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Research article

Existence and uniqueness of solutions for stochastic differential equations with locally one-sided Lipschitz condition

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Abstract: This paper investigated stochastic differential equations (SDEs) with locally one-sided Lipschitz coefficients. Apart from the local one-sided Lipschitz condition, a more general condition was introduced to replace the monotone condition. Then, in terms of Euler's polygonal line method, the existence and uniqueness of solutions for SDEs was established. In the meanwhile, the pth moment boundedness of solutions was also provided.

Keywords: stochastic differential equations; locally one-sided Lipschitz; existence and uniqueness; moment boundedness
Mathematics Subject Classification: 60H10, 34F99

1. Introduction

Stochastic differential equations (SDEs) have been widely applied in many fields, such as biology, economics and physics for modeling (see, e.g., [1–6]). More and more people have showed their interests in SDEs. So far, many results of solutions for SDEs have been obtained, such as the existence and uniqueness of solutions [7–12], Markov property [13], and even the long-term behavior [14]. In addition, to describe a wide variety of natural and man-made systems precisely, various types of SDEs are developed (see, e.g., [10, 15]), and the theory of these SDEs has always been a focus.

One of the popular topics of SDEs is the existence and uniqueness of solutions. Generally, the classical existence and uniqueness theorem for SDEs requires the coefficients to satisfy the global Lipschitz condition (see, e.g., [16, 17]). Under the local Lipschitz condition and the linear growth condition, Arnold [18] has showed the existence of the unique solutions for SDEs. However, there are many interesting SDEs such that their coefficients are only superlinear, for such SDEs, Mao [10] has derived that there exists a unique regular solution under the locally Lipschitz condition and the monotone condition. Based on these existence and uniqueness results for the classical SDEs, many authors have studied the existence and uniqueness problems for other types of SDEs. For instance, Zvonkin [19] has investigated the strong solutions of SDEs with singular coefficients. Mao and Yuan [20] have introduced the existence and uniqueness of solutions for SDEs with Markovian switching.

Furthermore, although many SDEs have been showed that they each have a unique solution, it is important to determine precisely under which conditions one obtains a unique solution for SDEs. Compared with more restrictive conditions, general conditions can provide the existence and uniqueness of solutions for a larger class of SDEs. By using the Euler method, Krylov [9] has established the existence and uniqueness theorem under the monotone condition and a more general condition which is known as the local one-sided Lipschitz condition. Then, Gyöngy and Sabanis [7] have developed this result to stochastic differential delay equations. Recently, Ji and Yuan [8] have established the existence and uniqueness result for neutral stochastic differential delay equations. In this paper, inspired by Li et al. [21] and Krylov [9], we aim to study the existence and uniqueness of solutions for SDEs under weaker conditions compared with what we have mentioned above. Also, we can obtain the pth moment boundedness. The main contribution of this paper is that we have included the case of $0 < p < 2$ in our conditions.

The rest organization of this paper is as follows: In Section 2, some notations and preliminaries are introduced. In Section 3, the existence and uniqueness of solutions is provided by deriving a localization lemma, and the pth moment is further estimated. An example is given to illustrate our results in Section 4.

2. Notations and preliminaries

In this paper, we let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). Denote $\mathbb{N}$ as the set of natural numbers and $m, d \in \mathbb{N}$. Let $\{B(t)\}_{t\geq 0}$ be a standard m-dimensional Brownian motion defined on the probability space. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}^d$ be d-dimensional Euclidean space, and $\mathbb{R}^{d\times m}$ be the space of real $d \times m$-matrices. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. For any matrix A, define its trace norm by $\|A\| = \sqrt{\text{trace}(AA^\top)}$, where $A^\top$ denotes its transpose. Moreover, for any $a \in \mathbb{R}$ and $b \in \mathbb{R}$, define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For a set G, let $I_G(x) = 1$ if $x \in G$ and otherwise 0. Let $\inf \emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the integer part of x. For any $p \in (0, +\infty)$, let $L^p = L^p(\Omega; \mathbb{R}^d)$ be the family of $\mathbb{R}^d$-valued random variables $Z(\omega)$ with $\mathbb{E}[|Z(\omega)|^p] < +\infty$, and $\mathcal{L} = \mathcal{L}([0, T]; \mathbb{R})$ denotes the set of $\mathbb{R}$-valued nonnegative integrable functions on $[0, T]$.

