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Research article

Local superderivations of Lie superalgebra osp(1, 2) to all simple modules

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Abstract: In this paper, we consider the orthogonal symplectic Lie superalgebra osp(1, 2) over an algebraically closed field of prime characteristic $p > 2$. Using the classification of the simple modules of the Lie superalgebra osp(1, 2), we prove that every local superderivation of osp(1, 2) to any simple module is a superderivation.

Keywords: Lie superalgebras; orthogonal symplectic Lie superalgebras; simple modules; superderivations; local superderivations
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1. Introduction

The concept of local derivations was originally proposed by Kadison, Larson, and Sourour in 1990 for the study of Banach algebras (see [6, 7]). In 2016, Ayupov and Kudaybergenov studied the local derivations of a Lie algebra. They asserted that every local derivation of a finite-dimensional semisimple Lie algebra over an algebraically closed field of characteristic zero is a derivation (see [1]). Many researchers have focused on studying local derivations of Lie algebras (see [2,12,14]). Motivated by [1], Chen et al. introduced the definition of local superderivations for a Lie superalgebra in 2017 (see [5]). More and more scholars have begun to study local superderivations of Lie superalgebras. In [4, 5, 13], Chen, Wang, and Yuan et al. studied local superderivations of a simple Lie superalgebra. They proved that every local superderivation is a superderivation for basic classical Lie superalgebras (except A(1, 1)), the strange Lie superalgebra q_n, and Cartan-type Lie superalgebras over the complex field. In [3, 11], Camacho and Wu et al. reached a similar conclusion for a particular class of solvable Lie superalgebras and the super-Virasoro algebras over the complex field. One can also consider local superderivations of a Lie superalgebra in their modules. When the simple modules of a Lie superalgebra are completely clear, it is possible to determine the local superderivations of a Lie superalgebra for all simple modules. In [10], Wang et al. determined the simple modules of the orthogonal symplectic Lie superalgebra osp(1, 2) over a field of prime characteristic. In [8, 9], Wang et al. studied the 2-local derivations of Lie algebra sl(2) for all simple modules and the first cohomology of osp(1, 2) with coefficients in simple modules over a field of prime characteristic.

In this paper, we are interested in determining all local superderivations of the Lie superalgebra osp(1, 2) for all simple modules over a field of prime characteristic. The paper is structured as follows: In Section 2, we recall the basic concepts and establish several lemmas. In Lemma 2.1, we show the connection between the bases of the simple module and the bases of the inner superderivation space. We introduce the notion of local superderivations for a Lie superalgebra to any finite-dimensional module (see Definition 2.1). By [10], any simple module of osp(1, 2) is isomorphic to some simple module $L_\chi(\lambda)$ for highest weight λ and p-character χ, and χ is either regular nilpotent, regular semisimple, or restricted. The first cohomology of osp(1, 2) with coefficients in $L_\chi(\lambda)$ was described in [9], from which we obtain the bases of the vector space of superderivations. We introduce the method to determine the local superderivations of osp(1, 2) to $L_\chi(\lambda)$ of parity α in Lemma 2.2. In Section 3 (resp. Section 4), we show that every local superderivation of osp(1, 2) to $L_\chi(\lambda)$ with χ being regular nilpotent or regular semisimple (resp. χ is restricted) is a superderivation.

2. Preliminaries

In this paper, the underlying field $\mathbb{F}$ is algebraically closed and of prime characteristic $p > 2$, and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ is the additive group of order two with addition, in which $\bar{1} + \bar{1} = \bar{0}$. Recall that a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is also called a superspace, where the elements of $V_{\bar{0}}$ (resp. $V_{\bar{1}}$) are said to be even (resp. odd). For $\alpha \in \mathbb{Z}_2$, any element v of V_α is said to be homogeneous of parity α, denoted by $|v| = \alpha$. Write $\{x_1, \ldots, x_p \mid y_1, \ldots, y_q\}$ implying that x_i is even and y_j is odd in a superspace. If $\{x_1, \ldots, x_p \mid y_1, \ldots, y_q\}$ is a $\mathbb{Z}_2$-homogeneous basis of a $\mathbb{Z}_2$-graded vector space V, we write $V = \langle x_1, \ldots, x_p \mid y_1, \ldots, y_q\rangle$. Denote by $\mathrm{Hom}(V, W)$ the set consisting of all the $\mathbb{F}$-linear maps from V to W, where V and W are $\mathbb{Z}_2$-graded vector spaces. We define the $\mathbb{Z}_2$-gradation on $\mathrm{Hom}(V, W)$ by $\mathrm{Hom}(V, W)_\alpha = \{\phi \in \mathrm{Hom}(V, W) \mid \phi(V_\beta) \subset W_{\alpha+\beta},\ \beta \in \mathbb{Z}_2\}$.

