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Research article

Triple solutions for a Leray-Lions $p(x)$-biharmonic operator involving Hardy potential and indefinite weight

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Abstract: Existence and multiplicity of three weak solutions for a Leray-Lions $p(x)$-biharmonic problem involving Hardy potential and indefinite weight were proved. Our main tools combined variational methods and some critical theorems.

Keywords: critical theorem; Leray-Lions operator; Hardy potential; variable exponent
Mathematics Subject Classification: 35J20, 35J35, 35J60, 35G30, 46E35

1. Introduction

Honoring Jean Leray and Jacques-Louis Lions [1], the Leray-Lions operator is an essential component of nonlinear partial differential equations (PDEs) that emerge in a variety of scientific and engineering fields such as electro-rheological fluids (Ružička [2]), elastic mechanics (Zhikov [3]), stationary thermo-rheological viscous flows of non-Newtonian fluids (Rajagopal-Ružička [4]), image processing (Chen-Levine-Rao [5]), and so on. For the purpose of examining the existence, uniqueness, regularity, and multiplicity of solutions to certain equations, this operator offers a strict framework.

The Leray-Lions operator gains more intricacy and richness when a Hardy potential is incorporated. The Hardy potential introduces singularities around the origin, which makes the operator's characteristics and the behavior of the solutions more delicate.

In this work, we will establish the existence of a triple weak solution to the following intriguing problem.

$$\begin{cases} \Delta\left(a(x,\Delta u)\right) + \theta(x)\frac{|u|^{s-2}u}{|x|^{2s}} = \lambda g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $g(x,u) = f(x)|u|^{r(x)-2}u$, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$), with boundary $\partial\Omega$ of class C^1, s is a constant, $\lambda > 0$ is a parameter, f is a function in a Lebesgue generalized space $L^{\gamma(x)}(\Omega)$, and functions r, p, and $\gamma \in C(\overline{\Omega})$ verify the following inequalities:

$$1 < s < \min_{x\in\overline{\Omega}} r(x) \leq \max_{x\in\overline{\Omega}} r(x) < \min_{x\in\overline{\Omega}} p(x) \leq \max_{x\in\overline{\Omega}} p(x) < \frac{N}{2} < \gamma(x),$$

for all $x \in \Omega$, where $\Delta\left(a(x,\Delta u)\right)$ is the fourth-order Leray-Lions operator, and a is a Carathéodory function that satisfies an appropriate additional requirements.

The study of higher-order, nonlinear PDEs with variable exponents is crucial as it can provide insights into the behavior of complex physical systems that exhibit nonlinear and anisotropic properties. Moreover, the inclusion of the fourth-order Leray-Lions operator $\Delta(a(x,\Delta u))$ in the problem adds an extra layer of complexity, making it an interesting and challenging problem. While the theory of variable exponent Lebesgue and Sobolev spaces has been extensively developed in the literature, the establishment of the existence of a triple weak solution to the given problem remains an open and non-trivial task. Previous studies have tackled similar problems, but the combination of the higher-order operator, the nonlinear terms, and the variable exponents introduces new challenges that require the development of specialized techniques and tools. This study aims to fill the existing research gap by establishing the existence of a triple weak solution to the Problem (1.1). To achieve this, the authors will employ advanced techniques from nonlinear analysis, such as variational methods, critical point theory, and fixed point theorems, along with a deep understanding of variable exponent Lebesgue and Sobolev spaces. The proposed approach will involve a careful analysis of the equation, the boundary conditions, and the parameter dependence, leading to the development of a comprehensive framework for the existence of a triple weak solution. The novelty and originality of this work lie in the fact that it combines the challenging aspects of higher-order PDEs, nonlinear terms, and variable exponents, which have not been fully addressed in the existing literature. The successful resolution of this problem will contribute to the advancement of the theory of nonlinear PDEs with variable exponents and may have further applications in various scientific and engineering disciplines. The main difficulties in establishing the existence of a triple weak solution to the given fourth-order, nonlinear problem with variable exponents lie in the inherent complexity of the equation. The presence of the higher-order Leray-Lions operator, the nonlinear terms with variable exponents, and the homogeneous Dirichlet boundary conditions on both the function and its Laplacian pose significant challenges. Addressing these challenges requires the development and application of advanced techniques from nonlinear analysis, such as variational methods, critical point theory, and fixed point theorems, coupled with a thorough understanding of variable exponent Lebesgue and Sobolev spaces. Furthermore, the parameter dependence of the problem adds an additional layer of complexity that must be carefully handled in the analysis. Recently, (Liu-Zhao [6]) established an existence and multiplicity result for Problem (1.1). Under certain appropriate conditions on nonlinearity g, we mention that our paper is an extension of that of Liu-Zhao; in fact, the nonlinearity g may change sign on Ω. Moreover the condition of type Ambrosetti-Rabinowitz is not needed. This paper is structured as follows: We give some background and preliminaries on the Sobolev spaces with variable exponents in Section 2, and Sections 3 and 4 include the proof of our results.

