Research article

Poisson-Lindley minification INAR process with application to financial data

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Abstract: This paper introduces the Poisson-Lindley minification integer-valued autoregressive (PL-MINAR) process, a novel statistical model for analyzing count time series data. The modified negative binomial thinning and the Poisson-Lindley (PL) marginal distribution served as the foundation for the model. The proposed model was examined in terms of its basic stochastic properties, especially related to conditional stochastic measures (e.g., transition probabilities, conditional mean and variance, autocorrelation function). Through comprehensive simulations, the effectiveness of various parameter estimation techniques was validated. The PL-MINAR model's practical utility was demonstrated in analyzing the number of Bitcoin transactions and stock trades, showing its superior or comparable performance to the established INAR model. By offering a robust tool for financial time series analysis, this research holds potential for significant improvements in forecasting and understanding market dynamics.

1. Introduction

Integer-valued time series models have been widely used to model count time series. The most represented models of this type are the so-called integer-valued autoregressive (INAR) time series models, for which there is extensive literature (see, as the most recent, e.g., [1–8]). On the other hand, there are real-world datasets that cannot be modeled by ordinary INAR models, and for this reason various modifications are introduced. One of them is represented by the so-called minification time series models, which first appeared in [9, 10], where a continuous, exponential marginal distribution is considered. Thereafter, the identical structure's minification model with Weibull marginal distribution was shown in [11]. The first discrete minification models were presented in [12], while different types of minification models with discrete marginal distributions were discussed in [13]. The situation of geometric marginal distributions was the main focus of [14–16], where some new results about discrete minification processes have recently appeared. Specifically, the findings in Aleksić & Ristić [14], where the minification INAR method with geometric marginal distribution was developed, should be highlighted. Motivated by this issue, a new minification integer-valued autoregressive process with Poisson-Lindley marginal distribution (abbr. PL-MINAR process) is proposed here. Some preliminary assumptions necessary to introduce this stochastic process, as well as its definition and some key stochastic features, are described in Section 2. Then, Section 3 presents some estimation techniques of the PL-MINAR process parameters and Monte Carlo simulations of the obtained estimators. Practical applications of the PL-MINAR process in modeling the dynamics and empirical distribution of real-world time series, related to the count-based financial data, are given in Section 4. Here, the PL-MINAR process has also been compared with ordinary INAR(1) processes and shown to have the same or even better prediction efficiency and accuracy. Section 5 contains some closing thoughts, while proofs for the stated proposition and theorems are given in the Appendix.

2. PL-MINAR process and its properties

2.1. Preliminaries

In this part, the definition of modified negative binomial thinning is given first. After that, some key properties of this operator are presented, as well as its specificities in the case of the Poisson-Lindley (PL) distribution.

Definition 1. *Let $\alpha > 0$ and $G_j \sim Geom(\alpha/(1+\alpha))$, $j = 1, 2, \ldots$ be independent identically distributed (IID) random variables (RVs) with a geometric distribution, whose probability mass function (PMF) is as follows:*

$$p_G(x; \alpha) := P\{G_j = x\} = \frac{\alpha^x}{(1 + \alpha)^{x+1}}, \quad x = 0, 1, 2, \ldots \tag{1}$$

The modified negative binomial thinning operator " $\circ$ " is given by the equality:

$$\alpha \circ X := \sum_{j=1}^{X+1} G_j, \tag{2}$$

where X is the integer-valued RV, also independent of the RVs (G_j), called counting series related to the mentioned operator.

It is worth noting that this form of thinning operator was proposed for the first time by Zhang et al. [17]. It differs from the usual negative binomial thinning operator, introduced in Ristić et al. [18], where the term "X" is used instead of "$X + 1$". The main reason is that the upper bound $X + 1$ prevents the RV $\alpha \circ X$ from "vanishing" in the case where $X = 0$, because then Eq (2) yields $\alpha \circ 0 = G_1 \neq 0$. Consequently, in line with Aleksić & Ristić [14], using this kind of modified operator can avoid the problem of constant behaviour of zero, which may appear when considering a minification model with an ordinary binomial or negative binomial thinning operator. Furthermore, for a given $X = y$, the RV $\alpha \circ X$ represents a sum of $y+1$ geometrically distributed RVs. Thus, as is well known (e.g., Pitman [19]), the RV $\alpha \circ X$ has the negative binomial (NB) distribution with "the number of successes" $y + 1$ and parameter $p = \alpha/(1 + \alpha)$. In this way, the conditional PMF of the RV $\alpha \circ X$ is as follows:

$$P\{\alpha \circ X = x \mid X = y\} = \binom{x + y}{y}\left(\frac{\alpha}{1 + \alpha}\right)^x \left(\frac{1}{1 + \alpha}\right)^{y+1}, \tag{3}$$

where $x = 0, 1, 2, \ldots$ In addition, using Equations (1) and (2) for the conditional probability generating function (PGF) of the RV $\alpha \circ X$, one obtains:

$$\Psi_{\alpha \circ X|X}(u; \alpha) = E[u^{\alpha \circ X} \mid X] = \left(E[u^{G_j}]\right)^{X+1} = \left[\frac{1}{1 + \alpha} \sum_{x=0}^{\infty}\left(\frac{\alpha u}{1 + \alpha}\right)^x\right]^{X+1}$$
$$= (1 + \alpha - \alpha u)^{-X-1}, \tag{4}$$

where $|u| < (1 + \alpha)/\alpha$.

According to this and the well-known properties of PGFs and negative binomial thinning operator (e.g., Ristić et al. [18]), the conditional measures of $\alpha \circ X$ can be easily obtained. Thus, the conditional mean is

$$E\left[\alpha \circ X \mid X\right] = \frac{\partial \Psi_{\alpha \circ X|X}(u; \alpha)}{\partial u}\bigg|_{u=1} = \alpha(X + 1)(1 + \alpha - \alpha u)^{-X-2}\bigg|_{u=1} = \alpha(X + 1), \tag{5}$$

as well as

$$E[\alpha \circ X(X - 1) \mid X] = \frac{\partial^2 \Psi_{\alpha \circ X|X}(u; \alpha)}{\partial u^2}\bigg|_{u=1} = \alpha^2(X + 1)(X + 2).$$

Based on this, the conditional second moment of $\alpha \circ X$ is

$$E\left[(\alpha \circ X)^2 \mid X\right] = \alpha^2(X + 1)^2 + \alpha(1 + \alpha)(X + 1),$$

and thus the conditional variance is obtained as:

$$\text{Var}[\alpha \circ X \mid X] = E\left[(\alpha \circ X)^2 \mid X\right] - (E[\alpha \circ X \mid X])^2 = \alpha(1 + \alpha)(X + 1). \tag{6}$$

Now, let us suppose that X is a RV with Poisson-Lindley (PL) distribution, whose PMF is given by:

$$p_X(x; \theta) := P\{X = x\} = \frac{\theta^2(\theta + x + 2)}{(\theta + 1)^{x+3}}, \qquad x = 0, 1, 2, \ldots, \tag{7}$$

where $\theta > 0$ is the (unknown) parameter. It should be noted that the PL distribution, firstly proposed by Sankaran [20], is obtained from the Poisson distribution, whose parameter is adapted to the Lindley

distribution (some more results about this distribution and its generalizations can be found, e.g., in [21–24]). Based on this, the PGF of the PL distribution is easily obtained:

$$\Psi_X(u; \theta) := E[u^X] = \frac{\theta^2(\theta - u + 2)}{(\theta + 1)(\theta - u + 1)^2}, \qquad |u| < \theta + 1, \tag{8}$$

and using a procedure similar to the previous one, its mean and variance are, respectively:

$$\mu_X := E[X] = \frac{\theta + 2}{\theta(\theta + 1)},$$

$$\sigma_X^2 := \text{Var}[X] = \frac{\theta^3 + 4\theta^2 + 6\theta + 2}{\theta^2(\theta + 1)^2} = \mu_X + \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta + 1)^2}. \tag{9}$$

Thus, the PL-distributed RV X is obviously over-dispersed. Finally, the survival function of X, which plays an important role in the further presentation, is obtained as follows:

$$S_X(x) := P\{X \geq x\} = \sum_{k=x}^{\infty} p_X(k; \theta) = \frac{\theta x + (\theta + 1)^2}{(\theta + 1)^{x+2}}. \tag{10}$$

Under the given assumptions, the following preliminary result can be proved.

Proposition 1. *Let X be the PL-distributed RV, with the PMF given by Eq (7), which is independent of RVs $G_j(\alpha/(1 + \alpha))$, given in Definition 1. Then, the RV $\alpha \circ X$ represents the mixture of the negative binomial distribution $NB(\beta/(1 + \beta), 2)$ and the geometric distribution $Geom(\beta/(1 + \beta))$, with the PMF:*

$$p_{\alpha \circ X}(x; \alpha, \theta) := P\{X = x\} = \frac{(x + 1)\beta^x}{(\theta + 1)^2(1 + \beta)^{x+2}} + \frac{\theta(\theta + 2)\beta^x}{(\theta + 1)^2(1 + \beta)^{x+1}}, \tag{11}$$

and the survival function:

$$S_{\alpha \circ X}(x; \alpha, \theta) := P\{\alpha \circ X \geq x\} = \frac{\alpha^x(\theta + 1)^{x-2}}{(\alpha\theta + \alpha + \theta)^{x+1}} \left[\theta\left(x + (\theta + 1)^2\right) + \alpha(\theta + 1)^3 \right], \tag{12}$$

where $x = 0, 1, 2, \ldots$ and $\beta = \alpha(\theta + 1)/\theta > 0$.

Proof. The proof of this proposition, as well as the following theorems, is given in the Appendix. □

Remark 1. *Using Proposition 1, as well as the facts previously stated, some more stochastic properties for the RV $\alpha \circ X$, when X has the $PL(\theta)$ distribution, can be highlighted. For instance, according to Equations (5), (6), and (9), as well as by applying the laws of total expectation and total variance, the mean and the variance of the RV $\alpha \circ X$ are, respectively:*

$$\mu_{\alpha \circ X} := E[\alpha \circ X] = E[E[\alpha \circ X \mid X]] = \alpha E[X + 1] = \alpha \frac{(\theta + 1)^2 + 1}{\theta(\theta + 1)}$$

$$\sigma_{\alpha \circ X}^2 := \text{Var}[\alpha \circ X] = E[\text{Var}[\alpha \circ X \mid X]] + \text{Var}[E[\alpha \circ X \mid X]]$$

$$= \alpha(\alpha + 1)E[X + 1] + \alpha^2 \text{Var}[X] = \alpha(\alpha + 1)\frac{(\theta + 1)^2 + 1}{\theta(\theta + 1)} + \alpha^2 \frac{\theta^3 + 4\theta^2 + 6\theta + 2}{\theta^2(\theta + 1)^2}$$

$$= \mu_{\alpha \circ X} + \alpha^2 \frac{\theta^4 + 4\theta^3 + 8\theta^2 + 8\theta + 2}{\theta^2(\theta + 1)^2}.$$

Note that the RV $\alpha \circ X$, as well as the PL-distributed RV X, are over-dispersed. Also, the same results can be obtained by applying Proposition 1, i.e., the PMF of the RV $\alpha \circ X$, given by Eq (11). Finally, observations of the RV $\alpha \circ X$ can be generated by applying the following simple algorithm:

Step 1. Generate a RV U from the uniform distribution over the interval $(0, 1)$.

Step 2. If $U < (\theta + 1)^{-2}$, generate the RV from the negative binomial distribution $NB(\beta/(1 + \beta), 2)$.

Step 3. Else, generate the RV from the geometric distribution $Geom(\beta/(1 + \beta))$.

2.2. Definition and key properties of the model

Similarly as in Aleksić & Ristić [14], a formal definition of the integer-valued minification process (of the first order) is given here. Thereafter, some stochastic properties of this process are considered in the case when it has a PL marginal distribution.

Definition 2. *A time series (X_t), $t \in \mathbb{Z}$, given by the equality:*

$$X_t := \min(\alpha \circ X_{t-1}, \varepsilon_t). \tag{13}$$

is a minification INAR model of the first order (abbr. MINAR(1) model) if it meets the following conditions:

(i) (ε_t), $t \in \mathbb{Z}$ is an innovation series, that is, the IID integer-valued RVs, independent from (X_t),

(ii) the counting series incorporated in $\alpha \circ X_{t-1}$, $t \in \mathbb{Z}$ are mutually independent and also independent of RVs X_{t-1} and ε_t,

(iii) the counting series incorporated in $\alpha \circ X_{t-1}$ and $\alpha \circ X_{k-1}$ are independent for all $t \neq k$,

(iv) the RVs X_{t-j} and ε_t are independent for all $j \in \mathbb{Z}$.