Let L be a Lie superalgebra and M an L-module. Recall that a $\mathbb{Z}_2$-homogeneous linear map of parity α, $\phi : L \to M$, is called a superderivation of parity α if

$$\phi([x, y]) = (-1)^{\alpha|x|} x\phi(y) - (-1)^{|y|(|x|+\alpha)} y\phi(x), \text{ for all } x,\ y \in L.$$

Write $\mathrm{Der}(L, M)_\alpha$ for the set of all superderivations of L to M of parity α. It is easy to verify that $\mathrm{Der}(L, M)_\alpha$ is a vector space. Denote

$$\mathrm{Der}(L, M) = \mathrm{Der}(L, M)_{\bar{0}} \oplus \mathrm{Der}(L, M)_{\bar{1}}.$$

For a $\mathbb{Z}_2$-homogeneous element $m \in M$, define the linear map D_m of L to M by $D_m(x) = (-1)^{|x||m|} x.m$, where $x \in L$. Then D_m is a superderivation of parity $|m|$. Let $\mathrm{Ider}(L, M)$ be the vector space spanned by all D_m with $\mathbb{Z}_2$-homogeneous elements $m \in M$. Then every element in $\mathrm{Ider}(L, M)$ is called an inner superderivation. It is easy to check that

$$\mathfrak{D} : M \to \mathrm{Ider}(L, M),\ m \mapsto D_m \tag{2.1}$$

is an even linear map. Then we have the following lemma, which is simple and useful.

Lemma 2.1. Let $\mathrm{H}^0(L, M) = 0$. Then the linear map $\mathfrak{D}$ (defined by Eq (2.1)) is a linear isomorphism. In particular, $\{D_{m_1}, D_{m_2}, \ldots, D_{m_k}\}$ is a basis of $\mathrm{Ider}(L, M)$ if and only if $\{m_1, m_2, \ldots, m_k\}$ is a basis of M.

Recall the well-known fact that the first cohomology of L with coefficients in L-module M is

$$\mathrm{H}^1(L, M) = \mathrm{Der}(L, M)/\mathrm{Ider}(L, M).$$

Obviously, $\mathrm{H}^1(L, M) = 0$ is equivalent to $\mathrm{Der}(L, M) = \mathrm{Ider}(L, M)$.

Definition 2.1. A $\mathbb{Z}_2$-homogeneous linear map ϕ_α of a Lie superalgbra L to L-mod M of parity α is called a local superderivation if, for any $x \in L$, there exists a superderivation $D_x \in \mathrm{Der}(L, M)_\alpha$ (depending on x) such that $\phi_\alpha(x) = D_x(x)$.

Let $B_\alpha = \{D_1, D_2, \ldots, D_m\}$ be a basis of $\mathrm{Der}(L, M)_\alpha$ and $T_\alpha \in \mathrm{Hom}(L, M)_\alpha$. For $x \in L$, we write $M(B_\alpha; x)$ for the matrix $(D_1x\ D_2x\ \ldots\ D_mx)$ and $M(B_\alpha, T_\alpha; x)$ for the matrix $(M(B_\alpha; x)\ T_\alpha x)$, where $\alpha \in \mathbb{Z}_2$. The following lemma can be easily verified by Definition 2.1.

Lemma 2.2. Let T_α be a homogeneous linear map of a Lie superalgbra L to L-mod M of parity α. Then T_α is a local superderivation of parity α if and only if the rank of $M(B_\alpha; x)$ is equal to the rank of $M(B_\alpha, T_\alpha; x)$ for any $x \in L$ and $\alpha \in \mathbb{Z}_2$.

Set $h := E_{22} - E_{33},\ e := E_{23},\ f := E_{32},\ E := E_{13} + E_{21},\ F := E_{12} - E_{31}$, where E_{ij} is the 3×3 matrix unit. Recall that $\{h, e, f, \mid E, F\}$ is the standard $\mathbb{Z}_2$-homogeneous basis of the Lie superalgebra $\mathrm{osp}(1, 2)$. Hereafter, we write L for $\mathrm{osp}(1, 2)$ over $\mathbb{F}$ and $L_\chi(\lambda)$ for the simple module of L with the highest weight λ and p-character χ. Recall the basic properties of $L_\chi(\lambda)$, which we discuss in this paper (see [10], Section 6). There are three orbits of $\chi \in L_{\bar{0}}^*$:

(1) regular nilpotent: $\chi(e) = \chi(h) = 0$ and $\chi(f) = 1$;

(2) regular semisimple: $\chi(e) = \chi(f) = 0$ and $\chi(h) = a^p$ for some $a \in \mathbb{F} \setminus \{0\}$;

(3) restricted: $\chi(e) = \chi(f) = \chi(h) = 0$.