2. Background and preliminaries

Throughout this paper, let

$$C_+(\overline{\Omega}) := \{\beta \mid \beta \in C(\overline{\Omega}), \beta(x) > 1, \text{ for all } \ x \in \overline{\Omega}\}.$$

Put

$$\beta^- := \inf_{x\in\overline{\Omega}} \beta(x) \quad \text{ and } \quad \beta^+ := \sup_{x\in\overline{\Omega}} \beta(x).$$

Moreover, for $\beta \in C_+(\overline{\Omega})$, put

$$[\kappa]^\beta := \max\{\kappa^{\beta^-}, \kappa^{\beta^+}\}, \ \ [\kappa]_\beta := \min\{\kappa^{\beta^-}, \kappa^{\beta^+}\},$$

which yields to $[\kappa]^{\frac{1}{\beta}} := \max\{\kappa^{\frac{1}{\beta^+}}, \kappa^{\frac{1}{\beta^-}}\}$ and $[\kappa]_{\frac{1}{\beta}} := \min\{\kappa^{\frac{1}{\beta^+}}, \kappa^{\frac{1}{\beta^-}}\}$.

In this study, we assume that $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded regular domain with C^1 boundary, $p, r, \gamma \in C_+(\overline{\Omega})$, which holds the following:

$$1 < s < r^- \le r^+ < p^- \le p^+ < \frac{N}{2} < \gamma^-.$$

We denote the variable exponent Lebesgue space [7] by

$$L^{p(x)}(\Omega) = \left\{\Omega \to \mathbb{R} : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}.$$

We define its Luxemburg norm as

$$|u|_{p(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

For every $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, one has the following Hölder type inequality

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) |u|_{p(x)} |v|_{p'(x)}.$$

Let's now recall the following proposition.

Proposition 2.1 ([8]). *For every $u \in L^{p(x)}(\Omega)$, one has*

$$[|u|_{p(x)}]_p \le \int_\Omega |u(x)|^{p(x)} dx \le [|u|_{p(x)}]^p.$$

Proposition 2.2 ([9]). *Let p and q be two functions in $C_+(\overline{\Omega})$ satisfying $q(x) \le p(x)$, a.e. on Ω, then $L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, and*

$$|u|_{q(x)} \le c_q |u|_{p(x)},$$

where c_q is a positive constant.

For for $k \in \{1, 2\}$, the Sobolev space with variable exponent $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index such that $|\alpha| = \Sigma_{i=1}^N \alpha_i$, $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_N} x_N}$. The above space endowed with the norm

$$\|u\|_{k,p(x)} = \Sigma_{|\alpha|\leq k} |D^\alpha u|_{p(x)},$$

is a reflexive separable Banach space. Let $W_0^{1,p(x)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, which has the norm $\|u\|_{1,p(x)} = |Du|_{p(x)}$. In the following, let

$$X := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$$

endowed with the norm

$$\|u\| := \inf\left\{\mu > 0 \int_\Omega \left|\frac{\Delta u}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

The modular on X is the mapping $\rho_{p(x)} : X \to \mathbb{R}$ defined by $\rho_{p(x)}(u) = \int_\Omega |\Delta u|^{p(x)} dx$. This mapping meets the same characteristics as Proposition 2.3. To be more specific, we have the following.

Proposition 2.3. *For every $u \in L^{p(x)}(\Omega)$, one has*

(1) $\|u\| < 1\ (resp, = 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (resp, = 1, > 1)$,
(2) $[\|u\|]_p \leq \rho_{p(x)}(u) \leq [\|u\|]^p$.

Proposition 2.4 ([10]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$, and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $w \in L^{q(x)}(\Omega)$, $w \neq 0$. Then*

$$[|w|_{p(x)q(x)}]_p \leq \||w|^{p(x)}|_{q(x)} \leq [|w|_{p(x)q(x)}]^p.$$

Remember that the critical Sobolev exponent is defined as:

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - 2p(x)}, & p(x) < \dfrac{N}{2}, \\ +\infty, & p(x) \geq \dfrac{N}{2}. \end{cases}$$

As a result of Proposition 2.2, if $q(x) \leq p(x)$ a.e on Ω, we have

$$W_0^{1,p(x)}(\Omega) \hookrightarrow W_0^{1,q(x)}(\Omega) \quad \text{and} \quad W^{2,p(x)}(\Omega) \hookrightarrow W^{2,q(x)}(\Omega).$$

In particular, one has

$$X \hookrightarrow W_0^{1,p^-}(\Omega) \cap W^{2,p^-}(\Omega).$$

The following definitions and assertions will be required in Section 3.