Based on this definition, it may seem at first glance that the minification process leads to relatively small values. However, due to the definition of the modified thinning operator, higher values are also possible. This behavior is similar to max-INAR models, discussed in [26–28], but there is a difference in how both models reach the extreme value (see for more details Aleksić & Ristić [14]). In the following, we assume that for all $t \in \mathbb{Z}$, the RVs X_t have a Poisson-Lindley distribution $PL(\theta)$, whose PMF is given by Eq (7). Thus, we say that the series (X_t) represents a PL-minification integer-valued autoregressive (abbr. PL-MINAR) process (of the first order). According to the definition of the PL-MINAR process and the results described in the previous section, we first determine the distribution of its innovation series (ε_t).

Theorem 1. *When the innovations (ε_t) have the survival function of the form:*

$$S_\varepsilon(x; \alpha, \theta) := P\{\varepsilon_t \geq x\} = \frac{(\alpha\theta + \alpha + \theta)^{x+1}(\theta(\theta + x + 2) + 1)}{\alpha^x(\theta + 1)^{2x}\left(\alpha(\theta + 1)^3 + \theta\left((\theta + 1)^2 + x\right)\right)} \tag{14}$$

then, the discrete time series (X_t), given by Eq (13), has a stationary Poisson-Lindley distribution $PL(\theta)$, whose PMF is given by Eq (7), if and only if the following inequality is valid:

$$\alpha \geq \frac{1}{2}\left(\frac{1 - \theta}{1 + \theta} + \sqrt{\frac{\theta^2 + 3\theta + 6}{(\theta + 1)(\theta + 2)}}\right). \tag{15}$$

The reverse statement is also true, and the PGF of RVs (ε_t) is given as follows:

$$\Psi_\varepsilon(u; \alpha, \theta) := E[u^{\varepsilon_t}] = 1 + \gamma(u - 1)\left(\frac{\lambda}{1 - \gamma u} - (\lambda - 1)\delta L_{\delta+1}(\gamma u)\right), \tag{16}$$

wherein $\lambda = \alpha\theta + \alpha + \theta$, $\gamma = \lambda/(\alpha(\theta + 1)^2)$, $\delta = \lambda(\theta + 1)^2/\theta$, $|u| < 1/\gamma$ and

$$L_\delta(u) := \sum_{k=0}^{\infty} \frac{u^k}{k + \delta}$$

is the Lerch transcendent (of the first order).

Remark 2. *Figure 1 shows distributions of the series (X_t), $(\alpha \circ X_t)$, and (ε_t), when the parameters $\alpha = 0.75$ and $\theta = 1$ are taken. It is easy to verify that both of the above conditions are fulfilled, and based on them, it can bee seen that increasing the value of parameter θ also increases the set of allowed values for α (see again Figure 6(b)). Also, it is worth noting that the series $(\alpha \circ X_t)$ and (X_t) have a quite similar distribution, which is expected, because according to the definition of the PL-MINAR process, given by Eq (13), their values partially match. On the other hand, the distribution of the innovation series (ε_t) is very specific and different compared to them, so its properties will be further investigated.*
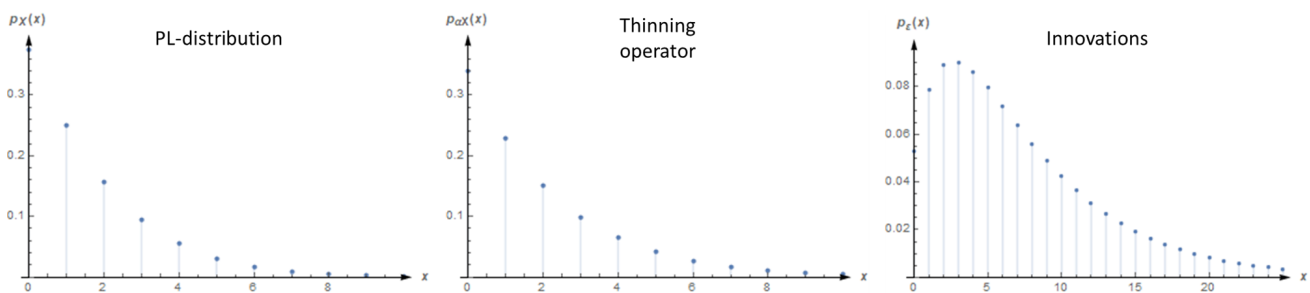

Figure 1. The PMFs of the RVs (X_t), $(\alpha \circ X_t)$, and (ε_t).

Remark 3. *By differentiating the PGF $\Psi_\varepsilon(u; \alpha, \theta)$ given by Eq (16) and then putting $u = 1$, the mean value of the innovations (ε_t) can be easily calculated as follows:*

$$\mu_\varepsilon := E[\varepsilon_t] = \frac{\partial \Psi_\varepsilon(u; \alpha, \theta)}{\partial u}\bigg|_{u=1} = \left[\frac{\lambda\gamma(1 - \gamma)}{(1 - \gamma u)^2} - (\lambda - 1)\gamma\delta\left(L_{\delta+1}(\gamma u) + \gamma(u - 1)L'_{\delta+1}(\gamma u)\right)\right]\bigg|_{u=1}$$

$$= \gamma\left[\frac{\lambda}{1 - \gamma} - (\lambda - 1)\delta L_{\delta+1}(\gamma)\right].$$

At the same time, note that under the condition $0 < \gamma < 1$ and the inequality:

$$L_{\delta+1}(\gamma) = \sum_{k=0}^{\infty} \frac{\gamma^k}{k + \delta + 1} < \frac{1}{\delta}\sum_{k=0}^{\infty}\gamma^k = \frac{1}{\delta(1 - \gamma)}$$

it follows that:

$$\mu_\varepsilon > \frac{\lambda\gamma}{1 - \gamma} - \frac{(\lambda - 1)\gamma}{1 - \gamma} = \frac{\gamma}{1 - \gamma} > 0.$$

Similarly, according to equality:

$$E[\varepsilon_t(\varepsilon_t - 1)] = \left. \frac{\partial^2 \Psi_\varepsilon(u; \alpha, \theta)}{\partial u^2} \right|_{u=1} = 2\gamma^2 \left[\frac{\lambda}{(1-\gamma)^2} - (\lambda - 1)\delta L'_{\delta+1}(\gamma) \right]$$

for the variance of the RVs (ε_t), after some computations, one obtains:

$$\mathrm{Var}[\varepsilon_t] = E[\varepsilon_t(\varepsilon_t - 1)] + E[\varepsilon_t] - \left(E[\varepsilon_t]\right)^2 = \frac{\gamma}{(1-\gamma)^2} \left[\lambda + (\lambda-1)(1-\gamma)^2 \delta L_{\delta+1}(\gamma) \left(\gamma \delta L_{\delta+1}(\gamma) + 1\right) \right.$$

$$\left. - \gamma \lambda (L_{\delta+1}(\gamma)\delta(1-\gamma) + 1)^2 \right] - 2\gamma^2 \delta(\lambda - 1)L'_{\delta+1}(\gamma).$$

Below are some important properties of the PL-MINAR series (X_t) related to its conditional stochastic measures. First, the Markov properties and the conditional PGF of the PL-MINAR series (X_t) are presented.

Theorem 2. *Let (X_t), $t \in \mathbb{Z}$, be a PL-MINAR process given by Eq (13). Then, (X_t) is a strictly stationary, homogeneous Markov process with one-step transition probabilities:*

$$P\{X_t = x \mid X_{t-1} = y\} = \mathcal{S}_\varepsilon(x; \alpha, \theta) \binom{x+y}{x} \frac{\alpha^x}{(1+\alpha)^{x+y+1}} + p_\varepsilon(x; \alpha, \theta) \sum_{k=x+1}^{\infty} \binom{k+y}{k} \frac{\alpha^k}{(1+\alpha)^{y+k+1}}, \quad (17)$$

where $x, y = 0, 1, 2, \ldots$. In addition, the conditional PGF of the PL-MINAR RVs (X_t), for a given value $X_{t-1} = y$, has the following form:

$$E[u^{X_t} \mid X_{t-1} = y] = 1 + \frac{\lambda \gamma (u-1)}{1 - \gamma u} \left(1 - \frac{1}{(1 + \alpha - \alpha \gamma u)^{y+1}} \right)$$

$$- (\lambda - 1)\gamma \delta(u-1) \left(L_{\delta+1}(\gamma u) - \sum_{j=0}^{\infty} \binom{y+j}{j} \frac{(\alpha \gamma u)^j L_{\delta+j+1}(\gamma u)}{(1+\alpha)^{y+j+1}} \right), \qquad (18)$$

where $|u| < 1/\gamma$ and the parameters λ, γ, δ are the same as in Theorem 1.

Remark 4. *Based on the previous theorem, in a similar way as in [21], some additional Markovian properties of the PL-MINAR process can be observed. Note that Eq (14), under the condition given by Eq (15), ensures $\mathcal{S}_\varepsilon(x; \alpha, \theta) > \mathcal{S}_\varepsilon(x+1; \alpha, \theta) > 0$, which implies $p_\varepsilon(x; \alpha, \theta) > 0$, for all $x = 0, 1, 2, \ldots$. Therefore, the transition probabilities given by Eq (17) are always positive, so the process (X_t) is an irreducible, aperiodic and (positively or null) recurrent Markov chain.*

Remark 5. *Based on the previous theorem and using the well-known properties of PGFs, the conditional mathematical expectation and the conditional variance of the PL-MINAR series (X_t) can be derived. Thus, by differentiating Eq (18) and setting $u = 1$, for the conditional mean of the series (X_t), after some computations, one obtains:*

$$E[X_t \mid X_{t-1}] = \left. \frac{\partial E[u^{X_t} \mid X_{t-1}]}{\partial u} \right|_{u=1} = \frac{\lambda \gamma}{1 - \gamma} \left[1 - \left(\frac{1}{1 + \alpha - \alpha \gamma} \right)^{X_{t-1}+1} \right]$$

$$- \delta(\lambda - 1)\gamma \left[L_{\delta+1}(\gamma) - \sum_{j=0}^{\infty} \binom{X_{t-1}+j}{j} \frac{(\alpha \gamma)^j L_{\delta+j+1}(\gamma)}{(1+\alpha)^{X_{t-1}+j+1}} \right]. \qquad (19)$$

Similarly, the conditional variance of (X_t) *can be obtained as:*

$$\mathrm{Var}[X_t \mid X_{t-1}] = E\left[X_t(X_t - 1) \mid X_{t-1}\right] + E\left[X_t \mid X_{t-1}\right] - \left(E\left[X_t \mid X_{t-1}\right]\right)^2,$$

where

$$E\left[X_t(X_t - 1) \mid X_{t-1}\right] = \left.\frac{\partial^2 E[u^{X_t} \mid X_{t-1}]}{\partial u^2}\right|_{u=1}.$$

However, due to the complexity of the calculation, a more detailed procedure will be omitted here.

Finally, the correlation structure of the PL-MINAR process is examined below, where it is presented in the next statement.