That is, the p-character χ is either regular nilpotent, regular semisimple, or 0. We have the following standard basis for $L_\chi(\lambda)$. For $\lambda < p$, we have $L_0(\lambda) = \langle v_0, v_2, \ldots, v_{2\lambda-2} \mid v_1, v_3, \ldots, v_{2\lambda-1}\rangle$. For $\chi \neq 0$, we have $L_\chi(\lambda) = \langle v_0, v_2, \ldots, v_{2p-2} \mid v_1, v_3, \ldots, v_{2p-1}\rangle$. The L-action is given by

$$h.v_i = (\lambda - i)v_i,$$

$$e.v_i = \begin{cases} -\frac{i}{2}(\lambda + 1 - \frac{i}{2})v_{i-2}, & \text{if } i \text{ is even,} \\ -\frac{i-1}{2}(\lambda - \frac{i-1}{2})v_{i-2}, & \text{if } i \text{ is odd,} \end{cases}$$

$$f.v_i = \begin{cases} -v_{i+2}, & 0 \leqslant i \leqslant 2p-3, \\ \chi_f^p v_0, & i = 2p-2, \\ \chi_f^p v_1, & i = 2p-1, \end{cases}$$

$$E.v_i = \begin{cases} -\frac{i}{2}v_{i-1}, & \text{if } i \text{ is even,} \\ (\lambda - \frac{i-1}{2})v_{i-1}, & \text{if } i \text{ is odd,} \end{cases}$$

$$F.v_i = \begin{cases} v_{i+1}, & 0 \leqslant i \leqslant 2p-2, \\ -\chi_f^p v_0, & i = 2p-1. \end{cases}$$

By [9, Theorem 1.2], we have $\mathrm{H}^1(L, L_\chi(\lambda)) = \langle 0 \mid \psi_1,\ \psi_2\rangle$ for $(\lambda,\chi) = (p-1, 0)$, where

$$\psi_1(e) = v_{2p-3},\ \psi_1(E) = -v_{2p-2},\ \psi_2(f) = v_1,\ \psi_2(F) = v_0,$$
$$\psi_1(h) = \psi_1(f) = \psi_1(F) = \psi_2(h) = \psi_2(e) = \psi_2(E) = 0.$$

Otherwise, $\mathrm{H}^1(L, L_\chi(\lambda)) = \langle 0 \mid 0\rangle$. By Lemma 2.1, we have

$$\mathrm{Der}(L, L_\chi(\lambda)) = \begin{cases} \langle D_{v_0}, D_{v_2}, \dots, D_{v_{2p-2}} \mid D_{v_1}, D_{v_3}, \dots, D_{v_{2p-1}}\rangle, & \text{if } \chi \neq 0, \\ \langle D_{v_0}, D_{v_2}, \dots, D_{v_{2\lambda}} \mid D_{v_1}, D_{v_3}, \dots, D_{v_{2\lambda-1}}\rangle, & \text{if } \chi = 0 \text{ and } \lambda \neq p-1, \\ \langle D_{v_0}, D_{v_2}, \dots, D_{v_{2p-2}} \mid D_{v_1}, D_{v_3}, \dots, D_{v_{2p-3}}, \psi_1, \psi_2\rangle, & \text{if } \chi = 0 \text{ and } \lambda = p-1. \end{cases}$$

3. The regular nilpotent or semisimple case

In this section, we shall characterize local superderivations of L to the simple module $L_\chi(\lambda)$, where $\chi \neq 0$. We have (see Section 2)

$$\mathrm{Der}(L, L_\chi(\lambda)) = \langle D_{v_0}, D_{v_2}, \dots, D_{v_{2p-2}} \mid D_{v_1}, D_{v_3}, \dots, D_{v_{2p-1}}\rangle.$$

Let D_i be the matrix of D_{v_i} under the standard ordered bases of L and $L_\chi(\lambda)$. That is,

$$(D_{v_i}(h), D_{v_i}(e), D_{v_i}(f), D_{v_i}(E), D_{v_i}(F)) = (v_0, v_2, \dots, v_{2p-2} \mid v_1, v_3, \dots, v_{2p-1})D_i.$$

By the definition of innner superderivations, we have

$$(D_{v_i}(h), D_{v_i}(e), D_{v_i}(f), D_{v_i}(E), D_{v_i}(F)) = \begin{cases} (h.v_i, e.v_i, f.v_i, E.v_i, F.v_i), & \text{if } i \text{ is even}, \\ (h.v_i, e.v_i, f.v_i, -E.v_i, -F.v_i), & \text{if } i \text{ is odd}. \end{cases}$$