Definition 2.1. *Let Φ and Ψ be two continuously Gâteaux differentiable functionals on a real Banach space X and let $d \in \mathbb{R}$. The functional $I := \Phi - \Psi$ verifies the Palais-Smale condition cut of upper at d ($(PS)^{[d]}$) if any sequence $\{u_n\}_{n\in\mathbb{N}} \in X$, which verifies*

- *$I(u_n)$ is bounded,*
- $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
- *$\Phi(u_n) < d$ for each $n \in \mathbb{N}$,*

has a convergent subsequence.

If $d = \infty$, the functional $I := \Phi - \Psi$ fulfills the Palais-Smale condition.

Our main existence result is due to the following theorem.

Theorem 2.1. *[11, Theorem 3.1] Let X be a real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{x\in X} \Phi = \Phi(0) = \Psi(0) = 0.$$

Suppose that there is a positive constant $d \in \mathbb{R}$ and $\overline{x} \in X$ with $0 < \Phi(\overline{x}) < d$ such that

$$\frac{\sup\limits_{x\in\Phi^{-1}(]-\infty,d])} \Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$$

and for any

$$\lambda \in \Lambda :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup\limits_{x\in\Phi^{-1}(]-\infty,d])} \Psi(x)}[,$$

$I_\lambda = \Phi - \lambda\Psi$ fulfills the $(PS)^{[d]}$-condition, so for every $\lambda \in \Lambda$, there is $x_\lambda \in \Phi^{-1}(]0, d])$ such that $I_\lambda(x_\lambda) \le I_\lambda(x)$ for all $x \in \Phi^{-1}(]0, d])$ and $I'_\lambda(u_\lambda) = 0$.

The multiplicity result is due to the following theorem.

Theorem 2.2 ([12])**.** *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable whose Gâteaux derivative is compact such that*

$$(a_0) \inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Suppose that there exist $d > 0$ and $\overline{x} \in X$, with $d < \Phi(\overline{x})$, such that

$$(a_1)\ \frac{\sup_{\Phi(x)<d} \Psi(x)}{d} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})},$$

(a_2) for each $\lambda \in \Lambda_d :=]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{d}{\sup_{\Phi(x)\le d} \Psi(x)}[$, $I_\lambda := \Phi - \lambda\Psi$ is coercive.

Then, for any $\lambda \in \Lambda_d$, $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Let $\delta(x) := \sup\{\delta > 0 \mid B(x, \delta) \subseteq \Omega\}$, for all $x \in \Omega$ where B is the ball centered at x and of radius δ. We can see easily that there exists $x^0 \in \Omega$ such that $B(x^0, R) \subseteq \Omega$, where $R = \sup\limits_{x\in\Omega} \delta(x)$. This paper will require the following hypotheses:

$A : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function with continuous derivative $a(x, \xi) = \partial_\xi A(x, \xi)$, satisfying $a(x, u + v) \leq c(a(x, u) + a(x, v)), \forall u, v \in W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega)$, for some positive constant c, and A satisfies the following assumptions:

$(A1)$ $A(x, 0) = 0, A(x, \xi) = A(x, -\xi)$ for all $x \in \Omega, \xi \in \mathbb{R}$;

$(A2)$ $|a(x, \xi)| \leqslant c_1 \left(\alpha(x) + |\xi|^{p(x)-1}\right)$ a.e. $(x, \xi) \in \Omega \times \mathbb{R}$, where $c_1 > 0, \alpha(x) \in L^{\frac{p(x)}{p(x)-1}}(\Omega), 1 < p(x) \in C(\overline{\Omega})$;

$(A3)$ $|\xi|^{p(x)} \leq a(x, \xi) \cdot \xi \leq p(x)A(x, \xi)$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}$;

$(A4)$ Assume that $f \in L^{\gamma(x)}(\Omega)$ satisfies the following:

$$f(x) := \begin{cases} \leq 0, & \text{for } x \in \Omega \setminus B(x^0, R), \\ \geq f_0, & \text{for } x \in B(x^0, \frac{R}{2}), \\ > 0, & \text{for } \ x \in B(x^0, R) \setminus B(x_0, \frac{R}{2}), \end{cases}$$

where $B(x^0, R)$ is the ball of radius R centered at x^0 and f_0 is a positive constant.

Remark 2.1. *By* $(A2)$*, one has*

$$|A(x, t)| \leq C \left(\alpha(x)|t| + |t|^{p(x)}\right) \text{ for a.e. } x \in \Omega, \quad \text{and all } \ t \in \mathbb{R},$$

for some constant $C > 0$.

Throughout this work, we will denote by $m := \dfrac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$, where Γ is the Gamma function.

3. Existence

This section is devoted to show some reacquired results needed to establish the existence and multiplicity result. We start by recalling the following remark.

Remark 3.1 (Kefi [13])**.** *The conjugate exponent of the function* $\gamma(x)$ *will be denoted by* $\gamma'(x)$ *and* $\beta(x) := \dfrac{\gamma(x)r(x)}{\gamma(x) - r(x)}$. *There are compact and continuous embeddings* $X \hookrightarrow L^{\gamma'(x)r(x)}(\Omega)$ *and* $X \hookrightarrow L^{\beta(x)}(\Omega)$*, as well as the best constant* $k > 0$ *such that*

$$|u|_{\gamma'(x)r(x)} \leq k\|u\|. \tag{3.1}$$

In what follows, we recall the Hardy-Rellich inequality [14].