Theorem 3. *The first-order autocorrelation of the PL-MINAR series* (X_t) *is given by the following expression:*

$$
\begin{aligned}
\rho_X(1) := \mathrm{Corr}(X_t, X_{t-1}) = {}& \frac{\theta^2(\theta+1)^2}{\theta^3 + 4\theta^2 + 6\theta + 2}\left[\frac{\theta+2}{\theta(\theta+1)}\left(\frac{\lambda\gamma}{1-\gamma-1} + (\lambda-1)\gamma\delta L_{\delta+1}(\gamma)\right)\right.\\
& + \frac{\gamma\delta(\lambda-1)}{(\alpha\theta+\alpha+\theta)^3}\left(\frac{\theta}{\theta+1}\right)^2\sum_{j=0}^{\infty}(j+1)\left(\frac{\alpha\gamma(\theta+1)}{(1+\alpha)(\theta+1)-1}\right)^j \qquad (20)\\
& \left. \times \left((1+\alpha)(\theta+1)(\theta+3) - \theta + j - 1\right)L_{j+\delta+1}(\gamma) - \frac{(\theta+2)^2}{\theta^2(\theta+1)^2}\right].
\end{aligned}
$$

Remark 6. *Let us examine in a nutshell the problem of finding the correlation dependence for RVs* X_t *and* X_{t-k}, *where* $k = 1, 2, \ldots$ *Using similar considerations as in the general minification process [12], we define a two-dimensional survival function:*

$$\mathcal{S}_k^{(2)}(x, y) := P\{X_t \geq x, X_{t-k} \geq y\} = P\{\min(\alpha \circ X_{t-1}, \varepsilon_t) \geq x, X_{t-k} \geq y\} = \mathcal{S}_\varepsilon(x)\, P\{\alpha \circ X_{t-1} \geq x, X_{t-k} \geq y\}.$$

From here, using the conditional probability and the conditional expectation, one obtains:

$$
\begin{aligned}
E[X_t X_{t-k}] &= \sum_{x=1}^{\infty}\sum_{y=1}^{\infty} \mathcal{S}_k^{(2)}(x, y) = \sum_{x=1}^{\infty} \mathcal{S}_\varepsilon(x) \sum_{y=1}^{\infty} P\{X_{t-k} \geq y \mid \alpha \circ X_{t-1} \geq x\}\, P\{\alpha \circ X_{t-1} \geq x\}\\
&= \sum_{x=1}^{\infty} \mathcal{S}_\varepsilon(x) P\{\alpha \circ X_{t-1} \geq x\}\, E[X_{t-k} \mid \alpha \circ X_{t-1} \geq x]\\
&= \sum_{x=1}^{\infty} P\{\min(\alpha \circ X_{t-1}, \varepsilon_t) \geq x\}\, E[X_{t-k} \mid \alpha \circ X_{t-1} \geq x]\\
&:= \Phi(X_{t-1}, X_{t-k}; \alpha),
\end{aligned}
$$

where the replacement of the order of the sums is ensured in a manner similar to the proof of Theorem 2 (see Appendix). Thus, the mixed moments $E[X_t X_{t-k}]$, *and therefore the autocorrelation function* $\rho_X(k) = \mathrm{Corr}(X_t, X_{t-k})$, $k = 1, 2, \ldots$, *can be expressed recursively, in terms of bivariate distribution of the RVs* (X_{t-1}, X_{t-k}).

3. Statistical analysis and simulation

3.1. Parameters estimation procedures

In this part, some procedures for estimating the unknown parameters $\alpha \in (0, 1)$ and $\theta > 0$ of the PL-MINAR model are discussed. For this purpose, based on previously obtained theoretical results, three different estimation methods are examined here, and their more detailed description will be given below. In doing so, as usual, it is assumed that the PL-MINAR series (X_t) is given by one of its observed realizations $X_1, \ldots, X_T$ of length $T > 0$.

3.1.1. Method of moments (MM)

This estimation method is based on equating the theoretical and empirical moments of the PL-MINAR series (X_t). According to the stationarity of the series (X_t) and the first of Eq (9), the MM-estimator of the parameter θ is easily obtained from the equality:

$$\widehat{\theta}_{MM} = \frac{1 - \overline{X}_T + \sqrt{\left(1 - \overline{X}_T\right)^2 + 8\overline{X}_T}}{2\overline{X}_T}, \tag{21}$$

where $\overline{X}_T := T^{-1} \sum_{t=1}^{T} X_t$ is the empirical mean of the realized series $X_1, \ldots, X_T$. By using some general results about the MM estimates of the PL distribution (e.g., [25]), the asymptotic normality of the estimator $\widehat{\theta}_{MM}$ can be proven. Thereafter, by replacing θ with $\widehat{\theta}_{MM}$, as well as the first-order correlation $\rho_X(1)$ with the empirical one

$$\widehat{\rho}_T(1) := \frac{\sum_{t=1}^{T-1}(X_t - \overline{X}_T)(X_{t+1} - \overline{X}_T)}{\sum_{t=1}^{T}(X_t - \overline{X}_T)^2},$$

solving Eq (20) with respect to α gives the estimator $\widehat{\alpha}_{MM}$ of this parameter. For this purpose, some numerical procedure must be used, although a particular problem is the complexity of this equation. Nevertheless, after substituting $\theta = \widehat{\theta}_{MM}$ and $\rho_X(1) = \widehat{\rho}_T(1)$ into Eq (20), it gains simplicity. Moreover, since the mean and variance of the PL-MINAR series (X_t) do not depend on $\alpha \in (0, 1)$, to obtain this parameter we can take Eq (A.8) (see Appendix), i.e., equality $E[X_t X_{t-1}] = \widehat{M}_T^{(2)}$, where

$$\widehat{M}_T^{(2)} = \frac{1}{T - 1} \sum_{t=2}^{T} X_t X_{t-1}$$

is the empirical mixed moment of observed series $X_1, \ldots, X_T$. Finally, a consistency and asymptotic normality of this estimator can be shown by using some general results of the MM estimators (see, for more detail, e.g., [29]).

3.1.2. Conditional maximum likelihood (CML) method

As is well-known, for the observed values $x_1, \ldots, x_T$ related to the mentioned realization, the CML method is based on maximizing the conditional log-likelihood function:

$$\ell(\alpha, \theta) = \log L(\alpha, \theta) = \log \prod_{t=2}^{T} P\{X_t = x_t \mid X_{t-1} = x_{t-1}\} = \sum_{t=2}^{T} \log P\{X_t = x_t \mid X_{t-1} = x_{t-1}\}. \tag{22}$$

Since the PL-MINAR process (X_t) has Markov properties, described in Theorem 2, its conditional log-likelihood function can be easily derived. Namely, by using previously obtained transitional probabilities, given by Eq (17), as well as their replacement in Eq (22), the following objective function on α, θ is obtained:

$$
\ell(\alpha, \theta) = \sum_{t=2}^{T} \log \left[\mathcal{S}_\varepsilon(x_t; \alpha, \theta) \binom{x_t + x_{t-1}}{x_t} \frac{\alpha^{x_t}}{(1 + \alpha)^{x_t + x_{t-1} + 1}} \right.
$$
$$
\left. + p_\varepsilon(x_t; \alpha, \theta) \left(1 - \sum_{k=0}^{x_t} \binom{k + x_{t-1}}{k} \frac{\alpha^k}{(1 + \alpha)^{x_{t-1} + k + 1}} \right) \right].
\tag{23}
$$

Note that CML estimates of the parameters α, θ can be computed by solving the coupled equations $\partial \ell(\alpha, \theta)/\partial \alpha = \partial \ell(\alpha, \theta)/\partial \theta = 0$. However, as is common for this estimation method, CML estimators $\widehat{\alpha}_{CLM}$ and $\widehat{\theta}_{CLM}$ cannot be obtained in closed form in this way. Therefore, it is necessary to use some numerical methods for their calculation, which will also be discussed below. It is worth noting that, similar to the previous method, the asymptotic properties of these estimators can be shown using general results related to CML estimators (see for more details, e.g., [30]).) Furthermore, using a procedure like that of the Poisson-Lindley INAR(1) model, introduced in [21], the asymptotic equivalence of the efficiency of the MM and CLM estimators can also be shown.

3.1.3. Conditional least squares (CLS) method

The third estimation method is the CLS method, where the estimates of the parameters $\alpha \in (0, 1)$ and $\theta > 0$ are obtained as values that minimize the following objective function:

$$
Q_T(\alpha, \theta) = \sum_{t=2}^{T} \left[X_t - E\left[X_t \mid X_{t-1} \right] \right]^2.
$$

By substituting here the conditional mean previously obtained by Equation (19), it follows:

$$
Q_T(\alpha, \theta) = \sum_{t=2}^{T} \left[X_t - \frac{\lambda \gamma}{1 - \gamma} \left[1 - \left(\frac{1}{1 + \alpha - \alpha \gamma} \right)^{X_{t-1}+1} \right] - \delta(\lambda - 1)\gamma \right.
$$
$$
\left. \times \left[L_{\delta+1}(\gamma) - \sum_{j=0}^{\infty} \binom{X_{t-1} + j}{j} \frac{(\alpha \gamma)^j L_{\delta+j+1}(\gamma)}{(1 + \alpha)^{X_{t-1}+j+1}} \right] \right]^2,
\tag{24}
$$

wherein $\lambda = \alpha\theta + \alpha + \theta$, $\gamma = \lambda/(\alpha(\theta + 1)^2)$ and $\delta = \lambda(\theta + 1)^2/\theta$. Therefore, by applying the usual procedure, that is, by solving coupled equations $\partial Q_T(\alpha, \theta)/\partial \alpha = \partial Q_T(\alpha, \theta)/\partial \theta = 0$, the CLS estimators can be computed. Note that, as in the previous case, the minimization of the function $Q_T(\alpha, \theta)$ can be performed using some numerical procedures, while the asymptotic properties of the obtained CLS estimates can be proved by applying some basic results of the CLS theory (see, for more detail, e.g., [31]). Finally, as before, the asymptotic efficiency equivalence of the CLS estimators with previous ones can be easily shown.

3.2. Numerical study

In this part, numerical simulations of previously described procedures for estimating the unknown parameters (α, θ) of the PL-MINAR process (X_t) were carried out, based on its observed realization $X_1, \ldots, X_T$. For this purpose, the innovations (ε_t) are first generated, so that their PMF, given by Eq (A.4) (see Appendix), is implemented using the R-package "discreteRV", authorized by Buja et al. [32]. Thereafter, using Eqs (2) and (13), the simulated values of the series $(\alpha \circ X_t)$ and (X_t), respectively, are simply generated. The procedure of generating the PL-MINAR process (X_t) can be presented by the following pseudo-code:

Step 1. For the innovation series (ε_t), by using Eq (14), define the appropriate survival function $\mathcal{S}_\varepsilon(x; \alpha, \theta) = P\{\varepsilon_t \geq x\}$, as well as its PMF $p_\varepsilon(x; \alpha, \theta) = \mathcal{S}_\varepsilon(x; \alpha, \theta) - \mathcal{S}_\varepsilon(x+1; \alpha, \theta)$, when $x = 0, 1, 2, \ldots$

Step 2. For $t = 0, 1, \ldots, T$ generate the values of ε_t, whose PMF is $p_\varepsilon(x; \alpha, \theta)$.

Step 3. Put the initial values $X_0 = \alpha \circ X_0 = \varepsilon_0$.

Step 4. For $t = 1, \ldots, T$, by using Eq (2), generate the values of $\alpha \circ X_{t-1}$, and then the values of $X_t = \min\{\alpha \circ X_{t-1}, \varepsilon_t\}$.

Note that the values of $\alpha \circ X_t$ can be alternatively generated according to the pseudo-code previously stated in Remark 1. As an illustration, Figure 2 shows the empirical frequency distributions of all three series, for different values of parameters (α, θ), obtained according to their realizations of length $T = 1500$. It can be noted, as explained in the previous theoretical part, that the empirical distributions of the series $(\alpha \circ X_t)$ and (X_t) are similar to each other. In contrast, the empirical distribution of innovations (ε_t) is significantly more flexible and more susceptible to changes in parameters, which is fully consistent with the procedure for obtaining these distributions, as stated in Theorem 1. Furthermore, by applying the described procedure, simulations of the PL-MINAR series (X_t) of different lengths $T \in \{500, 1500, 3000\}$ were conducted. Thereby, we point out that some of these lengths are taken in accordance with the size of real-world data, which are observed and analyzed in the next section. In addition, the previously presented estimation methods can be applied to each of these obtained realizations $\{X_1, \ldots, X_T\}$. To this end, for each of the mentioned lengths, we independently generated 1000 independent simulations of the series (X_t), and the results of applying the three previously described estimation procedures are shown in Table 1.

