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Research article

The influence of damping on the asymptotic behavior of solution for laminated beam

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Abstract: This paper dealt with a laminated beam system along with structural damping, past history, distributed delay, and in the presence of both temperatures and micro-temperatures effects. The damping terms left the system dissipative. Employing the semigroup approach, we established the existence and uniqueness of the solution. Additionally, with the help of convenient assumptions on the kernel, we demonstrated a general decay result for the solution of the considered system, with no constraints regarding the speeds of wave propagation. The main aim was to address how specific behaviors of the system were related to memory and delays. We aimed to investigate the joint impact of an infinite memory, distributed delay and micro-temperature effects on the system. We found a new relationship between the decay rate of solution and the growth of g at infinity. The objective was to find studies that use no- trivial results and their applications to relevant problems from mathematical physics.

Keywords: laminated beam; stability; well-posedness; micro-temperature effects; structural damping; Lyapunov functions; distributed delay; past history; dynamical systems
Mathematics Subject Classification: 35B40, 35L56, 93D15

1. Introduction and relevance of subject

Mathematical modeling is indispensable in engineering, natural science, and applied mathematics to capture the effects of both memory and delay ingrained in the studied actualities. To this end, the inclusion of both of them is often simplified for presentation purposes, as a specific description of basic operations can be intricate for mathematical manipulation. A key question to address is how certain behaviors are related to memory and delays. In this study, we investigate the joint impact of an infinite memory, distributed delay, and micro-temperature effects on the system (1.1).

In the current work, we study the following thermoelastic laminated beam, together with structural damping, infinite memory, distributed delay, and micro-temperatures effects:

$$\begin{cases} \varrho\varpi_{tt} + G(\phi - \varpi_x)_x + \gamma\theta_x = 0, \\ I_\varrho(3\psi - \phi)_{tt} - D(3\psi - \phi)_{xx} - G(\phi - \varpi_x) - m\theta + dr_x \\ \qquad\qquad + \displaystyle\int_0^\infty g(s)(3\psi - \phi)_{xx}(x, t - s)ds = 0, \\ 3I_\varrho\psi_{tt} - 3D\psi_{xx} + 3G(\phi - \varpi_x) + 4\delta\psi + 4\beta\psi_t + 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\psi_t(x, t - \varsigma)d\varsigma = 0, \\ c\theta_t - k_0\theta_{xx} + m(3\psi - \phi)_t + \gamma\varpi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + k_1\theta_x + d(3\psi - \phi)_{tx} = 0, \end{cases} \tag{1.1}$$

where

$$(x, \varsigma, t) \in (0, 1) \times (\varsigma_1, \varsigma_2) \times \mathbb{R}_+,$$

and the initial and boundary conditions are given by

$$\begin{cases} \varpi(x, 0) = \varpi_0,\ \psi(x, 0) = \psi_0,\ \phi(x, 0) = \phi_0,\ \theta(x, 0) = \theta_0,\ r(x, 0) = r_0,\ x \in (0, 1), \\ \varpi_t(x, 0) = \varpi_1,\ \psi_t(x, 0) = \psi_1,\ \phi_t(x, 0) = \phi_1, \quad x \in (0, 1), \\ \varpi_x(0, t) = \phi(0, t) = \psi(0, t) = \theta(0, t) = r(0, t) = 0, \quad t > 0, \\ \varpi(1, t) = \phi_x(1, t) = \psi_x(1, t) = \theta_x(1, t) = r(1, t) = 0,\ \psi_t(x, -t) = f_0(x, t) \quad t > 0. \end{cases} \tag{1.2}$$

Here, ϖ denotes the transverse displacement, ϕ represents the rotation angle, ψ is relative to the amount of slip occurring along the interface, θ is the temperature difference and r is the micro-temperature vector. The coefficients $\delta,\ \beta,\ \varrho,\ I_\varrho,\ G$, and D, are positive constants representing the adhesive stiffness, the adhesive damping parameter, the density, the shear stiffness, the flexible rigidity and the mass moment of inertia, respectively. We denote by the positive constants $c,\ k_0,\ k_1,\ k_2,\ k_3,\ d,\ \gamma,\ \alpha,\ m,$ the physical parameters describing the coupling between the various constituents of the materials.

Herein, $\varsigma_1,\ \varsigma_2$ are positive numbers such that $0 < \varsigma_1 < \varsigma_2$, and μ_2 is an L^∞ function satisfying the following assumption:

- The function $\mu_2 : [\varsigma_1, \varsigma_2] \to \mathbb{R}$ is bounded and it fulfills

$$\beta - \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|d\varsigma > 0.$$

To motivate our work, let us recall some earlier related results. For the problems with the Timoshenko system with/without thermal law, one can see the works [4, 6, 7, 16, 23, 24] and for problems related to thermoelasticity, we mention for instance [8, 10, 11, 13, 17, 18].

We start with the laminated beam model, which has become quite popular, and both scientists and engineers are interested in it. This model is a pertinent study topic, because of the wide industry applicability of such materials. Hansen and Spies in [12] were the first to introduce the following beam with two layers by developing this mathematical model:

$$\begin{cases} \rho_1 \varpi_{tt} + G(\phi - \varpi_x)_x = 0, \\ \rho_2(3\psi - \phi)_{tt} - G(\phi - \varpi_x) - D(3\psi - \phi)_{xx} = 0, \\ \rho_3 \psi_{tt} + G(\phi - \varpi_x) + \frac{4}{3}\gamma\psi + \frac{4}{3}\beta\psi_t - D\psi_{xx} = 0. \end{cases} \tag{1.3}$$

The laminated beam equations have produced some results so far, most of which are focused on the system's stability and existence. Provided that the assumption of equal wave speeds holds, it was demonstrated that system (1.3) is exponentially stable, when linear damping terms are incorporated in two of the three equations. However, if they are included in the three equations, then the system decays exponentially with no restriction on the speeds of wave propagations, see, for instance [1, 22].

Lately, a renewed focus on investigating the asymptotic behavior of the solutions of several thermoelastic laminated beams has grown. For more details about this topic the reader may consult [2, 9, 20].

The thermoelastic laminated beam problem together with nonlinear weights and time-varying delay was the study topic of Nonato et al. in [20], where the authors considered two cases (with and without the structural damping) and proved an exponential decay result for both of them. Distributed delay is one of the main damping factors in our model. It is used to model systems in which there is a delay of uncertain duration. The physical interpretation of this term differs from the delayed differential equation, as it can take several values. For example, in incoming signals, distributed delay shortens the setup and lengthens the hold time. Even moderate distributed delay likely makes setup time negative on those inputs that are directly connected to the register.

The infinite memory is a critical aspect in addressing problems, and it has been explored in various contexts such as the work of Liu and Zhao [14], in which they considered a thermoelastic laminated beam model with past history. The authors managed to establish both exponential and polynomial stabilities, depending on the kernel function for the system involving structural damping and with no constraint on the wave speeds. Moreover, concerning the system in the absence of structural damping, they were able to establish both exponential and polynomial stabilities, in case of equal wave speeds and lack of exponential stability in the opposite case.

The time delays problems are one of the most significant and active research areas recently. Numerous studies have demonstrated that delay can lead to instability unless certain conditions are incorporated, and it also can lead to distinct solutions that differ from those found in prior studies. Therefore, the issue of stability for systems that involve delay is highly crucial. To learn more about this term, we refer the reader to the following papers [3, 5, 21].

In [19], Nicaise and Pignotti made a study on the following wave equation, together with linear frictional damping and internal distributed delay:

$$u_{tt} - \Delta u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)\, ds, \quad \text{in } \Omega \times (0, \infty),$$

and assuming that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1,$$

the authors managed to prove that the solution is exponentially stable.

Problem (1.1) is considered as a delayed system, and it is also called hereditary systems, posteffect systems, and deviating argument. Distributed delay is a physical phenomenon which is found in a multitude of applications: Many real systems whose temporal evolution is not defined from a simple vector of state (expressed in the present tense) but depends irreducibly on the history of the system. This situation is encountered in the cases-numerous-where a transport of matter, energy or information generates a "dead time" in the reaction: in information and communication technologies (high-speed communication networks, control of networked systems, quality of service in Moving Picture Experts Group (MPEG) video transmissions, tele-operated systems, parallel computing, realtime computing in robotics), in population dynamics and epidemiology (gestation or incubation time), and in mechanics (viscoelasticity). Even if the process does not intrinsically contain a post-effect, its control chain can introduce distributed delays (for example, if the sensors require a significant acquisition/transmission time). For these reasons, it seems reasonable to consider distributed delay as a universal characteristic of the interaction between man and nature (hence, of sciences for engineers). The aim of our study then concerns the interaction between the different damping terms which intervene in the qualitative properties of the energy associated to the system. Before this analysis, we must ensure the existence of unique solution and then we can pass to see the asymptotic behavior of the solution with respect to damping terms. We used classical semigroup theory to find nontrivial results regarding the well-posedness of solutions. Then, under minimal restrictions on the kernel, we found qualitative properties of the solution by contracting an appropriate Lyapnov functional. The main goal is to present fundamental and new techniques for modern models applying science and technology that can stimulate research interest for exploration of mathematical applications in real life sciences.

