AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 22794–22812.
DOI: 10.3934/math.20241109
Received: 13 February 2024
Revised: 26 June 2024
Accepted: 03 July 2024
Published: 23 July 2024

Research article

Sturm-Liouville problem in multiplicative fractional calculus

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Abstract: Multiplicative calculus, or geometric calculus, is an alternative to classical calculus that relies on division and multiplication as opposed to addition and subtraction, which are the basic operations of classical calculus. It offers a geometric interpretation that is especially helpful for simulating systems that degrade or expand exponentially. Multiplicative calculus may be extended to fractional orders, much as classical calculus, which enables the analysis of systems having fractional scaling properties. So, in this paper, the well-known Sturm-Liouville problem in fractional calculus is reformulated in multiplicative fractional calculus. The considered problem consists of the Sturm-Liouville operator using multiplicative conformable derivatives on the equation and on boundary conditions. This research aimed to explore some of the problem's spectral aspects, like being self-adjointness of the operator, orthogonality of different eigenfunctions, and reality of all eigenvalues. In this specific situation, Green's function is also recreated.

Keywords: geometric calculus; multiplicative conformable fractional calculus; non-Newtonian Calculus; Sturm-Liouville equation
Mathematics Subject Classification: 11N05, 26A33, 34B24, 34L05

1. Introduction

1.1. Presentation of the problem

Consider the following multiplicative conformable fractional Sturm-Liouville equation of order $\alpha \in (0, 1]$ ($\alpha -{}^*$SL):

$$\boldsymbol{L}_\alpha [y] := \left(e^{-1} \odot \tau^2 y(x)\right) \oplus \left(e^{q(x)} \odot y(x)\right) = e^{\lambda} \odot y(x), \quad x \in [a, b], \tag{1.1}$$

with the conditions

$$U_1(y) := (e^{c_1} \odot y(a)) \oplus (e^{c_2} \odot \tau y(a)) = 1, \tag{1.2}$$

$$U_2(y) := \left(e^{d_1} \odot y(b)\right) \oplus \left(e^{d_2} \odot \tau y(b)\right) = 1, \tag{1.3}$$

where $q(x)$ is a real-valued continuous and multiplicative conformable fractional (CF) integrable function on $[a, b]$; λ is a spectral parameter; $\left(c_1^2 + c_2^2\right)\left(d_1^2 + d_2^2\right) \neq 0$, c_i, $d_i \in \mathbb{R}$ $(i = 1, 2)$. Throughout this study, $\tau\cdot$ denotes ${}^*T_\alpha \cdot = \frac{d_\alpha^* \cdot}{d_\alpha^* x}$ (the multiplicative CF derivative of order $\alpha \in (0, 1]$ with respect to x); $\tau^2\cdot$ denotes ${}^{*(2)}T_\alpha \cdot = {}^*T_\alpha {}^*T_\alpha \cdot = \frac{d_\alpha^{*2} \cdot}{d_\alpha^* x^2}$ (the second order multiplicative CF of order $\alpha \in (0, 1]$ with respect to x) for brevity.

Using the properties of multiplicative CF calculus [1], we can formally reduce this problem and arrive at the following $\alpha -{}^*$SL problem:

$$\boldsymbol{L}_\alpha [y] := (\tau^2 y)^{-1} y^{q(t)} = y^{\lambda}, \tag{1.4}$$

$$\begin{aligned} U_1(y) &:= (y(a))^{c_1} (\tau y(a))^{c_2} = 1, \\ U_2(y) &:= (y(b))^{d_1} (\tau y(b))^{d_2} = 1. \end{aligned} \tag{1.5}$$

A brief summary of the study is as follows:

The current section will examine several works that are directly linked to the current issue and provide some fundamental definitions and characteristics of the multiplicative, multiplicative CF calculus, and the theory of CF calculus. In Section 2, asymptotic estimations of the *eigenfunctions for the problems (1.1)–(1.3) will be computed. We will look into some of the problem's spectral aspects in Section 3, including the self-adjointness, the reality, the eigenvalues of the operator, the orthogonality of different eigenfunctions, etc. The Green's function of this problem will be reconstructed in Section 4.

1.2. Investigating research on the problem

Starting with the need to create and apply multiplicative (geometric) fractional calculus, let us emphasize the significance of polar coordinates in addition to Cartesian coordinates, which are previously known. Additionally, multiplicative and fractional calculus theories are combined to form multiplicative fractional calculus theories. It is therefore important to look at these two theories independently.

Let us start by talking about the theory of fractional calculus. Fractional calculus is an extension of classical calculus that is widely used in many scientific and technical domains with a wide range of applications [2–6]. Nearly every fractional derivative utilized in the literature, including the Riemann-Liouville, Caputo and Jumarie, Grünwald-Letnikov, Marchaud, and Riesz derivatives cannot satisfy some fundamental requirements. The conformable derivative, which is a new version of fractional calculus, was examined in our work because the derivative is local and behaves better in terms of the chain rule, the product rule, and the differentiation of a constant function compared to the previous Riemann-Liouville and Caputo fractional derivatives. Moreover, the definition of the conformable fractional derivative is simpler and contains no-delay, whereas other fractional derivatives are presented in terms of kernel integrals. For this reason, in this study, we favor the conformable fractional derivative. A relatively recent development in the field of fractional calculus, which means the differentiation and integration of an non-integer order for a given function, is conformable fractional calculus. Conformable fractional calculus can be applied in many domains, such as mathematical modeling, signal processing, physics, and engineering, where fractional operators are employed to characterize systems and phenomena that exhibit non-local behaviors and memory effects. Basic characteristics and primary findings on fractional derivatives can be found in [7, 8], while further findings can be found in [4, 9–14].

The multiplicative calculus theory will be discussed next. In [15, 16], Grossman and Katz initially introduced multiplicative calculus as a substitute for traditional calculus. Geometric calculus is a subfield of non-Newtonian calculus, also known by the same term. Numerous writers subsequently provided explanations of the fundamentals of multiplicative calculus, leading to the achievement of significant outcomes [17–20].

Because of the logarithmic features, this calculus modifies the roles of well-known operations like division and subtraction. For example, multiplication becomes addition instead of subtraction. In a roundabout way, it develops additive computations. Even though the application field of this calculus is rather limited (it only covers positive functions) several challenging issues from the usual calculus may be set up quite simply in this context. Certain principles in the multiplicative calculus allow for the definition of each feature in the usual calculus.

The multiplicative derivatives are used to explain many occurrences in which the logarithmic scale is present. Thus a better physical interpretation of these occurrences may be obtained by substituting multiplicative calculus for ordinary calculus. In many domains, including chaos theory [21, 22]; biology [23]; engineering [24]; demography, earthquakes [25]; economics [26, 27]; medicine [28]; business [29] and applied mathematics [30–35] this calculus produces better results than the normal case (see also [36, 37]).

Lastly, the article [1], which inspires us and provides the foundation for the multiplicative fractional calculus, is cited. Here, some of the characteristics of Riemann, Caputo, and multiplicative CF calculus is investigated.