In this paper, we consider a d-dimensional SDE described by

$$\mathrm{d}X(t) = f(X(t), t)\mathrm{d}t + g(X(t), t)\mathrm{d}B(t), \quad t \in [0, T], \tag{2.1}$$

with the initial value $X(0) = X_0 \in L^p$, where

$$f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d,\ g : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d\times m}$$

is Borel-measurable and has continuous mappings.

Moreover, in order for the main results, we impose the following assumptions.

Assumption 1. *For any $R,\ T \in [0, \infty)$, and $x \in \mathbb{R}^d$,*

$$\int_0^T \sup_{|x|\le R} \{|f(x,t)| \vee \|g(x,t)\|^2\} \mathrm{d}t < \infty.$$

Assumption 2. *(Locally one-sided Lipschitz condition) For any $R,\ T \in [0, \infty)$, there exists a $K_R(t) \in \mathcal{L}$, which is dependent on R such that*

$$2(x_1 - x_2)^\top (f(x_1, t) - f(x_2, t)) + \|g(x_1, t) - g(x_2, t)\|^2 \le K_R(t)|x_1 - x_2|^2,$$

for all $t \in [0, T]$, $x_1, x_2 \in \mathbb{R}^d$, and $|x_1| \vee |x_2| \le R$, where the $K_R(t)$ are satisfying $\int_0^T K_R(t)\mathrm{d}t < \infty$, for any $R,\ T \in [0, \infty)$.

For the regularity and pth moment boundedness of the exact solution, we make the following assumption.

Assumption 3. *For any $T \in [0, \infty)$ and $p \in (0, \infty)$, there exists a $K(t) \in \mathcal{L}$ such that*

$$(1 + |x|^2)(2x^\top f(x,t) + \|g(x,t)\|^2) - (2 - p)\left|x^T g(x,t)\right|^2 \le K(t)(1 + |x|^2)^2,$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^d$.

Remark 2.1. *If $p = 2$, we have*

$$2x^\top f(x,t) + \|g(x,t)\|^2 \le K(t)(1 + |x|^2),$$

which is the assumption of the paper [9]. In this paper, for any $p > 0$, the pth moment boundedness of solutions is also provided.

For the sake of simplicity, throughout the paper, we will fix $T \in [0, +\infty)$ arbitrarily, and unless otherwise stated, C denotes a generic positive real constant dependent on T, R etc. ***Please note that the values of C may change between occurrences***.

3. Existence and uniqueness of solution

In this section, we shall show that there exists a unique regular solution to (2.1). According to [8, 9, 12], we prepare a localization lemma below.

Lemma 3.1. *Suppose that $\{X^n(t)\}_{t\in[0,T]}$ are given continuous, $\mathbb{R}^d$-valued, and $\mathcal{F}_t$-adapted processes on $(\Omega,\ \mathcal{F},\ \mathbb{P})$. For $n \in \mathbb{N}$ such that $X^n(0) = X(0)$, and*

$$\mathrm{d}X^n(t) = f(X^n(t) + P^n(t), t)\mathrm{d}t + g(X^n(t) + P^n(t), t)\mathrm{d}B(t), \quad t \in [0, T],$$

where $P^n(t)$ is a progressively measurable process. Moreover, for $n \in \mathbb{N}$ and $R \in [0, \infty)$, suppose that there exists a nonrandom function $r : [0, \infty) \to [0, \infty)$ such that $\lim_{R\to\infty} r(R) = \infty$, and let $\tau_n(R)$ be $\mathcal{F}_t$-stopping times such that

