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Research article

On the nonlocal hybrid (k, φ)-Hilfer inverse problem with delay and anticipation

Abdelkrim Salim[1,2], **Sabri T. M. Thabet**[3,4,5,*], **Imed Kedim**[6] **and Miguel Vivas-Cortez**[7,*]

[1] Faculty of Technology, Hassiba Benbouali University of Chlef, P. O. Box 151, Chlef 02000, Algeria; a.salim@univ-chlef.dz

[2] Laboratory of Mathematics, Djillali Liabes University of Sidi Bel-Abbes, P. O. Box 89, Sidi Bel-Abbes 22000, Algeria; salim.abdelkrim@yahoo.com

[3] Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Chennai 602105, Tamil Nadu, India

[4] Department of Mathematics, Radfan University College, University of Lahej, Lahej, Yemen

[5] Department of Mathematics, College of Science, Korea University, Seoul 02814, South Korea

[6] Department of Mathematics, College of Science and Humanities in Al-Kharj, Prince Sattam Bin Abdulaziz University, Al-Kharj 11942, Saudi Arabia; i.kedim@psau.edu.sa

[7] Faculty of Exact and Natural Sciences, School of Physical Sciences and Mathematics, Pontifical Catholic University of Ecuador, Sede Quito, Ecuador

* **Correspondence:** Email: th.sabri@yahoo.com, mjvivas@puce.edu.ec.

Abstract: This paper focused on establishing results regarding the existence of solutions for a class of nonlocal terminal value problems involving hybrid implicit nonlinear fractional differential equations with the (k, φ)-Hilfer fractional derivative, which includes both finite delay and anticipation arguments. Our analysis was based on the Banach fixed point technique, and the Schauder and Krasnoselskii fixed point theorems. Moreover, illustrative examples were considered to support our new results.

Keywords: (k, φ)-Hilfer fractional derivative; hybrid equations; terminal value problem; retarded arguments; advanced arguments; existence; uniqueness; nonlocal condition
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1. Introduction

Fractional calculus has proven in recent years to be a helpful method of tackling the complexity of complex systems from various scientific and engineering branches. It involves the generalization of the integer order differentiation and integration of a function to non-integer order, see [1]. In recent years, there has been considerable interest in fractional differential equations, with numerous works dedicated to the topic. Notable examples include the books by Benchohra et al. [2, 3]. The authors of [4, 5] investigated the qualitative theorems of solutions to diverse fractional differential equations and inclusions about memory effects and predictive behavior arguments.

In a recent publication [6], Diaz introduced novel definitions for the special functions k-gamma and k-beta. Interested readers can refer to additional sources such as [7, 8] to delve deeper into this topic. Furthermore, in another work [9], Sousa et al. presented the φ-Hilfer derivative of fractional order and elucidated some crucial properties related to this type of fractional operator. Drawing inspiration from the various papers cited earlier, we have introduced a new extension of the renowned Hilfer fractional derivative [10–12].

On the other hand, delay differential equations are a type of functional differential equation that arise in various biological and physical applications and often require consideration of variable or state-dependent delays. The study of functional differential equations with delay has garnered significant attention in recent years due to their crucial applications in mathematical models of real-world phenomena. For examples, see [4, 5] and the references therein.

In [13], Krim et al. studied the problem

$$\begin{cases} \left({}^{\rho}D^{\vartheta_2}_{0^+} + \vartheta_1\right)(\varrho) = \mathfrak{g}\left(\varrho, \vartheta_1(\varrho), \left({}^{\rho}D^{\vartheta_2}_{0^+} + \vartheta_1\right)(\varrho)\right), \quad \varrho \in I := [0, T], \\ \vartheta_1(T) = \vartheta_{1T} \in \mathbb{R}, \end{cases}$$

where ${}^{\rho}D^{\vartheta_2}_{0^+}$ is the Katugampola derivative of fractional order $\vartheta_2 \in (0, 1]$, and

$$\mathfrak{g} : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a continuous function.

Using the Picard operator method, Krasnoselskii fixed point approach, and Gronwall's inequality lemma, Almalahi et al. [14] established the existence and stability theories for the problem:

$$\begin{cases} {}^{H}D^{\vartheta_1,\vartheta_2;\varphi}_{0^+}\mathsf{y}(\varrho) = \mathfrak{g}(\varrho, \mathsf{y}(\varrho), \mathsf{y}(\widehat{\mathfrak{g}}(\varrho)), \quad \varrho \in (0, \mathfrak{a}_2], \\ I^{1-\vartheta_3;\varphi}_{0^+}\mathsf{y}(0^+) = \sum_{\jmath=1}^{\mathsf{k}} c_\jmath \mathsf{y}\left(k_\jmath\right), \quad k_\jmath \in (0, \mathfrak{a}_2), \\ \mathsf{y}(\varrho) = \delta(\varrho), \quad \varrho \in [-\vartheta_2, 0], \end{cases}$$

where ${}^{H}D^{\vartheta_1,\vartheta_2;\varphi}_{0^+}(\cdot)$ is the φ-Hilfer derivative of fractional order $\vartheta_1 \in (0, 1)$ and type $\vartheta_2 \in [0, 1]$, $I^{1-\vartheta_3,\varphi}_{0^+}(\cdot)$ is the φ-Riemann-Liouville integral of fractional order $(1 - \vartheta_3)$,

$$\vartheta_3 = \vartheta_1 + \vartheta_2(1 - \vartheta_1), \quad 0 < \vartheta_3 < 1, \quad k_\jmath, \jmath = 1, 2, \ldots, \mathsf{k}$$

are prefixed points satisfying $0 < k_1 \leq k_2 \leq \ldots \leq k_\jmath < \mathfrak{a}_2$, and $c_\jmath \in \mathbb{R}, \delta \in C[-\vartheta_2, 0]$, the function

$$\mathfrak{g} : (0, \mathfrak{a}_2] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is continuous, and

$$\widehat{\mathfrak{g}} \in C(0, \mathfrak{a}_2] \to [-\vartheta_2, \mathfrak{a}_2]$$

with $\widehat{\mathfrak{g}}(\varrho) \leq \varrho, \vartheta_2 > 0$.

In light of the above studies, we focus on a terminal-valued hybrid problem governed by a nonlinear implicit (k, φ)-Hilfer fractional differential equation with mixed-type arguments (retarded and advanced):

$$\left({}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\psi\, \mathsf{y}\right)(\varrho) = \mathfrak{g}\left(\varrho, \mathsf{y}_\varrho(\cdot), \left({}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\psi\, \mathsf{y}\right)(\varrho)\right), \quad \varrho \in (\mathfrak{a}_1, \mathfrak{a}_2], \tag{1.1}$$

$$\mathsf{y}(\mathfrak{a}_2) = \sum_{\jmath=1}^{\tilde{n}} \alpha_\jmath \mathsf{y}(\epsilon_\jmath), \tag{1.2}$$

$$\mathsf{y}(\varrho) = \chi(\varrho), \quad \varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1], \ \mathsf{d} > 0, \tag{1.3}$$

$$\mathsf{y}(\varrho) = \tilde{\chi}(\varrho), \quad \varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right], \ \tilde{\mathsf{d}} > 0, \tag{1.4}$$

where ${}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}$ is the (k, φ)-Hilfer derivative of fractional order $\vartheta_1 \in (0, \mathsf{k})$ and type $\vartheta_2 \in [0, 1]$ defined in Section 2. Furthermore,

$$\vartheta_3 = \frac{1}{\mathsf{k}}(\vartheta_2(\mathsf{k} - \vartheta_1) + \vartheta_1), \ \ \mathsf{k} > 0,$$
$$\mathfrak{g} : [\mathfrak{a}_1, \mathfrak{a}_2] \times C\left(\left[-\mathsf{d}, \tilde{\mathsf{d}}\right], \mathbb{R}\right) \times \mathbb{R} \longrightarrow \mathbb{R}, \ \ \psi \in C\left([\mathfrak{a}_1, \mathfrak{a}_2], \mathbb{R}\backslash\{0\}\right), \ \ \epsilon_\jmath, \jmath = 1, \ldots, \tilde{n}$$

are pre-fixed points satisfying $\mathfrak{a}_1 < \epsilon_1 \leq \ldots \leq \epsilon_{\tilde{n}} < \mathfrak{a}_2$, and $\alpha_\jmath$, $\jmath = 1, \ldots, \tilde{n}$ are real numbers. For each function y defined on $\left[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$ and for any $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, we denote by y_ϱ the element defined by

$$\mathsf{y}_\varrho(s) = \mathsf{y}(\varrho + s), \quad s \in \left[-\mathsf{d}, \tilde{\mathsf{d}}\right].$$

The following are the primary novelties of the current paper:

- Given the diverse conditions imposed on problems (1.1)–(1.4), our study can be seen as both a continuation and a generalization of the studies mentioned above, such as the papers [13, 14].
- The introduced (k, φ)-Hilfer operator serves as an extension, encompassing previously established fractional derivatives such as the Caputo, Hadamard, and Hilfer fractional derivatives already present in the existing literature.
- The number of papers addressing a nonlocal condition combined with retarded and advanced arguments is very limited. Therefore, our work aims to fill this gap in the literature.
- The introduced (k, φ)-Hilfer operator serves as an extension, encompassing previously established fractional derivatives such as the Caputo, Hadamard, and Hilfer fractional derivatives already present in the existing literature.

The structure of this paper is as follows: Section 2 presents certain notations and preliminaries about the φ-Hilfer fractional derivative, the functions k-gamma and k-beta, and some auxiliary results. Further, we give the definition of the (k, φ)-Hilfer type fractional derivative and some essential theorems and lemmas. In Section 3, we present three existence and uniqueness results for the problems (1.1)–(1.4) that are founded on the Banach contraction principle, the Schauder and Krasnoselskii fixed point theorems. In the last section, illustrative examples are provided in support of the results obtained.

2. Preliminaries

First, we present the weighted spaces, notations, definitions, and preliminary facts that are used in this article. Please refer to [3] for all details on these spaces and notations.