Hereafter, write ε_i for the 5-dimensional column vector in which i entry is 1 and the other entries are 0 as well as $E_{i,j}$ (resp. $\tilde{E}_{i,j}$) for the $2p \times 5$ (resp. $p \times p$) matrix in which (i, j) entry is 1 and the other entries are 0. Then for $t \in \{0, 1, \dots, p-2\}$, we have

$$\begin{aligned} D_{2t} &= (\lambda - 2t)E_{t+1,1} - t(\lambda + 1 - t)E_{t,2} - E_{t+2,3} - tE_{p+t,4} + E_{p+t+1,5}, \\ D_{2t+1} &= (\lambda - 2t - 1)E_{p+t+1,1} - t(\lambda - t)E_{p+t,2} - E_{p+t+2,3} - (\lambda - t)E_{t+1,4} - E_{t+2,5}, \\ D_{2p-1} &= (\lambda + 1)E_{2p,1} + (\lambda + 1)E_{2p-1,2} + \chi(f)^p E_{p+1,3} - (\lambda + 1)E_{p,4} + \chi(f)^p E_{1,5}, \\ D_{2p-2} &= (\lambda + 2)E_{p,1} + (\lambda + 2)E_{p-1,2} + \chi(f)^p E_{1,3} + E_{2p-1,4} + E_{2p,5}. \end{aligned}$$

For convenience, put $I = \{1, 2, 3, 4\}$ and $Y = \{y_1, y_2, \dots, y_9\}$, where $y_i = \varepsilon_{i+1},\ y_{4+j} = \varepsilon_j + \varepsilon_4$, $y_{6+m} = \varepsilon_m + \varepsilon_5$ for $i \in I,\ j \in I \setminus \{3, 4\}$ and $m \in I \setminus \{4\}$. We introduce the following symbols for $k \in \{1, 2, \dots, p\}$:

$$M(B_{\bar{0}}; x)^1_k = \begin{pmatrix} \lambda x_1 & -\lambda x_2 & \cdots & 0 & 0 \\ -x_3 & (\lambda-2)x_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (\lambda - 2k + 4)x_1 & -(k-1)(\lambda - k + 2)x_2 \\ 0 & 0 & \cdots & -x_3 & (\lambda - 2k + 2)x_1 \end{pmatrix},$$

$$M(B_{\bar{1}};x)_k^1=\begin{pmatrix}-\lambda x_4 & 0 & \cdots & 0 & 0\\ -x_5 & -(\lambda-1)x_4 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -(\lambda-k+2)x_4 & 0\\ 0 & 0 & \cdots & -x_5 & -(\lambda-k+1)x_4\end{pmatrix},$$

$$M(B_{\bar{0}};x)_k^2=\begin{pmatrix}x_5 & -x_4 & \cdots & 0 & 0\\ 0 & x_5 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & x_5 & -(k-1)x_4\\ 0 & 0 & \cdots & 0 & x_5\end{pmatrix},$$

$$M(B_{\bar{1}};x)_k^2=\begin{pmatrix}(\lambda-1)x_1 & -(\lambda-1)x_2 & \cdots & 0 & 0\\ -x_3 & (\lambda-3)x_1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (\lambda-2k+3)x_1 & -(k-1)(\lambda-k+1)x_2\\ 0 & 0 & \cdots & -x_3 & (\lambda-2k+1)x_1\end{pmatrix}.$$

Proposition 3.1. Suppose that p-character $\chi \neq 0$. Let T_α be a homogeneous linear map of L to $L_\chi(\lambda)$ of parity α. Then the following statements hold:

(1) Suppose that χ is regular nilpotent. The matrices $M(B_\alpha, T_\alpha; x)$ and $M(B_\alpha; x)$ have the same rank for any $x \in L$ if and only if $M(B_\alpha, T_\alpha; y_i)$ and $M(B_\alpha; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_2, y_4, y_6\}$ if $\alpha = \bar{0}$ and $y_i \in Y \setminus \{y_3, y_4, y_6\}$ if $\alpha = \bar{1}$.

(2) Suppose that χ is regular semisimple. The matrices $M(B_\alpha, T_\alpha; x)$ and $M(B_\alpha; x)$ have the same rank for any $x \in L$ if and only if $M(B_\alpha, T_\alpha; y_i)$ and $M(B_\alpha; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_4, y_6\}$ if $\alpha = \bar{0}$ and $y_i \in Y \setminus \{y_3, y_8\}$ if $\alpha = \bar{1}$.

Proof. Set $T_{\bar{0}} = \begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}$ and $T_{\bar{1}} = \begin{pmatrix}0 & C\\ D & 0\end{pmatrix}$, where $A,\ D \in M_{p,3}$ and $B,\ C \in M_{p,2}$. Write a_{ij}, b_{ql}, c_{il} and d_{qj} for the elements of matrix blocks A, B, C, and D, respectively, where $i, j \in \{1, 2, 3\}$, $q, l \in \{1, 2\}$. Let $X = (x_1, x_2, \ldots, x_5)^{\mathrm{T}}$ be the coordinate of any element $x \in L$ under the standard basis of L. In this proof, we write $l \in I$, $k \in I \setminus \{p\}$, $m \in I \setminus \{p-1, p\}$, $t \in I \setminus \{1\}$, where $I = \{1, 2, \ldots, p\}$.