(*i*) $|X^n(t)| + |P^n(t)| \le R$ *for $t \in [0, \tau_n(R)]$ a.s.*

(ii) $\lim_{n\to\infty}\mathbb{E}[\int_0^{T\wedge\tau_n(R)}|P^n(t)|\mathrm{d}t]=0$ *for all* $R,\ T\in[0,\infty)$.
(iii) *For any* $T\in[0,\infty)$,

$$\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\,\mathbb{P}\Big\{\tau_n(R)\le T,\ \sup_{t\in[0,\tau_n(R)]}|X^n(t)|<r(R)\Big\}=0.$$

Then, also for any $T\in[0,\infty)$*, we have*

$$\sup_{t\in[0,T]}|X^n(t)-X^m(t)|\xrightarrow{\mathbb{P}}0,\quad as\ n,m\to\infty. \tag{3.1}$$

Proof. We borrow the techniques from [12] mainly and divide the proof into 2 steps.
Step 1. For $R\in[0,\infty)$ and $t\in[0,T]$, from Assumption 1 we assume that

$$\sup_{|x|\le R}\Big\{|f(x,t)|\vee\|g(x,t)\|^2\Big\}\le K_R(t),$$

(otherwise, we regard $K_R(t)$ as the maximum of $K_R(t)$ and the integrand in Assumption 1). Fix $R\in[0,\infty)$ and define the $\mathcal{F}_t$-stopping time

$$\tau(R,u)=\inf\{t\ge 0|\alpha_R(t)>u\},\ u\in(0,\infty),$$

where $\alpha_R(t)=\int_0^t K_R(s)\mathrm{d}s<\infty$. Clearly, $\tau(R,u)\uparrow\infty$ as $u\to\infty$. In particular, there exists $u(R)\in(0,\infty)$ such that

$$\mathbb{P}\{\tau(R,u(R))\le R\}\le\frac{1}{R}.$$

Now, we let $\tau(R)=\tau(R,u(R))$, then $\tau(R)\to\infty$ in probability as $R\to\infty$ and $\alpha_R(t\wedge\tau(R))\le u(R)$. Moreover, referring to [8,12], it is easy to prove that all three conditions (i)–(iii) still hold if we replace $\tau_n(R)$ by $\tau_n(R)\wedge\tau(R)$. So we can further assume that $\tau_n(R)\le\tau(R)$, then we have $\alpha_R(t\wedge\tau_n(R))\le u(R)$. For a fixed $R\in[0,\infty)$, we define

$$\lambda_n^R(t)=\int_0^t|P^n(s)|K_R(s)\mathrm{d}s,\quad t\in[0,T\wedge\tau_n(R)],\ n\in\mathbb{N},$$

and $\tau_{(n,m)}(R)=\tau_n(R)\wedge\tau_m(R)$ for $m,n\in\mathbb{N}$. Then we can obtain

$$\lim_{n\to\infty}\mathbb{E}[\lambda_n^R(T\wedge\tau_n(R))]=0. \tag{3.2}$$

Under Assumption 2, we have

$$\sup_{t\in[0,T\wedge\tau_{(n,m)}(R)]}|X^n(t)-X^m(t)|\xrightarrow{\mathbb{P}}0,\quad as\ n,m\to\infty. \tag{3.3}$$

We omit the proof of (3.2) and (3.3) there as the reader can refer to [8,12] for more details.
Step 2. In order for (3.1), we need to show that

$$\lim_{R\to\infty}\overline{\lim_{n\to\infty}}\,\mathbb{P}\{\tau_n(R)\le T\}=0,$$

for any given $T\in[0,\infty)$. For $t\in[0,T]$, let κ be a negative constant and define

$$\psi(t)=\exp\big(\kappa\beta(t)-|X(0)|\big),$$

where $\beta(t) = \int_0^t K(s)\mathrm{d}s$. For $t \in [0, T \wedge \tau_n(R)]$, applying the Itô formula, we have