From mathematical analysis to physics and engineering, the Sturm-Liouville (SL) operator offers a strong foundation for deriving solutions to boundary value issues and evaluating differential equations. It also helps to comprehend the behavior of linear operators. It is a fundamental idea in many branches of mathematics and science, including spectral theory, mathematical modeling, and quantum physics. The Schrödinger equation for a quantum system can frequently be expressed as a SL eigenvalue problem in quantum mechanics. The related quantum states are represented by the eigenfunctions, and the permitted energy levels of the system are indicated by the eigenvalues. A set of orthogonal functions with respect to a weight function is formed by the eigenfunctions of Sturm-Liouville operators, and any function may be expressed in terms of these orthogonal functions. In many branches of analysis and approximation theory, this feature is essential. As a natural result of these reasons, recent years have seen a significant increase in interest in SL theory as a promising area of study since it naturally arises in tackling several issues in the natural, engineering, physics, and social sciences. Considering the problems (1.1)–(1.3) form, the $\alpha-^{*}$SL problem may be generated by substituting the fractional derivative for the multiplicative CF derivative. Many writers have implemented this method in a similar way by substituting the ordinary derivative for the fractional derivative [38–42].

1.3. Preliminaries

This section includes some fundamental definitions and characteristics of the multiplicative, multiplicative CF, and CF calculus theories. These concepts will be used throughout the remainder of this study.

Let us first discuss a few of the arithmetic operations we performed throughout the course of the study. Multiplicative algebraic operations are the arithmetic operations that are performed by exponential functions. Using the arithmetic table for $q, q_1, q_2 \in \mathbb{R}^+$ below, let us indicate some characteristics of these operations.

$$q_1 \oplus q_2 = q_1 q_2, \quad q_1 \ominus q_2 = \frac{q_1}{q_2}, \quad q_1 \odot q_2 = q_1^{\ln q_2} = q_2^{\ln q_1}, \quad q^{2_G} = q \odot q = q^{\ln q}.$$

Many algebraic structures are constructed by the methods above. Given an operation $\oplus : D \times D \to D$ for $D \neq \emptyset$ and $D \subset \mathbb{R}^+$, then $(D, \oplus)$ is a * group. Comparably, in the multiplicative sense, $(D, \oplus, \odot)$ defines a ring [43].

Definition 1.1. *[7, 8] Take the function $\chi : [a, \infty) \to \mathbb{R}$ into consideration. Next, the following defines left- and right-sided CF derivatives of χ of order $\alpha \in (0, 1]$:*

$$T_\alpha^a \chi(t) := \lim_{k \to 0} \frac{\chi(t + k(t-a)^{1-\alpha}) - \chi(t)}{k},$$

$$^{b}_{\alpha}T\chi(t) := -\lim_{k \to 0} \frac{\chi(t + k(b-t)^{1-\alpha}) - \chi(t)}{k}.$$

The left-sided CF derivative is denoted by T_α when $a = 0$. It follows that $T_\alpha \chi(t) = t^{1-\alpha}\chi'(t)$ if χ is usually differentiable.

Definition 1.2. *[7, 8] Think about the function $\chi : [0, \infty) \to \mathbb{R}$. Next, the following defines left- and right-sided CF integrals of χ of order $\alpha \in (0, 1]$ for $t > 0$, respectively:*

$$I_\alpha^a \chi(t) := \int_a^t \chi(\zeta) d_\alpha(\zeta, a) = \int_a^t (\zeta - a)^{\alpha-1} \chi(\zeta) d\zeta,$$

$$^{b}I_\alpha \chi(t) := \int_t^b \chi(\zeta) d_\alpha(b, \zeta) = \int_t^b (b - \zeta)^{\alpha-1} \chi(\zeta) d\zeta.$$

Final integrals of these equations are the standard Riemann integrals. The left CF integral is expressed as I_α when $a = 0$.

Definition 1.3. *[1] The function $\chi : \mathbb{R} \to \mathbb{R}^+$ is under consideration. Next, we define the forward multiplicative derivative and the backward multiplicative derivative of $\chi(t)$, respectively, as follows:*

$$\frac{d^*}{dt^*}\chi(t) = \chi^*(t) := \lim_{h\to 0}\left(\frac{\chi(t+h)}{\chi(t)}\right)^{\frac{1}{h}},$$

$$\frac{d_*}{dt_*}\chi(t) = \chi_*(t) := \lim_{h\to 0}\left(\frac{\chi(t)}{\chi(t-h)}\right)^{\frac{1}{h}}.$$

It is simple to demonstrate that

$$\chi^{*(n)}(t) = \chi_*{}^{(n)}(t) = \exp\left(\frac{d^n}{dx^n}\ln\chi(t)\right).$$

Definition 1.4. *[1] Let $\chi : [a,b] \to \mathbb{R}^+$ be considered. Next, we define the forward and backward multiplicative integrals of $\chi(\zeta)$ as follows:*

$$\int_a^b \chi(\zeta)^{d\zeta} = \int_a^b \chi(\zeta)_{d\zeta} = \exp\left(\int_a^b \ln\chi(\zeta)d\zeta\right).$$

Definition 1.5. *[1] Let $\chi : [a,b] \to \mathbb{R}^+$. Then, the $\alpha \in (0,1]$ order multiplicative left- and right-sided CF derivatives of χ, respectively, are determined by*

$${}^*T^a_\alpha\chi(t) := \lim_{k\to 0}\left(\frac{\chi(t+k(t-a)^{1-\alpha})}{\chi(t)}\right)^{\frac{1}{k}},$$

$${}^b_\alpha T^*\chi(t) := \lim_{k\to 0}\left(\frac{\chi(t+k(b-t)^{1-\alpha})}{\chi(t)}\right)^{-\frac{1}{k}}.$$

Proposition 1.1. *[1] For $\alpha \in (0,1]$ and the function $\chi : [a,b] \to \mathbb{R}^+$,*

$$(i) \quad {}^*T^a_\alpha\chi(t) = \exp\left\{T^a_\alpha \ln\chi(t)\right\} = \exp\left\{\frac{T^a_\alpha\chi(t)}{\chi(t)}\right\}, \tag{1.6}$$

$$(ii) \quad {}^b_\alpha T^*\chi(t) = \exp\left\{{}^b_\alpha T \ln\chi(t)\right\} = \exp\left\{\frac{{}^b_\alpha T\chi(t)}{\chi(t)}\right\}$$

are satisfied.