Let
$$0 < \mathfrak{a}_1 < \mathfrak{a}_2 < \infty, \ \ \mathbb{T} = [\mathfrak{a}_1, \mathfrak{a}_2], \ \ \vartheta_1 \in (0, \mathsf{k}), \ \ \vartheta_2 \in [0, 1], \ \ \mathsf{k} > 0$$
and
$$\vartheta_3 = \frac{1}{\mathsf{k}}(\vartheta_2(\mathsf{k} - \vartheta_1) + \vartheta_1).$$
The Banach space of continuous functions is denoted by $C(\mathbb{T}, \mathbb{R})$ with the norm
$$\|\mathsf{y}\|_\infty = \sup\{|\mathsf{y}(\varrho)| : \varrho \in \mathbb{T}\}.$$
Let $AC^n(\mathbb{T}, \mathbb{R})$, $C^n(\mathbb{T}, \mathbb{R})$ be the spaces of continuous functions, n-times absolutely continuous, and n-times continuously differentiable functions on $\mathbb{T}$, respectively.

Let
$$C\left(\left[-\mathsf{d}, \tilde{\mathsf{d}}\right], \mathbb{R}\right), \ \ C = C([\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1], \mathbb{R})$$
and
$$\tilde{C} = C\left(\left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right], \mathbb{R}\right)$$
be the spaces gifted, respectively, with the norms
$$\begin{aligned} \|\mathsf{y}\|_{[-\mathsf{d}, \tilde{\mathsf{d}}]} &= \sup\left\{|\mathsf{y}(\varrho)| : \varrho \in \left[-\mathsf{d}, \tilde{\mathsf{d}}\right]\right\}, \\ \|\mathsf{y}\|_C &= \sup\{|\mathsf{y}(\varrho)| : \varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1]\}, \\ \|\mathsf{y}\|_{\tilde{C}} &= \sup\left\{|\mathsf{y}(\varrho)| : \varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]\right\}. \end{aligned}$$
Let $\varphi \in C([\mathfrak{a}_1, \mathfrak{a}_2], \mathbb{R})$ be an increasing function such that $\varphi'(\varrho) \neq 0$, for all $\varrho \in \mathbb{T}$.

Now, let the weighted Banach space be defined as
$$C_{\vartheta_3, \mathsf{k}; \varphi}(\mathbb{T}) = \left\{\mathsf{y} : (\mathfrak{a}_1, \mathfrak{a}_2] \to \mathbb{R} : \varrho \to \Psi^{\varphi}_{\vartheta_3}(\varrho, \mathfrak{a}_1)\mathsf{y}(\varrho) \in C(\mathbb{T}, \mathbb{R})\right\},$$
where
$$\Psi^{\varphi}_{\vartheta_3}(\varrho, \mathfrak{a}_1) = (\varphi(\varrho) - \varphi(\mathfrak{a}_1))^{1-\vartheta_3}$$
with the norm
$$\|\mathsf{y}\|_{C_{\vartheta_3, \mathsf{k}; \varphi}} = \sup_{\varrho \in \mathbb{T}} \left|\Psi^{\varphi}_{\vartheta_3}(\varrho, \mathfrak{a}_1)\mathsf{y}(\varrho)\right|,$$
and
$$\begin{aligned} &C^n_{\vartheta_3, \mathsf{k}; \varphi}(\mathbb{T}) = \left\{\mathsf{y} \in C^{n-1}(\mathbb{T}) : \mathsf{y}^{(n)} \in C_{\vartheta_3, \mathsf{k}; \varphi}(\mathbb{T})\right\}, \ \ n \in \mathbb{N}, \\ &C^0_{\vartheta_3, \mathsf{k}; \varphi}(\mathbb{T}) = C_{\vartheta_3, \mathsf{k}; \varphi}(\mathbb{T}) \end{aligned}$$
with the norm
$$\|\mathsf{y}\|_{C^n_{\vartheta_3, \mathsf{k}; \varphi}} = \sum_{\jmath=0}^{n-1} \|\mathsf{y}^{(\jmath)}\|_\infty + \|\mathsf{y}^{(n)}\|_{C_{\vartheta_3, \mathsf{k}; \varphi}}.$$

Next, let us define the Banach space

$$\mathbb{F} = \left\{ \mathsf{y} : \left[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right] \to \mathbb{R} : \mathsf{y}|_{[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1]} \in C, \mathsf{y}|_{\left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]} \in \tilde{C} \text{ and } \mathsf{y}|_{(\mathfrak{a}_1, \mathfrak{a}_2]} \in C_{\vartheta_3, \mathsf{k}; \varphi}(\mathbb{T}) \right\}$$

with the norm

$$\|\mathsf{y}\|_{\mathbb{F}} = \max\{\|\mathsf{y}\|_{C}, \|\mathsf{y}\|_{\tilde{C}}, \|\mathsf{y}\|_{C_{\vartheta_3,\mathsf{k};\varphi}}\}.$$

Denote $X^p_\varphi(\mathfrak{a}_1, \mathfrak{a}_2)$, $(1 \le p \le \infty)$ to the space of each real-valued Lebesgue measurable functions $\widehat{\mathfrak{g}}$ on $[\mathfrak{a}_1, \mathfrak{a}_2]$ such that $\|\widehat{\mathfrak{g}}\|_{X^p_\varphi} < \infty$, with the norm given as

$$\|\widehat{\mathfrak{g}}\|_{X^p_\varphi} = \left(\int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \varphi'(\varrho) |\widehat{\mathfrak{g}}(\varrho)|^p d\varrho \right)^{\frac{1}{p}},$$

where φ is a non-deceasing and non-negative function on $[\mathfrak{a}_1, \mathfrak{a}_2]$, such that φ' is continuous on $[\mathfrak{a}_1, \mathfrak{a}_2]$ with $\varphi(0) = 0$.

Definition 2.1. *[6] The* k*-gamma function is given as*

$$\Gamma_{\mathsf{k}}(\alpha) = \int_0^\infty \varrho^{\alpha-1} e^{-\frac{\varrho^{\mathsf{k}}}{\mathsf{k}}} d\varrho, \quad \alpha > 0,$$

where

$$\Gamma_{\mathsf{k}}(\alpha + \mathsf{k}) = \alpha \Gamma_{\mathsf{k}}(\alpha),$$
$$\Gamma_{\mathsf{k}}(\alpha) = \mathsf{k}^{\frac{\alpha}{\mathsf{k}} - 1} \Gamma\left(\frac{\alpha}{\mathsf{k}}\right),$$
$$\Gamma_{\mathsf{k}}(\mathsf{k}) = \Gamma(1) = 1,$$

and for $\mathsf{k} \to 1$*, then*

$$\Gamma(\alpha) = \Gamma_{\mathsf{k}}(\alpha).$$

Furthermore the k*-beta function is defined as follows:*

$$B_{\mathsf{k}}(\alpha, \mathfrak{a}) = \frac{1}{\mathsf{k}} \int_0^1 \varrho^{\frac{\alpha}{\mathsf{k}} - 1} (1 - \varrho)^{\frac{\mathfrak{a}}{\mathsf{k}} - 1} d\varrho,$$

so that

$$B_{\mathsf{k}}(\alpha, \mathfrak{a}) = \frac{1}{\mathsf{k}} B\left(\frac{\alpha}{\mathsf{k}}, \frac{\mathfrak{a}}{\mathsf{k}}\right)$$

and

$$B_{\mathsf{k}}(\alpha, \mathfrak{a}) = \frac{\Gamma_{\mathsf{k}}(\alpha)\Gamma_{\mathsf{k}}(\mathfrak{a})}{\Gamma_{\mathsf{k}}(\alpha + \mathfrak{a})}.$$

Definition 2.2. *[15] Let*

$$\widehat{\mathfrak{g}} \in X^p_\varphi(\mathfrak{a}_1, \mathfrak{a}_2), \quad \varphi(\varrho) > 0$$

be a non-decreasing function on $(\mathfrak{a}_1, \mathfrak{a}_2]$ *and* $\varphi'(\varrho) > 0$ *be continuous on* $(\mathfrak{a}_1, \mathfrak{a}_2)$ *and* $\vartheta_1 > 0$*. The generalized* k*-fractional integral operators of a function* $\widehat{\mathfrak{g}}$ *of order* ϑ_1 *are defined by*

$$\mathcal{J}^{\vartheta_1, \mathsf{k}; \varphi}_{\mathfrak{a}_1+} \widehat{\mathfrak{g}}(\varrho) = \int_{\mathfrak{a}_1}^{\varrho} \bar{\Psi}^{\mathsf{k}, \varphi}_{\vartheta_1}(\varrho, s) \varphi'(s) \widehat{\mathfrak{g}}(s) ds,$$
$$\mathcal{J}^{\vartheta_1, \mathsf{k}; \varphi}_{\mathfrak{a}_2-} \widehat{\mathfrak{g}}(\varrho) = \int_{\varrho}^{\mathfrak{a}_2} \bar{\Psi}^{\mathsf{k}, \varphi}_{\vartheta_1}(s, \varrho) \varphi'(s) \widehat{\mathfrak{g}}(s) ds$$

with $\mathsf{k} > 0$ *and*

$$\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(\varrho, s) = \frac{(\varphi(\varrho) - \varphi(s))^{\frac{\vartheta_1}{\mathsf{k}}-1}}{\mathsf{k}\Gamma_{\mathsf{k}}(\vartheta_1)}.$$

Additionally, the authors of the work [16] extended these operators and defined the generalized fractional integrals by

$$\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{G,\mathfrak{a}_1+}\widehat{\mathfrak{g}}(\varrho) = \frac{1}{\mathsf{k}\Gamma_{\mathsf{k}}(\vartheta_1)}\int_{\mathfrak{a}_1}^{\varrho}\frac{\varphi'(s)\widehat{\mathfrak{g}}(s)ds}{G(\varphi(\varrho) - \varphi(s), \frac{\vartheta_1}{\mathsf{k}})},$$
$$\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{G,\mathfrak{a}_2-}\widehat{\mathfrak{g}}(\varrho) = \frac{1}{\mathsf{k}\Gamma_{\mathsf{k}}(\vartheta_1)}\int_{\varrho}^{\mathfrak{a}_2}\frac{\varphi'(s)\widehat{\mathfrak{g}}(s)ds}{G(\varphi(s) - \varphi(\varrho), \frac{\vartheta_1}{\mathsf{k}})},$$

where $G(z, \vartheta_1) \in AC[\mathfrak{a}_1, \mathfrak{a}_2]$.

Theorem 2.3. *[16] Let* $\vartheta_1 > 0$, $\mathsf{k} > 0$, *and consider the integrable function* $\widehat{\mathfrak{g}}$*:* $[\mathfrak{a}_1, \mathfrak{a}_2] \to \mathbb{R}$*. Then* $\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{G,\mathfrak{a}_1+}\widehat{\mathfrak{g}}$ *exists for all* $\varrho \in [\mathfrak{a}_1, \mathfrak{a}_2]$.