More precisely, Table 1 contains, for all considered estimation methods and different lengths of simulated series, summary statistics of the computed estimates, i.e., their minimums (Min.), mean values (Mean), maximums (Max.), and standard deviations (SD). In addition, the mean squared errors of estimation (MSEE), as well as the Anderson-Darling normality test were performed. Note that in the case of the MM estimator, the parameter θ is easily estimated according to Eq (21). Afterwards, using the procedure specified in Subsection 3.1, the MM estimates of the parameter α are easily derived. To that cause, the R-procedure "nmilnb" for box-constrained optimization was used (see Gay [33] for more details), but the same results are obtained using the R-procedure "optimize" for unconstrained optimization. From the other side, CML and CLS estimates were performed using minimization of the objective functions $\ell(\alpha, \theta)$ and $Q(\alpha, \theta)$, given by Eqs (23) and (24), respectively. Thereby, the realizations from the uniform distribution were taken as the initial estimated values of the parameters, taking care to satisfy the conditions of Theorem 1.
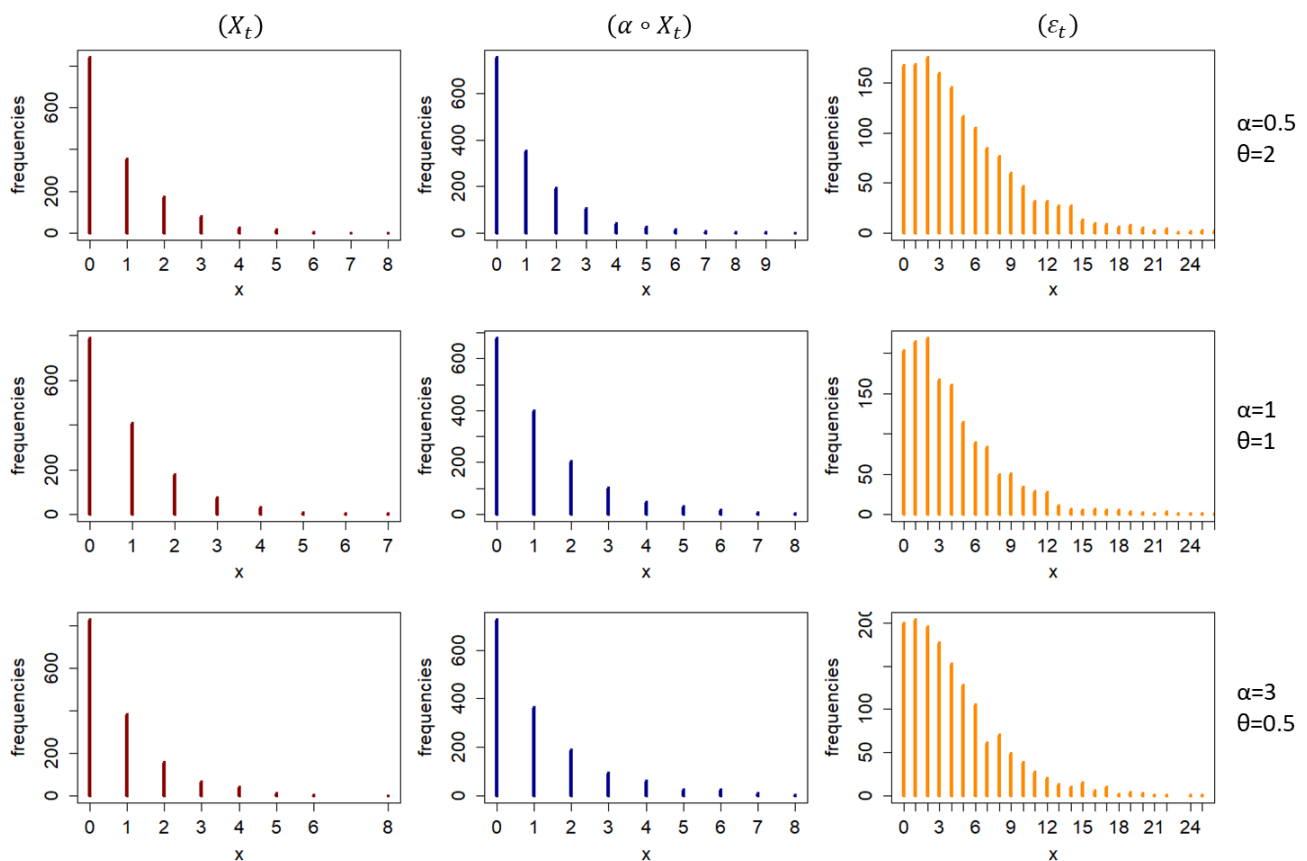
Figure 2. Empirical frequency distributions of three PL-MINAR series of length $T = 1500$, generated using Monte Carlo simulations for different values of parameters α, θ.

From the results given in Table 1, we can notice that the MM estimates of α have a large range, which is a consequence of their calculation using the reciprocal estimated autocorrelation. In contrast, the MM estimates of θ are significantly more efficient because they are obtained directly from Eq (21). More efficient estimates are also obtained for the other two estimation methods, although the estimates for θ obtained by the CML method have higher standard deviation values. By using similar considerations as Aleksić & Ristić [14], it should be noted that CML estimates require significantly more computing time, especially for large samples. Nevertheless, it is noticeable the decrease of SD and MSEE values with increasing series lengths, that is, there is a pronounced convergence of all estimation methods. The asymptotic normality (AN) test results are also shown in Table 1, where the Anderson–Darling normality test was conducted. The test statistic, denoted AD, along with the corresponding p-values, were calculated using the procedure from the R-package "nortest", authored by Gross [34]. According to the values obtained, it can be seen that the AN property is confirmed for almost all calculated estimates, at the significance level of $0.01 < p < 0.05$, which is also confirmed in Figure 3. Moreover, the AN property is the most pronounced in the case of CLS estimates, so this, along with the previously mentioned facts, indicates that these estimates may be the most adequate for practical application, which will be done below.

Table 1. Summary statistics, estimation errors, and AN testing of parameter estimates of the PL-MINAR process. (True parameters are: $\alpha = 0.5$, $\theta = 2$).

| Sample		MM		CML		CLS	
		$\widehat{\alpha}_{MM}$	$\widehat{\theta}_{MM}$	$\widehat{\alpha}_{CML}$	$\widehat{\theta}_{CML}$	$\widehat{\alpha}_{CLS}$	$\widehat{\theta}_{CLS}$
$T = 500$	Min.	0.1073	1.550	0.3600	1.360	0.3292	1.497
	Mean	0.5078	2.068	0.5121	2.035	0.4719	2.032
	Max.	0.8404	2.817	0.6713	2.620	0.6126	2.574
	SD	0.1007	0.1747	0.0705	0.3111	0.0519	0.1956
	MSEE	0.0102	0.0351	0.0712	0.0970	0.0588	0.0457
	AD	1.015*	0.6818	0.5395	0.7330	0.4749	0.5116
	(*p*-value)	(0.0113)	(0.0746)	(0.1585)	(0.0525)	(0.2355)	(0.1950)
$T = 1500$	Min.	0.1530	1.790	0.3810	1.436	0.3918	1.535
	Mean	0.5065	2.069	0.4924	2.038	0.5030	2.013
	Max.	0.8303	2.415	0.6397	2.597	0.5911	2.540
	SD	0.0977	0.0991	0.0617	0.2915	0.0314	0.1917
	MSEE	9.92E-3	0.0145	0.0615	0.0882	0.0315	0.0379
	AD	0.5357	0.3575	0.5089	0.6041	0.3380	0.2794
	(*p*-value)	(0.1694)	(0.4538)	(0.1940)	(0.1136)	(0.5033)	(0.6393)
$T = 3000$	Min.	0.2299	1.868	0.3747	1.778	0.4415	1.594
	Mean	0.5036	2.070	0.4954	2.006	0.4998	2.050
	Max.	0.8032	2.294	0.6200	2.549	0.5946	2.482
	SD	0.0990	0.0689	0.0584	0.2753	0.0271	0.1880
	MSEE	9.65E-3	0.0100	0.0586	0.0751	0.0269	0.0375
	AD	0.5506	0.2974	0.2089	0.6328	0.3202	0.1844
	(*p*-value)	(0.1558)	(0.5895)	(0.8599)	(0.0964)	(0.5319)	(0.9064)

* $0.01 < p < 0.05$
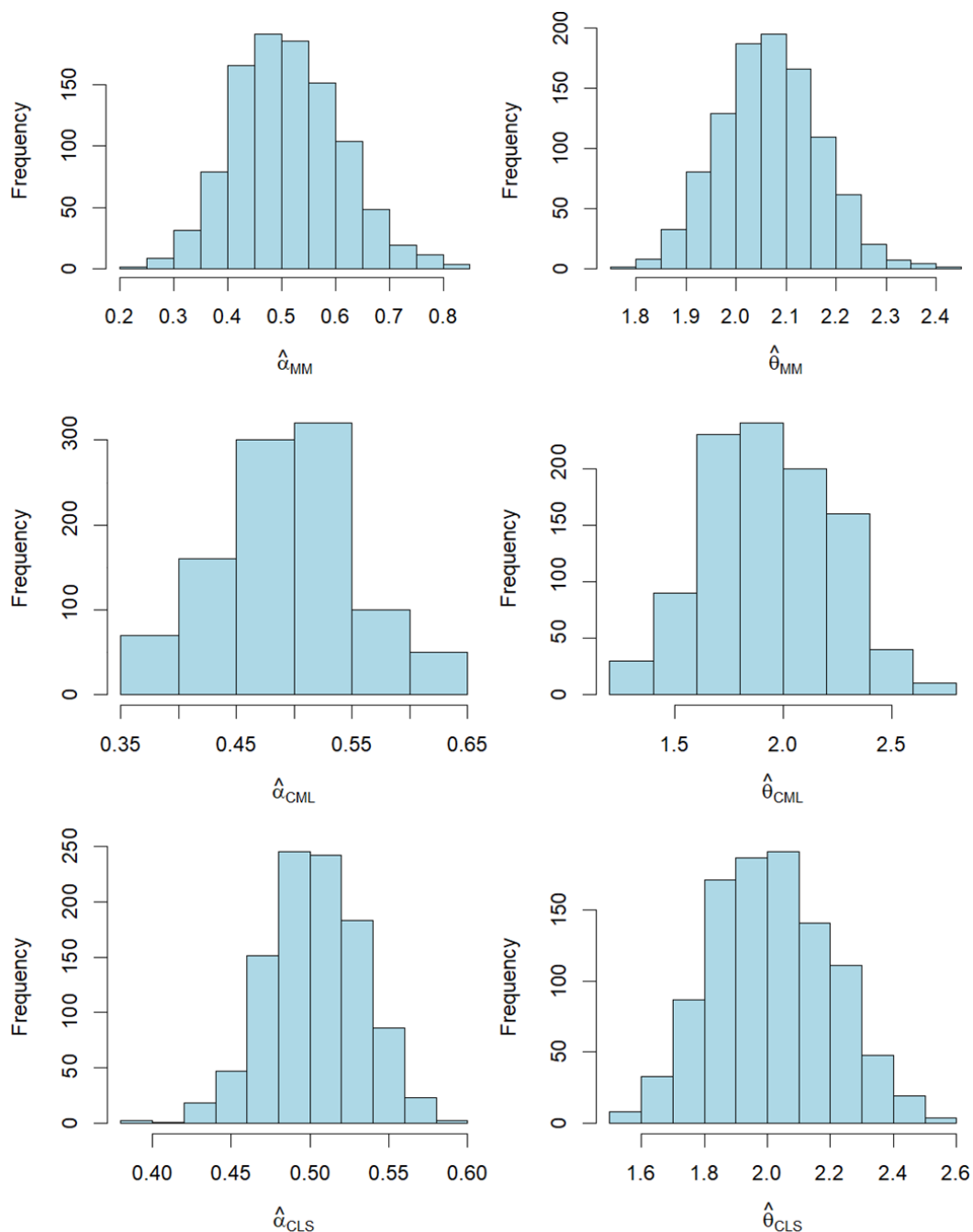
Figure 3. Histograms of the empirical distributions of the PL-MINAR process parameters estimates. (The length is $T = 1500$ and parameters are the same as in Table 1.)

4. Application to real-world data

This section discusses some practical applications of the PL-MINAR process in real-world data modeling. For this purpose, as well as to compare the efficiency of the proposed model with some well-known existing models, we used two datasets. The first one, labeled as Series A, represents daily bitcoin transaction volumes, calculated as the sum of all transaction outputs belonging to the blocks