(1) If χ is regular nilpotent, that is, $\chi(f) = 1$, $\chi(e) = \chi(h) = 0$. Then we have

$$M(B_{\bar{0}};x)=\begin{pmatrix}M(B_{\bar{0}};x)_p^1+x_3\tilde{E}_{1,p}\\ M(B_{\bar{0}};x)_p^2\end{pmatrix}\ \text{and}\ M(B_{\bar{1}};x)=\begin{pmatrix}M(B_{\bar{1}};x)_p^1+x_5\tilde{E}_{1,p}\\ M(B_{\bar{1}};x)_p^2+x_3\tilde{E}_{1,p}\end{pmatrix}.$$

Then, $M(B_{\bar{0}}, T_{\bar{0}}; x) = \begin{pmatrix}M(B_{\bar{0}}; x) & T_{\bar{0}}x\end{pmatrix}$ and $M(B_{\bar{1}}, T_{\bar{1}}; x) = \begin{pmatrix}M(B_{\bar{1}}; x) & T_{\bar{1}}x\end{pmatrix}$.

Since the matrices $M(B_{\bar{0}}, T_{\bar{0}}; y_i)$ and $M(B_{\bar{0}}; y_i)$ have the same rank for any $y_i \in Y\setminus\{y_2, y_4, y_6\}$, we have

$$a_{p,2}=b_{p,1}=0,\ a_{1,3}=b_{p,2},\ a_{l,1}=(\lambda-2l+2)b_{l,2},$$
$$a_{k,2}=-k(\lambda-k+1)b_{k+1,2},\ a_{k+1,3}=-b_{k,2},\ b_{k,1}=-kb_{k+1,2}.$$

It follows that for any $x \in L$,

$$T_{\bar{0}}x=b_{1,2}D_0x+b_{2,2}D_2x+\ldots+b_{p,2}D_{2p-2}x.$$

That is, $T_{\bar{0}}x$ is a linear combination of $\{D_0x, D_2x, \ldots, D_{2p-2}x\}$. Hence, $M(B_{\bar{0}}, T_{\bar{0}}; x)$ and $M(B_{\bar{0}}; x)$ have the same rank for any $x \in L$.

Since the matrices $M(B_{\bar{1}}, T_{\bar{1}}; y_i)$ and $M(B_{\bar{1}}; y_i)$ have the same rank for any $y_i \in Y\backslash\{y_3, y_4, y_6\}$, we have

$$\begin{aligned}
&d_{p,2} = 0,\ c_{p,1} = -(\lambda + 1)c_{1,2},\ d_{p-1,2} = d_{p,1} = (\lambda + 1)c_{1,2},\\
&c_{k,1} = (\lambda - k + 1)c_{k+1,2},\ d_{k,1} = -(\lambda - 2k + 1)c_{k+1,2},\\
&d_{m,2} = m(\lambda - m)c_{m+2,2},\ d_{l,3} = c_{l,2}.
\end{aligned}$$

It follows that for any $x \in L$,

$$T_{\bar{1}}x = -c_{2,2}D_1x - c_{3,2}D_3x - \ldots - c_{p,2}D_{2p-3}x + c_{1,2}D_{2p-1}x.$$

That is, $T_{\bar{1}}x$ is a linear combination of $\{D_1x, D_3x, \ldots, D_{2p-1}x\}$. Therefore, $M(B_{\bar{1}}, T_{\bar{1}}; x)$ and $M(B_{\bar{1}}; x)$ have the same rank for any $x \in L$.

(2) If χ is regular semisimple, that is, $\chi(e) = \chi(f) = 0, \chi(h) = a^p$, where $a \in \mathbb{F} \setminus \{0\}$. Then, for any $\alpha \in \mathbb{Z}_2$, we have

$$M(B_\alpha; x) = \begin{pmatrix} M(B_\alpha; x)_p^1 \\ M(B_\alpha; x)_p^2 \end{pmatrix}.$$

Therefore, $M(B_{\bar{0}}, T_{\bar{0}}; x) = \begin{pmatrix} M(B_{\bar{0}}; x) & T_{\bar{0}}x \end{pmatrix}$ and $M(B_{\bar{1}}, T_{\bar{1}}; x) = \begin{pmatrix} M(B_{\bar{1}}; x) & T_{\bar{1}}x \end{pmatrix}$.

A similar calculation, as in the case of regular nilpotent, shows that

$$\begin{aligned}
&a_{1,3} = a_{p,2} = b_{p,1} = 0,\ a_{l,1} = (\lambda - 2l + 2)b_{l,2},\\
&a_{k,2} = -k(\lambda - k + 1)b_{k+1,2},\ a_{k+1,3} = -b_{k,2},\ b_{k,1} = -kb_{k+1,2}.
\end{aligned}$$

It follows that for any $x \in L$,

$$T_{\bar{0}}x = b_{1,2}D_0x + b_{2,2}D_2x + \ldots + b_{p,2}D_{2p-2}x.$$

That is, $T_{\bar{0}}x$ is a linear combination of $\{D_0x, D_2x, \ldots, D_{2p-2}x\}$. Hence, $M(B_{\bar{0}}, T_{\bar{0}}; x)$ and $M(B_{\bar{0}}; x)$ have the same rank for any $x \in L$.