$$\begin{aligned}(1 + |X^n(t)|^2)^{\frac{p}{2}}\psi(t) =&(1 + |X(0)|^2)^{\frac{p}{2}}\psi(0) + \int_0^t \kappa K(s)(1 + |X^n(s)|^2)^{\frac{p}{2}}\psi(s)\mathrm{d}s + \frac{p}{2}\int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}}\\ &\times\Big\{(1 + |X^n(s)|^2)\Big(2(X^n(t))^T f(X^n(s) + P^n(s), s) + \|g(X^n(s) + P^n(s), s)\|^2\Big)\\ &- (2 - p)\Big|(X^n(s))^T g(X^n(s) + P^n(s), s)\Big|^2\Big\}\mathrm{d}s + J_n^R(t),\end{aligned}$$

where

$$J_n^R(t) = p\int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}}(X^n(s))^T g(X^n(s) + P^n(s), s)\mathrm{d}B(s).$$

Then, we further write that

$$\begin{aligned}(1 + |X^n(t)|^2)^{\frac{p}{2}}\psi(t) =& (1 + |X(0)|^2)^{\frac{p}{2}}\psi(0) + \int_0^t \kappa K(s)(1 + |X^n(s)|^2)^{\frac{p}{2}}\psi(s)\mathrm{d}s + \frac{p}{2}\int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}}\\ &\times\bigg\{\Big(1 + |X^n(s) + P^n(s)|^2 - 2(X^n(s) + P^n(s))^T P^n(s) + |P^n(s)|^2\Big)\\ &\times\Big(2(X^n(s) + P^n(s))^T f(X^n(s) + P^n(s), s) + \|g(X^n(s) + P^n(s), s)\|^2\\ &- 2(P^n(s))^T f(X^n(s) + P^n(s), s)\Big)\\ &- (2 - p)\Big[\Big|(X^n(s) + P^n(s))^T g(X^n(s) + P^n(s), s)\Big|^2\\ &- 2(P^n(s))^T (X^n(s) + P^n(s))\|g(X^n(s) + P^n(s), s)\|^2\\ &+ \Big|(P^n(s))^T g(X^n(s) + P^n(s), s)\Big|^2\Big]\bigg\}\mathrm{d}s + J_n^R(t)\\ =& (1 + |X(0)|^2)^{\frac{p}{2}}\psi(0) + \int_0^t \kappa K(s)(1 + |X^n(s)|^2)^{\frac{p}{2}}\psi(s)\mathrm{d}s + \sum_{i=1}^5 J_i(t) + J_n^R(t), \end{aligned}\tag{3.4}$$

where

$$\begin{aligned}J_1(t) =& \frac{p}{2}\int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}}\Big\{(1 + |X^n(s) + P^n(s)|^2)\Big[2(X^n(s) + P^n(s))^T f(X^n(s) + P^n(s), s)\\ &+ \|g(X^n(s) + P^n(s), s)\|^2\Big] - (2 - p)\Big|(X^n(s) + P^n(s))^T g(X^n(s) + P^n(s), s)\Big|^2\Big\}\mathrm{d}s,\\ J_2(t) =& \frac{p}{2}\int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}}\Big(-2(X^n(s) + P^n(s))^T P^n(s) + |P^n(s)|^2\Big)\\ &\times\Big[2(X^n(s) + P^n(s))^T f(X^n(s) + P^n(s), s) + \|g(X^n(s) + P^n(s), s)\|^2\Big]\mathrm{d}s,\\ J_3(t) =& \frac{p}{2}\int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}}(1 + |X^n(s) + P^n(s)|^2)\Big(-2(P^n(s))^T f(X^n(s) + P^n(s), s)\Big)\mathrm{d}s,\\ J_4(t) =& \frac{p}{2}\int_0^t \psi(s)(1 + |X^n(s)^2)^{\frac{p-4}{2}}\Big(-2(X^n(s) + P^n(s))^T P^n(s) + |P^n(s)|^2\Big)\\ &\times\Big(-2(P^n(s))^T f(X^n(s) + P^n(s), s)\Big)\mathrm{d}s,\end{aligned}$$

$$J_5(t) = \frac{p}{2}\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}}\Big\{-(2-p)\Big(-2(X^n(s)+P^n(s))^T P^n(s)\|g(X^n(s)+P^n(s),s)\|^2 + \Big|(P^n(s))^T g(X^n(s)+P^n(s),s)\Big|^2\Big)\Big\}\mathrm{d}s.$$