mined on a given day. The data refer to the first period of transactions with this cryptocurrency, from January 9, 2009 to April 3, 2010, based on the Bitcoin dashboard "Coinmetrics" [35]. Thus, a time series of length $T = 450$ was obtained, and similarly to [36, 37], the last 10%, that is, 45 observations, are used to test the predictive accuracy of PL-MINAR process. The second (Series B), considered also in [36], represents the number of transactions in shares of the Empire District Electric (EDE) company, measured at 5-minute intervals between 9:45 a.m. and 4:00 p.m. for the period from January 3 to February 18, 2005. The data was collected from the official website of the Wall Street Journal and in that way, a count time series of length $T = 2925$ was obtained, where the last 225 observations were left to check the validity of our model forecast. The dynamics of these two time series, along with their autocorrelation functions (ACFs), are shown in Figure 4. It is noticeable the decrease of both ACFs is a little bit slower than the standard INAR(1) models, especially for Series A. In accordance with previous theoretical results, this suggests the possibility of modeling the dynamics of both time series by the PL-MINAR model. Some confirmation of this can also be seen in Table 2, where the descriptive statistics of both series are presented. For instance, both observed series have a wide range of data, positive asymmetry, and significant overdispersion. It is somewhat more pronounced in the case of series A, where significant zero inflation is also present. Furthermore, the Augmented Dickey-Fuller (ADF) test was conducted using the R-package "aTSA" [38], and the alternative hypothesis that the observed series are stationary was clearly verified in both cases.
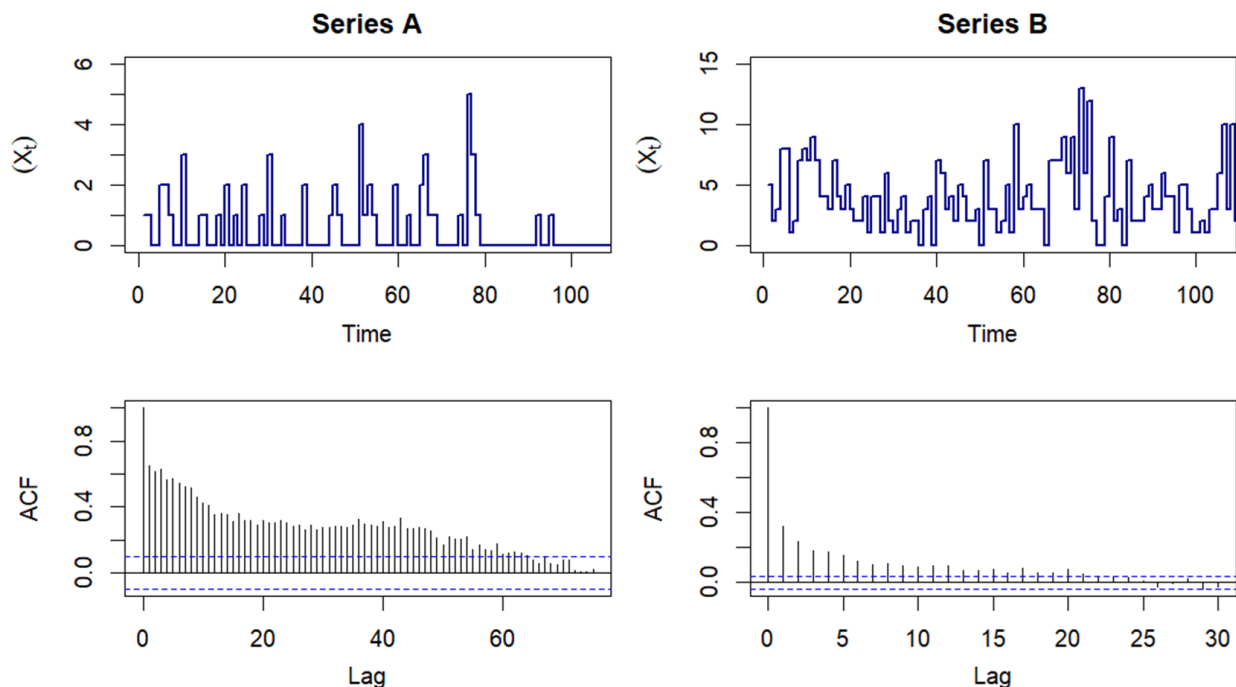

Figure 4. The dynamics of the real-world data series (plots above) and their appropriate ACFs (plots below).

Table 2. Summary statistics and stationarity testing of the observed data.

Statistics	Series A	Series B
Minimum	0	0
Maximum	16	25
Mode	0	1
Median	0	3
Mean	1.416	3.314
Variance	7.183	9.209
St. deviation	2.680	3.035
Skewness	2.875	1.658
Kurtosis	9.250	4.799
ADF-test	-13.46	-22.60
(p-value)	(< 0.01)	(< 0.01)

Further, by applying the previously described procedures, parameters were estimated for both series, assuming that their dynamics can be described by the PL-MINAR model. Additionally, to verify its effectiveness the proposed model is compared with the Poisson-Lindley INAR(1) model $X_t = \alpha * X_{t-1} + \varepsilon_t$, $t = 1, 2, \ldots, T$, introduced by Mohammadpour et al. [21], where $\alpha \in (0, 1)$ and $*$ is the binomial thinning operator (see, e.g., [39] for more details). Thus, both stochastic models, the PL-MINAR and the PL-INAR(1) process, have a Poisson-Lindley marginal distribution. To estimate the parameters of both models, according to the results obtained in the previous section, MM estimates were used as initial ones, while CLS estimates were used to fit and compare them with empirical data. The estimated values for both series, as well as the values of objective functions $Q_T(\alpha, \theta)$, are presented in Table 3. After that, using the estimated parameter values, 500 independent Monte Carlo simulations of both models were performed, and the distributions agreement of the real-world and fitted data was checked using MSEE statistics, Akaike's information criterion (AIC), and Bayesian information criterion (BIC).

Based on this, close estimated values of the θ parameter, as well as fit statistics and estimation errors, are noticeable for both estimation methods and both stochastic models. This is expected, because in both cases it is assumed that series (X_t) have a PL distribution. However, it is obvious that the error statistics are lower when the PL-MINAR model is applied. The forecast accuracy for both models is also analyzed below, where, as already mentioned, forecast length horizons of $h = 50$ (Series A) and $h = 100$ (Series B) are taken. The testing procedure was performed using a one-tailed Diebold–Marian test [40], with the null hypothesis being that both models have the same predictive accuracy, while the alternative is that the PL-MINAR model has better accuracy. The test statistic, labeled DM, as well as the appropriate p-values, were calculated within the R-package "forecast" [41] and presented in the lower part of Table 3. Based on this, it is clear that in both cases the PL-MINAR process has better predictive accuracy, at a significant level of $0.01 < p < 0.05$. Some confirmation of these facts can be seen in Figure 5, where empirical and fitted frequencies of both time series and both models are shown, as well as for observed and forecasted data.
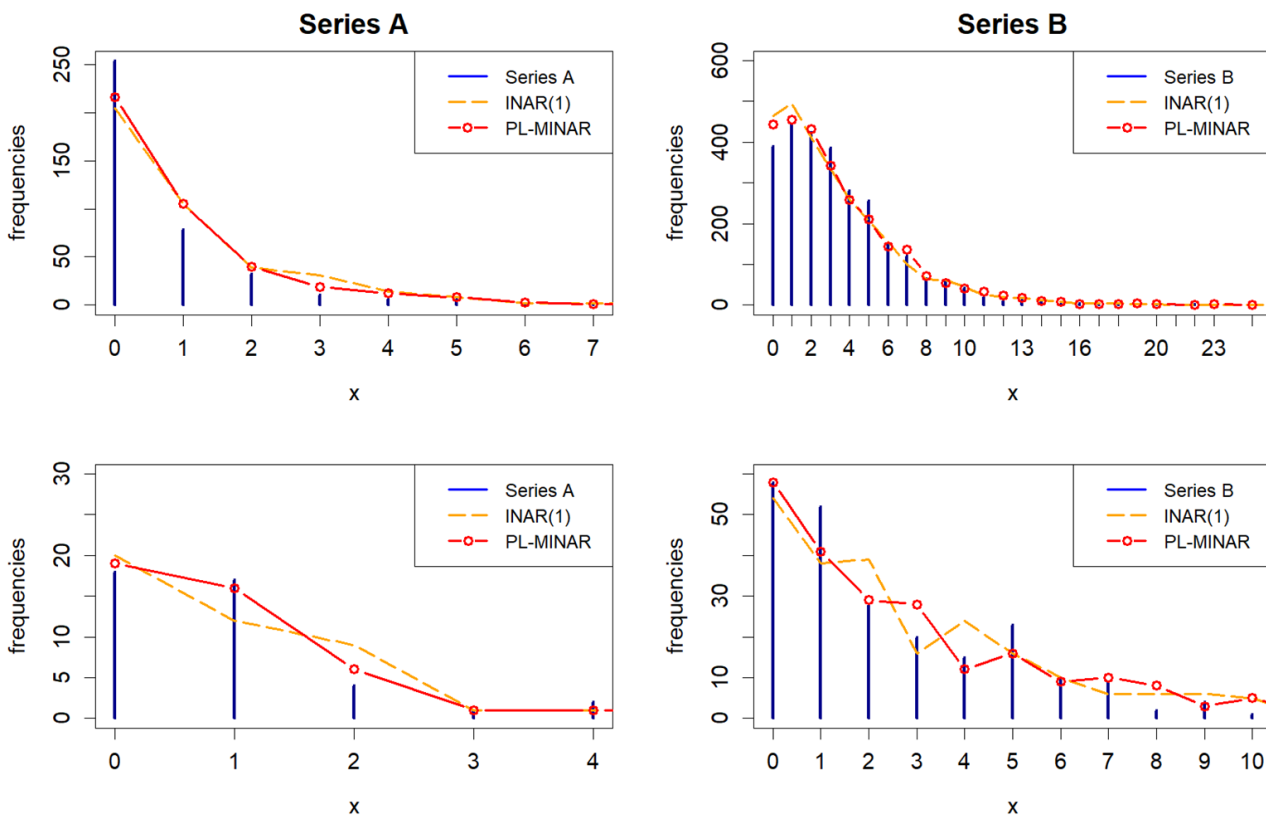
Figure 5. Frequency distributions (plots above) and prediction features (plots below) of the observed and fitted data.

Table 3. Estimated parameter values of the Poisson-Lindley INAR(1) and MINAR processes, along with the corresponding estimation errors and predictive test statistics.

Parameters/ Statistics	Series A		Series B	
	PL-INAR(1)	PL-MINAR	PL-INAR(1)	PL-MINAR
α	0.3337	1.5885	0.3211	1.2517
θ	1.5587	1.5034	0.4988	0.4990
Q_T	3.2697	5.6894	8.2238	13.235
MSEE	0.0580	0.0523	0.0485	0.0484
AIC	173.06	171.99	136.80	136.55
BIC	174.26	173.29	136.98	136.73
DM	1.7142*		1.6899*	
(p-value)	(0.0464)		(0.0471)	

*$0.01 < p < 0.05$

5. Conclusions

A novel count time series minification model, named the PL-MINAR process, is presented here. The key properties of the proposed model, as well as different procedures for estimating its parameters, are discussed in detail. Through Monte Carlo simulations, the consistency of these estimators was examined, and at the end, a practical application of the PL-MINAR process in fitting real-world data was given. To verify its effectiveness, the proposed model is applied to fit the distributions and predictions of two real-world time series, representing the number of bitcoin and stock transactions. Also, the PL-MINAR model was compared with the INAR(1) model, which is based on the same Poisson-Lindley (PL) marginal distribution. According to the obtained results, i.e., fitting errors and DM statistics, it can be seen that the PL-MINAR model has the same or even better fitting and forecasting capabilities within the observed data. These results can be motivating for further applications of the PL-MINAR process in some other fields, such as epidemiology, climate change, or cell counts. On the other hand, investigations of the minification count process, where some other and different count-based distributions are used as marginals, may also be a direction of some future research.

Appendix: Proofs

Proof of Proposition 1. According to the previously obtained conditional PGF of the RV $\alpha \circ X$, given by Eq (4), the PGF of this RV can be expressed as follows:

$$\Psi_{\alpha \circ X}(u; \alpha, \theta) = (1 + \alpha - \alpha u)^{-1} \Psi_X \left((1 + \alpha - \alpha u)^{-1} \right).$$

Now, for the PL-distributed RV X, i.e., using its PGF given by Eq (8), after some computation one obtains:

$$\Psi_{\alpha \circ X}(u; \alpha, \theta) = (1 + \alpha - \alpha u)^{-1} \frac{\theta^2(\theta - s + 2)}{(\theta + 1)(\theta - s + 1)^2} \bigg|_{s=(1+\alpha-\alpha u)^{-1}}$$

$$= \frac{1}{1 + \alpha - \alpha u} \cdot \frac{\theta^2 \dfrac{(\theta + 2)(1 + \alpha - \alpha u) - 1}{1 + \alpha - \alpha u}}{(\theta + 1)\left(\dfrac{(\theta + 1)(1 + \alpha - \alpha u) - 1}{1 + \alpha - \alpha u}\right)^2} \tag{A.1}$$

$$= \frac{\theta^2(\theta - \alpha(\theta + 2)(u - 1) + 1)}{(\theta + 1)(\theta - \alpha(\theta + 1)(u - 1))^2}, \qquad |u| \le 1.$$