The rest of the current paper is structured this way: In Section 2, we provide some resources required for our research, then highlight our major results. In Section 3, we establish the well-posedness of the system. In Section 4, we introduce some fundamental lemmas required in the proof later. In Section 5, we demonstrate our general decay result.

2. Preliminaries and main results

In this section, we provide some materials required in the proof later, then state our major results.

- $(\mathbf{A}_1)$ Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^1 function which satisfies

$$g(0) > 0, \;\; D - g_0 = \bar{l} > 0, \;\;\text{ where }\; g_0 := \int_0^\infty g(s)\, ds. \tag{2.1}$$

- $(\mathbf{A}_2)$ There exists a strictly increasing convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ which satisfies

$$\begin{cases} G(0) = G'(0) = 0, \\ \lim_{t\to+\infty} G'(t) = +\infty, \end{cases}$$

such that

$$\sup_{s\in\mathbb{R}_+}\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}\,ds+\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}\,ds<+\infty.$$

Now, we present the following useful inequalities.

Lemma 2.1. *The following inequalities are valid,*

$$\int_0^1\left[\int_0^\infty g(s)\,((3\psi-\psi)(t)-(3\psi-\phi)(t-s))\,ds\right]^2 dx\leq c_1(g\diamond(3\psi-\phi)_x)(t), \tag{2.2}$$

$$\int_0^1\left[\int_0^\infty g'(s)\,((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))\,ds\right]^2 dx\leq -g(0)(g'\diamond(3\psi-\phi)_x)(t), \tag{2.3}$$

$$\int_0^1\left[\int_0^\infty g(s)\,((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))\,ds\right]^2 dx\leq g_0(g\diamond(3\psi-\phi)_x)(t), \tag{2.4}$$

$$\int_0^1\left[\int_0^\infty g'(s)\,((3\psi-\phi)(t)-(3\psi-\phi)(t-s))\,ds\right]^2 dx\leq -c_2(g'\diamond(3\psi-\phi)_x)(t), \tag{2.5}$$

where $c_1,\ c_2>0$, *and*

$$(g\diamond v)(t)=\int_0^1\int_0^\infty g(s)(v(x,t)-v(x,t-s))^2dsdx.$$

Let us start by introducing (see [19])

$$\begin{cases}\eta^t(x,s)=(3\psi-\phi)(x,t)-(3\psi-\phi)(x,t-s),\\ \mathcal{S}(x,p,\varsigma,t)=\psi_t(x,t-\varsigma p),\end{cases} \tag{2.6}$$

where

$$(x,p,\varsigma,s,t)\in((0,1))^2\times(\varsigma_1,\varsigma_2)\times\mathbb{R}_+\times\mathbb{R}_+.$$

Then, the variables η^t and $\mathcal{S}$ surely satisfy

$$\begin{cases}\eta^t_t+\eta^t_s=(3\psi-\phi)_t,\\ \varsigma\mathcal{S}_t(x,p,\varsigma,t)+\mathcal{S}_p(x,p,\varsigma,t)=0,\\ \mathcal{S}(x,0,\varsigma,t)=\psi_t(x,t).\end{cases} \tag{2.7}$$

Hence, system (1.1) can be rewritten as

$$\begin{cases} \varrho\varpi_{tt} + G(\phi - \varpi_x)_x + \gamma\theta_x = 0, \\ I_\varrho(3\psi - \phi)_{tt} - D(3\psi - \phi)_{xx} - G(\phi - \varpi_x) - m\theta + dr_x + \int_0^\infty g(s)(3\psi - \phi)_{xx}(t-s)ds = 0, \\ 3I_\varrho\psi_{tt} - 3Ds_{xx} + 3G(\phi - \varpi_x) + 4\delta\psi + 4\beta\psi_t + 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathcal{S}(x, 1, \varsigma, t)d\varsigma = 0, \\ c\theta_t - k_0\theta_{xx} + m(3\psi - \phi)_t + \gamma\varpi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + k_1\theta_x + d(3\psi - \phi)_{tx} = 0, \\ \eta_t^t + \eta_s^t = (3\psi - \phi)_t, \\ \varsigma\mathcal{S}_t + \mathcal{S}_p = 0. \end{cases} \tag{2.8}$$

Certainly, system (2.8) is depending on the initial and boundary conditions below

$$\begin{cases} \varpi(x,0) = \varpi_0,\ \psi(x,0) = \psi_0,\ \phi(x,0) = \phi_0,\ \theta(x,0) = \theta_0,\ r(x,0) = r_0,\ x \in (0,1), \\ \varpi_t(x,0) = \varpi_1,\ \psi_t(x,0) = \psi_1,\ \phi_t(x,0) = \phi_1, \quad x \in (0,1), \\ \varpi_x(0,t) = \phi(0,t) = \psi(0,t) = \theta(0,t) = r(0,t) = 0, \quad t > 0, \\ \varpi(1,t) = \phi_x(1,t) = \psi_x(1,t) = \theta_x(1,t) = r(1,t) = 0,\ \psi_t(x,-t) = f_0(x,t) \quad t > 0, \\ \eta^t(0,s) = \eta_x^t(1,s) = 0,\ \eta^t(x,0) = 0,\ \eta^0(x,s) = \eta_0(x,s), \quad t,s > 0, \\ \mathcal{S}(x,p,\varsigma,0) = f_0(x,p\varsigma), \quad x,p \in (0,1),\ \varsigma \in (\varsigma_1,\varsigma_2),\ t,s > 0. \end{cases} \tag{2.9}$$

Now, let

$$\begin{cases} \zeta = 3\psi - \phi, \\ \zeta(0,t) = \zeta_x(1,t) = 0,\ \zeta(x,0) = \zeta_0,\ \zeta_t(x,0) = \zeta_1,\ (x,t) \in (0,1) \times \mathbb{R}_+. \end{cases}$$

Then, system (2.8) is equivalent to

$$\begin{cases} \varrho\varpi_{tt} + G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x = 0, \\ I_\varrho\zeta_{tt} - D\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + dr_x + \int_0^\infty g(s)\zeta_{xx}(t-s)ds = 0, \\ 3I_\varrho\psi_{tt} - 3D\psi_{xx} + 3G(3\psi - \zeta - \varpi_x) + 4\delta\psi + 4\beta\psi_t + 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathcal{S}(x,1,\varsigma,t)d\varsigma = 0, \\ c\theta_t - k_0\theta_{xx} + m\zeta_t + \gamma\varpi_{tx} + k_1 r_x = 0, \\ \alpha r_t - k_2 r_{xx} + k_3 r + k_1\theta_x + d\zeta_{tx} = 0, \\ \eta_t^t + \eta_s^t = \zeta_t, \\ \varsigma\mathcal{S}_t + \mathcal{S}_p = 0. \end{cases} \tag{2.10}$$

Taking advantage of (2.6), we can rewrite the second equation of (2.10) as

$$I_\varrho \zeta_{tt} - \bar{l}\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + dr_x - \int_0^\infty g(s)\eta^t_{xx}(x,s)ds = 0.$$

At this step, let us introduce the vector function $U = (\varpi, u, \zeta, \nu, \psi, y, \theta, r, \eta^t, \mathcal{S})^T$, with

$$\begin{aligned} u &= \varpi_t, \\ \nu &= \zeta_t, \\ y &= \psi_t, \end{aligned}$$

then, system (2.10) becomes

$$\begin{cases} \frac{d}{dt}U(t) = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0 = (\varpi_0, \varpi_1, \zeta_0, \zeta_1, \psi_0, \psi_1, \theta_0, r_0, \eta_0, f_0)^T, \end{cases} \tag{2.11}$$

here, $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} :\to \mathcal{H}$ stands for a linear operator indicated by

$$\mathcal{A}U = \begin{pmatrix} u \\ -\frac{1}{\varrho}\left(G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x\right) \\ \nu \\ \frac{1}{I_\varrho}\left(\bar{l}\zeta_{xx} + G(3\psi - \zeta - \varpi_x) + m\theta - dr_x + \int_0^\infty g(s)\eta^t_{xx}(x,s)ds\right) \\ y \\ \frac{1}{I_\varrho}\left(D\psi_{xx} - G(3\psi - \zeta - \varpi_x) - \frac{4}{3}\delta\psi - \frac{4}{3}\beta y - \frac{4}{3}\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathcal{S}(x,1,\varsigma,t)d\varsigma\right) \\ \frac{1}{c}\left(k_0\theta_{xx} - m\nu - \gamma u_x - k_1 r_x\right) \\ \frac{1}{\alpha}(k_2 r_{xx} - k_3 r - k_1\theta_x - d\nu_x) \\ \nu - \eta^t_s \\ -\frac{1}{\varsigma}\mathcal{S}_p \end{pmatrix}.$$