Theorem 2.4. *[16] Let* $\widehat{\mathfrak{g}} \in X^p_{\varphi}(\mathfrak{a}_1, \mathfrak{a}_2)$ *and take* $\vartheta_1 > 0$ *and* $\mathsf{k} > 0$*. Then* $\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{G,\mathfrak{a}_1+}\widehat{\mathfrak{g}} \in C([\mathfrak{a}_1, \mathfrak{a}_2], \mathbb{R})$.

Lemma 2.5. *[10, 11] Consider* $\vartheta_1 > 0$, $\vartheta_2 > 0$, *and* $\mathsf{k} > 0$*. Then, one has*

$$\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathcal{J}^{\vartheta_2,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathfrak{g}(\varrho) = \mathcal{J}^{\vartheta_1+\vartheta_2,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathfrak{g}(\varrho) = \mathcal{J}^{\vartheta_2,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathfrak{g}(\varrho)$$

and

$$\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_2-}\mathcal{J}^{\vartheta_2,\mathsf{k};\varphi}_{\mathfrak{a}_2-}\mathfrak{g}(\varrho) = \mathcal{J}^{\vartheta_1+\vartheta_2,\mathsf{k};\varphi}_{\mathfrak{a}_2-}\mathfrak{g}(\varrho) = \mathcal{J}^{\vartheta_2,\mathsf{k};\varphi}_{\mathfrak{a}_2-}\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_2-}\mathfrak{g}(\varrho).$$

Lemma 2.6. *[10, 11] Let* $\vartheta_1, \vartheta_2 > 0$ *and* $\mathsf{k} > 0$*. Then, we have*

$$\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_2}(\varrho, \mathfrak{a}_1) = \bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1+\vartheta_2}(\varrho, \mathfrak{a}_1)$$

and

$$\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_2-}\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_2}(\mathfrak{a}_2, \varrho) = \bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1+\vartheta_2}(\mathfrak{a}_2, \varrho).$$

Theorem 2.7. *[10, 11] Let* $0 < \mathfrak{a}_1 < \mathfrak{a}_2 < \infty, \vartheta_1 > 0, 0 \le \vartheta_3 < 1$, $\mathsf{k} > 0$, *and* $\mathsf{y} \in C_{\vartheta_3,\mathsf{k};\varphi}(\mathbb{T})$*. If*

$$\frac{\vartheta_1}{\mathsf{k}} > 1 - \vartheta_3,$$

then

$$\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathsf{y}\right)(\mathfrak{a}_1) = \lim_{\varrho\to\mathfrak{a}_1^+}\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathsf{y}\right)(\varrho) = 0.$$

Definition 2.8. *(*(k, φ)*-Hilfer derivative [10, 11]) Let*

$$n - 1 < \frac{\vartheta_1}{\mathsf{k}} \le n$$

with $n \in \mathbb{N}$, $\mathbb{T} = [\mathfrak{a}_1, \mathfrak{a}_2]$ *an interval such that*

$$-\infty \le \mathfrak{a}_1 < \mathfrak{a}_2 \le \infty$$

and

$$\widehat{\mathfrak{g}}, \varphi \in C^n([\mathfrak{a}_1, \mathfrak{a}_2], \mathbb{R})$$

are two functions such that φ *is increasing and* $\varphi'(\varrho) \neq 0$*, for all* $\varrho \in \mathbb{T}$*. The* (k, φ)*-Hilfer fractional derivative* ${}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}(\cdot)$ *and* ${}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_2-}(\cdot)$ *of a function* $\widehat{\mathfrak{g}}$ *of order* ϑ_1 *and type* $0 \leq \vartheta_2 \leq 1$*, with* $\mathsf{k} > 0$ *is defined by*

$$\begin{aligned} {}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\widehat{\mathfrak{g}}(\varrho) &= \left(\mathcal{J}^{\vartheta_2(\mathsf{k}n-\vartheta_1),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\left(\frac{1}{\varphi'(\varrho)}\frac{d}{d\varrho}\right)^n\left(\mathsf{k}^n\mathcal{J}^{(1-\vartheta_2)(\mathsf{k}n-\vartheta_1),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\widehat{\mathfrak{g}}\right)\right)(\varrho) \\ &= \left(\mathcal{J}^{\vartheta_2(\mathsf{k}n-\vartheta_1),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta^n_{\varphi}\left(\mathsf{k}^n\mathcal{J}^{(1-\vartheta_2)(\mathsf{k}n-\vartheta_1),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\widehat{\mathfrak{g}}\right)\right)(\varrho), \end{aligned}$$

where

$$\delta^n_{\varphi} = \left(\frac{1}{\varphi'(\varrho)}\frac{d}{d\varrho}\right)^n.$$

Lemma 2.9. *[10, 11] Let* $\varrho > \mathfrak{a}_1$, $\vartheta_1 > 0, 0 \leq \vartheta_2 \leq 1$, *and* $\mathsf{k} > 0$*. Thus, for*

$$0 < \vartheta_3 < 1, \quad \vartheta_3 = \frac{1}{\mathsf{k}}(\vartheta_2(\mathsf{k} - \vartheta_1) + \vartheta_1),$$

and one has

$$\left[{}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\left(\Psi^{\varphi}_{\vartheta_3}(s, \mathfrak{a}_1)\right)^{-1}\right](\varrho) = 0.$$

Theorem 2.10. *[10, 11] If*

$$\mathfrak{g} \in C^n_{\vartheta_3,\mathsf{k};\varphi}[\mathfrak{a}_1, \mathfrak{a}_2], \quad n - 1 < \frac{\vartheta_1}{\mathsf{k}} < n, \quad 0 \leq \vartheta_2 \leq 1,$$

where $n \in \mathbb{N}$ *and* $\mathsf{k} > 0$*, then*

$$\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\ {}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\mathfrak{g}\right)(\varrho) = \mathfrak{g}(\varrho) - \sum_{\jmath=1}^{n}\frac{(\varphi(\varrho) - \varphi(\mathfrak{a}_1))^{\vartheta_3-\jmath}}{\mathsf{k}^{\jmath-n}\Gamma_{\mathsf{k}}(\mathsf{k}(\vartheta_3 - \jmath + 1))}\left\{\delta^{n-\jmath}_{\varphi}\left(\mathcal{J}^{\mathsf{k}(n-\vartheta_3),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathfrak{g}(\mathfrak{a}_1)\right)\right\},$$

where

$$\vartheta_3 = \frac{1}{\mathsf{k}}(\vartheta_2(\mathsf{k}n - \vartheta_1) + \vartheta_1).$$

Particularly, for $n = 1$*, one gets*

$$\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\ {}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\mathfrak{g}\right)(\varrho) = \mathfrak{g}(\varrho) - \frac{(\varphi(\varrho) - \varphi(\mathfrak{a}_1))^{\vartheta_3-1}}{\Gamma_{\mathsf{k}}(\vartheta_2(\mathsf{k} - \vartheta_1) + \vartheta_1)}\mathcal{J}^{(1-\vartheta_2)(\mathsf{k}-\vartheta_1),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathfrak{g}(\mathfrak{a}_1).$$

Lemma 2.11. *[10, 11] Let* $\vartheta_1 > 0, 0 \leq \vartheta_2 \leq 1$, *and* $\mathsf{y} \in C^1_{\vartheta_3,\mathsf{k};\varphi}(\mathbb{T})$*, where* $\mathsf{k} > 0$*, then for* $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$*, we have*

$$\left({}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\ \mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathsf{y}\right)(\varrho) = \mathsf{y}(\varrho).$$

3. Main results

We start this section by taking the next fractional differential problem:

$$\left({}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\psi\,\mathsf{y}\right)(\varrho) = \delta(\varrho), \quad \varrho \in (\mathfrak{a}_1, \mathfrak{a}_2], \tag{3.1}$$

such that $0 < \vartheta_1 < \mathsf{k}, 0 \le \vartheta_2 \le 1$, subjected to the conditions

$$\mathsf{y}(\mathfrak{a}_2) = \sum_{\jmath=1}^{\tilde{n}} \alpha_\jmath \mathsf{y}(\epsilon_\jmath), \tag{3.2}$$

$$\mathsf{y}(\varrho) = \chi(\varrho), \quad \varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1], \ \mathsf{d} > 0, \tag{3.3}$$

$$\mathsf{y}(\varrho) = \tilde{\chi}(\varrho), \quad \varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right], \ \tilde{\mathsf{d}} > 0, \tag{3.4}$$

where

$$\vartheta_3 = \frac{\vartheta_2(\mathsf{k} - \vartheta_1) + \vartheta_1}{\mathsf{k}},$$

$\mathsf{k} > 0$, $\alpha_\jmath$, $\jmath = 1, \dots, \tilde{n}$, belong to $\mathbb{R}$, $\alpha_{\tilde{n}+1} = -1$ and $\epsilon_\jmath$, $\jmath = 1, \dots, \tilde{n}+1$, are pre-fixed points verifying

$$\mathfrak{a}_1 < \epsilon_1 \le \ldots \le \epsilon_{\tilde{n}} < \mathfrak{a}_2 = \epsilon_{\tilde{n}+1},$$

such that

$$\sum_{\jmath=1}^{\tilde{n}+1} \frac{\alpha_\jmath}{\Psi^{\varphi}_{\vartheta_3}(\epsilon_\jmath, \mathfrak{a}_1)} \neq 0,$$

and where $\delta(\cdot) \in C(\mathbb{T}, \mathbb{R})$, $\chi(\cdot) \in C$, $\psi \in C([\mathfrak{a}_1, \mathfrak{a}_2], \mathbb{R}\backslash\{0\})$, and $\tilde{\chi}(\cdot) \in \tilde{C}$.