By Assumption 3, we have

$$\begin{aligned} J_1(t) &\le \frac{p}{2}\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}} K(s)(1+|X^n(s)+P^n(s)|^2)^2\mathrm{d}s \\ &\le C\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}} K(s)\Big((1+|X^n(s)|^2)^2+|P^n(s)|^4\Big)\mathrm{d}s \\ &= C\int_0^t \psi(s)K(s)(1+|X^n(s)|^2)^{\frac{p}{2}}\mathrm{d}s + C\int_0^t \psi(s)K(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}}|P^n(s)|^4\mathrm{d}s. \end{aligned}$$

For $0 < p \le 4$, we have $(1+|X^n(t)|^2)^{\frac{p-4}{2}} \le 1$. For $t \in [0, T\wedge\tau_n(R)]$, using the condition (i), we have $|X^n(t)| + |P^n(t)| \le R$ *a.s.*. Then we can derive that

$$J_1(t) \le C\int_0^t \psi(s)K(s)(1+|X^n(s)|^2)^{\frac{p}{2}}\mathrm{d}s + C_R\int_0^t \psi(s)K(s)|P^n(s)|\mathrm{d}s, \tag{3.5}$$

where C_R denotes a generic positive constant related to R in this paper. ***Please note that the values of C_R may change between occurrences***.

While $p > 4$, using Young's inequality, we have

$$\begin{aligned} J_1(t) &\le C\Big(\int_0^t \psi(s)K(s)(1+|X^n(s)|^2)^{\frac{p}{2}}\mathrm{d}s + \int_0^t \psi(s)K(s)|P^n(s)|^p\mathrm{d}s\Big) \\ &\le C\int_0^t \psi(s)K(s)(1+|X^n(s)|^2)^{\frac{p}{2}}\mathrm{d}s + C_R\int_0^t \psi(s)K(s)|P^n(s)|\mathrm{d}s. \end{aligned} \tag{3.6}$$

For $p > 0$, combining (3.5) and (3.6), we have

$$J_1(t) \le C\int_0^t \psi(s)K(s)(1+|X^n(s)|^2)^{\frac{p}{2}}\mathrm{d}s + C_R\int_0^t \psi(s)K(s)|P^n(s)|\mathrm{d}s. \tag{3.7}$$

Next, we compute $J_2(t)$, that is,

$$\begin{aligned} J_2(t) &\le C\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)\Big(|X^n(s)+P^n(s)||P^n(s)| + \Big|P^n(s)\Big|^2\Big) \\ &\quad\times\Big(|X^n(s)+P^n(s)|+1\Big)\mathrm{d}s \\ &\le C\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)\Big(|X^n(s)+P^n(s)| + \Big|P^n(s)\Big|\Big)|P^n(s)| \\ &\quad\times\Big(|X^n(s)|+|P^n(s)|+1\Big)\mathrm{d}s \\ &\le C\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)\Big(1+|X^n(s)|+|P^n(s)|\Big)^2|P^n(s)|\mathrm{d}s \\ &\le C\int_0^t \psi(s)(1+|X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)\Big(1+|X^n(s)|^2+|P^n(s)|^2\Big)|P^n(s)|\mathrm{d}s. \end{aligned} \tag{3.8}$$

Obviously, we also need to consider (3.8) in two cases, respectively: $0 < p \le 4$ and $p > 4$. By the condition (i), for $p > 0$, it is easy to show that

$$J_2(t) \le (1 + C_R) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s. \tag{3.9}$$

For $J_3(t)$, we can write that

$$J_3(t) \le C \int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)(1 + |X^n(s) + P^n(s)|^2)|P^n(s)| \mathrm{d}s.$$