Lemma 3.1. *For* $1 < s < N/2$ *and* $u \in W_0^{1,s}(\Omega) \cap W^{2,s}(\Omega)$*, we have*

$$\int_\Omega \frac{|u(x)|^s}{|x|^{2s}} dx \leq \frac{1}{\mathcal{H}_s} \int_\Omega |\Delta u(x)|^s dx,$$

where

$$\mathcal{H}_s := \left(\frac{N(s-1)(N-2s)}{s^2}\right)^s.$$

Let's review what a weak solution to Problem (1.1) is

Definition 3.1. *$u \in X\backslash\{0\}$ is a weak solution of Problem* (1.1) *if $\Delta u = 0$ on $\partial\Omega$ and*

$$\int_\Omega a(x, \Delta u)\Delta v dx + \int_\Omega \theta(x)\frac{|u|^{s-2}}{|x|^{2s}}uv dx - \lambda \int_\Omega f(x)|u|^{r(x)-2}uv dx = 0, \quad \textit{for all} \;\; v \in X.$$

Let us denote by

$$\Psi(u) := \int_\Omega \frac{1}{r(x)} f(x)|u|^{r(x)} dx.$$

The Euler-Lagrange functional for Problem (1.1) is thus defined as $I_\lambda : X \to \mathbb{R}$,

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \text{ for all } u \in X,$$

where

$$\Phi(u) = \int_\Omega A(x, \Delta u)dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|u(x)|^s}{|x|^{2s}}dx.$$

It is obvious that assertion (a_0) of Theorem 2.2 holds, moreover Remark 2.1 assures that Φ is well-defined and by virtue of Proposition 2.4 and Remark 3.1, Ψ it the same as well, since we have for all $u \in X$,

$$|\Psi(u)| \leq \frac{1}{r^-}\int_\Omega |f(x)||u|^{r(x)}dx \leq \frac{1}{r^-}|f(x)|_{\gamma(x)}||u|^{r(x)}|_{\gamma'(x)} \leq \frac{1}{r^-}|f(x)|_{\gamma(x)}[|u|_{\gamma'(x)r(x)}]^r.$$

Furthermore, by inequality (3.1) in Remark 3.1, one has

$$|\Psi(u)| \leq \frac{1}{r^-}|f(x)|_{r(x)}|[k\|u\|]^r.$$

As a result, Ψ is a well-defined and

$$\langle \Psi'(u), v\rangle := \Psi'(u)[v] = \int_\Omega f(x)|u|^{r(x)-1}v dx,$$

for all $u, v \in X$ is compact (see [15]). Moreover, by using Proposition 2.3 and hypothesis ($A3$) for $u \in X$ with $\|u\| > 1$, one has

$$\Phi(u) \geq \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx \geq \frac{1}{p^+}\rho_{p(x)}(u) \geq \frac{1}{p^+}\|u\|^{p^-}, \tag{3.2}$$

and consequently, Φ is coercive. On the other hand, Φ is sequentially weakly lower semi-continuous, and of class C^1 on X (Liu-Zhao [6]), with

$$\Phi'(u)[v] = \int_\Omega (a(x, \Delta u)\cdot \Delta v + \theta(x)\frac{|u(x)|^{s-2}uv}{|x|^{2s}})dx.$$

Moreover, we have the following.

Proposition 3.1. *$\Phi' : X \to X^*$ is coercive and uniformly monotone and admits a continuous inverse in X^*.*

Proof. For the coercivity and due to assertion ($A3$) and Proposition 2.3, one has for any $u \in X$ with $||u|| \geq 1$,

$$\frac{\langle \Phi'(u), u\rangle}{||u||} \geq ||u||^{p^- -1},$$

so Φ' is coercive.

Using the assumption on θ, one has

$$\int_\Omega \frac{\theta(x)}{|x|^{2s}}(|u|^{s-2}u - |v|^{s-2}v)(u-v)dx \geq \frac{ess\inf_{x\in\overline{\Omega}} \theta(x)}{(diam(\Omega))^{2s}} \int_\Omega (|u|^{s-2}u - |v|^{s-2}v)(u-v)dx. \tag{3.3}$$