Theorem 3.1. *The function* y *verifies* (3.1)–(3.4) *if and only if*

$$\mathsf{y}(\varrho) = \begin{cases} \dfrac{1}{\psi(\varrho)}\left[\dfrac{-\displaystyle\sum_{\jmath=1}^{\tilde{n}+1} \dfrac{\alpha_\jmath}{\psi(\epsilon_\jmath)}\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta\right)(\epsilon_\jmath)}{\displaystyle\sum_{\jmath=1}^{\tilde{n}+1} \dfrac{\alpha_\jmath \Psi^{\varphi}_{\vartheta_3}(\varrho, \mathfrak{a}_1)}{\psi(\epsilon_\jmath)\Psi^{\varphi}_{\vartheta_3}(\epsilon_\jmath, \mathfrak{a}_1)}} + \left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta\right)(\varrho)\right], & \varrho \in (\mathfrak{a}_1, \mathfrak{a}_2], \\ \chi(\varrho), \quad \varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1], \\ \tilde{\chi}(\varrho), \quad \varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]. \end{cases} \tag{3.5}$$

Proof. Assume that y satisfies Eqs (3.1)–(3.4), and by implementing the integral operator $\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}(\cdot)$ of fractional order ϑ_1 on both sides of (3.1), we have

$$\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\ {}^{H}_{\mathsf{k}}\mathcal{D}^{\vartheta_1,\vartheta_2;\varphi}_{\mathfrak{a}_1+}\psi\,\mathsf{y}\right)(\varrho) = \left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta\right)(\varrho).$$

Using Theorem 2.10, we get

$$\mathsf{y}(\varrho) = \frac{1}{\psi(\varrho)}\left[\frac{\mathcal{J}^{\mathsf{k}(1-\vartheta_3),\mathsf{k};\varphi}_{\mathfrak{a}_1+}\mathsf{y}(\mathfrak{a}_1)}{\Psi^{\varphi}_{\vartheta_3}(\varrho, \mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)} + \left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta\right)(\varrho)\right]. \tag{3.6}$$

In what follows, by putting $\varrho = \epsilon_J$ into (3.6), and applying α_J to both sides, one gets

$$\alpha_J y(\epsilon_J) = \frac{1}{\psi(\epsilon_J)}\left[\frac{\alpha_J \mathcal{J}_{\mathfrak{a}_1+}^{\mathsf{k}(1-\vartheta_3),\mathsf{k};\varphi}\mathsf{y}(\mathfrak{a}_1)}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_J,\mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)} + \alpha_J\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_J)\right].$$

By (3.2) and (3.6) with $\varrho = \mathfrak{a}_2$, we have

$$\begin{aligned}
&\mathcal{J}_{\mathfrak{a}_1+}^{\mathsf{k}(1-\vartheta_3),\mathsf{k};\varphi}\mathsf{y}(\mathfrak{a}_1)\sum_{J=1}^{\tilde{n}}\frac{\alpha_J}{\psi(\epsilon_J)\Psi_{\vartheta_3}^{\varphi}(\epsilon_J,\mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)} + \sum_{J=1}^{\tilde{n}}\frac{\alpha_J}{\psi(\epsilon_J)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_J)\\
&= \frac{1}{\psi(\mathfrak{a}_2)}\left[\frac{\mathcal{J}_{\mathfrak{a}_1+}^{\mathsf{k}(1-\vartheta_3),\mathsf{k};\varphi}\mathsf{y}(\mathfrak{a}_1)}{\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)\right],
\end{aligned}$$

which implies

$$\begin{aligned}
\mathcal{J}_{\mathfrak{a}_1+}^{\mathsf{k}(1-\vartheta_3),\mathsf{k};\varphi}\mathsf{y}(\mathfrak{a}_1) &= \frac{\dfrac{-\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)}{\psi(\mathfrak{a}_2)} + \displaystyle\sum_{J=1}^{\tilde{n}}\frac{\alpha_J}{\psi(\epsilon_J)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_J)}{\dfrac{1}{\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)} - \displaystyle\sum_{J=1}^{\tilde{n}}\frac{\alpha_J}{\psi(\epsilon_J)\Psi_{\vartheta_3}^{\varphi}(\epsilon_J,\mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)}}\\
&= -\frac{\displaystyle\sum_{J=1}^{\tilde{n}+1}\frac{\alpha_J}{\psi(\epsilon_J)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_J)}{\displaystyle\sum_{J=1}^{\tilde{n}+1}\frac{\alpha_J}{\psi(\epsilon_J)\Psi_{\vartheta_3}^{\varphi}(\epsilon_J,\mathfrak{a}_1)\Gamma_{\mathsf{k}}(\mathsf{k}\vartheta_3)}}. \qquad (3.7)
\end{aligned}$$

Substituting (3.7) into (3.6), we obtain (3.5).

Now, we show that y verifies Eq (3.5), it follows that it also verifies (3.1)–(3.4). Applying ${}^{H}_{\mathsf{k}}\mathcal{D}_{\mathfrak{a}_1+}^{\vartheta_1,\vartheta_2;\varphi}(\cdot)$ on both sides of (3.5), we get

$$\left({}^{H}_{\mathsf{k}}\mathcal{D}_{\mathfrak{a}_1+}^{\vartheta_1,\vartheta_2;\varphi}\psi\,\mathsf{y}\right)(\varrho) = {}^{H}_{\mathsf{k}}\mathcal{D}_{\mathfrak{a}_1+}^{\vartheta_1,\vartheta_2;\varphi}\left(\frac{-\displaystyle\sum_{J=1}^{\tilde{n}+1}\frac{\alpha_J}{\psi(\epsilon_J)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_J)}{\displaystyle\sum_{J=1}^{\tilde{n}+1}\frac{\alpha_J\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)}{\psi(\epsilon_J)\Psi_{\vartheta_3}^{\varphi}(\epsilon_J,\mathfrak{a}_1)}}\right) + \left({}^{H}_{\mathsf{k}}\mathcal{D}_{\mathfrak{a}_1+}^{\vartheta_1,\vartheta_2;\varphi}\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\varrho).$$

In view of Lemmas 2.9 and 2.11, we find Eq (3.1). Now, taking $\varrho = \mathfrak{a}_2$ in Eq (3.5), we have

$$\psi(\mathfrak{a}_2)\mathsf{y}(\mathfrak{a}_2) = \frac{-\displaystyle\sum_{J=1}^{\tilde{n}+1}\frac{\alpha_J}{\psi(\epsilon_J)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_J)}{\displaystyle\sum_{J=1}^{\tilde{n}+1}\frac{\alpha_J\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)}{\psi(\epsilon_J)\Psi_{\vartheta_3}^{\varphi}(\epsilon_J,\mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2). \qquad (3.8)$$

Substituting $\varrho = \epsilon_\jmath$ into (3.5), we get

$$\psi(\epsilon_\jmath)\mathsf{y}(\epsilon_\jmath) = \frac{-\sum_{\jmath=1}^{\tilde{n}+1} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath).$$

Then, we have

$$\sum_{\jmath=1}^{\tilde{n}} \alpha_\jmath y(\epsilon_\jmath) = \frac{-\sum_{\jmath=1}^{\tilde{n}+1} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} \sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)} + \sum_{\jmath=1}^{\tilde{n}} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath),$$

and thus,

$$\begin{aligned}
\sum_{\jmath=1}^{\tilde{n}} \alpha_\jmath y(\epsilon_\jmath) &= \frac{\frac{\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)}{\psi(\mathfrak{a}_2)} - \sum_{\jmath=1}^{\tilde{n}} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{-\frac{1}{\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}+1} + \sum_{\jmath=1}^{\tilde{n}} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath) \\
&= \frac{\frac{\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)}{\psi(\mathfrak{a}_2)} - \frac{\sum_{\jmath=1}^{\tilde{n}} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}}{-\frac{1}{\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}+1} \\
&= \left(\frac{\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)}{\psi(\mathfrak{a}_2)} - \frac{\sum_{\jmath=1}^{\tilde{n}} \frac{\alpha_\jmath}{\psi(\epsilon_\jmath)} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}\right) \\
&\quad\times \left(\frac{\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}\right)
\end{aligned}$$

$$= \frac{\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}-\sum_{\jmath=1}^{\tilde{n}}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}.$$

Then,

$$\sum_{\jmath=1}^{\tilde{n}}\alpha_\jmath y(\epsilon_\jmath) = \frac{-\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath\psi(\mathfrak{a}_2)\Psi_{\vartheta_3}^{\varphi}(\mathfrak{a}_2,\mathfrak{a}_1)}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \frac{\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\mathfrak{a}_2)}{\psi(\mathfrak{a}_2)}. \tag{3.9}$$

From (3.8) and (3.9), we find that

$$\mathsf{y}(\mathfrak{a}_2) = \sum_{\jmath=1}^{\tilde{n}}\alpha_\jmath y(\epsilon_\jmath),$$

which implies that argument (3.2) holds. □

In sequel, we present the following finding as a consequence of Theorem 3.1.

Lemma 3.2. *Let*

$$\vartheta_3 = \frac{\vartheta_2(\mathsf{k}-\vartheta_1)+\vartheta_1}{\mathsf{k}},$$

such that $0<\vartheta_1<\mathsf{k}$ *and* $0\le\vartheta_2\le 1$*, and suppose that* $\chi(\cdot)\in C$*,* $\tilde{\chi}(\cdot)\in\tilde{C}$*, and*

$$\mathfrak{g}:\mathbb{T}\times C\left(\left[-\mathsf{d},\tilde{\mathsf{d}}\right],\mathbb{R}\right)\times\mathbb{R}\to\mathbb{R}$$

is a continuous function. Then, $\mathsf{y}\in\mathbb{F}$ *is a solution of problems* (1.1)–(1.4) *iff* y *is a fixed point of the mapping* $\mathbb{k}$*:* $\mathbb{F}\to\mathbb{F}$ *defined by*

$$(\mathbb{k}\mathsf{y})(\varrho) = \begin{cases} \frac{1}{\psi(\varrho)}\left[\frac{-\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)}{\psi(\epsilon_\jmath)\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\varrho)\right], & \varrho\in(\mathfrak{a}_1,\mathfrak{a}_2], \\ \chi(\varrho), \quad \varrho\in[\mathfrak{a}_1-\mathsf{d},\mathfrak{a}_1], \\ \tilde{\chi}(\varrho), \quad \varrho\in\left[\mathfrak{a}_2,\mathfrak{a}_2+\tilde{\mathsf{d}}\right], \end{cases} \tag{3.10}$$

where δ *is a function verifying*

$$\delta(\varrho) = \mathfrak{g}(\varrho,\mathsf{y}_\varrho(\cdot),\delta(\varrho))$$

and

$$\alpha_{\tilde{n}+1} = -1, \quad \epsilon_{\tilde{n}+1} = \mathfrak{a}_2.$$

Next, we present the following hypotheses for using in the sequel analysis:

(Ax1)

$$\mathfrak{g} : \mathbb{T} \times C\left(\left[-\mathsf{d}, \tilde{\mathsf{d}}\right], \mathbb{R}\right) \times \mathbb{R} \to \mathbb{R}$$

is a continuous function.