In the same way as discussed above, we have

$$J_3(t) \le (1 + C_R) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s. \tag{3.10}$$

Repeating the similar procedures, we also have

$$J_4(t) \le C \int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)\Big(|X^n(s) + P^n(s)||P^n(s)| + |P^n(s)|^2\Big)|P^n(s)| \mathrm{d}s,$$

and

$$\begin{aligned} J_5(t) &\le C \int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}} \Big(|X^n(s) + P^n(s)||P^n(s)||g(X^n(s) + P^n(s), s)|^2 \\ &\qquad + |(P^n(s))|^2 |g(X^n(s) + P^n(s), s)|^2\Big) \mathrm{d}s \\ &\le C \int_0^t \psi(s)(1 + |X^n(s)|^2)^{\frac{p-4}{2}} K_R(s)\Big(|X^n(s) + P^n(s)||P^n(s)| + |(P^n(s)|^2\Big) \mathrm{d}s. \end{aligned}$$

Therefore, for $t \in [0, T \wedge \tau_n(R)]$ and $p > 0$, by virtue of the condition (i), we derive that

$$J_4(t) \le (1 + C_R) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s, \tag{3.11}$$

and

$$J_5(t) \le (1 + C_R) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s. \tag{3.12}$$

Substituting (3.7) and (3.9)–(3.12) into (3.4), we have

$$\begin{aligned} (1 + |X^n(t)|^2)^{\frac{p}{2}} \psi(t) &\le (1 + |X(0)|^2)^{\frac{p}{2}} \psi(0) + \int_0^t \kappa K(s)(1 + |X^n(s)|^2)^{\frac{p}{2}} \psi(s) \mathrm{d}s \\ &\quad + C \int_0^t \psi(s) K(s)(1 + |X^n(t)|^2)^{\frac{p}{2}} \mathrm{d}s + C_R \int_0^t \psi(s) K(s) |P^n(s)| \mathrm{d}s \\ &\quad + (1 + C_R) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s + J_n^R(t). \end{aligned} \tag{3.13}$$

Choosing $\kappa = -C$ and then replacing $K_R(s)$ by $K(s) \vee K_R(s)$, we have

$$(1 + |X^n(t)|^2)^{\frac{p}{2}}\psi(t) \leq (1 + |X(0)|^2)^{\frac{p}{2}}\psi(0) + (1 + C_R)\int_0^t \psi(s)K_R(s)|P^n(s)|\mathrm{d}s + J_n^R(t). \tag{3.14}$$

Furthermore, since $\psi(t) \leq 1$ and $J_n^R(t)$ is a continuous local $\mathcal{F}_t$-martingale with $J_n^R(0) = 0$, according to [10], for any $R,\ T \in [0, \infty)$, taking expectations on both sides of (3.14), it is easy to see that

$$\mathbb{E}\Big[(1 + |X^n(\varsigma)|^2)^{\frac{p}{2}}\psi(\varsigma)] \leq \psi(0)\mathbb{E}[(1 + |X(0)|^2)^{\frac{p}{2}}] + (1 + C_R)\mathbb{E}[\lambda_n^R(T \wedge \tau_n(R)],$$

where ς represents any $\mathcal{F}_t$-stopping time satisfying $\varsigma \leq T \wedge \tau_n(R)$. Then, based on [9, p. 584, Lemma 1], for any $l \in (0, \infty)$, we have

$$l\mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_n(R)]} |X^n(t)|^p\psi(t) \geq l\Big\} \leq (1 + C_R)\Big(1 + \mathbb{E}[\lambda_n^R(T \wedge \tau_n(R))]\Big).$$

We then have

$$\mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_n(R)]} |X^n(t)|^p\psi(t) \geq l\Big\} \leq \frac{(1 + C_R)\Big(1 + \mathbb{E}[\lambda_n^R(T \wedge \tau_n(R))]\Big)}{l}.$$

Thanks to (3.2), it is easy to derive that

$$\lim_{l\to\infty} \sup_{R\in[0,\infty)} \overline{\lim_{n\to\infty}}\, \mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_n(R)]} |X^n(t)|^p\psi(t) \geq l\Big\} = 0. \tag{3.15}$$