Definition 1.6. *[1] Consider the function $\chi : [a,b] \to \mathbb{R}^+$, $\alpha \in (n, n+1]$ and $\beta = \alpha - n$. Then, the higher order multiplicative left and right CF derivatives of χ, respectively, are defined by*

$$\left({}^*\boldsymbol{T}^a_\alpha\chi\right)(t) := \left({}^*T^a_\beta\chi^{(n)}_*\right)(t) = \exp\left\{T^a_\beta \ln(\chi^{(n)}_*(t))\right\} = \exp\left\{T^a_\beta \frac{d^n}{dt^n}\ln(\chi(t))\right\},$$

$$\left({}^b_\alpha\boldsymbol{T}^*\chi\right)(t) := \left({}^b_\beta T^*\chi^{(n)}_*\right)(t) = \exp\left\{{}^b_\beta T \ln(\chi^{(n)}_*(t))\right\} = \exp\left\{{}^b_\beta T \frac{d^n}{dt^n}\ln(\chi(t))\right\}.$$

For $\alpha \in (0,1]$ and $n \in \mathbb{Z}^{+}$,

$$ {}^{*(n)}T_{\alpha}^{a}\chi(t) = \underbrace{{}^{*}T_{\alpha}^{a}\,{}^{*}T_{\alpha}^{a}\ldots{}^{*}T_{\alpha}^{a}}_{n-times}\chi(t), $$

$$ {}_{\alpha}^{b}T^{*(n)}\chi(t) = \underbrace{{}_{\alpha}^{b}T^{*}\,{}_{\alpha}^{b}T^{*}\ldots{}_{\alpha}^{b}T^{*}}_{n-times}\chi(t) $$

define the sequential multiplicative left and right CF derivatives of nth order, respectively.

Definition 1.7. *[1] Consider the function $\chi : [a,b] \to \mathbb{R}^{+}$. Then, the $\alpha \in (0,1]$ order multiplicative left and right CF integrals of χ, respectively, are defined as follows for $t > 0$:*

$$ \begin{aligned} ({}^{*}I_{\alpha}^{a}\chi)(t) :&= \int_{a}^{t} \chi(\zeta)_{d_{\alpha}^{*}(\zeta,a)} = \exp\left\{\int_{a}^{t} \ln\chi(\zeta)d_{\alpha}(\zeta,a)\right\} \\ &= \int_{a}^{t} \chi(\zeta)_{d\zeta}^{(\zeta-a)^{\alpha-1}} = \exp\left\{\int_{a}^{t} (\zeta-a)^{\alpha-1}\ln\chi(\zeta)d\zeta\right\}, \end{aligned} \tag{1.7} $$

$$ \begin{aligned} ({}_{\alpha}^{b}I^{*}\chi)(t) :&= \int_{t}^{b} \chi(\zeta)_{d_{\alpha}^{*}(b,\zeta)} = \exp\left\{\int_{t}^{b} \ln\chi(\zeta)d_{\alpha}(b,\zeta)\right\} \\ &= \int_{t}^{b} \chi(\zeta)_{d\zeta}^{(b-\zeta)^{\alpha-1}} = \exp\left\{\int_{t}^{b} (b-\zeta)^{\alpha-1}\ln\chi(\zeta)d\zeta\right\}. \end{aligned} $$

The multiplicative left CF integral may be expressed as ${}^{*}I_{\alpha}$, and $d_{\alpha}^{*}(\zeta,a) = d_{\alpha}^{*}\zeta$ for $a = 0$.

Proposition 1.2. *[1] The following properties are satisfied for $\alpha \in (0,1]$ and $\chi : [a,b] \to \mathbb{R}^{+}$:*

$$ \begin{aligned} &(i) \quad ({}^{*}T_{\alpha}^{a}\,{}^{*}I_{\alpha}^{a}\chi)(t) = \chi(t), \quad if\,\chi\,is\ continuous, \\ &(ii) \quad ({}_{\alpha}^{b}T^{*}\,{}_{\alpha}^{b}I^{*}\chi)(t) = \chi(t), \quad if\,\chi\,is\ continuous, \\ &(iii) \quad ({}^{*}I_{\alpha}^{a}\,{}^{*}T_{\alpha}^{a}\chi)(t) = \frac{\chi(t)}{\chi(a)}, \\ &(iv) \quad ({}_{\alpha}^{b}I^{*}\,{}_{\alpha}^{b}T^{*}f)(t) = \frac{\chi(t)}{\chi(b)}. \end{aligned} \tag{1.8} $$

Definition 1.8. *[1] Consider the function $\chi : [a,b] \to \mathbb{R}^{+}$, $\alpha \in (n, n+1]$ and $\beta = \alpha - n$. Then, the higher order multiplicative left and right CF integrals of χ, respectively, are defined by:*

$$ ({}^{*}I_{\alpha}^{a}\chi)(t) = {}_{a}I_{*}^{n+1}\left(\chi(t)^{(t-a)^{\beta-1}}\right), $$

$$ ({}_{\alpha}^{b}I^{*}\chi)(t) = {}_{*}I_{b}^{n+1}\left(\chi(t)^{(b-t)^{\beta-1}}\right). $$

Theorem 1.3. *[38] Assume that ξ is CF differentiable of order $\alpha \in (0,1]$ at t and that $\chi, \chi_1, \chi_2 : [0,b] \to \mathbb{R}^{+}$ is multiplicative (left) CF differentiable of the same order. For a positive constant c,*

$$ (i) \quad \tau\,(c\chi)\,(t) = \tau\chi(t), $$

$$
\begin{aligned}
&(ii) \quad \tau\left(\chi_1\chi_2\right)(t) = \tau\chi_1(t)\,\tau\chi_2(t),\\
&(iii) \quad \tau\left(\frac{\chi_1}{\chi_2}\right)(t) = \frac{\tau\chi_1(t)}{\tau\chi_2(t)},\\
&(iv) \quad \tau\left(\chi^{\xi}\right)(t) = \{\tau\chi(t)\}^{\xi(t)}\,\chi(t)^{T_\alpha\xi(t)},\\
&(v) \quad \tau\left(\chi\circ\xi\right)(t) = \{(\tau\chi)\left(\xi(t)\right)\}^{T_\alpha\xi(t)\xi(t)^{\alpha-1}},\\
&(vi) \quad \tau\left(\chi_1+\chi_2\right)(t) = [\tau\chi_1(t)]^{\frac{\chi_1(t)}{\chi_1(t)+\chi_2(t)}}[\tau\chi_2(t)]^{\frac{\chi_2(t)}{\chi_1(t)+\chi_2(t)}}.
\end{aligned}
\tag{1.9}
$$

Theorem 1.4. *[38] Let $\chi, \chi_1, \chi_2 : [0, b] \to \mathbb{R}^+$ be multiplicative (left) CF integrable of order $\alpha \in (0, 1]$ at ζ. Thus, the following properties are given:*

$$
\begin{aligned}
&(i) \quad \int_0^b [\chi(\zeta)]^k_{d^*_\alpha\zeta} = \left[\int_0^b \chi(\zeta)_{d^*_\alpha\zeta}\right]^k, \; k\in\mathbb{R},\\
&(ii) \quad \int_0^b [\chi_1(\zeta)\chi_2(\zeta)]_{d^*_\alpha\zeta} = \int_0^b \chi_1(\zeta)_{d^*_\alpha\zeta} \int_0^b \chi_2(\zeta)_{d^*_\alpha\zeta},\\
&(iii) \quad \int_0^b \left[\frac{\chi_1(\zeta)}{\chi_2(\zeta)}\right]_{d^*_\alpha\zeta} = \frac{\int_0^b \chi_1(\zeta)_{d^*_\alpha\zeta}}{\int_0^b \chi_2(\zeta)_{d^*_\alpha\zeta}},\\
&(iv) \quad \int_0^b \chi(\zeta)_{d^*_\alpha\zeta} = \int_0^c \chi(\zeta)_{d^*_\alpha\zeta} \int_c^b \chi(\zeta)_{d^*_\alpha\zeta}, \; c\in[a,b] \text{ is a constant},\\
&(v) \quad \int_0^b [\tau\chi_1(\zeta)]^{\chi_2(\zeta)}_{d^*_\alpha\zeta} = \frac{\chi_1(b)^{\chi_2(b)}}{\chi_1(0)^{\chi_2(0)}}\left\{\int_0^b \chi_1(\zeta)^{T_\alpha\chi_2(\zeta)}_{d^*_\alpha\zeta}\right\}^{-1}.
\end{aligned}
\tag{1.10}
$$

The last formula is known as integration by parts of $\alpha-^*$.