Using the partial fraction decomposition method, Eq (A.1) can be expressed in an equivalent way as:

$$\Psi_{\alpha \circ X}(u; \alpha, \theta) = \frac{\theta^2}{(\theta + 1)^2(\theta - \alpha(\theta + 1)(u - 1))^2} + \frac{\theta^3 + 2\theta^2}{(\theta + 1)^2(\theta - \alpha(\theta + 1)(u - 1))}$$

$$= \frac{1}{(\theta + 1)^2(1 + \beta - \beta u)^2} + \frac{\theta(\theta + 2)}{(\theta + 1)^2(1 + \beta - \beta u)}, \tag{A.2}$$

where $\beta = \alpha(\theta + 1)/\theta$ and $|u| < (1 + \beta)/\beta$. It is obvious that functions

$$\Psi_1(u; \beta) := \frac{1}{(1 + \beta - \beta u)^2}, \qquad \Psi_2(u; \beta) := \frac{1}{1 + \beta - \beta u}$$

are, respectively, the PGFs of the negative binomial distribution $NB(\beta/(1 + \beta), 2)$ and the geometric distribution $Geom(\beta/(1+\beta))$. Finally, using the PMF of the RV $\alpha \circ X$, given by Eq (11), and after some computation, the survival function of $\alpha \circ X$ is obtained as follows:

$$S_{\alpha \circ X}(x; \alpha, \theta) := P\{\alpha \circ X \geq x\} = \frac{1}{(\theta + 1)^2} \sum_{k=x}^{\infty} \frac{(k+1)\beta^k}{(1 + \beta)^{k+2}} + \frac{\theta(\theta + 2)}{(\theta + 1)^2} \sum_{k=x}^{\infty} \frac{\beta^k}{(1 + \beta)^{k+1}}$$

$$= \frac{1}{(\theta + 1)^2} \cdot \frac{(1 + \beta + x)\beta^x}{(1 + \beta)^{x+1}} + \frac{\theta(\theta + 2)}{(\theta + 1)^2} \cdot \frac{\beta^x}{(1 + \beta)^x} = \frac{\beta^x \left((1 + \beta)(\theta + 1)^2 + x\right)}{(\theta + 1)^2(1 + \beta)^{x+1}}$$

$$= \frac{\alpha^x(\theta + 1)^{x-2}}{(\alpha\theta + \alpha + \theta)^{x+1}} \left[\theta\left(x + (\theta + 1)^2\right) + \alpha(\theta + 1)^3\right].$$

Therefore, this statement is fully proved. □

Proof of Theorem 1. Let us observe an arbitrary non-negative integer $x = 0, 1, 2, \dots$. Based on the definition of the PL-MINAR process, as well as the independence of the RVs (X_t) and (ε_t), it follows that:

$$P\{X_t \geq x\} = P\{\min(\alpha \circ X_{t-1}, \varepsilon_t) \geq x\} = P\{\alpha \circ X_{t-1} \geq x\} P\{\varepsilon_t \geq x\}.$$

From here, the survival function for RVs (ε_t) is directly obtained:

$$S_\varepsilon(x) := P\{\varepsilon_t \geq x\} = \frac{P\{X_t \geq x\}}{P\{\alpha \circ X_{t-1} \geq x\}},$$

and using the previously obtained survival functions, given by Eqs (10) and (12), as well as after some computation, Eq (14) follows. At the same time, note that this expression is well defined if and only if the following two conditions are met:

I condition: The exponential part of Eq (14) must converge to 0 and 1, when $x \to +\infty$ and $x \to 0$, respectively. According to $\alpha, \theta > 0$, this is obviously fulfilled if and only if:

$$\frac{\alpha\theta + \alpha + \theta}{\alpha(\theta + 1)^2} < 1 \quad \Longleftrightarrow \quad \alpha(\theta + 1)^2 - \alpha(\theta + 1) - \theta > 0$$

$$\Longleftrightarrow \quad \alpha\theta(\theta + 1) > \theta \tag{A.3}$$

$$\Longleftrightarrow \quad \alpha > \frac{1}{\theta + 1}.$$

II condition: The survival function $S_\varepsilon(x) = P\{\varepsilon_t \geq x\}$ is monotonically non-increasing, i.e., for arbitrary $x = 0, 1, 2, \dots$ the PMF of the RVs (ε_t) must satisfy the condition:

$$p_\varepsilon(x; \alpha, \theta) := S_\varepsilon(x) - S_\varepsilon(x + 1) \geq 0. \tag{A.4}$$

Applying Eq (14) yields the following inequality:

$$\frac{(\alpha\theta + \alpha + \theta)^{x+1}}{\alpha^x(\theta + 1)^{2x}} \left(\frac{\theta(\theta + x + 2) + 1}{\alpha(\theta + 1)^3 + \theta\left((\theta + 1)^2 + x\right)} \right.$$

$$\left. - \frac{(\alpha\theta + \alpha + \theta)(\theta(\theta + x + 3) + 1)}{\alpha(\theta + 1)^2 \left(\alpha(\theta + 1)^3 + \theta\left((\theta + 1)^2 + x + 1\right)\right)} \right) \geq 0,$$

and after some computation, it can be expressed in the following equivalent way:

$$a(\alpha, \theta)x^2 + b(\alpha, \theta)x + c(\alpha, \theta) \geq 0, \tag{A.5}$$

where:

$$a(\alpha, \theta) := \theta^3(\alpha(\theta + 1) - 1)$$

$$b(\alpha, \theta) := \theta^2(\alpha(\theta + 1) - 1)\left(\alpha(\theta + 1)^3 + \theta^3 + 3\theta^2 + 4\theta + 1\right)$$

$$c(\alpha, \theta) := \theta^2(\theta + 1)^2\left(\alpha^2(\theta + 2)(\theta + 1)^2 + \alpha(\theta + 2)(\theta^2 - 1) - \theta(\theta + 3) - 1\right).$$

Note that from the first condition, that is, when Eq (A.3) holds, the inequalities $a(\alpha, \theta) > 0$ and $b(\alpha, \theta) > 0$ obviously follow. Therefore, the second condition is fulfilled if and only if it holds:

$$\widetilde{c}(\alpha, \theta) := \frac{c(\alpha, \theta)}{\theta^2(\theta + 1)^2} = (\theta + 2)(\theta + 1)^2\,\alpha^2 + (\theta + 2)(\theta^2 - 1)\,\alpha - \theta(\theta + 3) - 1 \geq 0. \tag{A.6}$$

To that end, notice that $\widetilde{c}(\alpha, \theta)$ is a convex quadratic function with respect to $\alpha > 0$, with coefficients:

$$a_1(\theta) := (\theta + 2)(\theta + 1)^2 > 0, \quad c_1(\theta) := -(\theta^2 + 3\theta + 1) < 0,$$

and discriminant:

$$D(\theta) = (\theta + 1)^3(\theta + 2)(\theta^2 + 3\theta + 6) > 0.$$

Therefore, equation $\widetilde{c}(\alpha, \theta) = 0$ has two real solutions $\alpha_1 = \alpha_1(\theta)$ and $\alpha_2 = \alpha_2(\theta)$, with different signs because, according to Vieta formulas, $\alpha_1 \cdot \alpha_2 = c_1(\theta)/a_1(\theta) < 0$. Thus, this implies that the inequality (A.6) holds if and only if $\alpha > 0$ is greater than or equal to the larger solution of the equation $\widetilde{c}(\alpha, \theta) = 0$ (see Figure 6(a)). In that way, the second condition is equivalent to the inequality:

$$\alpha \geq \frac{(1 - \theta)(\theta + 1)(\theta + 2) + \sqrt{(\theta + 1)^3(\theta + 2)(\theta^2 + 3\theta + 6)}}{2(\theta + 2)(\theta + 1)^2},$$

which yields, after some rearrangement, Eq (15). Finally, let us notice that, for arbitrary $\theta > 0$, the following expressions hold:

$$\frac{1}{2}\left(\frac{1 - \theta}{1 + \theta} + \sqrt{\frac{\theta^2 + 3\theta + 6}{\theta^2 + 3\theta + 2}}\right) > \frac{1}{2}\left(\frac{1 - \theta}{1 + \theta} + 1\right) = \frac{1}{\theta + 1}.$$

This means that the second condition, given by Eq (15), implies the first condition, given by Eq (A.3), as can also be seen in Figure 6(b).
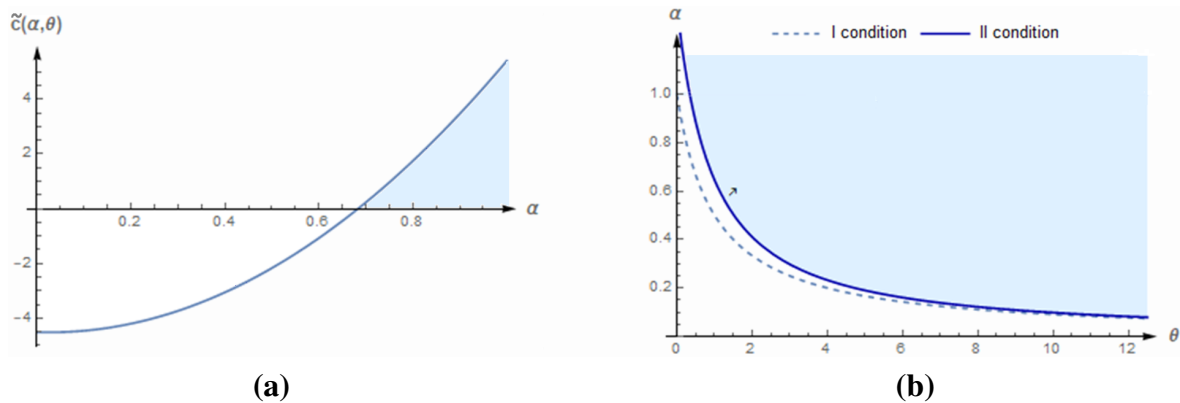
Figure 6. **(a)** Dependence of parameters α and θ given by quadratic function $\widetilde{c}(\alpha, \theta)$. **(b)** Conditions for a well-defined distribution of innovations (ε_t) depending on the parameters α and θ.

Finally, using Eqs (14) and (A.4), for the PGF of the RVs (ε_t) one obtains:

$$
\begin{aligned}
\Psi_\varepsilon(u; \alpha, \theta) &= \sum_{x=0}^{\infty} u^x p_\varepsilon(x; \alpha, \theta) = \sum_{x=0}^{\infty} u^x \left[\mathcal{S}_\varepsilon(x; \alpha, \theta) - \mathcal{S}_\varepsilon(x+1; \alpha, \theta) \right] \\
&= \sum_{x=0}^{\infty} u^x \mathcal{S}_\varepsilon(x; \alpha, \theta) - \sum_{x=1}^{\infty} u^{x-1} \mathcal{S}_\varepsilon(x; \alpha, \theta) = 1 + \sum_{x=1}^{\infty} \left(u^x - u^{x-1} \right) \mathcal{S}_\varepsilon(x; \alpha, \theta) \\
&= 1 + \left(1 - u^{-1} \right) \sum_{x=1}^{\infty} u^x \frac{(\alpha\theta + \alpha + \theta)^{x+1}}{\alpha^x (\theta + 1)^{2x}} \cdot \frac{\theta x + (\theta + 1)^2}{\theta x + (\theta + 1)^2 (\alpha\theta + \alpha + \theta)} \\
&= 1 + \left(1 - \frac{1}{u} \right) (\alpha\theta + \alpha + \theta) \sum_{x=1}^{\infty} \left(\frac{u(\alpha\theta + \alpha + \theta)}{\alpha(\theta + 1)^2} \right)^x \left(1 - \frac{(\theta + 1)^2 (\alpha\theta + \alpha + \theta - 1)}{\theta x + (\theta + 1)^2 (\alpha\theta + \alpha + \theta)} \right) \\
&= 1 + \frac{u - 1}{u} \lambda \left(\sum_{x=1}^{\infty} (\gamma u)^x - (\theta + 1)^2 (\lambda - 1) \sum_{x=1}^{\infty} \frac{(\gamma u)^x}{\theta x + \lambda(\theta + 1)^2} \right) \\
&= 1 + \frac{u - 1}{u} \lambda \sum_{x=1}^{\infty} (\gamma u)^x - \frac{u - 1}{u} \frac{\lambda(\theta + 1)^2}{\theta} (\lambda - 1) \sum_{x=1}^{\infty} \frac{(\gamma u)^x}{x + \lambda(\theta + 1)^2 / \theta} \\
&= 1 + \frac{u - 1}{u} \cdot \frac{\lambda \gamma u}{1 - \gamma u} - \frac{u - 1}{u} \delta(\lambda - 1) \sum_{x=1}^{\infty} \frac{(\gamma u)^x}{x + \delta} \\
&= 1 + \frac{\lambda \gamma (u - 1)}{1 - \gamma u} - \gamma \delta(\lambda - 1)(u - 1) \sum_{x=0}^{\infty} \frac{(\gamma u)^x}{x + \delta + 1}.
\end{aligned}
$$

The last expression is obviously the same as in Eq (16), whereby according to Eq (A.3) it follows:

$$
0 < \gamma = \frac{\alpha\theta + \alpha + \theta}{\alpha(\theta + 1)^2} < 1.
$$

Thus, under the condition $|u| < 1/\gamma$, the PGF $\Psi_\varepsilon(u; \alpha, \theta)$ is properly defined, which completely proves this theorem. $\square$