(Ax2) There exist real numbers $\zeta_1 > 0$ and $0 < \zeta_2 < 1$, where

$$|\mathfrak{g}(\varrho, \mathsf{y}_1, \widehat{\mathsf{y}}_1) - \mathfrak{g}(\varrho, \mathsf{y}_2, \widehat{\mathsf{y}}_2)| \leq \zeta_1 \|\mathsf{y}_1 - \mathsf{y}_2\|_{[-\mathsf{d},\tilde{\mathsf{d}}]} + \zeta_2 |\widehat{\mathsf{y}}_1 - \widehat{\mathsf{y}}_2|$$

for any

$$\mathsf{y}_1, \mathsf{y}_2 \in C\left(\left[-\mathsf{d}, \tilde{\mathsf{d}}\right], \mathbb{R}\right), \quad \widehat{\mathsf{y}}_1, \widehat{\mathsf{y}}_2 \in \mathbb{R},$$

and $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$.

(Ax3) There exist functions $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3 \in C(\mathbb{T}, \mathbb{R}_+)$ with

$$\mathfrak{m}_1^* = \sup_{\varrho \in \mathbb{T}} \mathfrak{m}_1(\varrho), \;\; \mathfrak{m}_2^* = \sup_{\varrho \in \mathbb{T}} \mathfrak{m}_2(\varrho), \;\; \mathfrak{m}_3^* = \sup_{\varrho \in \mathbb{T}} \mathfrak{m}_3(\varrho) < 1,$$

such that

$$|\mathfrak{g}(\varrho, \mathsf{y}, \widehat{\mathsf{y}})| \leq \mathfrak{m}_1(\varrho) + \mathfrak{m}_2(\varrho)\|\mathsf{y}\|_{[-\mathsf{d},\tilde{\mathsf{d}}]} + \mathfrak{m}_3(\varrho)|\widehat{\mathsf{y}}|$$

for any

$$\mathsf{y} \in C\left(\left[-\mathsf{d}, \tilde{\mathsf{d}}\right], \mathbb{R}\right), \quad \widehat{\mathsf{y}} \in \mathbb{R}$$

and $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$.

(Ax4) The function ψ is continuous on $\mathbb{T}$ and there exists $\mathfrak{G} > 0$ such that

$$|\psi(\varrho)| \geq \mathfrak{G}.$$

Now, we will study the uniqueness theorem for problems (1.1)–(1.4) by utilizing the Banach fixed point technique [17].

Theorem 3.3. *Suppose that* (Ax1)*,* (Ax2)*, and* (Ax4) *are satisfied. If*

$$\mathcal{L} = \frac{2\zeta_1 \left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})(1 - \zeta_2)} < 1, \tag{3.11}$$

then, problems (1.1)–(1.4) *have a unique solution in* $\mathbb{F}$*.*

Proof. In order to prove that the mapping $\Bbbk$ given in (3.10) possesses one fixed point in $\mathbb{F}$. Let us take $\mathsf{y}, \widehat{\mathsf{y}} \in \mathbb{F}$, thus for any

$$\varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1] \cup \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right],$$

we have

$$|\Bbbk\mathsf{y}(\varrho) - \Bbbk\widehat{\mathsf{y}}(\varrho)| = 0.$$

Thus

$$\|\Bbbk\mathsf{y} - \Bbbk\widehat{\mathsf{y}}\|_C = \|\Bbbk\mathsf{y} - \Bbbk\widehat{\mathsf{y}}\|_{\tilde{C}} = 0. \tag{3.12}$$

Further, for $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, we have

$$|\mathbb{k}\mathsf{y}(\varrho) - \mathbb{k}\widehat{\mathsf{y}}(\varrho)| \leq \frac{1}{|\psi(\varrho)|}\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}|\delta_1(s)-\delta_2(s)|\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)}{|\psi(\epsilon_\jmath)|\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}|\delta_1(s)-\delta_2(s)|\right)(\varrho)\right],$$

where δ_1 and δ_2 are functions satisfying the functional equations

$$\delta_1(\varrho) = \mathfrak{g}(\varrho, \mathsf{y}_\varrho(\cdot), \delta_1(\varrho)),$$
$$\delta_2(\varrho) = \mathfrak{g}(\varrho, \widehat{\mathsf{y}}_\varrho(\cdot), \delta_2(\varrho)).$$

By (Ax2), we have

$$\begin{aligned}|\delta_1(\varrho) - \delta_2(\varrho)| &= |\mathfrak{g}(\varrho, \mathsf{y}_\varrho, \delta_1(\varrho)) - \mathfrak{g}(\varrho, \widehat{\mathsf{y}}_\varrho, \delta_2(\varrho))| \\ &\leq \zeta_1\|\mathsf{y}_\varrho - \widehat{\mathsf{y}}_\varrho\|_{[-\mathsf{d},\tilde{\mathsf{d}}]} + \zeta_2|\delta_1(\varrho)-\delta_2(\varrho)|.\end{aligned}$$

Then,

$$|\delta_1(\varrho) - \delta_2(\varrho)| \leq \frac{\zeta_1}{1-\zeta_2}\|\mathsf{y}_\varrho - \widehat{\mathsf{y}}_\varrho\|_{[-\mathsf{d},\tilde{\mathsf{d}}]}.$$

Therefore, for each $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$,

$$\begin{aligned}|\mathbb{k}\mathsf{y}(\varrho) - \mathbb{k}\widehat{\mathsf{y}}(\varrho)| &\leq \frac{\zeta_1\sum_{\jmath=1}^{\tilde{n}+1}|\alpha_\jmath|\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\left\|\mathsf{y}_s - \widehat{\mathsf{y}}_s\right\|_{[-\mathsf{d},\tilde{\mathsf{d}}]}\right)(\epsilon_\jmath)}{\mathfrak{G}(1-\zeta_2)\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \frac{\zeta_1}{\mathfrak{G}(1-\zeta_2)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\left\|\mathsf{y}_s - \widehat{\mathsf{y}}_s\right\|_{[-\mathsf{d},\tilde{\mathsf{d}}]}\right)(\varrho) \\ &\leq \frac{\zeta_1\|\mathsf{y}-\widehat{\mathsf{y}}\|_{\mathbb{F}}}{\mathfrak{G}(1-\zeta_2)}\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}|\alpha_\jmath|\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}(1)\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}(1)\right)(\varrho)\right].\end{aligned}$$

By Lemma 2.6, we have

$$|\mathbb{k}\mathsf{y}(\varrho) - \mathbb{k}\widehat{\mathsf{y}}(\varrho)| \leq \frac{\zeta_1\|\mathsf{y}-\widehat{\mathsf{y}}\|_{\mathbb{F}}}{\mathfrak{G}(1-\zeta_2)}\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}|\alpha_\jmath|\left(\varphi(\epsilon_\jmath)-\varphi(\mathfrak{a}_1)\right)^{\frac{\vartheta_1}{\mathsf{k}}}}{\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \frac{(\varphi(\varrho)-\varphi(\mathfrak{a}_1))^{\frac{\vartheta_1}{\mathsf{k}}}}{\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})}\right].$$

Hence,

$$\left|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)\left(\mathbb{k}\mathsf{y}(\varrho) - \mathbb{k}\widehat{\mathsf{y}}(\varrho)\right)\right| \leq \frac{\zeta_1\|\mathsf{y}-\widehat{\mathsf{y}}\|_{\mathbb{F}}}{\mathfrak{G}(1-\zeta_2)}\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}|\alpha_\jmath|\left(\varphi(\epsilon_\jmath)-\varphi(\mathfrak{a}_1)\right)^{\frac{\vartheta_1}{\mathsf{k}}}}{\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}} + \frac{(\varphi(\varrho)-\varphi(\mathfrak{a}_1))^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})}\right],$$

which implies that

$$\|\Bbbk \mathsf{y} - \Bbbk \widehat{\mathsf{y}}\|_{C_{\vartheta_3,\mathsf{k};\varphi}} \le \frac{2\zeta_1 \left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})(1 - \zeta_2)} \|\mathsf{y} - \widehat{\mathsf{y}}\|_{\mathbb{F}}.$$

Thus,

$$\|\Bbbk \mathsf{y} - \Bbbk \widehat{\mathsf{y}}\|_{C_{\vartheta_3,\mathsf{k};\varphi}} \le \mathcal{L}\|\mathsf{y} - \widehat{\mathsf{y}}\|_{\mathbb{F}}. \tag{3.13}$$

By (3.12) and (3.13), we obtain

$$\|\Bbbk \mathsf{y} - \Bbbk \widehat{\mathsf{y}}\|_{\mathbb{F}} \le \mathcal{L}\|\mathsf{y} - \widehat{\mathsf{y}}\|_{\mathbb{F}}.$$

Based on (3.11), the mapping $\Bbbk$ is a contraction on $\mathbb{F}$. Therefore, by the Banach fixed point technique, $\Bbbk$ owns one fixed point $\mathsf{y} \in \mathbb{F}$, which is a unique solution for problems (1.1)–(1.4). □

Our subsequent existence theorem for problems (1.1)–(1.4) will be proved by the Schauder fixed point technique [17].

Theorem 3.4. *Suppose that* $(Ax1)$*,* $(Ax3)$*, and* $(Ax4)$ *are verified. If*

$$\ell = \frac{2\mathfrak{m}_2^* \left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}(1 - \mathfrak{m}_3^*)\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})} < 1, \tag{3.14}$$

then, problems (1.1)–(1.4) *have at least one solution in* $\mathbb{F}$.

Proof. We will split the proof into several steps.
Step 1. The mapping $\Bbbk$ is continuous.

Consider $\{\mathsf{y}_n\}$ to be a convergent sequence to y in $\mathbb{F}$. For each

$$\varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1] \cup \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right],$$

we have

$$|\Bbbk \mathsf{y}_n(\varrho) - \Bbbk \mathsf{y}(\varrho)| = 0.$$

For $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, we have

$$|\Bbbk \mathsf{y}(\varrho) - \Bbbk \widehat{\mathsf{y}}(\varrho)| \le \frac{1}{|\psi(\varrho)|} \left[\frac{\sum_{\jmath=1}^{\tilde{n}+1} \frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|} \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi} |\delta_n(s) - \delta(s)|\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1} \frac{|\alpha_\jmath| \Psi_{\vartheta_3}^{\varphi}(\varrho, \mathfrak{a}_1)}{|\psi(\epsilon_\jmath)| \Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath, \mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi} |\delta_n(s) - \delta(s)|\right)(\varrho) \right],$$

where δ and δ_n are functions satisfying the functional equations

$$\delta(\varrho) = \mathfrak{g}(\varrho, \mathsf{y}_\varrho(\cdot), \delta(\varrho)),$$
$$\delta_n(\varrho) = \mathfrak{g}(\varrho, \mathsf{y}_{n\varrho}(\cdot), \delta_n(\varrho)).$$

Since $\mathsf{y}_n \to \mathsf{y}$, then we get $\delta_n(\varrho) \to \delta(\varrho)$ as $n \to \infty$ for each $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, and since $\mathfrak{g}$ is continuous, then we have

$$\|\Bbbk \mathsf{y}_n - \Bbbk \mathsf{y}\|_{\mathbb{F}} \to 0 \text{ as } n \to \infty.$$

Step 2. We show $\Bbbk(B_M) \subset B_M$.