Recalling that $r(R) \to \infty\ as\ R \to \infty$ and choosing $l = r^p(R)\psi(t)$ in (3.15) , we have

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_n(R)]} |X^n(t)| \geq r(R)\Big\} = 0,$$

which implies

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\Big\{\sup_{t\in[0,\tau_n(R)]} |X^n(t)| \geq r(R), \tau_n(R) \leq T\Big\} = 0.$$

Under condition (*iii*), we obtain

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\{\tau_n(R) \leq T\} = 0.$$

Hence for any $\varepsilon > 0$, thanks to (3.3), we have

$$\begin{aligned}
\mathbb{P}\Big\{\sup_{t\in[0,T]} |X^n(t) - X^m(t)| > \varepsilon\Big\} &= \mathbb{P}\Big\{\sup_{t\in[0,T]} |X^n(t) - X^m(t)| > \varepsilon, \tau_{(n,m)}(R) \leq T\Big\} \\
&\quad + \mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_{(n,m)}(R)]} |X^n(t) - X^m(t)| > \varepsilon, \tau_{(n,m)}(R) > T\Big\} \\
&\leq \mathbb{P}\Big\{\tau_n(R) \leq T\Big\} + \mathbb{P}\Big\{\tau_m(R) \leq T\Big\} + \mathbb{P}\Big\{\sup_{t\in[0,T\wedge\tau_{(n,m)}(R)]} |X^n(t) - X^m(t)| > \varepsilon\Big\},
\end{aligned}$$

which leads to (3.1). □

We now give the theorem about the existence and uniqueness of the exact solution to (2.1).

Theorem 3.1. *Let Assumptions* 1–3 *hold with $p > 0$. Then, for any $T \in [0, \infty)$, there exists a unique process $\{X(t)\}_{t\in[0,T]}$ that satisfies Eq* (2.1) *with the property*

$$\sup_{t\in[0,T]} \mathbb{E}[|X(t)|^p] < C. \tag{3.16}$$

Proof. Based on Euler's method, we construct a sequence $\{X^n(\cdot)\}, n \in \mathbb{N}$. For $n \in \mathbb{N}$, we define $\{X^n(t)\}_{t\geq 0}$ as follows:

$$\begin{cases} X^n(0) = X(0), \\ X^n(t) = X^n(\frac{k}{n}) + \int_{\frac{k}{n}}^t f(X^n(\frac{k}{n}), s)\mathrm{d}s + \int_{\frac{k}{n}}^t g(X^n(\frac{k}{n}), s)\mathrm{d}B(s), \quad t \in [\frac{k}{n}, \frac{k+1}{n}), k \in \{0\} \cup \mathbb{N}. \end{cases}$$

We further define $\iota(n,t) = \lfloor nt \rfloor / n$. Then, for $t \geq 0$, we have

$$X^n(t) = X^n(\iota(n,t)) + \int_{\iota(n,t)}^t f(X^n(\iota(n,s)), s)\mathrm{d}s + \int_{\iota(n,t)}^t g(X^n(\iota(n,s)), s)\mathrm{d}B(s),$$

which can be written as

$$X^n(t) = X^n(0) + \int_0^t f(X^n(\iota(n,s)), s)\mathrm{d}s + \int_0^t g(X^n(\iota(n,s)), s)\mathrm{d}B(s). \tag{3.17}$$