Proof of Theorem 2. For arbitrary non-negative integers x, y, the transition probabilities can be expressed in terms of conditional survival functions as follows:

$$P\{X_t = x \mid X_{t-1} = y\} = P\{X_t \geq x \mid X_{t-1} = y\} - P\{X_t \geq x+1 \mid X_{t-1} = y\}.$$

Using the independence of the RVs $\alpha \circ X_{t-1}$ and ε_t, as well as the fact that RVs $\alpha \circ X_{t-1}$, when $X_{t-1} = y$, have a negative binomial distribution $NB(\alpha/(1+\alpha), y+1)$, given by Eq (3), it follows that:

$$P\{X_t \geq x \mid X_{t-1} = y\} = P\{\min(\alpha \circ X_{t-1}, \varepsilon_t) \geq x \mid X_{t-1} = y\} = P\{\alpha \circ X_{t-1} \geq x \mid X_{t-1} = y\} P\{\varepsilon_t \geq x\}$$

$$= \mathcal{S}_\varepsilon(x; \alpha, \theta) \sum_{k=x}^{\infty} P\{\alpha \circ X_{t-1} = k \mid X_{t-1} = y\} = \mathcal{S}_\varepsilon(x; \alpha, \theta) \sum_{k=x}^{\infty} \binom{k+y}{k} \frac{\alpha^k}{(1+\alpha)^{k+y+1}}.$$

According to this, one obtains:

$$P\{X_t = x \mid X_{t-1} = y\} = \mathcal{S}_\varepsilon(x; \alpha, \theta) \sum_{k=x}^{\infty} \binom{k+y}{k} \frac{\alpha^k}{(1+\alpha)^{k+y+1}} - \mathcal{S}_\varepsilon(x+1; \alpha, \theta) \sum_{k=x+1}^{\infty} \binom{k+y}{k} \frac{\alpha^k}{(1+\alpha)^{k+y+1}}$$

$$= \mathcal{S}_\varepsilon(x; \alpha, \theta) \binom{x+y}{x} \frac{\alpha^x}{(1+\alpha)^{x+y+1}} + \left[\mathcal{S}_\varepsilon(x; \alpha, \theta) - \mathcal{S}_\varepsilon(x+1; \alpha, \theta)\right] \sum_{k=x+1}^{\infty} \binom{k+y}{k} \frac{\alpha^k}{(1+\alpha)^{k+y+1}},$$

and the last expression is obviously the same as in Eq (17).

To prove the strict stationarity of our model, we use a similar procedure as Aleksić & Ristić [14], that is, we show that for all $h \in \mathbb{N}_0$ and $n \in \mathbb{N}$ holds:

$$P\{X_1 = x_1, \ldots, X_n = x_n\} = P\{X_{1+h} = x_1, \ldots, X_{n+h} = x_n\}.$$

According to the proven Markov's property, this equality is equivalent to the following:

$$P\{X_1 = x_1\} \prod_{j=2}^{n} P\{X_j = x_j \mid X_{j-1} = x_{j-1}\} = P\{X_{1+h} = x_1\} \prod_{j=2}^{n} P\{X_{j+h} = x_j \mid X_{j+h-1} = x_{j-1}\},$$

where (X_t) is marginally stationary, and proven Eq (17) ensures that the above conditional probabilities do not depend on the time $t \in \mathbb{Z}$. Thus, it follows that the left and right sides of the above equation are really equal, i.e., (X_t) is indeed a strictly stationary stochastic process.

Now, by using Eq (17), the conditional PGF of the RVs (X_t) can be expressed as follows:

$$\begin{aligned}
E[u^{X_t} \mid X_{t-1} = y] &= \sum_{x=0}^{\infty} u^x P\{X_t = x \mid X_{t-1} = y\} \\
&= u^x \left(P\{X_t \geq x \mid X_{t-1} = y\} - P\{X_{t+1} \geq x \mid X_{t-1} = y\}\right) \\
&= \sum_{x=0}^{\infty} u^x \mathcal{S}_\varepsilon(x; \alpha, \theta) \sum_{j=x}^{\infty} \binom{y+j}{j} \frac{\alpha^j}{(1+\alpha)^{y+j+1}} \\
&\qquad - \sum_{x=0}^{\infty} u^x \mathcal{S}_\varepsilon(x+1; \alpha, \theta) \sum_{j=x+1}^{\infty} \binom{y+j}{j} \frac{\alpha^j}{(1+\alpha)^{y+j+1}} \\
&= 1 + \left(1 - \frac{1}{u}\right) \sum_{x=1}^{\infty} u^x \mathcal{S}_\varepsilon(x; \alpha, \theta) \mathcal{S}(x, y; \alpha).
\end{aligned} \tag{A.7}$$

In the last summation, we refer to:

$$\mathcal{S}(x, y; \alpha) := \sum_{j=x}^{\infty} \binom{y + j}{j} \frac{\alpha^j}{(1 + \alpha)^{y+j+1}}$$

as the survival function of the RV with negative binomial distribution $NB(\alpha/(1 + \alpha), y + 1)$, which implies that $\mathcal{S}(x, y; \alpha)$ is monotonically decreasing on x and $\mathcal{S}(x, y; \alpha) \leq 1$, for all $x, y = 0, 1, 2, \ldots$. In addition, similarly as in the proof of Theorem 1, we have:

$$T_\varepsilon(j; \alpha, \theta) := \sum_{k=1}^{j} u^k \mathcal{S}_\varepsilon(k; \alpha, \theta) = \sum_{k=1}^{j} \frac{u^k (\alpha\theta + \alpha + \theta)^{k+1}}{\alpha^k(\theta + 1)^{2k}} \cdot \frac{\theta k + (\theta + 1)^2}{\theta k + (\theta + 1)^2(\alpha\theta + \alpha + \theta)}$$

$$= (\alpha\theta + \alpha + \theta) \sum_{k=1}^{j} \left(\frac{u(\alpha\theta + \alpha + \theta)}{\alpha(\theta + 1)^2} \right)^k \left(1 - \frac{(\theta + 1)^2(\alpha\theta + \alpha + \theta - 1)}{\theta k + (\theta + 1)^2(\alpha\theta + \alpha + \theta)} \right)$$

$$= (\alpha\theta + \alpha + \theta) \left(\sum_{k=1}^{j} (\gamma u)^k - (\theta + 1)^2(\alpha\theta + \alpha + \theta - 1) \sum_{k=1}^{j} \frac{(\gamma u)^k}{\theta k + (\theta + 1)^2(\alpha\theta + \alpha + \theta)} \right)$$

$$= \lambda \sum_{k=1}^{j} (\gamma u)^k - \delta(\lambda - 1) \sum_{k=1}^{j} \frac{(\gamma u)^k}{k + \delta}$$

$$= \lambda \gamma u \frac{1 - (\gamma u)^j}{1 - \gamma u} - \delta(\lambda - 1) \gamma u \left(L_{\delta+1}(u\gamma) - (\gamma u)^j L_{\delta+j+1}(u\gamma) \right),$$

by which, under the condition $|\gamma u| < 1$, it follows:

$$\lim_{j \to +\infty} T_\varepsilon(j; \alpha, \theta) = \frac{\lambda \gamma u}{1 - \gamma u} - \delta(\lambda - 1) \gamma u \, L_{\delta+1}(u\gamma) < +\infty.$$

Therefore, according to Abel's convergence criterion (see, e.g., [42]), the double sum in Eq (A.7) is convergent, and the order of sums can be changed, which yields:

$$E[u^{X_t} \mid X_{t-1} = y] = 1 + \left(1 - \frac{1}{u} \right) \sum_{j=1}^{\infty} T_\varepsilon(j; \alpha, \theta) \binom{y + j}{j} \frac{\alpha^j}{(1 + \alpha)^{y+j+1}}$$

$$= 1 + \left(1 - \frac{1}{u} \right) \left[\frac{\lambda \gamma u}{1 - \gamma u} \sum_{j=0}^{\infty} \left(1 - (\gamma u)^j \right) \binom{y + j}{j} \frac{\alpha^j}{(1 + \alpha)^{y+j+1}} \right.$$

$$\left. - \delta(\lambda - 1)\gamma u \sum_{j=0}^{\infty} \left(L_{\delta+1}(u\gamma) - (\gamma u)^j L_{\delta+j+1}(u\gamma) \right) \binom{y + j}{j} \frac{\alpha^j}{(1 + \alpha)^{y+j+1}} \right]$$

$$= 1 + \frac{\lambda\gamma(u - 1)}{1 - \gamma u} \left(1 - (1 + \alpha - \alpha\gamma u)^{-y-1} \right)$$

$$- \delta(\lambda - 1)\gamma(u - 1) \left(L_{\delta+1}(\gamma u) - \sum_{j=0}^{\infty} L_{\delta+j+1}(\gamma u) \binom{y + j}{j} \frac{(\alpha\gamma u)^j}{(1 + \alpha)^{y+j+1}} \right).$$

Thus, the obtained expression obviously represents Eq (18), which proves this theorem. □

Proof of Theorem 3. First, we have calculated the joint two-dimensional PGF of RVs X_t and X_{t-1}. Using the law of total expectation, the previously obtained PGF of the PL-distributed RVs, given by Eq (8), and the conditional PGF of the first order, given by Eq (18), one obtains:

$$\Psi_X^{(2)}(u,v) := E\left[u^{X_t}v^{X_{t-1}}\right] = E\left[v^{X_{t-1}}E[u^{X_t} \mid X_{t-1}]\right] = E\left[v^{X_{t-1}}\left[1 + \frac{\lambda\gamma(u-1)}{1-\gamma u}\left(1 - \frac{1}{(1+\alpha-\alpha\gamma u)^{X_{t-1}+1}}\right)\right.\right.$$

$$\left.\left. - (\lambda-1)\gamma\delta(u-1)\left(L_{\delta+1}(\gamma u) - \sum_{j=0}^{\infty}\binom{X_{t-1}+j}{j}\frac{(\alpha\gamma u)^j L_{\delta+j+1}(\gamma u)}{(1+\alpha)^{X_{t-1}+j+1}}\right)\right]\right]$$

$$= A(u;\lambda,\gamma,\delta)\,\Psi_X(v;\theta) - B(u;\alpha,\lambda,\gamma)\,K(u,v;\alpha,\gamma) + C(u;\alpha,\lambda,\gamma,\delta)\sum_{j=0}^{\infty}\eta_j(u;\alpha,\gamma,\delta)\,M_j(v;\alpha,\theta),$$

where:

$$A(u;\lambda,\gamma,\delta) = 1 + \frac{\lambda\gamma(u-1)}{1-\gamma u} - (\lambda-1)\gamma\delta(u-1)L_{\delta+1}(\gamma u),$$

$$B(u;\alpha,\lambda,\gamma) = \frac{\lambda\gamma(u-1)}{(1-\gamma u)(1+\alpha-\alpha\gamma u)},$$

$$C(u;\alpha,\gamma,\delta) = \frac{(\lambda-1)\gamma\delta(u-1)}{1+\alpha},$$

$$\Psi_X(v;\theta) = E\left[v^{X_{t-1}}\right] = \frac{\theta^2(\theta-v+2)}{(\theta+1)(\theta-v+1)^2},$$

$$K(u,v;\alpha,\gamma,\theta) = E\left[\left(\frac{v}{1+\alpha-\alpha\gamma u}\right)^{X_{t-1}}\right] = \Psi_X\left(\frac{v}{1+\alpha-\alpha\gamma u}\right) = \left.\frac{\theta^2(\theta-w+2)}{(\theta+1)(\theta-w+1)^2}\right|_{w=\frac{v}{1+\alpha-\alpha\gamma u}}$$

$$= \frac{\theta^2(1+\alpha-\alpha\gamma u)\left[(\theta+2)(1+\alpha-\alpha\gamma u)-v\right]}{(\theta+1)\left[(\theta+1)(1+\alpha-\alpha\gamma u)-v\right]^2},$$

$$\eta_j(u;\alpha,\gamma,\delta) = \left(\frac{\alpha\gamma u}{1+\alpha}\right)^j L_{\delta+j+1}(\gamma u),$$

$$M_j(v;\alpha,\theta) = E\left[\binom{X_{t-1}+j}{j}\left(\frac{v}{1+\alpha}\right)^{X_{t-1}}\right] = \frac{1}{j!}E\left[(X_{t-1}+j)(X_{t-1}+j-1)(X_{t-1}+1)\left(\frac{v}{1+\alpha}\right)^{X_{t-1}}\right]$$

$$= \left.\frac{1}{j!}\frac{\partial^j E[z^{X_{t-1}+j}]}{\partial z^j}\right|_{z=\frac{v}{1+\alpha}} = \left.\frac{1}{j!}\frac{\partial^j}{\partial z^j}\left(z^j\,\Psi_X(z;\theta)\right)\right|_{z=\frac{v}{1+\alpha}}$$

$$= \left.\frac{\theta^2(\theta+1)^{j-2}\left[(\theta+1)(\theta+2)-z(\theta-j+1)\right]}{(\theta-z+1)^{j+2}}\right|_{z=\frac{v}{1+\alpha}}$$

$$= \theta^2(\theta+1)^{j-2}(1+\alpha)^{j+1}\frac{(\theta+1)(\theta+2)(1+\alpha)-v(\theta-j+1)}{\left[(\theta+1)(1+\alpha)-v\right]^{j+2}}.$$