Now, we shall consider the ensuing energy space

$$\begin{aligned} \mathcal{H} = \tilde{\mathbb{J}}^1_*(0,1) \times L^2(0,1) \times \mathbb{J}^1_*(0,1) \times L^2(0,1) \times \mathbb{J}^1_*(0,1) \times L^2(0,1) \times L^2(0,1) \times L^2(0,1) \\ \times \mathbb{L}_g \times L^2\left((0,1) \times (0,1) \times (\varsigma_1, \varsigma_2)\right), \end{aligned}$$

where

$$\begin{aligned} \mathbb{J}^1_*(0,1) &= \left\{\varphi \in H^1(0,1) : \ \varphi(0) = 0\right\}, \\ \tilde{\mathbb{J}}^1_*(0,1) &= \left\{\varphi \in H^1(0,1) : \ \varphi(1) = 0\right\}, \\ \mathbb{J}^2_*(0,1) &= H^2(0,1) \cap \mathbb{J}^1_*(0,1), \\ \tilde{\mathbb{J}}^2_*(0,1) &= H^2(0,1) \cap \tilde{\mathbb{J}}^1_*(0,1), \end{aligned}$$

and

$$\mathbb{L}_g = \left\{\varphi : \mathbb{R}_+ \to \mathbb{J}^1_*(0,1),\ \int_0^1 \int_0^\infty g(s)\varphi_x^2\, dsdx < \infty\right\}.$$

For the space $\mathbb{L}_g$, we take the following inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\mathbb{L}_g} = \int_0^1 \int_0^\infty g(s)\varphi_{1x}\varphi_{2x}\, dsdx.$$

Furthermore, we consider the following domain

$$\mathcal{L}_g\left(\mathbb{R}_+, \mathbb{J}^1_*(0,1)\right) = \left\{\eta^t \in \mathbb{L}_g,\ \eta^t_s \in \mathbb{L}_g,\ \eta^t(x,0) = 0\right\}.$$

Then, we introduce

$$\begin{aligned}
\langle U, \bar{U} \rangle_{\mathcal{H}} &= \varrho \int_0^1 u\bar{u}\, dx + I_\varrho \int_0^1 v\bar{v}\, dx + 3I_\varrho \int_0^1 y\bar{y}\, dx + c\int_0^1 \theta\bar{\theta}\, dx + \alpha \int_0^1 r\bar{r}\, dx \\
&+ \bar{l} \int_0^1 \zeta_x\bar{\zeta}_x\, dx + G\int_0^1 (3\psi - \zeta - \varpi_x)(3\bar{\psi} - \bar{\zeta} - \bar{\varpi}_x)dx + 4\delta \int_0^1 \psi\bar{\psi}\, dx \\
&+ 3D\int_0^1 \psi_x\bar{\psi}_x\, dx + \int_0^1 \int_0^1 g(s)\eta^t_x(x,t)\bar{\eta}^t_x(x,s)\, dsdx \\
&+ 4\int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathcal{S}\bar{\mathcal{S}}\, d\varsigma dpdx. \qquad (2.12)
\end{aligned}$$

We deduce that $\mathcal{H}$ together with (2.12) is a Hilbert space, once we do that, we define $D(\mathcal{A})$ by

$$D(\mathcal{A}) = \left\{\begin{array}{c}
U \in \mathcal{H} :\ \varpi \in \tilde{\mathbb{J}}^2_*(0,1);\ \zeta,\ \psi \in \mathbb{J}^2_*(0,1); \\
u \in \tilde{\mathbb{J}}^1_*(0,1);\ v,\ y \in \mathbb{J}^1_*(0,1),\ \theta \in \mathbb{J}^1_*(0,1),\ \theta_t \in L^2(0,1); \\
r \in H^2(0,1) \cap H^1_0(0,1),\ \eta^t \in \mathcal{L}_g\left(\mathbb{R}_+, \mathbb{J}^1_*(0,1)\right); \\
\mathcal{S},\ \mathcal{S}_p \in L^2\left((0,1)\times(0,1)\times(\varsigma_1,\varsigma_2)\right),\ \mathcal{S}(x,0,\varsigma,t) = y \\
\varpi_x(0,t) = \zeta_x(1,t) = \psi_x(1,t) = \theta_x(1,t) = \eta^t_x(1,s) = 0.
\end{array}\right\}.$$

Obviously, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Now, we are ready to state our results.

Theorem 2.1. *Let $U_0 \in D(\mathcal{A})$, then problem (2.9)-(2.10) admits a unique solution*

$$U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

In addition, if $U_0 \in \mathcal{H}$, then

$$U \in C(\mathbb{R}_+, \mathcal{H}).$$

We give the energy of the solution of problem (2.8)-(2.9) by

$$\begin{aligned}
E(t) = \frac{1}{2}\int_0^1 &\left\{\varrho\varpi_t^2 + G(\phi - \varpi_x)^2 + I_\varrho(3\psi_t - \phi_t)^2 + \bar{l}(3\psi_x - \phi_x)^2 + 3I_\varrho\psi_t^2\right. \\
&\left. +3D\psi_x^2 + 4\delta\psi^2 + c\theta^2 + \alpha r^2\right\} dx + \frac{1}{2}\left(g \diamond (3\psi - \phi)_x\right)(t) \qquad (2.13) \\
&+2\int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathcal{S}^2(x,p,\varsigma,t)\, d\varsigma dpdx.
\end{aligned}$$

Then, we have the following stability result.

Theorem 2.2. *Let $(\varpi, \phi, \psi, \theta, r, \eta^t, \mathcal{S})$ be the solution of (2.8)-(2.9), suppose that (**T**), (**A**$_1$) and (**A**$_2$) hold. Then, for any initial data $U_0 \in D(\mathcal{A})$ satisfying, for some $p_0 \geq 0$,*

$$\int_0^1 \eta_{0x}^2(x,s)dx \leq p_0, \quad \textit{for all} \;\; s > 0, \tag{2.14}$$

there exist positive constants α_1, α_2, and α_3, such that

$$E(t) \leq \alpha_1 G_*^{-1}(\alpha_2 t + \alpha_3), \tag{2.15}$$

where

$$G_*^{-1}(t) = \int_t^\infty \frac{ds}{G_0(s)}, \quad G_0(t) = tG'(\epsilon_0 t), \quad \textit{for all} \;\; \epsilon_0 \geq 0.$$

3. Existence and uniqueness

In this part, we utilize the semigroup approach to prove our well-posedness result.

Proof of Theorem 2.1. Let's us establish the dissipativity of $\mathcal{A}$. By (2.12) and for any $U \in D(\mathcal{A})$, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -4\beta \int_0^1 y^2\, dx - k_3 \int_0^1 r^2\, dx - k_2 \int_0^1 r_x^2\, dx - k_0 \int_0^1 \theta_x^2\, dx \\ -4\int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| y\mathcal{S}(x,1,\varsigma,t)\, d\varsigma dx - 4\int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}_p \mathcal{S}\, d\varsigma dp dx \\ +\frac{1}{2}(g' \diamond \zeta_x)(t) \leq 0.\end{aligned}$$

One can notice that

$$\begin{aligned}-4\int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}_p \mathcal{S}\, d\varsigma dp dx &= -2\int_0^1 \int_{\varsigma_1}^{\varsigma_2} \int_0^1 |\mu_2(\varsigma)| \partial_p \mathcal{S}^2\, dp d\varsigma dx \\ &= -2\int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}^2(x,1,\varsigma,t)\, d\varsigma dx \\ &\quad + 2\int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}^2(x,0,\varsigma,t)\, d\varsigma dx.\end{aligned} \tag{3.1}$$