Definition 1.9. *[38] Let $\chi : [0, b] \to \mathbb{R}^+$ and $\alpha \in (0, 1]$. The *inner product space*

$$
{}^*L^2_\alpha[0,b] = \left\{\chi : \int_0^b [\chi(\zeta)\odot\chi(\zeta)]_{d^*_\alpha\zeta} < \infty\right\},
$$

has

$$
<,>_*: {}^*L^2_\alpha[0,b]\times{}^*L^2_\alpha[0,b] \to \mathbb{R}^+,
$$

$$
<\chi_1,\chi_2>_* = \int_0^b [\chi_1(\zeta)\odot\chi_2(\zeta)]_{d^*_\alpha\zeta},
$$

where $\chi_1, \chi_2 \in {}^*L^2_\alpha[0, b]$ are positive functions.

2. Asymptotic estimates of *eigenfunctions

By setting $c_1/c_2 = -h$ and $d_1/d_2 = H$, such that neither h nor H is infinite, the boundary conditions (1.5) are converted to

$$(y(a))^{-h}(\tau y(a)) = 1, \qquad (y(b))^{H}(\tau y(b)) = 1.$$

Denote the solutions of (1.1) by $\kappa(t,\lambda)$ and $\varsigma(t,\lambda)$, which satisfy

$$\kappa(0,\lambda) = e, \quad \tau\kappa(0,\lambda) = e^{h}, \tag{2.1}$$

and

$$\varsigma(0,\lambda) = 1, \quad \tau\varsigma(0,\lambda) = e,$$

respectively.

Theorem 2.1. *Let $\lambda = \mu^2$. The *eigenfunctions of the problems (1.1)–(1.3) have the following asymptotic estimates:*

$$\kappa(t,\lambda) = e^{\cos\left(\mu\frac{t^\alpha}{\alpha}\right)+\frac{h}{\mu}\sin\left(\mu\frac{t^\alpha}{\alpha}\right)} \int_0^t \left[\kappa(s,\lambda)^{q(s)\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}^{\frac{1}{\mu}}, \tag{2.2}$$

$$\varsigma(t,\lambda) = e^{\frac{1}{\mu}\sin\left(\mu\frac{t^\alpha}{\alpha}\right)} \int_0^t \left[\varsigma(s,\lambda)^{q(s)\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}^{\frac{1}{\mu}}. \tag{2.3}$$

Proof. The asymptotic estimate (2.2) will be proved. The same method may be used to get the asymptotic estimate (2.3).

Since $\kappa(x,\lambda)$ satisfies (1.4), we get

$$\begin{aligned} &\int_0^t \left[\kappa(s,\lambda)^{q(s)\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s} \\ &= \int_0^t \left[\left\{\tau^2\kappa(s,\lambda)\right\}^{\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s} \int_0^t \left[\kappa(s,\lambda)^{\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}^{\mu^2}. \end{aligned} \tag{2.4}$$

The equality

$$\begin{aligned} &\int_0^t \left[\left\{\tau^2\kappa(s,\lambda)\right\}^{\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s} \\ &= \frac{\{\kappa(t,\lambda)\}^{\mu}}{\{\tau\kappa(0,\lambda)\}^{\sin\left(\mu\frac{t^\alpha}{\alpha}\right)}\{\kappa(0,\lambda)\}^{\mu\cos\left(\mu\frac{t^\alpha}{\alpha}\right)}} \left[\int_0^t \kappa(s,\lambda)^{\sin\left\{\mu\left(\frac{t^\alpha}{\alpha}-\frac{s^\alpha}{\alpha}\right)\right\}}\right]_{d_\alpha^* s}^{-\mu^2}, \end{aligned}$$

is produced if the first multiplier on the right side of the last equality is twice subjected to the $\alpha-$*integration by parts (1.10). Then, by considering the conditions (2.1) in (2.4) with the above relation, this completes the proof. □

3. Certain spectral characteristics of $\alpha-^{*}$SL problem

In this part, we look at a few qualities related to the $\alpha-^{*}$SL problem, including self-adjointness, orthogonality, reality, and simplicity.

Lemma 3.1. *($\alpha-^{*}$Lagrange Identity) Let $\kappa, \varsigma \in {}^{*}L^{2}_{\alpha}[0,b]$. Thus,*

$$(\boldsymbol{L}_{\alpha}[\kappa] \odot \varsigma) \ominus (\kappa \odot \boldsymbol{L}_{\alpha}[\varsigma]) = \tau([\kappa, \varsigma]_{t}), \tag{3.1}$$

where

$$[\kappa, \varsigma]_{t} = (\kappa(t) \odot \tau\varsigma(t)) \ominus (\varsigma(t) \odot \tau\kappa(t)). \tag{3.2}$$

Proof. Let $\kappa, \varsigma \in {}^{*}L^{2}_{\alpha}[0,b]$. From (1.4), we have

$$\begin{aligned}(\boldsymbol{L}_{\alpha}[\kappa] \odot \varsigma) \ominus (\kappa \odot \boldsymbol{L}_{\alpha}[\varsigma]) &= \left(\left\{(\tau^{2}\kappa)^{-1}\, \kappa^{q(t)}\right\} \odot \varsigma\right) \ominus \left(\kappa \odot \left\{(\tau^{2}\varsigma)^{-1}\, \varsigma^{q(t)}\right\}\right) \\ &= \left(\kappa \odot \tau^{2}\varsigma\right) \ominus \left(\varsigma \odot \tau^{2}\kappa\right) \\ &= \tau\left\{(\kappa \odot \tau\varsigma) \ominus (\varsigma \odot \tau\kappa)\right\},\end{aligned}$$

which establishes the outcome. □

Lemma 3.2. *($\alpha-^{*}$Green's Formula) Let $\kappa, \varsigma \in {}^{*}L^{2}_{\alpha}[0,b]$. Then,*

$$\int_{0}^{b} [(\boldsymbol{L}_{\alpha}[\kappa] \odot \varsigma) \ominus (\kappa \odot \boldsymbol{L}_{\alpha}[\varsigma])]_{d^{*}_{\alpha}t} = [\kappa, \varsigma]_{t}\Big|_{0}^{b}. \tag{3.3}$$

Proof. By multiplicative CF integration on $[0,b]$ on both sides of (3.1), the proof can be readily demonstrated. □

Theorem 3.3. *Formally, on ${}^{*}L^{2}_{\alpha}[0,b]$ the $\alpha-^{*}$SL operator $\boldsymbol{L}_{\alpha}$ in (1.1) is self-adjoint .*

Proof. We obtain

$$\begin{aligned}[\kappa, \varsigma]_{b} &= (\kappa(b) \odot \tau\varsigma(b)) \ominus (\varsigma(b) \odot \tau\kappa(b)) \\ &= \left(\{\tau\kappa(b)\}^{\frac{-d_{2}}{d_{1}}} \odot \tau\varsigma(b)\right) \ominus \left(\{\tau\varsigma(b)\}^{\frac{-d_{2}}{d_{1}}} \odot \tau\kappa(b)\right) = 1,\end{aligned}$$

from the boundary conditions (1.2). Similarly, we get $[\kappa, \varsigma]_{0} = 1$ from (1.3).