Since $M(B_{\bar{1}}, T_{\bar{1}}; y_i)$ and $M(B_{\bar{1}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_3, y_8\}$, we have

$$\begin{aligned}
&d_{p,2} = d_{1,3} = c_{1,2} = 0, d_{p,1} = d_{p-1,2} = -c_{p,1}, c_{k,1} = (\lambda - k + 1)c_{k+1,2},\\
&d_{k,1} = -(\lambda - 2k + 1)c_{k+1,2}, d_{m,2} = m(\lambda - m)c_{m+2,2}, d_{t,3} = c_{t,2}.
\end{aligned}$$

It follows that for any $x \in L$,

$$T_{\bar{1}}x = -c_{2,2}D_1x - c_{3,2}D_3x - \ldots - c_{p,2}D_{2p-3}x - \frac{1}{\lambda + 1}c_{p,1}D_{2p-1}x.$$

That is, $T_{\bar{1}}x$ is a linear combination of $\{D_1x, D_3x, \ldots, D_{2p-1}x\}$. Therefore, $M(B_{\bar{1}}, T_{\bar{1}}; x)$ and $M(B_{\bar{1}}; x)$ have the same rank for any $x \in L$. □

By Lemma 2.2, as a direct consequence of Proposition 3.1, we have the following theorem:

Theorem 3.1. Let $L_\chi(\lambda)$ be the simple module of $\mathrm{osp}(1, 2)$ with the highest weight λ and p-character χ. Suppose that p-character χ is regular nilpotent or regular semisimple. Then every local superderivation of $\mathrm{osp}(1, 2)$ to $L_\chi(\lambda)$ is a superderivation.

4. The restricted case

In this section, we shall characterize local superderivations of osp(1, 2) to the simple module $L_0(\lambda)$. We have (see Section 2)

$$\mathrm{Der}(L, L_0(\lambda)) = \begin{cases} \langle D_{v_0}, D_{v_2}, \ldots, D_{v_{2\lambda}} \mid D_{v_1}, D_{v_3}, \ldots, D_{v_{2\lambda-1}} \rangle, & \text{if } \lambda \neq p-1, \\ \langle D_{v_0}, D_{v_2}, \ldots, D_{v_{2p-2}} \mid D_{v_1}, D_{v_3}, \ldots, D_{v_{2p-3}}, \psi_1, \psi_2 \rangle, & \text{if } \lambda = p-1. \end{cases}$$

Let D_i be the matrix of D_{v_i} under the standard ordered bases of L and $L_0(\lambda)$. That is,

$$(D_{v_i}(h), D_{v_i}(e), D_{v_i}(f), D_{v_i}(E), D_{v_i}(F)) = (v_0, v_2, \ldots, v_{2\lambda} \mid v_1, v_3, \ldots, v_{2\lambda-1}) D_i.$$

By the definition of inner superderivations, we have

$$(D_{v_i}(h), D_{v_i}(e), D_{v_i}(f), D_{v_i}(E), D_{v_i}(F)) = \begin{cases} (h.v_i, e.v_i, f.v_i, E.v_i, F.v_i), & \text{if } i \text{ is even}, \\ (h.v_i, e.v_i, f.v_i, -E.v_i, -F.v_i), & \text{if } i \text{ is odd}. \end{cases}$$

Write $\tilde{\varepsilon}_i$ for the λ-dimensional column vector in which i entry is 1 and the other entries are 0 as well as $\hat{E}_{i,j}$ for the $(2\lambda+1) \times 5$ matrix in which (i, j) entry is 1 and the other entries are 0. Then for $m \in \{0, 1, \ldots, \lambda-1\}$, $n \in \{0, 1, \ldots, \lambda-2\}$, we have

$$\begin{aligned} D_{2m} &= (\lambda - 2m)\hat{E}_{m+1,1} - m(\lambda - m + 1)\hat{E}_{m,2} - \hat{E}_{m+2,3} - m\hat{E}_{\lambda+m+1,4} + \hat{E}_{\lambda+m+2,5}, \\ D_{2n+1} &= (\lambda - 2n - 1)\hat{E}_{\lambda+n+2,1} - n(\lambda - n)\hat{E}_{\lambda+n+1,2} - \hat{E}_{\lambda+n+3,3} - (\lambda - n)\hat{E}_{n+1,4} - \hat{E}_{n+2,5}, \\ D_{2\lambda-1} &= -(\lambda - 1)\hat{E}_{2\lambda+1,1} - (\lambda - 1)\hat{E}_{2\lambda,2} - \hat{E}_{\lambda,4} - \hat{E}_{\lambda+1,5}, \\ D_{2\lambda} &= -\lambda\hat{E}_{\lambda+1,1} - \lambda\hat{E}_{\lambda,2} - \lambda\hat{E}_{2\lambda+1,4}. \end{aligned}$$