Consider M to be a positive real number, where

$$M \geq \max\left\{\frac{\mathfrak{m}_1^* \ell}{\mathfrak{m}_2^*(1-\ell)}, \|\chi\|_C, \|\tilde{\chi}\|_{\tilde{C}}\right\}.$$

Now, we present the next closed bounded ball

$$B_M = \{\mathsf{y} \in \mathbb{F} : \|\mathsf{y}\|_{\mathbb{F}} \leq M\}.$$

Then, for each $\varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1]$, we have

$$|\Bbbk \mathsf{y}(\varrho)| \leq \|\chi\|_C,$$

and for each $\varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$, we have

$$|\Bbbk \mathsf{y}(\varrho)| \leq \|\tilde{\chi}\|_{\tilde{C}}.$$

Further, for each $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, (3.10) implies that

$$|\Bbbk \mathsf{y}(\varrho)| \leq \frac{1}{|\psi(\varrho)|}\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1} \frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi} |\mathfrak{g}(s, \mathsf{y}_s, \delta(s))|\right)(\epsilon_\jmath)}{\sum_{\jmath=1}^{\tilde{n}+1} \frac{|\alpha_\jmath| \Psi_{\vartheta_3}^{\varphi}(\varrho, \mathfrak{a}_1)}{|\psi(\epsilon_\jmath)| \Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath, \mathfrak{a}_1)}} + \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi} |\mathfrak{g}(s, \mathsf{y}_s, \delta(s))|\right)(\varrho)\right]. \tag{3.15}$$

By hypothesis (Ax3), for $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, we have

$$\begin{aligned} |\delta(\varrho)| &= |\mathfrak{g}(\varrho, \mathsf{y}_\varrho, \delta(\varrho))| \\ &\leq \mathfrak{m}_1(\varrho) + \mathfrak{m}_2(\varrho)\|\mathsf{y}_\varrho\|_{[-\mathsf{d}, \tilde{\mathsf{d}}]} + \mathfrak{m}_3(\varrho)|\delta(\varrho)|, \end{aligned}$$

which implies that

$$|\delta(\varrho)| \leq \mathfrak{m}_1^* + \mathfrak{m}_2^* M + \mathfrak{m}_3^* |\delta(\varrho)|,$$

then

$$|\delta(\varrho)| \leq \frac{\mathfrak{m}_1^* + \mathfrak{m}_2^* M}{1 - \mathfrak{m}_3^*} := \Delta.$$

Thus for $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, from (3.15) we get

$$|\Psi_{\vartheta_3}^{\varphi}(\varrho, \mathfrak{a}_1) \Bbbk \mathsf{y}(\varrho)| \leq \frac{\Delta \sum_{\jmath=1}^{\tilde{n}+1} |\alpha_\jmath| \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}(1)\right)(\epsilon_\jmath)}{\mathfrak{G} \sum_{\jmath=1}^{\tilde{n}+1} \frac{|\alpha_\jmath|}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath, \mathfrak{a}_1)}} + \frac{\Delta}{\mathfrak{G}} \Psi_{\vartheta_3}^{\varphi}(\varrho, \mathfrak{a}_1) \left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}(1)\right)(\varrho).$$

By Lemma 2.6, we have

$$|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)\Bbbk\mathsf{y}(\varrho)| \le \frac{\Delta}{\mathfrak{G}}\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}|\alpha_\jmath|\left(\varphi(\epsilon_\jmath)-\varphi(\mathfrak{a}_1)\right)^{\frac{\vartheta_1}{\mathsf{k}}}}{\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}+\frac{(\varphi(\varrho)-\varphi(\mathfrak{a}_1))^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})}\right].$$

Thus

$$\begin{aligned}|\Psi_{\vartheta_3}^{\varphi}(\varrho,\mathfrak{a}_1)\Bbbk\mathsf{y}(\varrho)| &\le \frac{2\Delta\left(\varphi(\mathfrak{a}_2)-\varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})}\\ &\le M.\end{aligned}$$

Then, for each

$$\varrho\in\left[\mathfrak{a}_1-\mathsf{d},\mathfrak{a}_2+\tilde{\mathsf{d}}\right],$$

we obtain

$$\|\Bbbk\mathsf{y}\|_{\mathbb{F}}\le M.$$

Step 3. We prove that the set $\Bbbk(B_M)$ is relatively compact.

Let

$$k_1,k_2\in(\mathfrak{a}_1,\mathfrak{a}_2],\quad k_1<k_2$$

and let $\mathsf{y}\in B_M$. Then,

$$\begin{aligned}\left|\Psi_{\vartheta_3}^{\varphi}(k_1,\mathfrak{a}_1)\Bbbk\mathsf{y}(k_1)-\Psi_{\vartheta_3}^{\varphi}(k_2,\mathfrak{a}_1)\Bbbk\mathsf{y}(k_2)\right| &\le \left|\frac{1}{\psi(k_1)}-\frac{1}{\psi(k_2)}\right|\times\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|}\left|\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)\right|}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}\right]\\ &\quad+\left|\frac{\Psi_{\vartheta_3}^{\varphi}(k_1,\mathfrak{a}_1)}{\psi(k_1)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta(s)\right)(k_1)-\frac{\Psi_{\vartheta_3}^{\varphi}(k_2,\mathfrak{a}_1)}{\psi(k_2)}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta(s)\right)(k_2)\right|\\ &\le \left|\frac{1}{\psi(k_1)}-\frac{1}{\psi(k_2)}\right|\times\left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|}\left|\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\epsilon_\jmath)\right|}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|\Psi_{\vartheta_3}^{\varphi}(\epsilon_\jmath,\mathfrak{a}_1)}}\right]\\ &\quad+\int_{\mathfrak{a}_1}^{k_1}\left|\frac{\Psi_{\vartheta_3}^{\varphi}(k_1,\mathfrak{a}_1)\bar{\Psi}_{\vartheta_1}^{\mathsf{k},\varphi}(k_1,s)}{\psi(k_1)}-\frac{\Psi_{\vartheta_3}^{\varphi}(k_2,\mathfrak{a}_1)\bar{\Psi}_{\vartheta_1}^{\mathsf{k},\varphi}(k_2,s)}{\psi(k_2)}\right||\varphi'(s)\mathsf{y}(s)|ds\\ &\quad+\left|\frac{\Psi_{\vartheta_3}^{\varphi}(k_2,\mathfrak{a}_1)}{\psi(k_2)}\left(\mathcal{J}_{k_1^+}^{\vartheta_1,\mathsf{k};\varphi}|\mathsf{y}(s)|\right)(k_2)\right|.\end{aligned}$$

By Lemma 2.6, we get

$$\left|\Psi^{\varphi}_{\vartheta_3}(k_1,\mathfrak{a}_1)\Bbbk \mathsf{y}(k_1)-\Psi^{\varphi}_{\vartheta_3}(k_2,\mathfrak{a}_1)\Bbbk \mathsf{y}(k_2)\right| \leq \left|\frac{1}{\psi(k_1)}-\frac{1}{\psi(k_2)}\right| \times \left[\frac{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|}\left|\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta\right)(\epsilon_\jmath)\right|}{\sum_{\jmath=1}^{\tilde{n}+1}\frac{|\alpha_\jmath|}{|\psi(\epsilon_\jmath)|\Psi^{\varphi}_{\vartheta_3}(\epsilon_\jmath,\mathfrak{a}_1)}}\right]$$
$$+\Delta\int_{\mathfrak{a}_1}^{k_1}\left|\frac{\Psi^{\varphi}_{\vartheta_3}(k_1,\mathfrak{a}_1)\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(k_1,s)}{\psi(k_1)}-\frac{\Psi^{\varphi}_{\vartheta_3}(k_2,\mathfrak{a}_1)\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(k_2,s)}{\psi(k_2)}\right||\varphi'(s)|ds$$
$$+\frac{\Delta\Psi^{\varphi}_{\vartheta_3}(k_2,\mathfrak{a}_1)\left(\varphi(k_2)-\varphi(k_1)\right)^{\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})}.$$

As $k_1 \to k_2$, the right side of the above inequality tends to zero. From Steps 1–3, using the Arzela-Ascoli theorem, we infer that $\Bbbk$: $\mathbb{F} \to \mathbb{F}$ is a continuous and compact mapping. Consequently, we deduce that $\Bbbk$ owns at least one fixed point, which is a solution for problems (1.1)–(1.4). □

Our third outcome depends on the Krasnoselskii fixed point technique [17].

Theorem 3.5. *Suppose that* $(Ax1)$–$(Ax4)$ *are verified. If*

$$\frac{\zeta_1\left(\varphi(\mathfrak{a}_2)-\varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})(1-\zeta_2)}<1, \tag{3.16}$$

then, problems (1.1)–(1.4) *have a solution in* $\mathbb{F}$.