Thus, we get

$$\int_{0}^{b} [(\boldsymbol{L}_{\alpha}[\kappa] \odot \varsigma) \ominus (\kappa \odot \boldsymbol{L}_{\alpha}[\varsigma])]_{d^{*}_{\alpha}t} = \frac{[\kappa, \varsigma]_{b}}{[\kappa, \varsigma]_{0}} = 1,$$

or

$$< \boldsymbol{L}_{\alpha}[\kappa], \varsigma >_{*} = < \kappa, \boldsymbol{L}_{\alpha}[\varsigma] >_{*}, \tag{3.4}$$

by (3.3), which validates the theorem. □

Theorem 3.4. *For $\alpha-^{*}$SL problems (1.1)–(1.3), all of the eigenvalues are real.*

Proof. Assume that the eigenvalue λ has the eigenfunction $\kappa = \kappa(t, \lambda)$. Thus,

$$< \boldsymbol{L}_\alpha [\kappa], \kappa >_* = < e^\lambda \odot \kappa, \kappa >_* = e^\lambda \odot < \kappa, \kappa >_* \tag{3.5}$$

and

$$< \kappa, \boldsymbol{L}_\alpha [\kappa] >_* = < \kappa, e^\lambda \odot \kappa >_* = e^{\bar{\lambda}} \odot < \kappa, \kappa >_* . \tag{3.6}$$

We get

$$e^\lambda \odot < \kappa, \kappa >_* = e^{\bar{\lambda}} \odot < \kappa, \kappa >_* \quad \text{or} \quad < \kappa, \kappa >_*^{\lambda - \bar{\lambda}} = 1,$$

from (3.4)–(3.6). $\kappa(t) \neq 1$ is the result of $\lambda = \bar{\lambda}$, which validates the theory. □

Theorem 3.5. *For the $\alpha-{}^*SL$ problems (1.1)–(1.3), the *eigenfunctions $\kappa = \kappa(t, \lambda_1)$ and $\varsigma = \varsigma(t, \lambda_2)$, which correspond to the distinct *eigenvalues λ_1 and λ_2, are orthogonal, meaning that*

$$\int_0^b [\kappa(t, \lambda_1) \odot \varsigma(t, \lambda_2)]_{d_\alpha^* t} = 1.$$

Proof. Upon considering $\boldsymbol{L}_\alpha [\kappa] = e^{\lambda_1} \odot \kappa$ and $\boldsymbol{L}_\alpha [\varsigma] = e^{\lambda_2} \odot \varsigma$ from (1.4) in the equality (3.4), we get

$$< e^{\lambda_1} \odot \kappa, \varsigma >_* = < \kappa, e^{\lambda_2} \odot \varsigma >_* \quad \text{or} \quad < \kappa, \varsigma >_*^{\lambda_1 - \lambda_2} = 1.$$

Given that $\lambda_1 \neq \lambda_2$, $< \kappa, \varsigma >_* = 1$. Therefore, we see that $\kappa(t)$ and $\varsigma(t)$ are orthogonal. □

The $\alpha-{}^*$Wronskian of $\kappa(t)$ and $\varsigma(t)$ will now be defined using the formula (3.2).

Theorem 3.6. *Any solution to Eq (1.1) has an $\alpha-{}^*$Wronskian that is independent of t.*

Proof. Assume that $\kappa(t)$ and $\varsigma(t)$ are the two solutions of (1.1). By using (3.2), $\boldsymbol{L}_\alpha [\kappa] = e^\lambda \odot \kappa$, and $\boldsymbol{L}_\alpha [\bar{\varsigma}] = e^{\bar{\lambda}} \odot \bar{\varsigma}$, we can get

$$\int_0^t [\kappa(x, \lambda) \odot \varsigma(x, \mu)]_{d_\alpha^* x}^{\lambda - \bar{\lambda}} = \frac{[\kappa, \bar{\varsigma}]_t}{[\kappa, \bar{\varsigma}]_0}.$$

However, due to $\lambda = \bar{\lambda}$ (obtained using theorem 3.4) and $[\kappa, \bar{\varsigma}]_0 = 1$, and thus, ${}^*W_\alpha (\kappa, \varsigma)(t) = [\kappa, \varsigma]_t = 1$, namely; the $\alpha-{}^*$Wronskian is independent of t. □

Theorem 3.7. *If the $\alpha-{}^*$Wronskian of any two solutions to Eq (1.1) equals one, then the solutions are multiplicatively linearly dependent.*

Proof. If the $\alpha-{}^*$Wronskian of any two solutions to the Eq (1.1) equals one, then the solutions are multiplicatively linearly dependent. Hence,

$$\begin{aligned} {}^*W_\alpha (\kappa, \varsigma)(t) = [\kappa, \varsigma]_t &= (\kappa(t) \odot \tau\varsigma(t)) \ominus (\varsigma(t) \odot \tau\kappa(t)) \\ &= (\varsigma(t)^c \odot \tau\varsigma(t)) \ominus (\varsigma(t) \odot \tau\varsigma(t)^c) = 1. \end{aligned}$$

On the other hand, $\kappa(t) = \varsigma(t)^c$ since ${}^*W_\alpha (\kappa, \varsigma)(t) = 1$, meaning that $\kappa(t)$ and $\varsigma(t)$ are the two multiplicatively linearly dependent ones. □

Lemma 3.8. *From a geometric perspective, all of the eigenvalues of the $\alpha-{}^*SL$ problems (1.1)–(1.3) are simple.*

Proof. Consider μ as an eigenvalue with $\kappa(t)$ and $\varsigma(t)$ as its *eigenfunctions.