Proposition 4.1. Let $T_{\bar{0}}$ be a homogeneous linear map of L to $L_0(\lambda)$ of parity $\bar{0}$, where $\lambda \in \{0, 1, \ldots, p-1\}$. Then the matrices $M(B_{\bar{0}}, T_{\bar{0}}; x)$ and $M(B_{\bar{0}}; x)$ have the same rank for any $x \in L$ if and only if $M(B_{\bar{0}}, T_{\bar{0}}; y_i)$ and $M(B_{\bar{0}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_3, y_4, y_8\}$.

Proof. Set $T_{\bar{0}} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A \in M_{\lambda+1,3}$ and $B \in M_{\lambda,2}$. Denote by a_{ij} and b_{ql} the elements of matrix blocks A and B, respectively, where $i, j \in \{1, 2, 3\}$, $q, l \in \{1, 2\}$. Let $X = (x_1, x_2, \ldots, x_5)^{\mathrm{T}}$ be the coordinate of any element $x \in L$ under the standard basis of L. In this proof, we write $k \in \{1, 2, \ldots, \lambda\}$, $t \in \{1, 2, \ldots, \lambda - 1\}$.

It is obviously true that the proposition holds for $\lambda = 0$. In the following, we assume that λ is not equal to 0. Denote

$$M^*(B_{\bar{0}}; x)^2_\lambda = \begin{pmatrix} M(B_{\bar{0}}; x)^2_\lambda & -\lambda x_4 \tilde{\varepsilon}_\lambda \end{pmatrix}_{\lambda \times (\lambda+1)}.$$

Then we have

$$M(B_{\bar{0}}; x) = \begin{pmatrix} M(B_{\bar{0}}; x)^1_{\lambda+1} \\ M^*(B_{\bar{0}}; x)^2_\lambda \end{pmatrix}.$$

Therefore, $M(B_{\bar{0}}, T_{\bar{0}}; x) = \begin{pmatrix} M(B_{\bar{0}}; x) & T_{\bar{0}}x \end{pmatrix}$. Since the matrices $M(B_{\bar{0}}, T_{\bar{0}}; y_i)$ and $M(B_{\bar{0}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_3, y_4, y_8\}$, we have

$$a_{1,3} = a_{\lambda+1,2} = 0, a_{\lambda,2} = b_{\lambda,1}, a_{k,1} = (\lambda - 2k + 2)b_{k,2},$$

$$a_{t,2} = -t(\lambda - t + 1)b_{t+1,2}, a_{k+1,3} = -b_{k,2}, b_{t,1} = -tb_{t+1,2}.$$

Therefore, for any $x \in L$, we have

$$T_{\bar{0}}x = b_{1,2}D_0x + b_{2,2}D_2x + \ldots + b_{\lambda,2}D_{2\lambda-2}x - \frac{1}{\lambda}b_{2\lambda,2}D_{2\lambda}x.$$

That is, $T_{\bar{0}}x$ is a linear combination of $\{D_0x, D_2x, \ldots, D_{2\lambda}x\}$. Therefore, $M(B_{\bar{0}}, T_{\bar{0}}; x)$ and $M(B_{\bar{0}}; x)$ have the same rank for any $x \in L$. □

Proposition 4.2. Let $T_{\bar{1}}$ be a homogeneous linear map of L to $L_0(\lambda)$ of parity $\bar{1}$, where $\lambda \in \{0, 1, \ldots, p-1\}$. Then the following statements hold:

(1) Suppose that $\lambda \neq p - 1$. The matrices $M(B_{\bar{1}}, T_{\bar{1}}; x)$ and $M(B_{\bar{1}}; x)$ have the same rank for any $x \in L$ if and only if $M(B_{\bar{1}}, T_{\bar{1}}; y_i)$ and $M(B_{\bar{1}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_8\}$.

(2) Suppose that $\lambda = p - 1$. The matrices $M(B_{\bar{1}}, T_{\bar{1}}; x)$ and $M(B_{\bar{1}}; x)$ have the same rank for any $x \in L$ if and only if $M(B_{\bar{1}}, T_{\bar{1}}; y_i)$ and $M(B_{\bar{1}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_1, y_2, y_3, y_4, y_8\}$.

Proof. Let $T_{\bar{1}} = \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix}$, where $C \in M_{\lambda+1,2}$ and $D \in M_{\lambda,3}$. Denote by c_{il} and d_{kj} the elements of matrix blocks C and D, respectively, where $i, j \in \{1, 2, 3\}$, $k, l \in \{1, 2\}$. Let $X = (x_1, x_2, \ldots, x_5)^{\mathrm{T}}$ be the coordinate of any element $x \in L$ under the standard basis of L. In this proof, we write $k \in J$, $m \in J \setminus \{1\}$, $t \in J \setminus \{\lambda\}$, where $J = \{1, 2, \ldots, \lambda\}$.