Applying Youg's inequality, we obtain

$$\begin{aligned}-4\int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| y\mathcal{S}(x,1,\varsigma,t)\, d\varsigma dx &\leq 2\left(\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\, d\varsigma\right) \int_0^1 y^2\, dx \\ &\quad + 2\int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}^2(x,1,\varsigma,t)\, d\varsigma dx,\end{aligned}$$

therefore, by (**T**) and given $\mathcal{S}(x,0) = y$, we end up with

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -4\left(\beta - \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\, d\varsigma\right) \int_0^1 y^2\, dx - k_3 \int_0^1 r^2\, dx - k_2 \int_0^1 r_x^2\, dx$$

$$-k_0 \int_0^1 \theta_x^2\, dx + \frac{1}{2}(g' \diamond \zeta_x)(t) \leq 0.$$

Thereby, $\mathcal{A}$ is dissipative.
Thereafter, we establish the surjectivity of $(I - \mathcal{A})$, that is, we show that

$$\begin{aligned} \forall H = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10})^T \in \mathcal{H},\ \exists U \in D(\mathcal{A}) : \\ (I - \mathcal{A})U = H. \end{aligned} \tag{3.2}$$

Now, we have

$$\begin{cases} \varpi - u = h_1, \\ \varrho u + G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x = \varrho h_2, \\ \zeta - \nu = h_3, \\ I_\varrho \nu - \bar{l}\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + dr_x - \int_0^\infty g(s)\eta^t_{xx}(x,s)ds = I_\varrho h_4, \\ \psi - y = h_5, \\ 3I_\varrho y - 3D\psi_{xx} + 3G(3\psi - \zeta - \varpi_x) + 4\delta\psi + 4\beta y + 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}(x, 1, \varsigma, t) d\varsigma = 3I_\varrho h_6, \\ c\theta - k_0\theta_{xx} + m\nu + \gamma u_x + k_1 r_x = ch_7, \\ (\alpha + k_3)r - k_2 r_{xx} + k_1\theta_x + d\nu_x = \alpha h_8, \\ \eta^t - \nu + \eta^t_s = h_9, \\ \varsigma\mathcal{S} + \mathcal{S}_p = \varsigma h_{10}. \end{cases} \tag{3.3}$$

Solving $(3.3)_{10}$ and using $\mathcal{S}(x, 0, \varsigma, t) = y(x, t)$, we find

$$\mathcal{S}(x, p, \varsigma, t) = y(x,t)e^{-\varsigma p} + \varsigma e^{-\varsigma p} \int_0^p e^{\varsigma\sigma} h_{10}(x, \sigma, \varsigma, t)\, d\sigma.$$

Hence,

$$\mathcal{S}(x, 1, \varsigma, t) = y(x,t)e^{-\varsigma} + \varsigma e^{-\varsigma} \int_0^1 e^{\varsigma\sigma} h_{10}(x, \sigma, \varsigma, t)\, d\sigma. \tag{3.4}$$

Now, we solve Eq $(3.3)_9$, and we find

$$\eta^t = e^{-s} \int_0^s e^{\sigma}(\nu + h_9(\sigma))\, d\sigma. \tag{3.5}$$

Inserting (3.5), (3.4), and

$$\begin{cases} u = \varpi - h_1, \\ \nu = \zeta - h_3, \\ y = \psi - h_5, \end{cases}$$

into $(3.3)_2$, $(3.3)_4$, $(3.3)_6$, $(3.3)_7$ and $(3.3)_8$, we get

$$\begin{cases} \varrho\varpi + G(3\psi - \zeta - \varpi_x)_x + \gamma\theta_x = \varrho(h_1 + h_2), \\ I_\varrho\zeta - \left(\bar{l} + \int_0^\infty (1 - e^{-s})g(s)ds\right)\zeta_{xx} - G(3\psi - \zeta - \varpi_x) - m\theta + dr_x = I_\varrho(h_3 + h_4) + \tilde{h}, \\ \mu_1\psi - 3D\psi_{xx} + 3G(3\psi - \zeta - \varpi_x) = \tilde{\mu}_1 h_5 + 3I_\varrho h_6 - 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\varsigma e^{-\varsigma} \int_0^1 e^{\varsigma\sigma} h_{10}\, d\sigma d\varsigma, \\ c\theta - k_0\theta_{xx} + m\zeta + \gamma\varpi_x + k_1 r_x = \gamma h_{1x} + ch_7 + mh_3, \\ (\alpha + k_3)r - k_2 r_{xx} + k_1\theta_x + d\zeta_x = \alpha h_8 + dh_{3x}, \end{cases} \tag{3.6}$$

where

$$\tilde{h} = \int_0^\infty g(s) \int_0^s e^{\sigma - s}(h_9 - h_3)_{xx}\, d\sigma ds,$$

$$\mu_1 = 3I_\varrho + 4\delta + 4\beta + 4\int_{\varsigma_1}^{\varsigma_2} e^{-\varsigma}|\mu_2(\varsigma)|\, d\varsigma,$$

and

$$\tilde{\mu}_1 = 3I_\varrho + 4\beta + 4\int_{\varsigma_1}^{\varsigma_2} e^{-\varsigma}|\mu_2(\varsigma)|\, d\varsigma.$$

We take the following variational formulation to solve (3.6):

$$Q((\varpi,\zeta,\psi,\theta,r),(\bar{\varpi},\bar{\zeta},\bar{\psi},\bar{\theta},\bar{r})) = L(\bar{\varpi},\bar{\zeta},\bar{\psi},\bar{\theta})), \quad \forall(\bar{\varpi},\bar{\zeta},\bar{\psi},\bar{\theta},\bar{r})) \in X, \tag{3.7}$$

where,

$$X = \tilde{\mathbb{J}}_*^1(0,1) \times \mathbb{J}_*^1(0,1) \times \mathbb{J}_*^1(0,1) \times L^2(0,1) \times H_0^1(0,1),$$

is a Hilbert space endowed with

$$\|(\varpi,\zeta,\psi,\theta,r)\|_X^2 = \|3\psi - \zeta - \varpi_x\|_2^2 + \|\varpi\|_2^2 + \|\zeta_x\|_2^2 + \|\psi_x\|_2^2 + \|\theta_x\|_2^2 + \|r\|_2^2 + \|r_x\|_2^2.$$

As a part of this step, we provide definitions for both the bilinear form $Q : X \times X \to \mathbb{R}$ and the linear form $L : X \to \mathbb{R}$ as follows:

$$\begin{aligned}
&Q((\varpi,\zeta,\psi,\theta,r),(\bar{\varpi},\bar{\zeta},\bar{\psi},\bar{\theta},\bar{r}))\\
=&\varrho\int_0^1 \varpi\bar{\varpi}\,dx + I_\varrho\int_0^1 \zeta\bar{\zeta}\,dx + \mu_1\int_0^1 \psi\bar{\psi}\,dx + c\int_0^1 \theta\bar{\theta}\,dx\\
&+(\alpha + k_3)\int_0^1 r\bar{r}\,dx + k_2\int_0^1 r_x\bar{r}_x\,dx + \gamma\int_0^1 (\theta_x\bar{\varpi} + \varpi_x\bar{\theta})\,dx\\
&+k_0\int_0^1 \theta_x\bar{\theta}_x\,dx + G\int_0^1 (3\psi - \zeta - \varpi_x)(3\bar{\psi} - \bar{\zeta} - \bar{\varpi}_x)\,dx\\
&+\left(\bar{l} + \int_0^\infty (1 - e^{-s})g(s)\,ds\right)\int_0^1 \zeta_x\bar{\zeta}_x\,dx + 3D\int_0^1 \psi_x\bar{\psi}_x\,dx\\
&+d\int_0^1 (r_x\bar{\zeta} + \zeta_x\bar{r})\,dx + k_1\int_0^1 (r_x\bar{\theta} + \bar{r}\theta_x)\,dx\\
&+m\int_0^1 (\zeta\bar{\theta} - \bar{\zeta}\theta)\,dx,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
L(\bar{\varpi},\bar{\zeta},\bar{\psi},\bar{\theta},\bar{r}) = &\varrho\int_0^1 \bar{\varpi}(h_1 + h_2)\,dx + I_\varrho\int_0^1 \bar{\zeta}(h_3 + h_4)\,dx + \int_0^1 \bar{\zeta}\tilde{h}\,dx\\
&+ \int_0^1 \bar{\theta}(\gamma h_{1x} + mh_3 + ch_7)\,dx + \int_0^1 \bar{r}(\alpha h_8 + dh_{3x})\,dx\\
&+ \int_0^1 \bar{\psi}\left[\tilde{\mu}_1 h_5 + 3I_\varrho h_6 - 4\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\varsigma e^{-\varsigma}\int_0^1 e^{\varsigma\sigma}h_{10}\,d\sigma d\varsigma\right]\,dx.
\end{aligned}$$