Now, let $U_\beta = \{x \in \Omega : \beta(x) \geq 2\}$ and $V_\beta = \{x \in \Omega : 1 < \beta(x) < 2\}$. By using the elementary inequality [16], for $\beta > 1$ there exists a positive constant C_β, such that if $\beta \geq 2$, then

$$\left\langle |x|^{\beta-2}x - |y|^{\beta-2}y, x-y \right\rangle \geq C_\beta |x-y|^\beta, \text{ for } \beta \geq 2 \tag{3.4}$$

and if $1 < \beta < 2$, then

$$\left\langle |x|^{\beta-2}x - |y|^{\beta-2}y, x-y \right\rangle \geq C_\beta \frac{|x-y|^2}{(|x|+|y|)^{2-\beta}}, \text{ for } 1 < \beta < 2, \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$. Due to the fact that for any $u, v \in X$,

$$a(x, u+v) \leq c(a(x,u) + a(x,v)),$$

for some $c > 0$, and by assumptions (A1) and (A3), we have

$$\begin{aligned}
\langle \Phi'(u) - \Phi'(v), u-v \rangle &= \int_\Omega (a(x,\Delta u) - a(x,\Delta v)) \cdot (\Delta u - \Delta v)dx \\
&+ \int_\Omega \frac{\theta(x)}{|x|^{2s}}(|u|^{s-2}u - |v|^{s-2}v)(u-v)dx, \\
&= \int_\Omega (a(x,\Delta u) + a(x,-\Delta v)) \cdot (\Delta u - \Delta v)dx \\
&+ \int_\Omega \frac{\theta(x)}{|x|^{2s}}(|u|^{s-2}u - |v|^{s-2}v)(u-v)dx \\
&\geq \frac{1}{c}\int_\Omega a(x, \Delta u - \Delta v) \cdot (\Delta u - \Delta v)dx \\
&\geq \frac{1}{c}\int_\Omega |\Delta u - \Delta v|^{p(x)}dx.
\end{aligned}$$

By Proposition 2.3 and taking into account inequalities (3.3)–(3.5), one has, for any $u, v \in X$,

$$\langle \Phi'(u) - \Phi'(v), u-v \rangle \geq \frac{1}{c}[||u-v||]_p, \tag{3.6}$$

which assures that Φ' is uniformly monotone. Theorem 26.(A)d of [17] ends the proof. □

Proposition 3.2. Φ' *satisfies the condition* $(S)_+$*, which means: If* $u_n \rightharpoonup u$ *and*

$$\overline{\lim}_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0,$$

then $u_n \to u$ *(strongly).*

Proof. Since $u_n \rightharpoonup u$ and $\overline{lim}_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0$, then due to (3.6), one has

$$0 \leq C_p \overline{\lim}_{n\to+\infty}[||u_n - v||]_p \leq \overline{lim}_{n\to+\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \leq 0,$$

which ends the proof. □

Remark 3.2. *Under assumptions* $(A2)$ *and* $(A3)$*, one has*

$$\frac{1}{p^+}[||u||]_p \leq \Phi(u) \leq K\left([||u||]^p + ||u||^s + ||u||\right),$$

where

$$K = \max\left\{C, C||\alpha||_{\frac{p(x)}{p(x)-1}}, \frac{|\theta|_\infty}{s\mathcal{H}_s}\right\}.$$

Proof. By using assumptions $(A2)$ and $(A3)$ and Proposition 2.3, we have

$$\begin{aligned}\frac{1}{p^+}[||u||]_p &\leq \int_\Omega \frac{1}{p(x)}|\Delta u|^{p(x)}dx\\ &\leq \Phi(u)\\ &\leq C\int_\Omega \alpha(x)|\Delta u|dx + C\int_\Omega |\Delta u|^{p(x)}dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|u(x)|^s}{|x|^{2s}}dx.\\ &\leq C||\alpha||_{\frac{p(x)}{p(x)-1}}||u|| + C\int_\Omega |\Delta u|^{p(x)}dx + \int_\Omega \theta(x)\frac{|u(x)|^s}{|x|^{2s}}dx.\end{aligned}$$

By Hardy's inequality, we deduce

$$\frac{1}{p^+}[||u||]_p \leq \Phi(u) \leq K\left([||u||]^p + ||u||^s + ||u||\right)$$

where

$$K = \max\left\{C, C||\alpha||_{\frac{p(x)}{p(x)-1}}, \frac{|\theta|_\infty}{s\mathcal{H}_s}\right\},$$

and this ends the proof. □

Remark 3.3. *If* $I'_\lambda(u) = 0$*, we have*

$$\int_\Omega \left(a(x, \Delta u)\cdot\Delta v + \theta(x)\frac{|u|^{s-2}uv}{|x|^{2s}}\right)dx - \lambda\int_\Omega f(x)|u|^{r(x)-2}uvdx = 0$$

for any $u, v \in X$*, which assures that the critical points of* I_λ *are exactly weak solutions of Problem* (1.1).

Lemma 3.2. I_λ *fulfills the Palais-Smale condition for any* $\lambda > 0$.