By differentiating the function $\Psi_X^{(2)}(u, v)$ with respect to u, v, we get:

$$\frac{\partial^2 \Psi_X^{(2)}(u, v)}{\partial u\, \partial v} = \frac{\partial A(u; \lambda, \gamma, \delta)}{\partial u} \frac{\partial \Psi_X(v; \theta)}{\partial v} - \frac{\partial B(u; \alpha, \lambda, \gamma)}{\partial u} \frac{\partial K(u, v; \alpha, \gamma, \theta)}{\partial v}$$

$$- B(u; \alpha, \lambda, \gamma) \frac{\partial^2 K(u, v; \alpha, \gamma, \theta)}{\partial u\, \partial v} + \sum_{j=0}^{\infty} \left[\frac{\partial C(u; \alpha, \lambda, \gamma, \delta)}{\partial u} \eta_j(u, \alpha, \gamma) + C(u; \alpha, \lambda, \gamma, \delta) \frac{\partial \eta_j(u, \alpha, \gamma)}{\partial u} \right]$$

$$\times \frac{\partial M_j(v; \alpha, \theta)}{\partial v}.$$

Thereafter, putting $u = v = 1$ and after some computations, the mixed moment for RVs X_t and X_{t-1} can be obtained as follows:

$$E[X_t X_{t-1}] = \frac{\partial^2 \Psi_X^{(2)}(u, v)}{\partial u\, \partial v} \bigg|_{u=v=1} = \frac{(\theta + 2)\gamma \left(\lambda - (\lambda - 1)(1 - \gamma)\delta L_{\delta+1}(\gamma)\right)}{\theta(\theta + 1)(1 - \gamma)}$$

$$- \frac{\lambda \gamma \theta^2 (\alpha(1 - \gamma)(\theta + 3) + \theta + 2)}{(1 - \gamma)(\theta + 1)(\alpha(1 - \gamma)(\theta + 1) + \theta)^3} + \frac{\gamma \delta(\lambda - 1)}{(\alpha\theta + \alpha + \theta)^3} \left(\frac{\theta}{\theta + 1}\right)^2 \tag{A.8}$$

$$\times \sum_{j=0}^{\infty} (j + 1) \left(\frac{\alpha\gamma(\theta + 1)}{(1 + \alpha)(\theta + 1) - 1}\right)^j ((1 + \alpha)(\theta + 1)(\theta + 3) - \theta + j - 1) L_{j+\delta+1}(\gamma).$$

According to this and by applying Eq (9), for the first-order correlation of the series (X_t) one obtains:

$$\rho_X(1) = \text{Corr}(X_t, X_{t-1}) = \frac{E[X_t X_{t-1}] - E[X_t]\, E[X_{t-1}]}{\text{Var}[X_t]} = \left(E[X_t X_{t-1}] - \frac{(\theta + 2)^2}{\theta^2(\theta + 1)^2}\right) \frac{\theta^2(\theta + 1)^2}{\theta^3 + 4\theta^2 + 6\theta + 2},$$

which after some rearrangement yields Eq (20). □

Author contributions

Vladica S. Stojanović: Methodology; Conceptualization, Writing – Original Draft; Visualization; Hassan S. Bakouch: Supervision; Writing – Original Draft; Formal Analysis; Radica Bojičić: Writing – Review & Editing; Validation; Visualization; Gadir Alomair: Resources; Writing – Review & Editing; Funding Acquisition; Shuhrah A. Alghamdi: Formal Analysis, Investigation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

References

1. L. Guan, X. Wang, A discrete-time dual risk model with dependence based on a poisson INAR(1) process, *AIMS Math.,* **7** (2022), 20823–20837. http://dx.doi.org/10.3934/math.20221141

2. R. Maya, C. Chesneau, A. Krishna, M. R. Irshad, Poisson extended exponential distribution with associated INAR(1) process and applications, *Stats*, **5** (2022), 755–772.

3. W. C. Khoo, S. H. Ong, B. Atanu, Coherent forecasting for a mixed integer-valued time series model, *Mathematics*, **10** (2022), Article No. 2961.

4. K. Yu, T. Tao, An Observation-Driven random parameter INAR(1) model based on the poisson thinning operator, *Entropy*, **25** (2023), Article No. 859. https://doi.org/10.3390/e25060859

5. V. S. Stojanović, H. S. Bakouch, E. Ljajko, N. Qarmalah, Zero-and-One integer-valued AR(1) time series with power series innovations and probability generating function estimation approach, *Mathematics*, **11** (2023), Article No. 1772. https://doi.org/10.3390/math11081772

6. J. Huang, F. Zhu, D. Deng, A mixed generalized poisson INAR model with applications, *J. Stat. Comput. Sim.*, **93** (2023), 1851–1878.

7. V. S. Stojanović, H. S. Bakouch, Z. Gajtanović, F. E. Almuhayfith, K. Kuk, Integer-valued Split-BREAK process with a general family of innovations and application to accident count data modeling, *Axioms*, **13** (2024), Article No. 40. https://doi.org/10.3390/axioms13010040

8. Y. Kang, F. Zhu, D. Wang, S. Wang, A zero-modified geometric INAR(1) model for analyzing count time series with multiple features, *Can. J. Stat.*, (2024). https://doi.org/10.1002/cjs.11774

9. L. V. Tavares, An exponential Markovian stationary process, *J. Appl. Probab.*, **17** (1980), 1117–1120. https://doi.org/10.2307/3213224

10. L. V. Tavares, A Non-Gaussian Markovian model to simulate hydrologic processes, *J. Hydrol.*, **46** (1980), 281–287. https://doi.org/10.1016/0022-1694(80)90081-5

11. C. H. Sim, Simulation of Weibull and Gamma autoregressive stationary process, *Commun. Stat. Simul. Comput.*, **15** (1986), 1141–1146. `https://www.tandfonline.com/doi/abs/10.1080/03610918608812565`

12. P. A. Lewis, E. D. McKenzie, Minification processes and their transformations, *J. Appl. Probab.*, **28** (1991), 45–57. https://doi.org/10.2307/3214739

13. V. A. Kalamkar, Minification processes with discrete marginals, *J. Appl. Probab.*, **32** (1995), 692–706. https://doi.org/10.2307/3215123

14. M. Aleksić, M. Ristić, A geometric minification integer-valued autoregressive model, *Appl. Math. Model.*, **90** (2021), 265–280. https://doi.org/10.1016/j.apm.2020.08.047

15. M. Stojanović, An EM algorithm for estimation of the parameters of the geometric minification INAR model, *J. Stat. Comput. Simul.*, **92** (2022). https://doi.org/10.1080/00949655.2022.2053125

16. L. Qian, F. Zhu, A new minification integer-valued autoregressive process driven by explanatory variables, *Aust. N. Z. J. Stat.*, **64** (2022), 478–494.

17. Q. Zhang, D. Wang, X. Fan, A negative binomial thinning-based bivariate INAR(1) process, *Stat. Neerl.*, **74** (2020), 517–537.

18. M. S. Ristić, H. S. Bakouch, A. S. Nastić, A new geometric first-order integer-valued autoregressive (NGINAR(1)) process, *J. Stat. Plann. Infer.*, **139** (2009), 2218–2226. https://doi.org/10.1016/j.jspi.2008.10.007 .

19. J. Pitman, Probability, New York, NY: Springer New York. p. 372, 1993. https://doi.org/10.1007/978-1-4612-4374-8. ISBN 978-0-387-94594-1

20. M. Sankaran, The discrete Poisson-Lindley distribution, *Biometrics*, **26** (1970), 145–149.

21. M. Mohammadpour, H. S. Bakouch, M. Shirozhan, Poisson–Lindley INAR(1) Model with Applications, *Braz. J. Probab. Stat.*, **32** (2018), 262–280. https://doi.org/10.1214/16-BJPS341 .

22. Z. Mohammadi, Z. Sajjadnia, H. S. Bakouch, M. Sharafi, Zero-and-One inflated Poisson–Lindley INAR(1) process for modelling count time series with extra zeros and ones, *J. Stat. Comput. Sim.*, **92** (2022), 2018–2040.

23. W. A. H. Al-Nuaami, A. A. Heydari, H. J. Khamnei, The Poisson–Lindley distribution: Some characteristics, with its application to SPC, *Mathematics*, **11** (2023), Article No. 2428. https://doi.org/10.3390/math11112428

24. H. S. Bakouch, F. Gharari, K. Karakaya, Y. Akdoğan, Fractional Lindley distribution generated by time scale theory, with application to discrete-time lifetime data, *Math. Popul. Stud.*, **31** (2024), 116–146. https://doi.org/10.1080/08898480.2024.2301865

25. M. Ghitany, D. Al-Mutairi, Estimation methods for the discrete Poisson–Lindley distribution, *J. Stat. Comput. Simul.*, **79** (2009), 1—9.

26. M. G. Scotto, C. H. Weiß, T. A. Möller, S. Gouveia, The Max-INAR(1) model for count processes, *Test*, **27** (2018), 850–870. https://doi.org/10.1007/s11749-017-0573-z

27. L. Qian, G. Li, A Class of Max-INAR (1) processes with explanatory variables, *J. Stat. Comput. Simul.*, **92** (2022), 1898–1919.

28. M. G. Scotto, S. Gouveia, On the extremes of the Max-INAR (1) process for time series of counts, *Commun. Stat. Theory M.*, **52** (2023), 1136–1154.

29. V. L. Martin, A. R. Tremayne, R. C. Jung, Efficient method of moments estimators for integer time series models, *J. Time Series Anal.*, **35** (2014), 491–516.

30. Y. Cui, Q. Zheng, Conditional maximum likelihood estimation for a class of observation-driven time series models for count data, *Stat. Probab. Lett.*, **123** (2017), 193–201. https://doi.org/10.1016/j.spl.2016.11.002.

31. R. Azrak, G. Mélard, Asymptotic properties of conditional Least-Squares estimators for array time series, *Stat. Inference Stoch. Process*, **24** (2021), 525–547. https://doi.org/10.1007/s11203-021-09242-8

32. A. Buja, E. Hare, H. Hofmann, Create and manipulate discrete random variables, R package version 1.2.2, (2015). `https://CRAN.R-project.org/package=discreteRV` (accessed on 20 February 2024).

33. D. M. Gay, *Usage Summary for Selected Optimization Routines*, Computing Science, Technical Report 153, AT&T Bell Laboratories, Murray Hill, 1990. (accessed on 25 February 2024).

34. L. Gross, *Tests for Normality*, R package Version: 1.0.4, (2013). `http://CRAN.R-project.org/package=nortest` (accessed on 25 February 2024)

35. COINMETRICS, `https://coinmetrics.io/`

36. A. Aknouche, B. S. Almohaimeed, S. Dimitrakopoulos, Forecasting transaction counts with integer-valued GARCH models, *Stud. Nonlinear Dyn. E.*, **26** (2021), 529–539. https://doi.org/10.1515/snde-2020-0095

37. C. H. Weiß, F. Zhu, Conditional-Mean multiplicative operator models for count time series, *Comput. Stat. Data Anal.*, **191** (2024), Article No. 107885.

38. D. Qiu, *Alternative Time Series Analysis*, R package Version: 3.1.2.1. (2015). Available from: `https://rdocumentation.org/packages/aTSA/versions/3.1.2.1`. (accessed on 29 February 2024)

39. O. Kella, A. Löpker, On Binomial Thinning and Mixing, *Indag. Math.*, **5** (2023), 1121–1145.

40. F. Diebold, R. Mariano, Comparing Predictive Accuracy, *J. Bus. Econ. Stat.* **13** (1995), 253–263.

41. R. Hyndman, *Forecasting Functions for Time Series and Linear Models*, R Package Version 7.1. (2016). Available from: `http://CRAN.R-project.org/package=forecast` (accessed on 3 March 2024).

42. T. M. Apostol, *Mathematical Analysis* (2nd ed.), Addison-Wesley, 1974. ISBN 978-0-201-00288-1