We can easily prove the continuity of Q and L. Moreover, from (3.8) together with integration by parts, we arrive at

$$\begin{aligned}
&Q((\varpi,\zeta,\psi,\theta,r),(\varpi,\zeta,\psi,\theta,r))\\
=&\varrho\int_0^1 \varpi^2\,dx + I_\varrho\int_0^1 \zeta^2\,dx + \mu_1\int_0^1 \psi^2\,dx + c\int_0^1 \theta^2\,dx\\
&+(\alpha+k_3)\int_0^1 r^2\,dx + k_2\int_0^1 r_x^2\,dx + k_0\int_0^1 \theta_x^2\,dx\\
&+G\int_0^1 (3\psi-\zeta-\varpi_x)^2\,dx + 3D\int_0^1 \psi_x^2\,dx\\
&+\left(\bar{l}+\int_0^\infty (1-e^{-s})g(s)\,ds\right)\int_0^1 \zeta_x^2\,dx\\
&\geq M\|(\varpi,\zeta,\psi,\theta,r)\|_X^2,\ M>0.
\end{aligned}$$

From this, we conclude the coercivity of Q. It follows from the Lax-Milgram lemma that (3.6) admits a unique solution satisfying

$$\begin{gathered}
\varpi \in \tilde{\mathbb{J}}_*^1(0,1),\\
\zeta,\ \psi \in \mathbb{J}_*^1(0,1),\\
\theta \in L^2(0,1),
\end{gathered}$$

and

$$r \in H_0^1(0,1).$$

If we substitute ϖ, ζ, and ψ into $(3.3)_1$, $(3.3)_3$ and $(3.3)_5$, we find

$$u \in \tilde{\mathbb{J}}_*^1(0,1),$$

and

$$v,\ y \in \mathbb{J}_*^1(0,1).$$

In addition, taking $(\bar{\zeta},\bar{\psi},\bar{\theta},\bar{r}) \equiv (0,0,0,0) \in \left(\mathbb{J}_*^1(0,1)\right)^2 \times L^2(0,1) \times H_0^1(0,1)$, (3.7) becomes

$$G\int_0^1 \bar{\varpi}\varpi_{xx}\,dx = \int_0^1 \bar{\varpi}\left(\varrho\varpi + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1+h_2)\right)dx, \tag{3.9}$$

for all $\bar{\varpi} \in \tilde{\mathbb{J}}_*^1(0,1)$, which indicates that

$$G\varpi_{xx} = \varrho\varpi + 3G\psi_x - G\zeta_x + \gamma\theta_x - \varrho(h_1+h_2) \in L^2(0,1). \tag{3.10}$$

The standard elliptic regularity implies that

$$\varpi \in \tilde{\mathbb{J}}_*^2(0,1).$$

We note that (3.9) remains valid for $\bar{\varphi} \in C^1([0,1]) \subset \tilde{\mathbb{J}}_*^1(0,1)$, that is $\bar{\varphi}(1) = 0$. Then, we obtain

$$G\int_0^1 \bar{\varphi}_x\varpi_x\,dx = \int_0^1 \bar{\varphi}\left(-\varrho\varpi - 3G\psi_x + G\zeta_x - \gamma\theta_x + \varrho(h_1+h_2)\right)dx.$$

Integrating by parts, it follows that

$$\varpi_x(0)\bar{\varphi}(0) = 0, \quad \text{for all } \bar{\varphi} \in C^1([0,1]).$$

Hence

$$\varpi_x(0) = 0.$$

Likewise, we show that

$$(\zeta,\psi) \in \left(\mathbb{J}_*^2(0,1)\right)^2, \quad \theta \in \mathbb{J}_*^1(0,1), \quad r \in H^2(0,1) \cap H_0^1(0,1),$$

$$\text{and} \quad \zeta_x(1) = \psi_x(1) = \theta_x(1) = 0.$$

The standard elliptic regularity guarantees the existence of a unique $U \in D(\mathcal{A})$ which fulfills (3.2). Thereby, $\mathcal{A}$ is surjective.

As a consequence, we infer that $\mathcal{A}$ is a maximal dissipative operator. Then, the well-posedness result follows using Lumer-Philips theorem [15]. □

4. Technical lemmas

The main purpose of this section is to establish the essential practical lemmas required to prove our stability results. To attain this goal, we apply a specific approach known as the multiplier technique, which enables us to prove the stability results of problem (2.8). Nevertheless, this method necessitates creating an appropriate Lyapunov functional equivalent to the energy and we will clarify on this in the next section. To simplify matters, we will employ $\chi_* > 0$ to represent a generic constant.

Lemma 4.1. *Let $(\varpi, \phi, \psi, \theta, r, \eta^t, \mathcal{S})$ be the solution of (2.8) and (2.9), then, the energy functional satisfies*

$$\frac{d}{dt}E(t) \leq -m_0 \int_0^1 \psi_t^2 dx - k_0 \int_0^1 \theta_x^2 dx - k_2 \int_0^1 r_x^2 dx - k_3 \int_0^1 r^2 dx \tag{4.1}$$
$$+\frac{1}{2}(g' \diamond (3\psi_x - \phi_x))(t) \leq 0, \quad \textit{where } m_0 > 0.$$

Proof. As a start, we multiply $(2.8)_1$, $(2.8)_2$, $(2.8)_3$, $(2.8)_4$ and $(2.8)_5$ by ϖ_t, $(3\psi_t - \phi_t)$, ψ_t, θ and r respectively, then, we integrate over $(0,1)$ and use integration by parts together with boundary conditions (2.9) and (2.6) to find

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Big\{\varrho\varpi_t^2 + G(\phi - \varpi_x)^2 + I_\varrho(3\psi_t - \phi_t)^2 + \bar{l}(3\psi_x - \phi_x)^2 + 3I_\varrho\psi_t^2 + 3D\psi_x^2$$
$$+4\delta\psi^2 + c\theta^2 + \alpha r^2\Big\} dx + 4\beta \int_0^1 \psi_t^2 dx + k_0 \int_0^1 \theta_x^2 dx + k_2 \int_0^1 r_x^2 dx$$
$$+ k_3 \int_0^1 r^2 dx - \int_0^1 (3\psi - \phi)_t \int_0^\infty g(s)\eta_{xx}^t(x,s) ds dx \tag{4.2}$$
$$+4 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \psi_t |\mu_2(\varsigma)| \mathcal{S}(x,1,\varsigma,t)\, d\varsigma dx = 0.$$

It follows from the sixth equation in (2.8) and the integration by parts that

$$\begin{aligned}
&\int_0^1 (3\psi-\phi)_t \int_0^\infty g(s)\eta^t_{xx}(x,s)dsdx \\
&= \int_0^\infty g(s)\left(\int_0^1 \eta^t_t \eta^t_{xx}(x,s)dx\right)ds \\
&+ \int_0^\infty g(s)\left(\int_0^1 \eta^t_s \eta^t_{xx}(x,s)dx\right)ds \\
&= -\frac{1}{2}\frac{d}{dt}(g\diamond(3\psi_x-\phi_x))(t) \\
&+ \frac{1}{2}(g'\diamond(3\psi_x-\phi_x))(t).
\end{aligned} \tag{4.3}$$

Applying Young's inequality, we find

$$\begin{aligned}
\int_0^1\int_{\varsigma_1}^{\varsigma_2} \psi_t|\mu_2(\varsigma)|\mathcal{S}(x,1,\varsigma,t)\, d\varsigma dx \le{}& \frac{1}{2}\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathcal{S}^2(x,1,\varsigma,t)\, d\varsigma dx \\
&+ \frac{1}{2}\left(\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|d\varsigma\right)\int_0^1 \psi_t^2\, dx.
\end{aligned} \tag{4.4}$$