Proof. Let $\{u_n\} \subseteq X$ be a Palais-Smale sequence, so, one has

$$\sup_n I_\lambda(u_n) < +\infty \quad \text{and} \quad \left\|I'_\lambda(u_n)\right\|_{X^*} \longrightarrow 0. \tag{3.7}$$

Let us show that $\{u_n\} \subseteq X$ contains a convergent subsequence. By the Hölder inequality, Proposition 2.4 and Remark 3.1, we have

$$\begin{aligned}\langle \Psi'(u), u\rangle &= \int_\Omega f(x)|u|^{r(x)}dx\\ &\le |f|_{\gamma(x)}||u|^{r(x)}|_{\gamma'(x)}\\ &\le [k]^r|f|_{\gamma(x)}[||u||]^r.\end{aligned}$$

So, for n large enough, by assumption $(A3)$ and Proposition 2.3, one has

$$\begin{aligned}\left\langle I'_\lambda(u_n), u_n\right\rangle &= \left\langle \Phi'_\lambda(u_n), u_n\right\rangle - \lambda\left\langle \Psi'_\lambda(u_n), u_n\right\rangle\\ &\ge [||u_n||]_p - \lambda[k]^r|f|_{\gamma(x)}[||u_n||]^r.\end{aligned}$$

Moreover, using (3.7), we have

$$\left[||u_n||\right]_p \le \lambda[k]^r|f|_{\gamma(x)}[||u_n||]^r,$$

since $r^+ < p^-$, then $\{u_n\}$ is bounded and passing to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$. By Proposition 3.2, $u_n \to u$ (strongly) in X, and so I_λ fulfills the Palais-Smale condition. □

Our existence result is the following:

Theorem 3.1. *Suppose that there exist $d, \delta > 0$ such that*

$$K\left(\left[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right]^p + \left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)^s + \left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)\right) m\left(R^N - \left(\frac{R}{2}\right)^N\right) < d \tag{3.8}$$

where $m := \frac{\pi^{N/2}}{N/2\Gamma(N/2)}$ is the measure of the unit ball of $\mathbb{R}^N$, and Γ is the Gamma function. So, for any $\lambda \in]A_\delta, B_d[$, with

$$A_\delta := \frac{K\left(\left[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right]^p + \left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)^s + \left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)\right)\left(2^N-1\right)}{\frac{f_0}{r^+}[\delta]_r}$$

and

$$B_d := \frac{d}{\frac{(p^+)^{\frac{r^+}{p^-}}}{r^-}|f|_{\gamma(x)}\left[k\right]^r\left[[d]^{\frac{1}{p}}\right]^r},$$

and Problem (1.1) has at least one nontrivial weak solution.

Proof. We try to prove our existence result using Theorem 2.1. For this purpose, we have to show that all conditions of Theorem 2.1 are met.

To begin, for a given $\lambda > 0$, we mention that from Lemma 3.2, the functional I_λ fulfills the $(PS)^{[d]}$ condition. Let $d > 0, \delta > 0$ be as in (3.8) and let $w \in X$ be defined by

$$w(x) := \begin{cases} 0, & x \in \Omega\backslash B\left(x^0, R\right),\\ \delta, & x \in B\left(x^0, \frac{R}{2}\right),\\ \frac{\delta}{R^2-\left(\frac{R}{2}\right)^2}\left(R^2 - \sum_{i=1}^N \left(x_i - x_i^0\right)^2\right), & x \in B\left(x^0, R\right)\backslash B\left(x^0, \frac{R}{2}\right),\end{cases}$$

where $x = (x_1, \ldots, x_N) \in \Omega$. Then,

$$\sum_{i=1}^{N} \frac{\partial^2 w}{\partial x_i^2}(x) = \begin{cases} 0, & x \in \left(\Omega \backslash B\left(x^0, R\right)\right) \cup B\left(x^0, \frac{R}{2}\right), \\ -\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}, & x \in B\left(x_0, R\right) \backslash B\left(x^0, \frac{R}{2}\right). \end{cases}$$

So, by applying Remark 3.2, one has

$$\begin{aligned} &\frac{1}{p^+}\Big[\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\Big]_p m\left(R^N - \left(\frac{R}{2}\right)^N\right) \\ &< \Phi(w) \\ &\leq K\left(\Big[\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\Big]^p + \left(\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\right)^s + \left(\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\right)\right) m\left(R^N - \left(\frac{R}{2}\right)^N\right), \end{aligned}$$

so, $\Phi(w) < d$. On the other hand, one has

$$\Psi(w) \geq \int_{B\left(x^0, \frac{R}{2}\right)} \frac{f(x)}{r(x)} |w|^{r(x)} dx \geq \frac{f_0}{r^+}[\delta]_r m(\frac{R}{2})^N, \tag{3.9}$$

then, we deduce that

$$\frac{\Psi(w)}{\Phi(w)} > \frac{\frac{f_0}{r^+}[\delta]_r}{K\left(\Big[\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\Big]^p + \left(\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\right)^s + \left(\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\right)\right)(2^N - 1)}.$$

Using Remark 2.3, for any $u \in \Phi^{-1}((-\infty, d])$, we have

$$\frac{1}{p^+}\Big[\|u\|\Big]_p \leq \Phi(u) \leq d.$$

Hence, from Proposition 2.4 and Remark 3.2, we deduce

$$\Psi(u) \leq \frac{1}{r^-}|f|_{\gamma(x)}\Big[|u|_{r(x)\gamma'(x)}\Big]^r \leq \frac{1}{r^-}|f|_{\gamma(x)}\Big[k\|u\|\Big]^r. \tag{3.10}$$