Proof. Let us assume that the ball

$$B_\omega=\{\mathsf{y}\in\mathbb{F}:\|\mathsf{y}\|_{\mathbb{F}}\leq\omega\},\quad \omega\geq r_1+r_2$$

with

$$r_1:=\frac{(\mathfrak{m}_1^*+\mathfrak{m}_2^*\omega)\left(\varphi(\mathfrak{a}_2)-\varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}(1-\mathfrak{m}_3^*)\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})},$$
$$r_2:=\max\left\{\|\chi\|_{C},\|\tilde{\chi}\|_{\tilde{C}},\frac{(\mathfrak{m}_1^*+\mathfrak{m}_2^*\omega)\left(\varphi(\mathfrak{a}_2)-\varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}(1-\mathfrak{m}_3^*)\Gamma_{\mathsf{k}}(\vartheta_1+\mathsf{k})}\right\}.$$

Next, we introduce the mappings ∇_1 and ∇_2 on B_ω as follows:

$$\nabla_1\mathsf{y}(\varrho)=\begin{cases}\dfrac{-\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath}{\psi(\epsilon_\jmath)}\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta\right)(\epsilon_\jmath)}{\psi(\varrho)\sum_{\jmath=1}^{\tilde{n}+1}\frac{\alpha_\jmath\Psi^{\varphi}_{\vartheta_3}(\varrho,\mathfrak{a}_1)}{\psi(\epsilon_\jmath)\Psi^{\varphi}_{\vartheta_3}(\epsilon_\jmath,\mathfrak{a}_1)}}, & \varrho\in(\mathfrak{a}_1,\mathfrak{a}_2],\\ 0,\ \varrho\in[\mathfrak{a}_1-\mathsf{d},\mathfrak{a}_1],\\ 0,\ \varrho\in\left[\mathfrak{a}_2,\mathfrak{a}_2+\tilde{\mathsf{d}}\right],\end{cases} \tag{3.17}$$

and

$$\nabla_2 \mathsf{y}(\varrho) = \begin{cases} \dfrac{\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}\delta\right)(\varrho)}{\psi(\varrho)}, & \varrho \in (\mathfrak{a}_1, \mathfrak{a}_2], \\ \chi(\varrho), \quad \varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1], \\ \tilde{\chi}(\varrho), \quad \varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right], \end{cases} \tag{3.18}$$

where δ is a function verifying

$$\delta(\varrho) = \mathfrak{g}(\varrho, \mathsf{y}_\varrho(\cdot), \delta(\varrho)).$$

Then (3.10) can be written as

$$\Bbbk\mathsf{y}(\varrho) = \nabla_1 \mathsf{y}(\varrho) + \nabla_2 y(\varrho), \quad \mathsf{y} \in \mathbb{F}.$$

Step 1. We prove that

$$\nabla_1 \mathsf{y} + \nabla_2 \widehat{\mathsf{y}} \in B_\omega$$

for any $\mathsf{y}, \widehat{\mathsf{y}} \in B_\omega$.

By (Ax3) and from (3.10), for $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, we have

$$\begin{aligned} |\delta(\varrho)| &= |\mathfrak{g}(\varrho, \mathsf{y}_\varrho, \delta(\varrho))| \\ &\leq \mathfrak{m}_1(\varrho) + \mathfrak{m}_2(\varrho)\|\mathsf{y}_\varrho\|_{[-\mathsf{d},\tilde{\mathsf{d}}]} + \mathfrak{m}_3(\varrho)|\delta(\varrho)|, \end{aligned}$$

which implies that

$$|\delta(\varrho)| \leq \mathfrak{m}_1^* + \mathfrak{m}_2^*\omega + \mathfrak{m}_3^*|\delta(\varrho)|,$$

and then

$$|\delta(\varrho)| \leq \frac{\mathfrak{m}_1^* + \mathfrak{m}_2^*\omega}{1 - \mathfrak{m}_3^*} := \mathcal{A}.$$

Thus, for $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$ and by (3.17), we have

$$|\Psi_{\vartheta_3}^{\varphi}(\varrho, \mathfrak{a}_1)\nabla_1 \mathsf{y}(\varrho)| \leq \frac{\mathcal{A}\left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})}.$$

Then, for each $\varrho \in \left[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$, we obtain

$$\|\nabla_1 \mathsf{y}\|_{\mathbb{F}} \leq \frac{\mathcal{A}\left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})}. \tag{3.19}$$

For $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$ and by (3.18), we have

$$|\Psi_{\vartheta_3}^{\varphi}(\varrho, \mathfrak{a}_1)\nabla_2 \widehat{\mathsf{y}}(\varrho)| \leq \frac{\mathcal{A}\left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})}.$$

For each $\varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1]$, we have

$$|\nabla_2\widehat{\mathsf{y}}(\varrho)| \leq \|\chi\|_C,$$

and for each $\varrho \in \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$, we have

$$|\nabla_2\widehat{\mathsf{y}}(\varrho)| \leq \|\tilde{\chi}\|_{\tilde{C}}.$$

Then, for each $\varrho \in \left[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$, we get

$$\|\nabla_2\widehat{\mathsf{y}}\|_{\mathbb{F}} \leq \max\left\{\|\chi\|_C, \|\tilde{\chi}\|_{\tilde{C}}, \frac{\mathcal{A}\left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})}\right\}. \tag{3.20}$$

From (3.19) and (3.20), for each $\varrho \in \left[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$, we have

$$\begin{aligned}\|\nabla_1\mathsf{y} + \nabla_2\widehat{\mathsf{y}}\|_{\mathbb{F}} &\leq \|\nabla_1\mathsf{y}\|_{\mathbb{F}} + \|\nabla_2\widehat{\mathsf{y}}\|_{\mathbb{F}} \\ &\leq r_1 + r_2 \\ &\leq \omega,\end{aligned}$$

which infers that

$$\nabla_1\mathsf{y} + \nabla_2\widehat{\mathsf{y}} \in B_\omega.$$

Step 2. The mapping ∇_1 is a contraction.

In view of the condition (3.16) and Theorem 3.3, the mapping ∇_1 is a contraction on $\mathbb{F}$ with the norm $\|\cdot\|_{\mathbb{F}}$.

Step 3. ∇_2 is continuous and compact.

Let $\{\mathsf{y}_n\}$ be a sequence such that $\mathsf{y}_n \longrightarrow \mathsf{y}$ in $\mathbb{F}$. For each

$$\varrho \in [\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_1] \cup \left[\mathfrak{a}_2, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right],$$

we have

$$|\nabla_2\mathsf{y}_n(\varrho) - \nabla_2\mathsf{y}(\varrho)| = 0.$$

For $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, we have

$$|\nabla_2\mathsf{y}_n(\varrho) - \nabla_2\mathsf{y}(\varrho)| \leq \frac{1}{|\psi(\varrho)|}\left(\mathcal{J}_{\mathfrak{a}_1+}^{\vartheta_1,\mathsf{k};\varphi}|\delta_n(s) - \delta(s)|\right)(\varrho),$$

such that δ and δ_n are functions verifying the functional equations

$$\begin{aligned}\delta(\varrho) &= \mathfrak{g}(\varrho, \mathsf{y}_\varrho(\cdot), \delta(\varrho)), \\ \delta_n(\varrho) &= \mathfrak{g}(\varrho, \mathsf{y}_{n\varrho}(\cdot), \delta_n(\varrho)).\end{aligned}$$

Since $\mathsf{y}_n \to \mathsf{y}$, then we get $\delta_n(\varrho) \to \delta(\varrho)$ as $n \to \infty$ for each $\varrho \in (\mathfrak{a}_1, \mathfrak{a}_2]$, and since $\mathfrak{g}$ is continuous, then we have

$$\|\nabla_2\mathsf{y}_n - \nabla_2\mathsf{y}\|_{\mathbb{F}} \to 0 \text{ as } n \to \infty.$$

Then ∇_2 is continuous. Next we prove that ∇_2 is uniformly bounded on B_ω. For each

$$\varrho \in \left[\mathfrak{a}_1 - \mathsf{d}, \mathfrak{a}_2 + \tilde{\mathsf{d}}\right]$$

and any $\widehat{\mathsf{y}} \in B_\omega$, we get

$$\|\nabla_2 \widehat{\mathsf{y}}\|_{\mathbb{F}} \leq \max\left\{\|\chi\|_C, \|\tilde{\chi}\|_{\tilde{C}}, \frac{\mathcal{A}\left(\varphi(\mathfrak{a}_2) - \varphi(\mathfrak{a}_1)\right)^{1-\vartheta_3+\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})}\right\}.$$

This implies that the mapping ∇_2 is uniformly bounded on B_ω. In order to show the compactness of ∇_2, we take $k_1, k_2 \in (\mathfrak{a}_1, \mathfrak{a}_2]$ such that $k_1 < k_2$, and $\mathsf{y} \in B_\omega$. Then

$$\begin{aligned}
\left|\Psi^{\varphi}_{\vartheta_3}(k_1, \mathfrak{a}_1)\nabla_2 \mathsf{y}(k_1) - \Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)\nabla_2 \mathsf{y}(k_2)\right| &\leq \left|\frac{\Psi^{\varphi}_{\vartheta_3}(k_1, \mathfrak{a}_1)}{\psi(k_1)}\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta(s)\right)(k_1) - \frac{\Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)}{\psi(k_2)}\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{\mathfrak{a}_1+}\delta(s)\right)(k_2)\right| \\
&\leq \int_{\mathfrak{a}_1}^{k_1}\left|\frac{\Psi^{\varphi}_{\vartheta_3}(k_1, \mathfrak{a}_1)\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(k_1, s)}{\psi(k_1)} - \frac{\Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(k_2, s)}{\psi(k_2)}\right| |\varphi'(s)\mathsf{y}(s)| ds \\
&\quad + \left|\frac{\Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)}{\psi(k_2)}\left(\mathcal{J}^{\vartheta_1,\mathsf{k};\varphi}_{k_1^+}|\mathsf{y}(s)|\right)(k_2)\right|.
\end{aligned}$$

By Lemma 2.6, we get

$$\begin{aligned}
\left|\Psi^{\varphi}_{\vartheta_3}(k_1, \mathfrak{a}_1)\nabla_2 \mathsf{y}(k_1) - \Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)\nabla_2 \mathsf{y}(k_2)\right| &\leq \mathcal{A}\int_{\mathfrak{a}_1}^{k_1}\left|\frac{\Psi^{\varphi}_{\vartheta_3}(k_1, \mathfrak{a}_1)\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(k_1, s)}{\psi(k_1)} - \frac{\Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)\bar{\Psi}^{\mathsf{k},\varphi}_{\vartheta_1}(k_2, s)}{\psi(k_2)}\right| |\varphi'(s)| ds \\
&\quad + \frac{\mathcal{A}\Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)\left(\varphi(k_2) - \varphi(k_1)\right)^{\frac{\vartheta_1}{\mathsf{k}}}}{\mathfrak{G}\Gamma_{\mathsf{k}}(\vartheta_1 + \mathsf{k})}.
\end{aligned}$$

Note that

$$\left|\Psi^{\varphi}_{\vartheta_3}(k_1, \mathfrak{a}_1)\nabla_2 \mathsf{y}(k_1) - \Psi^{\varphi}_{\vartheta_3}(k_2, \mathfrak{a}_1)\nabla_2 \mathsf{y}(k_2)\right| \to 0 \quad \text{as} \quad k_1 \to k_2.$$

This proves that $\nabla_2 B_\omega$ is equicontinuous on $(\mathfrak{a}_1, \mathfrak{a}_2]$. Therefore, ∇_2 is compact. Thus, based on the Krasnoselskii fixed point technique, we conclude that $\mathbb{k}$ possesses a fixed point, which satisfies problems (1.1)–(1.4). □

4. Applications

We give various examples of (1.1)–(1.4), with

$$\begin{aligned}
\mathbb{T} = [1, \pi], \quad \vartheta_3 &= \frac{1}{\mathsf{k}}(\vartheta_2(\mathsf{k} - \vartheta_1) + \vartheta_1), \\
\mathfrak{g}(\varrho, \mathsf{y}, \widehat{\mathsf{y}}) &= \frac{1}{105 + 125e^{\pi-\varrho}}\left[1 + \frac{\widehat{\mathsf{y}}}{3 + |\widehat{\mathsf{y}}|} - \frac{\mathsf{y}}{1 + \mathsf{y}}\right], \\
\psi(\varrho) &= \frac{3}{13e^{-5}}(\varrho + \sin(\varrho) + 2),
\end{aligned}$$

where $\varrho \in \mathbb{T}$, $\mathsf{y} \in C\left(\left[-\mathsf{d}, \tilde{\mathsf{d}}\right], \mathbb{R}\right)$, and $\widehat{\mathsf{y}} \in \mathbb{R}$.