Next, we multiply $(2.8)_7$ by $\mathcal{S}|\mu_2(\varsigma)|$ and integrate the result over $(0,1)\times(0,1)\times(\varsigma_1,\varsigma_2)$. We get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2}\varsigma|\mu_2(\varsigma)|\mathcal{S}^2(x,p,\varsigma,t)\, d\varsigma dp dx \\
&= -\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathcal{S}_p\mathcal{S}(x,p,\varsigma,t)\, d\varsigma dp dx \\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\partial_p\mathcal{S}^2(x,p,\varsigma,t)\, d\varsigma dp dx \\
&= \frac{1}{2}\left(\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|d\varsigma\right)\int_0^1\psi_t^2\, dx \\
&- \frac{1}{2}\int_0^1\int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|\mathcal{S}^2(x,1,\varsigma,t)\, d\varsigma dx,
\end{aligned} \tag{4.5}$$

which, together with (4.2)–(4.4) and (**T**) gives us

$$\frac{d}{dt}E(t) \le -4\left(\beta - \int_{\varsigma_1}^{\varsigma_2}|\mu_2(\varsigma)|d\varsigma\right)\int_0^1\psi_t^2dx - k_0\int_0^1\theta_x^2dx - k_2\int_0^1 r_x^2dx - k_3\int_0^1 r^2dx$$

$$+\frac{1}{2}(g'\diamond(3\psi_x-\phi_x))(t) \le 0.$$

We have then reached the desired result. □

Lemma 4.2. *Consider the functional*

$$I_1(t) := -I_\varrho\int_0^1(3\psi_t-\phi_t)\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx, \tag{4.6}$$

then, it satisfies

$$
\begin{aligned}
I_1'(t) \le & \frac{-I_\varrho g_0}{2}\int_0^1 (3\psi_t-\phi_t)^2dx+\epsilon_1\int_0^1(3\psi_x-\phi_x)^2dx \\
&+\epsilon_1\int_0^1(\phi-\varpi_x)^2dx+\epsilon_1\int_0^1\theta_x^2dx+\chi_*\int_0^1 r^2dx \\
&+\chi_*\left(1+\frac{1}{\epsilon_1}\right)(g\diamond(3\psi_x-\phi_x))(t)-\chi_*(g'\diamond(3\psi_x-\phi_x))(t),\ \forall\epsilon_1>0.
\end{aligned}
\tag{4.7}
$$

Proof. First, we notice that

$$
\begin{aligned}
&\partial_t\left(\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))ds\right)\\
&=\partial_t\left(\int_{-\infty}^t g(t-s)((3\psi-\phi)(t)-(3\psi-\phi)(s))ds\right)\\
&=\int_{-\infty}^t g'(t-s)((3\psi-\phi)(t)-(3\psi-\phi)(s))ds\\
&+\int_{-\infty}^t g(t-s)(3\psi-\phi)_t(t)ds\\
&=\int_0^\infty g'(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))ds\\
&+g_0(3\psi-\phi)_t(t).
\end{aligned}
\tag{4.8}
$$

Next, we proceed by differentiating $I_1(t)$ and using both $(2.8)_2$ and relation (4.8), then, integrating by parts, we get

$$
\begin{aligned}
F_1'(t)=&-I_\varrho\int_0^1(3\psi_{tt}-\phi_{tt})\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx\\
&-I_\varrho\int_0^1(3\psi_t-\phi_t)\frac{\partial}{\partial t}\left(\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx\right)\\
&=D\int_0^1(3\psi_x-\phi_x)\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))dsdx\\
&-G\int_0^1(\phi-\varpi_x)\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx\\
&-m\int_0^1\theta\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx-I_\varrho g_0\int_0^1(3\psi_t-\phi_t)^2dx\\
&-I_\varrho\int_0^1(3\psi_t-\phi_t)\int_0^\infty g'(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx\\
&-d\int_0^1 r\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))dsdx\\
&-\int_0^1\left(\int_0^\infty g(s)(3\psi-\phi)_x(x,t-s)ds\right)\\
&\times\left(\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))ds\right)\,dx.
\end{aligned}
\tag{4.9}
$$

The last term in (4.9) can be rewritten as

$$-\int_0^1\left(\int_0^\infty g(s)(3\psi-\phi)_x(x,t-s)ds\right)\left(\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))ds\right)dx$$

$$=\int_0^1\left(\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))ds\right)^2dx \tag{4.10}$$
$$-g_0\int_0^1(3\psi-\phi)_x\left(\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))ds\right)dx.$$

Now, replacing (4.10) into (4.9), leads to

$$\begin{aligned}F_1'(t)=&\,\bar{l}\int_0^1(3\psi_x-\phi_x)\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))dsdx\\&-G\int_0^1(\phi-\varpi_x)\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx\\&-m\int_0^1\theta\int_0^\infty g(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx-I_\varrho g_0\int_0^1(3\psi_t-\phi_t)^2dx\\&-I_\varrho\int_0^1(3\psi_t-\phi_t)\int_0^\infty g'(s)((3\psi-\phi)(t)-(3\psi-\phi)(t-s))dsdx\\&-d\int_0^1 r\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))dsdx\\&+\int_0^1\left(\int_0^\infty g(s)((3\psi-\phi)_x(t)-(3\psi-\phi)_x(t-s))ds\right)^2dx.\end{aligned}$$

Finally, applying Young's inequality and making use of Lemma 2.1, we obtain (4.7). □

Lemma 4.3. *Consider the functional*

$$I_2(t):=-c\varrho\int_0^1\varpi_t\left(\int_x^1\theta(y)dy\right)dx,$$

then, it satisfies

$$I_2'(t)\le\frac{-\gamma\varrho}{2}\int_0^1\varpi_t^2\,dx+\chi_*\int_0^1(3\psi_t-\phi_t)^2\,dx+\epsilon_2\int_0^1(\phi-\varpi_x)^2\,dx \tag{4.11}$$
$$+\chi_*\int_0^1 r^2\,dx+\chi_*\left(1+\frac{1}{\epsilon_2}\right)\int_0^1\theta_x^2\,dx,\ \forall\epsilon_2>0.$$

Proof. Simple calculations, using $(2.8)_1$, $(2.8)_4$ and integration by parts, we get

$$\begin{aligned}I_2'(t)=&-c\varrho\int_0^1\varpi_{tt}\left(\int_x^1\theta(y)dy\right)dx-c\varrho\int_0^1\varpi_t\left(\int_x^1\theta_t(y)dy\right)dx\\=&\,cG\int_0^1(\phi-\varpi_x)\theta\,dx+k_0\varrho\int_0^1\theta_x\varpi_t\,dx+\gamma c\int_0^1\theta^2\,dx\\&-\gamma\varrho\int_0^1\varpi_t^2\,dx-k_1\varrho\int_0^1 r\varpi_t\,dx+m\varrho\int_0^1\varpi_t\int_x^1(3\psi_t-\phi_t)(y)dydx.\end{aligned}$$

Now, thanks to Young, Poincaré's and Cauchy–Schwarz inequalities, we get, for any $\epsilon_2 > 0$,

$$I_2'(t) \leq \frac{-\gamma\varrho}{2}\int_0^1 \varpi_t^2\,dx + \chi_*\int_0^1 (3\psi_t-\phi_t)^2\,dx + \epsilon_2\int_0^1 (\phi-\varpi_x)^2\,dx$$
$$+\chi_*\int_0^1 r^2\,dx + \chi_*\left(1+\frac{1}{\epsilon_2}\right)\int_0^1 \theta_x^2\,dx.$$

The proof is then completed. □

Lemma 4.4. *Consider the functional*

$$I_3(t) := \varrho\int_0^1 \varpi_t\varpi\,dx + \varrho\int_0^1 \phi\left(\int_0^x \varpi_t(y)dy\right)\,dx, \tag{4.12}$$

then, it satisfies

$$I_3'(t) \leq -\frac{G}{2}\int_0^1 (\phi-\varpi_x)^2\,dx + \varrho\int_0^1 (3\psi_t-\phi_t)^2\,dx + \frac{3\varrho}{2}\int_0^1 \varpi_t^2\,dx \tag{4.13}$$
$$+\chi_*\int_0^1 \theta_x^2\,dx + 9\varrho\int_0^1 \psi_t^2 dx.$$

Proof. We differentiate I_3, using $(2.8)_1$ together with integration by parts, to get

$$\begin{aligned}
I_3'(t) &= \varrho\int_0^1 \varpi_t^2\,dx + \varrho\int_0^1 \varpi_{tt}\varpi\,dx + \varrho\int_0^1 \phi_t\left(\int_0^x \varpi_t(y)dy\right)\,dx \\
&+ \varrho\int_0^1 \phi\left(\int_0^x \varpi_{tt}(y)dy\right)\,dx \\
&= \varrho\int_0^1 \varpi_t^2\,dx - G\int_0^1 (\phi-\varpi_x)_x\varpi\,dx - \gamma\int_0^1 \varpi\theta_x\,dx \\
&+ \varrho\int_0^1 \phi_t\left(\int_0^x \varpi_t(y)dy\right)\,dx - G\int_0^1 (\phi-\varpi_x)\phi\,dx - \gamma\int_0^1 \theta\phi\,dx \\
&= \varrho\int_0^1 \varpi_t^2\,dx - G\int_0^1 (\phi-\varpi_x)^2\,dx - \gamma\int_0^1 (\phi-\varpi_x)\theta\,dx \\
&+ \varrho\int_0^1 \phi_t\left(\int_0^x \varpi_t(y)dy\right)\,dx.
\end{aligned}$$

Notice that

$$\int_0^1 \phi_t^2\,dx \leq 2\int_0^1 (3\psi_t-\phi_t)^2\,dx + 18\int_0^1 \psi_t^2\,dx.$$