We obtain ${}^*W_\alpha(\kappa, \varsigma)(0) = [\kappa, \varsigma]_0 = 1$ by the condition (1.2), which indicates that the set $\{\kappa(t), \varsigma(t)\}$ is linearly dependent. This means that there is a matching one *eigenvalue and *eigenfunction. □

We now have to explain how we were able to determine the *eigenvalues and *eigenfunctions of the given problem. Assume that $\phi_1(\cdot, \lambda)$ and $\phi_2(\cdot, \lambda)$ are linearly independent solutions of (1.1) that fulfill the condition

$$\tau^{j-1}\phi_i(0, \lambda) = \delta^*_{ij}, \quad i, j = 1, 2,$$

where $\delta^*_{ij} = \begin{cases} e, & i = j \\ 1, & i \neq j \end{cases}$ represents the *Kronecker delta. Consequently, any solution to Eq (1.1) will have the following form:

$$y(t, \lambda) = \phi_1(t, \lambda)^{A_1}\,\phi_2(t, \lambda)^{A_2},$$

where the constants A_1 and A_2 are independent on t. In this case, the solution given Eq (1.1) will be the *eigenfunction of the associated problem if it provides conditions (1.2) and (1.3). To put it another way, if a non-trivial solution to

$$\begin{aligned} A_1 \ln U_1(\phi_1) + A_2 \ln U_1(\phi_2) &= 0, \\ A_1 \ln U_2(\phi_1) + U_2 \ln L_2(\phi_2) &= 0, \end{aligned}$$

can be found, it will be an *eigenfunction, with U_1 and U_2 specified by (1.5). Thus, λ is an *eigenvalue of the given problem iff

$${}^*\Delta_\alpha(\lambda) = \begin{vmatrix} \ln U_1(\phi_1) & \ln U_1(\phi_2) \\ \ln U_2(\phi_1) & \ln U_2(\phi_2) \end{vmatrix} = 0.$$

In this case, zeros of ${}^*\Delta_\alpha(\lambda)$ are *eigenvalues of (1.1)–(1.3), and the function ${}^*\Delta_\alpha(\lambda)$ is referred to as the $\alpha-{}^*$SL characteristic determinant, denoted by (1.1)–(1.3).

Theorem 3.9. *For the $\alpha-{}^*SL$ problems (1.1)–(1.3), all of the eigenvalues are simple zeros of ${}^*\Delta_\alpha(\lambda)$.*

Proof. Assume that $\theta_1(\cdot, \lambda)$ and $\theta_2(\cdot, \lambda)$ are given by the following equalities:

$$\theta_1(t, \lambda) = [U_1(\phi_2) \odot \phi_1(t, \lambda)] \ominus [U_1(\phi_1) \odot \phi_2(t, \lambda)], \tag{3.7}$$

$$\theta_2(t, \lambda) = [U_2(\phi_2) \odot \phi_1(t, \lambda)] \ominus [U_2(\phi_1) \odot \phi_2(t, \lambda)]. \tag{3.8}$$

According to this definition, they can be written as

$$\theta_1(t, \lambda) = \frac{[(\phi_2(0))^{c_1}\,(\tau\phi_2(0))^{c_2}]^{\ln \phi_1(t,\lambda)}}{[(\phi_1(0))^{c_1}\,(\tau\phi_1(0))^{c_2}]^{\ln \phi_2(t,\lambda)}},$$

and

$$\theta_2(t, \lambda) = \frac{\left[(\phi_2(b))^{d_1}\,(\tau\phi_2(b))^{d_2}\right]^{\ln \phi_1(t,\lambda)}}{\left[(\phi_1(b))^{d_1}\,(\tau\phi_1(b))^{d_2}\right]^{\ln \phi_2(t,\lambda)}}.$$

Consequently, the below conditions are satisfied.

$$\begin{aligned}\theta_1(0,\lambda) = e^{c_2},\ \ \tau\theta_1(0,\lambda) = e^{-c_1},\\ \theta_2(b,\lambda) = e^{d_2},\ \ \tau\theta_2(b,\lambda) = e^{-d_1}.\end{aligned}\tag{3.9}$$

However, if we use the $\alpha-{}^*$Wronskian definition, we have

$${}^*W_\alpha(\theta_1(t,\lambda),\theta_2(t,\lambda)) = \{{}^*W_\alpha(\phi_1(t,\lambda),\phi_2(t,\lambda))\}^{{}^*\Delta_\alpha(\lambda)} = e^{{}^*\Delta_\alpha(\lambda)}.\tag{3.10}$$

For the problems (1.1)–(1.3), let $\tilde{\lambda}$ be an *eigenvalue. As *eigenfunctions with multiplicative linear dependence, $\theta_1\left(t,\tilde{\lambda}\right)$ and $\theta_2\left(t,\tilde{\lambda}\right)$ are obtained from (3.10). Thus,

$$\theta_1\left(t,\tilde{\lambda}\right) = \theta_2\left(t,\tilde{\lambda}\right)^{\xi},$$

is satisfied by the existence of a nonzero constant, ξ. Hence, by (3.8) and (3.9), we arrive at

$$\theta_1\left(b,\tilde{\lambda}\right) = \theta_2(b,\lambda)^{\xi},\quad \tau\theta_1\left(b,\tilde{\lambda}\right) = \tau\theta_2(b,\lambda)^{-\xi}.$$

By setting $\kappa(t) = \theta_1(t,\lambda)$, $\varsigma(t) = \theta_1\left(t,\tilde{\lambda}\right)$ and (3.3), we get

$$\int_0^b \left[\theta_1(t,\lambda)\odot\theta_1\left(t,\tilde{\lambda}\right)\right]_{d_\alpha^* t}^{\lambda-\bar{\lambda}} = {}^*\Delta_\alpha^{\xi}(\lambda).$$

Since ${}^*\Delta_\alpha(\lambda)$ is an *entire function of λ, we arrive at

$${}^*\Delta_\alpha^*\left(\tilde{\lambda}\right) = \lim_{\lambda\to\tilde{\lambda}}\left({}^*\Delta_\alpha(\lambda)\right)^{\frac{1}{\lambda-\tilde{\lambda}}} = \int_0^b \left[\theta_1(t,\lambda)\odot\theta_1\left(t,\tilde{\lambda}\right)\right]_{d_\alpha^* t}^{\frac{1}{\xi}} \neq 1,$$

where ${}^*\Delta_\alpha^*\left(\tilde{\lambda}\right)$ is the multiplicative derivative of ${}^*\Delta_\alpha\left(\tilde{\lambda}\right)$. Hence, $\tilde{\lambda}$ is a simple zero of ${}^*\Delta_\alpha(\lambda)$. □

4. $\alpha-{}^*$Green's function

This part will describe $\alpha-{}^*$Green's function for non-homogeneous $\alpha-{}^*$SL and list some of its characteristics. We take up the problem

$$(\tau^2 y)^{-1} y^{q(t)-\lambda} = e^{f(t)},\tag{4.1}$$

with the condition (1.5), where $q(t)$ is real-valued continuous and multiplicative conformable fractional integrable function on $[0,b]$; λ is a spectral parameter; $\alpha\in(0,1]$; $\left(c_1^2+c_2^2\right)\left(d_1^2+d_2^2\right)\neq 0$, c_i, $d_i\in\mathbb{R}$ $(i=1,2)$, $f(t)\in {}^*L_\alpha^2[0,b]$.