(1) Let $\lambda \neq p - 1$. Denote

$$M^*(B_{\bar{1}}; x)^1_\lambda = \begin{pmatrix} M(B_{\bar{1}}; x)^1_\lambda \\ -x_5(\tilde{\varepsilon}_\lambda)^{\mathrm{T}} \end{pmatrix}_{(\lambda+1)\times\lambda}.$$

Then we have

$$M(B_{\bar{1}}; x) = \begin{pmatrix} M^*(B_{\bar{1}}; x)^1_\lambda \\ M(B_{\bar{1}}; x)^2_\lambda \end{pmatrix}.$$

Therefore, $M(B_{\bar{1}}, T_{\bar{1}}; x) = \begin{pmatrix} M(B_{\bar{1}}; x) & T_{\bar{1}}x \end{pmatrix}$.

Since the matrices $M(B_{\bar{1}}, T_{\bar{1}}; y_i)$ and $M(B_{\bar{1}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_8\}$, we have

$$d_{1,3} = d_{\lambda,2} = c_{1,2} = c_{\lambda+1,1} = 0, d_{k,1} = -(\lambda - 2k + 1)c_{k+1,2},$$
$$d_{t,2} = -t(\lambda - t)c_{t+2,2}, d_{m,3} = c_{m,2}, c_{k,1} = (\lambda - k + 1)c_{k+1,2}.$$

Then, for any $x \in L$, we have

$$T_{\bar{1}}x = c_{2,2}D_1x + c_{3,2}D_3x + \ldots + c_{\lambda,2}D_{2\lambda-3}x + c_{\lambda+1,2}D_{2\lambda-1}x.$$

That is, $T_{\bar{1}}x$ is a linear combination of $\{D_1x, D_3x, \ldots, D_{2\lambda-1}x\}$. Therefore, $M(B_{\bar{1}}, T_{\bar{1}}; x)$ and $M(B_{\bar{1}}; x)$ have the same rank for any $x \in L$.

(2) Let $\lambda = p - 1$. Using the fact that $M(B_{\bar{1}}, T_{\bar{1}}; y_i)$ and $M(B_{\bar{1}}; y_i)$ have the same rank for any $y_i \in Y \setminus \{y_1, y_2, y_3, y_4, y_8\}$, we have

$$c_{k,1} = -kc_{k+1,2}, d_{k,1} = 2kc_{k+1,2}, d_{t,2} = -(t + 1)c_{t+2,1},$$
$$d_{k,3} = c_{k,2}, d_{p-1,2} = -c_{p,1}.$$

Therefore, for any $x \in L$, we have

$$T_{\bar{1}}x = -c_{2,2}D_1x - c_{3,2}D_3x - \ldots - c_{p,2}D_{2p-3}x - c_{p,1}\psi_1 x + c_{1,2}\psi_2 x.$$

That is, $T_{\bar{1}}x$ is a linear combination of $\{D_1x, D_3x, \ldots, D_{2p-1}x, \psi_1 x, \psi_2 x\}$. Therefore, $M(B_{\bar{1}}, T_{\bar{1}}; x)$ and $M(B_{\bar{1}}; x)$ have the same rank for any $x \in L$. □

By Lemma 2.2, as a direct consequence of Propositions 4.1 and 4.2, we have the following result:

Theorem 4.1. Let $L_\chi(\lambda)$ be the simple module of $\mathrm{osp}(1,2)$ with the highest weight λ and p-character χ. Suppose that p-character χ is restricted. Then every local superderivation of $\mathrm{osp}(1,2)$ to $L_\chi(\lambda)$ is a superderivation.

5. Conclusions

Let L be the orthogonal symplectic Lie superalgebra $\mathrm{osp}(1,2)$ over an algebraically closed field of prime characteristic $p > 2$. By [10], any simple module of L is isomorphic to some simple module $L_\chi(\lambda)$ for highest weight λ and p-character χ, and χ is either regular nilpotent, regular semisimple, or restricted. According to Theorems 3.1 and 4.1, the following conclusion can be summarized: Every local superderivation of L to any simple module is a superderivation over an algebraically closed field of prime characteristic $p > 2$.

We give an example. Every local superderivation of L to a 1-dimensional trivial module of L is a superderivation. In fact, according to the definition of superderivations, we can obtain that every superderivation of L to a 1-dimensional trivial module is equal to 0. According to Definition 2.1, we know that in this case, every local superderivation is equal to 0.

Author contributions

Shiqi Zhao: Writing-original draft, Editing; Wende Liu and Shujuan Wang: Supervision, Writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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