By Young, Poincaré's, and Cauchy-Schwarz inequalities, we easily prove (4.13). □

Lemma 4.5. *Consider the functional*

$$I_4(t) := I_\varrho\int_0^1 (3\psi-\phi)_t(3\psi-\phi)\,dx, \tag{4.14}$$

then, it satisfies

$$I_4'(t) \le -\frac{\bar{l}}{2}\int_0^1 (3\psi_x - \phi_x)^2\,dx + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2\,dx + \chi_* \int_0^1 (r^2 + \theta_x^2)\,dx$$
$$+\chi_* \int_0^1 (\phi - \varpi_x)^2\,dx + \chi_*(g \diamond (3\psi_x - \phi_x))(t). \tag{4.15}$$

Proof. We proceed by differentiating the functional I_4 and using Eq $(2.8)_2$ together with integration by parts, which leads to

$$\begin{aligned} I_4'(t) =& I_\varrho \int_0^1 (3\psi - \phi)_{tt}(3\psi - \phi)\,dx + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2\,dx \\ =& I_\varrho \int_0^1 (3\psi_t - \phi_t)^2\,dx - \bar{l}\int_0^1 (3\psi_x - \phi_x)^2\,dx \\ &+ G\int_0^1 (3\psi - \phi)(\phi - \varpi_x)\,dx + m\int_0^1 (3\psi - \phi)\theta\,dx \\ &+ d\int_0^1 (3\psi - \phi)_x r\,dx - \int_0^1 (3\psi - \phi)_x \int_0^\infty g(s)((3\psi - \phi)_x(t) - (3\psi - \phi)_x(t-s))\,dsdx. \end{aligned} \tag{4.16}$$

By virtue of Young's inequality and (2.4), we have

$$\begin{aligned} I_4'(t) \le& -\frac{\bar{l}}{2}\int_0^1 (3\psi_x - \phi_x)^2\,dx + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2\,dx + \chi_* \int_0^1 (r^2 + \theta_x^2)\,dx \\ &+ \chi_* \int_0^1 (\phi - \varpi_x)^2\,dx + C^1 \int_0^1 \left[\int_0^\infty g(s)\,((3\psi - \phi)_x(t) - (3\psi - \phi)_x(t-s))\,ds\right]^2 dx \\ \le& -\frac{\bar{l}}{2}\int_0^1 (3\psi_x - \phi_x)^2\,dx + I_\varrho \int_0^1 (3\psi_t - \phi_t)^2\,dx + \chi_* \int_0^1 (r^2 + \theta_x^2)\,dx \\ &+ \chi_* \int_0^1 (\phi - \varpi_x)^2\,dx + \chi_*(g \diamond (3\psi_x - \phi_x))(t). \end{aligned}$$

This completes the proof of (4.15). □

Lemma 4.6. *Consider the functional*

$$I_5(t) := 3I_\varrho \int_0^1 \psi_t \psi\,dx + 2\beta \int_0^1 \psi^2\,dx, \tag{4.17}$$

then, it satisfies the estimate

$$I_5'(t) \le -2\delta \int_0^1 \psi^2\,dx - 3D\int_0^1 \psi_x^2\,dx + 3I_\varrho \int_0^1 \psi_t^2\,dx + \chi_* \int_0^1 (\phi - \varpi_x)^2\,dx$$
$$+ \chi_* \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}^2(x,1,p,t)\,d\varsigma dx. \tag{4.18}$$

Proof. Simple computations using Eq $(2.8)_3$ and integration by parts, yield

$$I_5'(t) = 3I_\varrho \int_0^1 \psi_t^2\, dx - 3D \int_0^1 \psi_x^2\, dx - 4\delta \int_0^1 \psi^2\, dx - 3G \int_0^1 (\phi - \varpi_x)\psi\, dx \tag{4.19}$$

$$-4 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \psi |\mu_2(\varsigma)| \mathcal{S}(x, 1, p, t)\, d\varsigma dx.$$

Employing Young's inequality, we conclude (4.18). □

Lemma 4.7. *Consider the functional*

$$I_6(t) := \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p} |\mu_2(\varsigma)| \mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx, \tag{4.20}$$

then, it satisfies

$$\begin{aligned} I_6'(t) \le \beta \int_0^1 \psi_t^2\, dx - m_1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \mathcal{S}^2(x, 1, \varsigma, t)\, d\varsigma dx \\ - m_1 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma |\mu_2(\varsigma)| \mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx, \end{aligned} \tag{4.21}$$

where m_1 is a positive constant.

Proof. Taking the derivative of I_6 and using $(2.8)_7$ and $\mathcal{S}(x, 0, t) = \psi_t$, we have

$$\begin{aligned} I_6'(t) &= -2 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} e^{-\varsigma p} |\mu_2(\varsigma)| \mathcal{S}_p \mathcal{S}(x, p, \varsigma, t)\, d\varsigma dp dx \\ &= - \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p} |\mu_2(\varsigma)| \mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx \\ &\quad - \int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)| \left\{ e^{-\varsigma} \mathcal{S}^2(x, 1, \varsigma, t) - \psi_t^2(x, t) \right\} d\varsigma dx. \end{aligned}$$

From $e^{-\varsigma} \le e^{-\varsigma p} \le 1$, where $0 < p < 1$, we arrive at

$$\begin{aligned} I_6'(t) \le & - \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma} |\mu_2(\varsigma)| \mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx \\ & + \left(\int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\, d\varsigma \right) \int_0^1 \psi_t^2(x, t)\, dx \\ & - \int_0^1 \int_{\varsigma_1}^{\varsigma_2} e^{-\varsigma} |\mu_2(\varsigma)| \mathcal{S}^2(x, 1, \varsigma, t) d\varsigma dx. \end{aligned}$$

Since $-e^{-\varsigma}$ is an increasing function, then

$$-e^{-\varsigma} \le -e^{-\varsigma_2}, \quad \text{forall } \varsigma \in [\varsigma_1, \varsigma_2].$$

Hence, if we denote $m_1 = e^{-\varsigma_2}$ and use (**T**), we easily prove (4.21). □

5. Stability results

Let us now prove our stability result by using the lemmas in Section 4.

Proof of Theorem 2.2. We proceed by introducing a Lyapunov functional

$$L(t) = NE(t) + \sum_{j=1}^{6} N_j I_j(t), \tag{5.1}$$

where constants $N, N_j > 0,\ \ j = 1, \cdots, 6$, will be chosen later.
From (5.1), we write

$$\begin{aligned}
|L(t) - NE(t)| \le{}& I_\varrho N_1 \int_0^1 \left|(3\psi - \phi)_t \int_0^\infty g(s)((3\psi - \phi)(t) - (3\psi - \phi)(t-s))ds\right| dx \\
&+ c\varrho N_2 \int_0^1 \left|\varpi_t \int_x^1 \theta(y)dy\right| dx + \varrho N_3 \int_0^1 |\varpi_t \varpi|\, dx + \varrho N_3 \int_0^1 \left|\phi \int_0^x \varpi_t(y)dy\right| dx \\
&+ I_\varrho N_4 \int_0^1 |(3\psi - \phi)_t (3\psi - \phi)|\, dx + 3I_\varrho N_5 \int_0^1 |\psi_t \psi| dx + 2\beta N_5 \int_0^1 \psi^2 dx \\
&+ N_6 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p} |\mu_2(\varsigma)| \mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx.
\end{aligned}$$

Thanks to Young, Cauchy-Schwarz and Poincaré's inequalities, we get

$$|L(t) - NE(t)| \le \vartheta_1 E(t), \qquad \text{where } \ \vartheta_1 > 0,$$

i.e.,

$$(N - \vartheta_1)E(t) \le L(t) \le (N + \vartheta_1)E(t). \tag{5.2}$$

Now, differentiating the Lyapunov functional $L(t)$, using (4.1), (4.7), (4.11), (4.13), (4.15), (4.18), and (4.21), and fixing

$$N_4 = N_5 = 1,\ \epsilon_1 = \frac{\bar{l}}{4N_1},\ \epsilon_2 = \frac{GN_3}{4N_2}.$$