So

$$\sup_{\Phi(u)\leq d} \Psi(u) \leq \frac{(p^+)^{\frac{r^+}{p^-}}}{r^-}|f|_{\gamma(x)}\Big[k\Big]^r\Big[[d]^{\frac{1}{p}}\Big]^r.$$

As a result, the criteria of Theorem 2.1 are confirmed. So, for any

$$\lambda \in]A_\delta, B_d[\subseteq]\frac{\Phi(w)}{\Psi(w)}, \frac{d}{\sup_{u\in\Phi^{-1}(]-\infty,d])}\Psi(u)}[,$$

I_λ admits at least one nonzero critical point, which is the problem's weak solution. □

4. Triple solutions

Theorem 4.1. *For any $\lambda \in]A_\delta, B_d[$, A_δ and B_d are those of Theorem 3.1,*

$$A_\delta := \frac{K\left(\left[\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right]^p + \left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)^s + \left(\frac{2\delta N}{R^2-\left(\frac{R}{2}\right)^2}\right)\right)\left(2^N - 1\right)}{\frac{f_0}{r^+}[\delta]_r}$$

and

$$B_d := \frac{d}{\frac{(p^+)^{\frac{r^+}{p^-}}}{r^-}|f|_{\gamma(x)}\left[k\right]^r\left[[d]^{\frac{1}{p}}\right]^r},$$

and Problem (1.1) admits at least three weak solutions.

Proof. Note that Φ and Ψ fulfill the regularity assumptions of Theorem 2.2. Let us verify conditions (i) and (ii) of this theorem. For this purpose, let

$$\frac{1}{p^+}\Big[\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\Big]_p m\left(R^N - \left(\frac{R}{2}\right)^N\right) = d$$

and let $w \in X$ be as mentioned above, that is,

$$w(x) := \begin{cases} 0, & x \in \Omega \backslash B\left(x^0, R\right), \\ \delta, & x \in B\left(x^0, \frac{R}{2}\right), \\ \frac{\delta}{R^2-\left(\frac{R}{2}\right)^2}\left(R^2 - \sum_{i=1}^{N}\left(x_i - x_i^0\right)^2\right), & x \in B\left(x^0, R\right) \backslash B\left(x^0, \frac{R}{2}\right). \end{cases}$$

So, by applying assumption $(A3)$ and Remark 3.2, one has

$$\Phi(w) = \int_\Omega A(x, \Delta w)dx + \frac{1}{s}\int_\Omega \theta(x)\frac{|w(x)|^s}{|x|^{2s}}dx > \frac{1}{p^+}\Big[\frac{2\delta N}{R^2 - \left(\frac{R}{2}\right)^2}\Big]_p m\left(R^N - \left(\frac{R}{2}\right)^N\right) = d.$$

Therefore, the assumption (i) of Theorem 2.2 holds. Let us show that I_λ is coercive for any $\lambda > 0$. By using (3.10), one has

$$\Psi(u) \leq \frac{1}{r^-}|f|_{\gamma(x)}\left[k\|u\|\right]^r,$$

additionally, from Remark 3.2,

$$\frac{1}{p^+}\left[\|u\|\right]_p \leq \Phi(u).$$

So,

$$I_\lambda(u) \geq \frac{1}{p^+}\left[\|u\|\right]_p - \frac{1}{r^-}|f|_{\gamma(x)}\left[k\|u\|\right]^r,$$

and by using $p^- > r^+ > 1$, we deduce that I_λ is coercive and, consequently, condition (ii) is fulfilled, which assures that all assumptions of Theorem 4.1 are satisfied. So, for any $\lambda \in]A_\delta, B_d[$, I_λ has at least three distinct critical points, which represents the weak solutions of Problem (1.1). □

Example 4.1. *Let* $\Omega = B(0,1) \subset \mathbb{R}^3$ *be the unit ball and put* s *such that* $1 < s < \frac{11}{10}$ *and* $a(x,t) = \vartheta(x)|t|^{p(x)-2}t$, $\vartheta \equiv 1$; *moreover, let* $\gamma(x) = \gamma$ *be a constant such that* $\gamma > \frac{3}{2}$ *and define the variable exponents and weight function as,*

$$p(x) = \frac{12}{10} + \frac{1}{4}|x|^2, \quad r(x) = \frac{11}{10} + \frac{1}{8}|x|^2.$$

It's obvious that

$$1 < s < r^- = \frac{11}{10} \le r^+ = \frac{49}{40} < p^- = \frac{12}{10} \le p^+ = \frac{29}{20} < \frac{3}{2} < \gamma.$$

$$f(x) = \begin{cases} \frac{1}{|x|^2}, & \text{if } |x| \le \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} < |x| \le 1. \end{cases}$$

Problem (1.1) *can be written as*

$$\begin{cases} \Delta^2_{p(x)}u + \theta(x)\frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x)|u|^{q(x)-2}u, & \text{in } B(0,1), \\ u = \Delta u = 0, & \text{on } \partial B(0,1), \end{cases}$$

and admits three distinct weak solutions, one of which may be the trivial one.