Theorem 4.1. *Let us admit that λ is not an eigenvalue of the problems (4.1) and (1.5). In addition, $\psi(\cdot,\lambda)$ satisfies Eq (4.1) and the boundary conditions (1.5). Then,*

$$\psi(x,\lambda) = \int_0^b \left({}^*\mathbb{G}_\alpha(x,\zeta,\lambda)\odot e^{f(\zeta)}\right)_{d_\alpha^*\zeta},\ \zeta\in[0,b],\tag{4.2}$$

where $^*\mathbb{G}_\alpha(x,\zeta,\lambda)$ is $\alpha-^*$Green's function for (4.1), (1.5) defined by

$$^*\mathbb{G}_\alpha(x,\zeta,\lambda) = e^{\frac{-1}{^*\Delta_\alpha(\lambda)}} \odot \begin{cases} \theta_1(\zeta,\lambda)\odot\theta_2(x,\lambda), & 0\le\zeta\le x \\ \theta_1(x,\lambda)\odot\theta_2(\zeta,\lambda), & x\le\zeta\le b \end{cases}. \tag{4.3}$$

On the other hand, the problems (4.1) and (1.5) are satisfied by the function $\varphi(\cdot,\lambda)$ as stated by (4.2). Moreover, $^*\mathbb{G}_\alpha(x,\zeta,\lambda)$ is unique. Here, θ_1 and θ_2 are multiplicative linearly independent solutions of the problems (1.1)–(1.2) and (1.1)–(1.3), respectively.

Proof. The $\alpha-^*$Green's function definition leads us to

$$^*\mathbb{G}_\alpha(x,\zeta,\lambda)\odot e^{f(\zeta)} = \begin{cases} \left\{\theta_1(\zeta)^{f(\zeta)}\right\}^{\ln\theta_2(x)\frac{-1}{^*\Delta_\alpha(\lambda)}}, & 0\le\zeta\le x \\ \\ \left\{\theta_2(\zeta)^{f(\zeta)}\right\}^{\ln\theta_1(x)\frac{-1}{^*\Delta_\alpha(\lambda)}}, & x\le\zeta\le b \end{cases}. \tag{4.4}$$

From (4.2), performing multiplicative CF integration of (4.4) with regard to ζ on $[0,b]$, we get

$$\psi(x,\lambda) = \left\{\int_0^x \left[\theta_1(\zeta)^{f(\zeta)}\right]_{d_\alpha^*\zeta}\right\}^{\frac{-\ln\theta_2(x)}{^*\Delta_\alpha(\lambda)}} \left\{\int_x^b \left[\theta_2(\zeta)^{f(\zeta)}\right]_{d_\alpha^*\zeta}\right\}^{\frac{-\ln\theta_1(x)}{^*\Delta_\alpha(\lambda)}}. \tag{4.5}$$

Now, we get

$$\begin{aligned}\tau^2\psi(x,\lambda) = &\left\{\int_0^x \left[\theta_1(\zeta)^{f(\zeta)}\right]_{d_\alpha^*\zeta}\right\}^{\frac{-T_\alpha^2\ln\theta_2(x)}{^*\Delta_\alpha(\lambda)}} \left\{\int_x^b \left[\theta_2(\zeta)^{f(\zeta)}\right]_{d_\alpha^*\zeta}\right\}^{\frac{-T_\alpha^2\ln\theta_1(x)}{^*\Delta_\alpha(\lambda)}} \\ &\times\left[^*W_\alpha(\theta_1,\theta_2)(x)\right]^{-\frac{f(x)}{^*\Delta_\alpha(\lambda)}},\end{aligned}$$

after the twice multiplicative CF derivative with regard to x on both sides of the Eq (4.5). Then, since $\theta_1(x)$ and $\theta_2(x)$ are solutions of Eq (1.1), from (3.10), we obtain

$$\begin{aligned}\tau^2\psi(x,\lambda) &= \left\{\int_0^x \left[\theta_1(\zeta)^{f(\zeta)}\right]_{d_\alpha^*\zeta}\right\}^{\frac{-(q(x)-\lambda)\ln\theta_2(x)}{^*\Delta_\alpha(\lambda)}} \left\{\int_x^b \left[\theta_2(\zeta)^{f(\zeta)}\right]_{d_\alpha^*\zeta}\right\}^{\frac{-(q(x)-\lambda)\ln\theta_1(x)}{^*\Delta_\alpha(\lambda)}} \\ &= \{\psi(x,\lambda)\}^{(q(x)-\lambda)}\, e^{-f(x)}.\end{aligned}$$

So, this proves the validity of (4.1) for $\psi(x,\lambda)$ defined by (4.2).

Let us now demonstrate the uniqueness of $\alpha-^*$Green's function for the problems (4.1) and (1.5). Admittedly, for the identical issue, there exists another $\alpha-^*$Green's function $^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda)$. Then,

$$\psi(x,\lambda) = \int_0^b \left[^*\mathbb{G}_\alpha(x,\zeta,\lambda)\odot e^{f(\zeta)}\right]_{d_\alpha^*\zeta}$$

and

$$\psi(x,\lambda) = \int_0^b \left[^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda)\odot e^{f(\zeta)}\right]_{d_\alpha^*\zeta}$$

are obtained. Thence, we get the by multiplicative subtraction

$$\int_0^b \left[\left\{{}^*\mathbb{G}_\alpha(x,\zeta,\lambda)\ominus{}^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda)\right\}\odot e^{f(\zeta)}\right]_{d^*_\alpha\zeta}=1$$

all functions $f(x)\in{}^*L^2_\alpha[0,b]$. By establishing $f(x)=\ln\left({}^*\mathbb{G}_\alpha(x,\zeta,\lambda)\ominus{}^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda)\right)$, we get

$$\int_0^b \left[{}^*\mathbb{G}_\alpha(x,\zeta,\lambda)\ominus{}^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda)\right]^{2_{\mathbb{G}}}_{d^*_\alpha\zeta}=1,$$

and in this case,

$${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)\ominus{}^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda)=1.$$

Finally, we obtain

$${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)={}^*\widetilde{\mathbb{G}}_\alpha(x,\zeta,\lambda).$$

□

Theorem 4.2. *The features of* $\alpha-{}^*$*Green's function in (1.1)–(1.3) are as follows:*

(i) ${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)$ *is continuous at* $(0,0)$ *.*

(ii) ${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)={}^*\mathbb{G}_\alpha(\zeta,x,\lambda)$.

(iii) For any $x\in\mathbb{R}$ *as a function of t,* ${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)$ *fulfills (1.2), (1.3), and Eq (1.1).*

(iv) Consider an eigenvalue of ${}^*\Delta_\alpha(\lambda)$ *to be* λ_0*. Hence,*

$${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)=\left[\psi_0(x)^{-\psi_0(\zeta)}\right]^{\frac{1}{\lambda-\lambda_0}}\,{}^*\breve{\mathbb{G}}_\alpha(x,\zeta,\lambda),$$

and λ_0 *is the simple pole point of* ${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)$*. In this case, for* λ *in the neighborhood of* λ_0, ${}^*\breve{\mathbb{G}}_\alpha(x,\zeta,\lambda)$ *is a type of holomorphic function. The normalized eigenfunction associated with* λ_0 *is* ψ.