We find

$$
\begin{aligned}
\frac{d}{dt}L(t) \leq & -\left(\frac{\gamma\varrho}{2}N_2 - \frac{3\varrho}{2}N_3\right)\int_0^1 \varpi_t^2 dx \\
& -\left(\frac{I_\varrho g_0}{2}N_1 - \chi_* N_2 - \varrho N_3 - I_\varrho\right)\int_0^1 (3\psi_t - \phi_t)^2 dx \\
& -\left(m_0 N - 9\varrho N_3 - \beta N_6 - 3I_\varrho\right)\int_0^1 \psi_t^2 dx - \frac{\bar{l}}{4}\int_0^1 (3\psi_x - \phi_x)^2 dx \\
& -\left(\frac{G}{4}N_3 - \left(\frac{\bar{l}}{4} + 2\chi_*\right)\right)\int_0^1 (\phi - \varpi_x)^2 dx \\
& - 2\delta \int_0^1 \psi^2 dx - \left(k_0 N - \chi_*\left(1 + \frac{N_2}{N_3}\right)N_2 - \chi_* N_3 - \chi_* - \frac{\bar{l}}{4}\right)\int_0^1 \theta_x^2 dx \\
& - 3D\int_0^1 \psi_x^2 dx - k_2 N\int_0^1 r_x^2 dx - (k_3 N - \chi_* N_1 - \chi_* N_2 - \chi_*)\int_0^1 r^2 dx \\
& - m_1 N_6 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma|\mu_2(\varsigma)|\mathcal{S}^2(x,p,\varsigma,t)\, d\varsigma dp dx \\
& - (m_1 N_6 - \chi_*)\int_0^1 \int_{\varsigma_1}^{\varsigma_2} |\mu_2(\varsigma)|\mathcal{S}^2(x,1,\varsigma,t)\, d\varsigma dx \\
& + \left(\frac{N}{2} - \chi_* N_1\right)(g' \diamond (3\psi_x - \phi_x))(t) + \left(\chi_*\left(1 + \frac{4N_1}{\bar{l}}\right)N_1 + \chi_*\right)(g \diamond (3\psi_x - \phi_x))(t).
\end{aligned}
\tag{5.3}
$$

Next, we choose our coefficients in (5.3), in a way that, they all except the last two become negative. We start by selecting N_6 big enough so that

$$m_1 N_6 - \chi_* > 0,$$

then, we take N_3 fairly wide, such that

$$\frac{G}{4}N_3 - \left(\frac{\bar{l}}{4} + 2\chi_*\right) > 0,$$

after that, we choose N_2 large enough, so that

$$\frac{\gamma\varrho}{2}N_2 - \frac{3\varrho}{2}N_3 > 0,$$

now, we select N_1 sufficiently large such that

$$\frac{I_\varrho g_0}{2}N_1 - \chi_* N_2 - \varrho N_3 - I_\varrho > 0.$$

We can now select N large enough so that we have (5.2) and

$$
\begin{cases}
\frac{1}{2}N - \chi_* N_1 > 0, \\
m_0 N - 9\varrho N_3 - \beta N_6 - 3I_\varrho > 0, \\
k_3 N - \chi_* N_1 - \chi_* N_2 - \chi_* > 0, \\
k_0 N - \chi_*\left(1 + \frac{N_2}{N_3}\right)N_2 - \chi_* N_3 - \chi_* - \frac{\bar{l}}{4} > 0.
\end{cases}
$$

Hence, relation (5.3) becomes

$$\frac{d}{dt}L(t) \leq -\vartheta_2 \int_0^1 \Big\{\varpi_t^2 + (\phi - \varpi_x)^2 + (3\psi_t - \phi_t)^2 + \psi^2 + (3\psi_x - \phi_x)^2 + \psi_x^2$$
$$+\psi_t^2 + \theta^2 + r^2\Big\}\, dx - \vartheta_2 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p}|\mu_2(\varsigma)|\mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx$$
$$+ \vartheta_3(g \diamond (3\psi_x - \phi_x))(t), \quad \vartheta_2, \vartheta_3 > 0. \tag{5.4}$$

Now, exploiting (2.13) and Poincaré's inequality, we obtain

$$E(t) \leq \vartheta_4 \int_0^1 \Big\{\varpi_t^2 + (\phi - \varpi_x)^2 + (3\psi_t - \phi_t)^2 + \psi^2 + (3\psi_x - \phi_x)^2 + \psi_x^2$$
$$+\psi_t^2 + \theta^2 + r^2\Big\}\, dx + \vartheta_4 \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p}|\mu_2(\varsigma)|\mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx$$
$$+ \vartheta_4(g \diamond (3\psi_x - \phi_x))(t), \quad \text{where } \vartheta_4 > 0,$$

from which

$$-\int_0^1 \Big\{\varpi_t^2 + (\phi - \varpi_x)^2 + (3\psi_t - \phi_t)^2 + \psi^2 + (3\psi_x - \phi_x)^2 + \psi_x^2$$
$$+\psi_t^2 + \theta^2 + r^2\Big\}\, dx - \int_0^1 \int_0^1 \int_{\varsigma_1}^{\varsigma_2} \varsigma e^{-\varsigma p}|\mu_2(\varsigma)|\mathcal{S}^2(x, p, \varsigma, t)\, d\varsigma dp dx$$
$$- (g \diamond (3\psi_x - \phi_x))(t) \leq -\vartheta_5 E(t), \tag{5.5}$$

where $\vartheta_5 > 0$. Thereby, if we combine (5.5) and (5.4), we have

$$\frac{d}{dt}L(t) \leq -\vartheta_6 E(t) + \vartheta_7(g \diamond (3\psi_x - \phi_x))(t), \quad \text{where } \vartheta_6, \vartheta_7 > 0. \tag{5.6}$$

Next, we multiply (5.6), by

$$G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right),$$

we find

$$G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right)\frac{d}{dt}L(t) \leq -\vartheta_6 G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right)E(t) + \vartheta_7 G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right)(g \diamond (3\psi_x - \phi_x))(t). \tag{5.7}$$

Now, we estimate the last term in (5.7) and use both ($\mathbf{A}_2$) and (2.14), we find

$$\vartheta_7 G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right)(g \diamond (3\psi_x - \phi_x))(t) \leq -\vartheta_7' E'(t) + \vartheta_7'\epsilon_0 G_0\left(\frac{E(t)}{E(0)}\right), \ \vartheta_7' > 0. \tag{5.8}$$

We insert (5.8) in (5.7) and set $\epsilon_0 = \frac{\vartheta_6 E(0)}{2\vartheta_7'}$, we get

$$G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right)\frac{d}{dt}L(t) + \vartheta_7' E'(t) \leq -\Gamma G_0\left(\frac{E(t)}{E(0)}\right), \quad \Gamma > 0. \tag{5.9}$$

We consider now the functional

$$L_1(t) := G'\left(\frac{\epsilon_0 E(t)}{E(0)}\right)L(t) + \vartheta_7' E(t).$$

It is clear that

$$L_1(t) \sim E(t),$$

moreover, noticing that $E'(t) \leq 0,\ G''(t) > 0$, we obtain

$$\frac{d}{dt}L_1(t) \leq -\Gamma G_0\left(\frac{E(t)}{E(0)}\right). \tag{5.10}$$

Next, we present the functional

$$L_2(t) := b_1 \frac{L_1(t)}{E(0)} \sim E(t), \qquad \text{such that}$$

$$\begin{cases} L_2(t) \leq 1, \\ \frac{d}{dt}L_2(t) \leq -\alpha_2 G_0(L_2(t)), \end{cases}$$

where, α_2 is a positive constant, therefore,

$$G_*'(L_2(t)) \geq \alpha_2.$$

We integrate over $(0, t)$ to find

$$L_2(t) \leq G_*^{-1}(\alpha_2 t + \alpha_3),$$

from which, we deduce that

$$E(t) \leq \alpha_1 G_*^{-1}(\alpha_2 t + \alpha_3),$$

where, α_1 and α_3 are positive constants. The proof is then completed. □

6. Conclusions

The article is about the laminated beam system along with structural damping, past history, distributed delay, and in the presence of both temperatures and micro-temperature effects introduced in (1.1). By the semigroup approach, we established the existence and uniqueness of the solution which can be considered as the first main result. In addition, as a second novelty, a general decay result for the solution unusually with no constraints regarding the speeds of wave propagation is found. This last new result is considered, as far as we know, the first similar result in the literature for such a system, where we succeed to improve the earlier works known for the case of finite history, to the case of infinite history. The relaxation function becomes intended to satisfy a broader class of relaxation functions.

We mention here that the distributed delay in our system makes a good interaction between the past history and the other damping terms of system (1.1). This type of damping gives more information and qualitative properties on the solution and also its impact on stability is very important as it is shown in the requirement of Theorem 2.2. Of course, the other terms (both temperatures and micro-temperature effects) act as balances in the stability of the system.

Author contributions

Fares Yazid and Fatima Siham Djerad: Writing—original draft preparation; Abdelkader Moumen and Moheddine Imsatfia, Tayeb Mahrouz: Writing—review and editing; Keltoum Bouhali: Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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