5. Conclusions

The paper focuses on the existence and multiplicity of three weak solutions for a Leray-Lions $p(x)$-biharmonic problem involving constant Hardy potential. We may extend the obtained results to the existence of generalized solutions to Kirchhoff type elliptic equations involving variable Hardy singular coefficients such as the following sixth-order variable exponent Kirchhoff type $p(x)$-triharmonic equations involving $(h, r(x))$-Hardy singular coefficients:

$$\begin{cases} M(J(u))(\Delta^3_{p(x)}u + a(x)|u|^{p(x)-2}u) + \dfrac{b(x)|u|^{h-2}u}{|x|^{3h}} + \dfrac{c(x)|u|^{q(x)-2}u}{|x|^{r(x)}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Furthermore, we can reduce the regularity of corresponding weighted functions $a(x), b(x), c(x)$.

Author contributions

Conceptualization: Kefi formulated the initial research problem and developed the overarching mathematical framework. Validation: Liu independently verified the correctness of the mathematical results. Both authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. J. Leray, J. L. Lions, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France*, **93** (1965), 97–107. https://doi.org/10.24033/bsmf.1617
2. M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, Vol. 1748, Berlin: Springer, 2000. https://doi.org/10.1007/BFb0104029
3. V. V. Zhikov, Lavrentiev phenomenon and homogenization for some variational problems, *C. R. Acad. Sci. Paris Sér. I Math.*, **316** (1993), 435–439.
4. K. Rajagopal, M. Ružička, Mathematical modeling of electrorheological materials, *Contin. Mech. Thermodyn.*, **13** (2001), 59–78. https://doi.org/10.1007/s001610100034
5. Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image processing, *SIAM J. Appl. Math.*, **66** (2006), 1383–1406. https://doi.org/10.1137/050624522
6. J. Liu, Z. Zhao, Leray-Lions type $p(x)$-biharmonic equations involving Hardy potentials, *Appl. Math. Lett.*, **149** (2024), 108907. https://doi.org/10.1016/j.aml.2023.108907
7. X. Fan, Q. Zhang, Existence of solutions for $p(x)$-Laplacian Dirichlet problem, *Nonlinear Anal.*, **12** (2003), 1843–1852. https://doi.org/10.1016/S0362-546X(02)00150-5
8. Y. Karagiorgos, N. Yannakaris, A Neumann problem involving the $p(x)$-Laplacian with $p = \infty$ in a subdomain, *Adv. Calc. Var.*, **9** (2016), 65–76. https://doi.org/10.1515/acv-2014-0003
9. X. L. Fan, D. Zhao, On the generalized Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$, *J. Gansu Educ. College*, **12** (1998), 1–6.
10. D. Edmunds, J. Rákosnik, Sobolev embeddings with variable exponent, *Studia Math.*, **143** (2000), 267–293. https://doi.org/10.4064/sm-143-3-267-293
11. G. Bonanno, P. Candito, G. D'Aguí, Variational methods on finite dimensional Banach spaces and discrete problems, *Adv. Nonlinear Stud.*, **14** (2014), 915–939. https://doi.org/10.1515/ans-2014-0406
12. G. Bonanno, S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, *Appl. Anal.,* **89** (2010), 1–10. https://doi.org/10.1080/00036810903397438

13. K. Kefi, $p(x)$-Laplacian with indefinite weight, *Proc. Amer. Math. Soc.*, **139** (2011), 4351–4360. https://doi.org/10.1515/10.1090/S0002-9939-2011-10850-5

14. E. B. Davis, A. M. Hinz, Explicit constants for Rellich inequalities in $L_p(\Omega)$, *Math. Z.*, **227** (1998), 511–523. https://doi.org/10.1007/PL00004389

15. K. Kefi, N. Irzi, M. M. Al-Shomrani, D. D. Repovš, On the fourth-order Leray-Lions problem with indefinite weight and nonstandard growth conditions, *Bull. Math. Sci.*, **12** (2022), 2150008, https://doi.org/10.1142/S1664360721500089

16. J. Simon, Régularité de la solution d'une équation non linéaire dans $\mathbb{R}^N$, In: P. Bénilan, J. Robert, *Journées d'analyse non linéaire*, Lecture Notes in Mathematics, Berlin: Springer, **665** (1978), 205–227. https://doi.org/10.1007/BFb0061807

17. E. Zeilder, *Nonlinear functional analysis and its applications: II/B: nonlinear monotone operators*, Berlin: Springer, 1990. https://doi.org/10.1007/978-1-4612-0981-2

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