Example 4.1. *Taking* $\vartheta_2 \to \frac{1}{2}$, $\vartheta_1 = \frac{1}{2}$, $\mathsf{k} = 1$, $\varphi(\varrho) = \pi^{\varrho}$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 3$, $\epsilon_1 = \frac{5}{4}$, $\epsilon_2 = \frac{4}{3}$, $\epsilon_3 = \frac{3}{2}$, $\tilde{n} = 3$, $\mathsf{d} = \tilde{\mathsf{d}} = \frac{1}{3}$, *and* $\vartheta_3 = \frac{3}{4}$, *we have the system below:*

$$\left({}_{1}^{H}\mathcal{D}_{1+}^{\frac{1}{2},\frac{1}{2};\varphi}\psi\,\mathsf{y}\right)(\varrho) = \mathfrak{g}\left(\varrho, \mathsf{y}_{\varrho}(\cdot), \left({}_{1}^{H}\mathcal{D}_{1+}^{\frac{1}{2},\frac{1}{2};\varphi}\psi\,\mathsf{y}\right)(\varrho)\right), \quad \varrho \in (1,\pi], \tag{4.1}$$

$$\mathsf{y}(\pi) = \mathsf{y}(\tfrac{5}{4}) + 2x(\tfrac{4}{3}) + 3x(\tfrac{3}{2}), \tag{4.2}$$

$$\mathsf{y}(\varrho) = \chi(\varrho), \quad \varrho \in \left[\frac{2}{3}, 1\right], \tag{4.3}$$

$$\mathsf{y}(\varrho) = \tilde{\chi}(\varrho), \quad \varrho \in \left[\pi, \pi + \frac{1}{3}\right]. \tag{4.4}$$

We have

$$C_{\vartheta_3,\mathsf{k};\varphi}(\mathbb{T}) = C_{\frac{3}{4},1;\varphi}(\mathbb{T}) = \left\{\mathsf{y} : (1,\pi] \to \mathbb{R} : (\pi^{\varrho} - \pi)^{\frac{1}{4}}\mathsf{y} \in C(\mathbb{T},\mathbb{R})\right\},$$

and then

$$\mathbb{F} = \left\{\mathsf{y} : \left[\frac{2}{3}, \pi + \frac{1}{3}\right] \to \mathbb{R} : \mathsf{y}|_{[\frac{2}{3},1]} \in C,\ \mathsf{y}|_{[\pi,\pi+\frac{1}{3}]} \in \tilde{C} \text{ and } \mathsf{y}|_{(1,\pi]} \in C_{\frac{3}{4},1;\varphi}(\mathbb{T})\right\}.$$

By continuity of the function $\mathfrak{g}$, *the hypothesis* $(Ax1)$ *holds. For every*

$$\mathsf{y} \in C\left(\left[-\frac{1}{3}, \frac{1}{3}\right], \mathbb{R}\right),\ \widehat{\mathsf{y}} \in \mathbb{R} \quad \text{and} \quad \varrho \in \mathbb{T},$$

one has

$$|\mathfrak{g}(\varrho, \mathsf{y}, \widehat{\mathsf{y}})| \le \frac{1}{105 + 125e^{\pi-\varrho}}\left(1 + \|\mathsf{y}\|_{[-\mathsf{d},\tilde{\mathsf{d}}]} + |\widehat{\mathsf{y}}|\right).$$

Then, the condition $(Ax3)$ *is satisfied with*

$$\mathfrak{m}_1(\varrho) = \mathfrak{m}_2(\varrho) = \mathfrak{m}_3(\varrho) = \frac{1}{105 + 125e^{\pi-\varrho}}$$

and

$$\mathfrak{m}_1^* = \mathfrak{m}_2^* = \mathfrak{m}_3^* = \frac{1}{230}.$$

The condition $(Ax4)$ *is verified since we have that*

$$|\psi(\varrho)| \ge \frac{6}{13e^{-5}}.$$

We have

$$\ell = \frac{52e^{-5}\,(\pi^{\pi} - \pi)^{\frac{3}{4}}}{1374\sqrt{\pi}} \approx 0.001995278633 < 1.$$

Hence, in view of Theorem 3.4, we infer that problems (4.1)–(4.4) *possess a solution in* $\mathbb{F}$.

Example 4.2. *Considering* $\vartheta_2 \to 0$, $\vartheta_1 = \frac{1}{2}$, $\mathsf{k} = 1$, $\varphi(\varrho) = \varrho^{\rho}$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 5$, $\epsilon_1 = \frac{3}{2}$, $\epsilon_2 = 2$, $\epsilon_3 = \frac{5}{2}$, $\tilde{n} = 3$, $\mathsf{d} = \tilde{\mathsf{d}} = \frac{1}{2}$, $\rho = \frac{1}{2}$, *and* $\vartheta_3 = \frac{1}{2}$, *we have the next system:*

$$\left({}_{1}^{H}\mathcal{D}_{1+}^{\frac{1}{2},0;\varphi}\psi\,\mathsf{y}\right)(\varrho) = \left({}^{\rho}\mathbb{D}_{1^+}^{\frac{1}{2},0}\mathsf{y}\right)(\varrho) = \mathfrak{g}\left(\varrho, \mathsf{y}_{\varrho}(\cdot), \left({}^{\rho}\mathbb{D}_{1^+}^{\frac{1}{2},0}\psi\,\mathsf{y}\right)(\varrho)\right), \quad \varrho \in (1,3], \tag{4.5}$$

$$\mathsf{y}(3) = \mathsf{y}(\tfrac{3}{2}) + \mathsf{y}(2) + 5x(\tfrac{5}{2}), \tag{4.6}$$

$$\mathsf{y}(\varrho) = e^{\varrho}, \quad \varrho \in \left[\frac{1}{2}, 1\right], \tag{4.7}$$

$$\mathsf{y}(\varrho) = e^{\varrho}, \quad \varrho \in \left[3, \frac{7}{2}\right]. \tag{4.8}$$

We have

$$C_{\vartheta_3,\mathsf{k};\varphi}(\mathbb{T}) = C_{\frac{1}{2},1;\varphi}(\mathbb{T}) = \left\{\mathsf{y} : (1,3] \to \mathbb{R} : \sqrt{(\sqrt{\varrho} - 1)}\mathsf{y} \in C(\mathbb{T}, \mathbb{R})\right\},$$

and then

$$\mathbb{F} = \left\{\mathsf{y} : \left[\frac{1}{2}, \frac{7}{2}\right] \to \mathbb{R} : \mathsf{y}|_{[\frac{1}{2},1]} \in C,\ \mathsf{y}|_{[3,\frac{7}{2}]} \in \tilde{C} \ \textit{and} \ \ \mathsf{y}|_{(1,3]} \in C_{\frac{3}{4},1;\varphi}(\mathbb{T})\right\}.$$

Additionally, for every

$$\mathsf{y}_1, \widehat{\mathsf{y}_1} \in C\left(\left[-\frac{1}{2}, \frac{1}{2}\right], \mathbb{R}\right),\ \ \mathsf{y}_2, \widehat{\mathsf{y}_2} \in \mathbb{R},\ \ \textit{and}\ \ \varrho \in \mathbb{T},$$

one has

$$|\mathfrak{g}(\varrho, \mathsf{y}_1, \mathsf{y}_2) - \mathfrak{g}(\varrho, \widehat{\mathsf{y}_1}, \widehat{\mathsf{y}_2})| \le \frac{1}{105 + 125e^{\pi-\varrho}}\left(\|\mathsf{y}_1 - \widehat{\mathsf{y}_1}\|_{[-\mathsf{d},\tilde{\mathsf{d}}]} + |\mathsf{y}_2 - \widehat{\mathsf{y}_2}|\right).$$

Then, the condition $(Ax2)$ *holds with*

$$\zeta_1 = \zeta_2 = \frac{1}{230}.$$

Since

$$\mathcal{L} \approx 0.000105319912 \ < 1.$$

Hence, all of the hypotheses of Theorem 3.3 are verified. It follows that problems (4.5)–(4.8) *possess one solution in* $\mathbb{F}$.

5. Conclusions

Our research considered a class of problems involving nonlinear implicit (k, φ)-Hilfer hybrid fractional differential equations with nonlocal terminal conditions. We achieved this by proving the existence and uniqueness of solutions for these equations. Our strategy hinged on powerful mathematical tools: the Banach contraction principle, Schauder's fixed point theorem, and Krasnoselskii's fixed point techniques. To showcase the practical applications of our findings and the ease of using our theorems, we presented some illustrative examples. These illustrations effectively highlight the flexibility and wide-reaching impact of the studied operator across various cases. It is noteworthy that the introduced (k, φ)-Hilfer operator operates as an extension, encompassing previously established fractional derivatives such as the Caputo, Hadamard, and Hilfer fractional derivative already present in the existing literature. This broader conceptual framework substantially contributes to the ongoing advancement of fractional calculus, thus laying the groundwork for promising directions of future exploration within this ever-evolving and dynamic domain.

Author contributions

A. Salim: conceptualization, data curation, formal analysis, investigation, methodology, writing-original draft; S. T. M. Thabet: conceptualization, data curation, formal analysis, methodology, writing-original draft; I. Kedim: data curation, formal analysis, investigation, methodology, writing-review and editing; M. Vivas-Cortez: investigation, writing-review and editing. All authors have read and agreed to the published version of the article.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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