Proof. (i) For any $\lambda\in\mathbb{C}$, continuity of $\theta_1(.,\lambda)$ and $\theta_2(.,\lambda)$ gives proof.

Then, if some fundamental concepts from multiplicative CF calculus are used, (ii) and (iii) may be demonstrated with ease.

(iv) Assume that ${}^*\mathcal{R}(x,\zeta)$ is the residue of ${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)$ at $\lambda=\lambda_0$ and that λ_0 is the pole point of ${}^*\mathbb{G}_\alpha(x,\zeta,\lambda)$. Then, we get

$${}^*\mathcal{R}(x,\zeta)=\psi_0(x,\lambda_0)^{-\psi_0(\zeta,\lambda_0)}.$$

The proof is finished. □

Now, we give an example to illustrate the validity of the main results.

Example 4.1. *Let us consider the below* $\alpha-{}^*SL$*:*

$$(\tau^2 y)^{-1}=y^\lambda,\ \ x\in[0,1],\tag{4.6}$$

$$U_1(y):=y(0)=1,\quad U_2(y):=\tau y(1)=1.\tag{4.7}$$

It is clear that the functions

$$\phi_1(t,\lambda) = e^{\cos\left(\sqrt{\lambda}\frac{t^\alpha}{\alpha}\right)},$$

$$\phi_2(t,\lambda) = \begin{cases} e^{\frac{\sin\left(\sqrt{\lambda}\frac{t^\alpha}{\alpha}\right)}{\sqrt{\lambda}}}, & \lambda \neq 0 \\ e^{\frac{t^\alpha}{\alpha}}, & \lambda = 0 \end{cases}$$

are solutions of (4.6) [1,40,44]. Moreover, from (3.7), (3.8), we have

$$\begin{aligned} \theta_1(t,\lambda) &= [U_1(\phi_2) \odot \phi_1(t,\lambda)] \ominus [U_1(\phi_1) \odot \phi_2(t,\lambda)] \\ &= [\phi_2(0) \odot \phi_1(t,\lambda)] \ominus [\phi_1(0) \odot \phi_2(t,\lambda)] \\ &= e^{-\frac{\sin\left(\sqrt{\lambda}\frac{t^\alpha}{\alpha}\right)}{\sqrt{\lambda}}} \end{aligned}$$

and

$$\begin{aligned} \theta_2(t,\lambda) &= [U_2(\phi_2) \odot \phi_1(t,\lambda)] \ominus [U_2(\phi_1) \odot \phi_2(t,\lambda)] \\ &= [\tau\phi_2(1) \odot \phi_1(t,\lambda)] \ominus [\tau\phi_1(1) \odot \phi_2(t,\lambda)] \\ &= e^{\cos\left(\frac{\sqrt{\lambda}}{\alpha}\right)\cos\left(\sqrt{\lambda}\frac{t^\alpha}{\alpha}\right)+\sin\left(\frac{\sqrt{\lambda}}{\alpha}\right)\sin\left(\sqrt{\lambda}\frac{t^\alpha}{\alpha}\right)} \\ &= e^{\cos\left(\frac{\sqrt{\lambda}}{\alpha}(1-t^\alpha)\right)} \end{aligned}$$

respectively. Furthermore, we give the $\alpha-{}^*$SL characteristic determinant

$${}^*\Delta_\alpha(\lambda) = \begin{vmatrix} \ln U_1(\phi_1) & \ln U_1(\phi_2) \\ \ln U_2(\phi_1) & \ln U_2(\phi_2) \end{vmatrix} = \cos\left(\frac{\sqrt{\lambda}}{\alpha}\right).$$

So, zeros of $\cos\left(\frac{\sqrt{\lambda}}{\alpha}\right)$ are *eigenvalues of (4.6) and (4.7).

Hence, $\alpha-{}^*$Green's function is obtained by

$$\begin{aligned} {}^*\mathbb{G}_\alpha(x,\zeta,\lambda) &= e^{\frac{-1}{{}^*\Delta_\alpha(\lambda)}} \odot \begin{cases} \theta_1(\zeta,\lambda) \odot \theta_2(x,\lambda), & 0 \leq \zeta \leq x \\ \theta_1(x,\lambda) \odot \theta_2(\zeta,\lambda), & x \leq \zeta \leq 1 \end{cases} \\ &= e^{\frac{-1}{\cos\left(\frac{\sqrt{\lambda}}{\alpha}\right)}} \odot \begin{cases} e^{-\frac{\sin\left(\sqrt{\lambda}\frac{\zeta^\alpha}{\alpha}\right)}{\sqrt{\lambda}}} \odot e^{\cos\left(\frac{\sqrt{\lambda}}{\alpha}(1-x^\alpha)\right)}, & 0 \leq \zeta \leq x \\ e^{-\frac{\sin\left(\sqrt{\lambda}\frac{x^\alpha}{\alpha}\right)}{\sqrt{\lambda}}} \odot e^{\cos\left(\frac{\sqrt{\lambda}}{\alpha}(1-\zeta^\alpha)\right)}, & x \leq \zeta \leq 1 \end{cases} \\ &= \begin{cases} e^{\frac{\sin\left(\sqrt{\lambda}\frac{\zeta^\alpha}{\alpha}\right)\cos\left(\frac{\sqrt{\lambda}}{\alpha}(1-x^\alpha)\right)}{\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{\alpha}\right)}}, & 0 \leq \zeta \leq x \\ e^{\frac{\sin\left(\sqrt{\lambda}\frac{x^\alpha}{\alpha}\right)\cos\left(\frac{\sqrt{\lambda}}{\alpha}(1-\zeta^\alpha)\right)}{\sqrt{\lambda}\cos\left(\frac{\sqrt{\lambda}}{\alpha}\right)}}, & x \leq \zeta \leq 1. \end{cases} \end{aligned}$$

5. Conclusions

The multiplicative conformable Sturm-Liouville problem was established. In reality, this problem is a fractional extension of the Sturm-Liouville problem in multiplicative form for the situation

$\alpha = 1$ [36]. Initially, we were able to derive the *eigenfunctions of the problem. We later demonstrated that the *eigenfunctions are orthogonal in $^*L^2_\alpha[0, b]$-space and that the *eigenvalues are real and simple. Green's function was established for the multiplicative case. We believe that this problem will greatly contribute to mathematical physics in multiplicative situations, since it is of utmost importance for quantum physics and effective in both fractional and classical cases. As it turns out, the problem we looked at in the multiplicative case matches one that requires a lot more work and effort to evaluate in fractional or classical calculus. The significance of the outcomes and the many calculations we employed were amplified in this case.

Author contributions

Tuba Gulsen: Methodology, Writing–original draft, Writing–review & editing; Sertac Goktas: Methodology, Resources, Validation, Writing–original draft, Writing–review & editing; Thabet Abdeljawad: Resources, Supervision, Writing-review and editing; Yusuf Gurefe: Investigation, Methodology, Project administration. All authors have read and agreed to the published version of this manuscript.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

Acknowledgments

Thabet Abdeljawad would like to thank Prince Sultan University for paying APC and for support through TAS research lab.

Conflict of interest

The authors declare no conflicts of interest.

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