde{t}>0$

By $J(u_0)>0$ and the continuity of $J(u)$ on t, we can find a time $t^*\in(0,T_{max})$ that satisfies $J(u(t)<0$ for $t>t^*$ and $J(u(t^*))=0$. Thus, we choose $u(t^*)$ to regard as a new initial datum. Further, it follows from Lemma 6.1 that $I(u)<0$ for $t>t^*$. As to the proof of Theorem 4.4, the result of finite time blow-up of the weak solutions is proved.

Based on the above two situations, we can infer the blow-up result of weak solutions in finite time with supercritical initial energy. □

Author contributions

Y. X. Zhao: Methodology, Writing-original draft; X. L. Wu: Methodology, Writing-original draft, Writing-review and editing. Both of authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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