This is equivalent to

$$X^n(t) = X^n(0) + \int_0^t f(X^n(s) + P^n(s), s)\mathrm{d}s + \int_0^t g(X^n(s) + P^n(s), s)\mathrm{d}B(s),$$

where $P^n(t) = X^n(\iota(n,t)) - X^n(t) = -\int_{\iota(n,t)}^t f(X^n(\iota(n,s)), s)\mathrm{d}s - \int_{\iota(n,t)}^t g(X^n(\iota(n,s)), s)\mathrm{d}B(s)$. In order for the existence and uniqueness, we need to show that there exist an $\mathcal{F}_t$-adapted continuous process $\{X(t)\}_{t\in[0,T]}$ and

$$X(t) = X(0) + \int_0^t f(X(s), s)\mathrm{d}s + \int_0^t g(X(s), s)\mathrm{d}B(s) \qquad \mathbb{P} - a.s.$$

after taking limits on both sides of (3.17). Also, define $\tau_n(R)$ as the first exit time of $X^n(t)$ from the sphere ($|x| < \frac{R}{3}$), and let nonrandom function $r(R) = \frac{R}{4}$. Then, $|P^n(t)| \leq \frac{2R}{3}$, $|X^n(t)| \leq \frac{R}{3}$ for $t \in [0, \tau_n(R)]$ *a.s.* and in terms of Lemma 3.1, the proofs of these are same as [9, 12], so we omit it there. Thus, there exists a unique solution of Eq (2.1).

It remains to prove the pth moment boundedness. In fact, for an application of the Itô formula to Eq (2.1), we have

$$\begin{aligned}(1 + |X(t)|^2)^{\frac{p}{2}} = (1 + |X(0)|^2)^{\frac{p}{2}} + \frac{p}{2}\int_0^t (1 + |X(s)|^2)^{\frac{p-4}{2}}\Big\{(1 + |X(s)|^2)\big(2(X(s))^\top f(X(s), s) + \|g(X(s), s)\|^2\big) \\ - (2-p)\big|(X(s))^\top g(X(s), s)\big|^2\Big\}\mathrm{d}s + H(t),\end{aligned}$$

where

$$H(t) = p\int_0^t (1 + |X(s)|^2)^{\frac{p-4}{2}}(X(s))^\top g(X(s), s)\mathrm{d}B(s).$$

We recall that $(1 + |X(t)|^2)^{\frac{p-4}{2}} \leq 1$ for $0 < p \leq 4$, and Young's inequality can be used in the case of $p > 4$. Therefore, by Assumption 3, (3.16) follows directly from [21, p. 851, Theorem 2.3]. $\square$

4. Example

In this section, we consider an example that is a scalar SDE as follows:

$$\mathrm{d}X(t) = \left(X(t)\sin t + |X(t)|^2 - X(t)^3 - |X(t)|^{\frac{1}{2}}\right)\mathrm{d}t + X(t)\sin t\mathrm{d}B(t), \quad t \in [0, T], \tag{4.1}$$

with the initial data $X(0) = X_0 \in \mathbb{R}$, where

$$f(x,t) = x\sin t + |x|^2 - x^3 - |x|^{\frac{1}{2}}, \quad g(x,t) = x\sin t,$$

and $B(t)$ is a Brownian motion. Clearly, Assumption 1 holds for any $R,\ T \in [0,\infty)$ and $x \in \mathbb{R}$, and $f(x,t)$ does not satisfy the local Lipschitz condition. Therefore, the techniques in [10, 21] can't be applied to the existence and uniqueness of the solution for (4.1). However, by the Young inequality, we then have

$$\begin{aligned}(x_1 - x_2)\left(f(x_1,t) - f(x_2,t)\right) &= (x_1 - x_2)\left(x_1\sin t + |x_1|^2 - x_1^3 - |x_1|^{\frac{1}{2}} - x_2\sin t - |x_2|^2 + x_2^3 + |x_2|^{\frac{1}{2}}\right)\\ &\le (|\sin t| + |x_1| + |x_2|)\,|x_1 - x_2|^2 - (x_1 - x_2)(|x_1|^{\frac{1}{2}} - |x_2|^{\frac{1}{2}})\\ &\le (|\sin t| + 2R)\,|x_1 - x_2|^2,\end{aligned}$$

for any $x_1, x_2 \in \mathbb{R}$, and $|x_1| \vee |x_2| \le R$. This means that $f(x,t)$ satisfies Assumption 2 (i.e., locals one-sided Lipschitz condition). Moreover, it should be noted that the monotone condition requiring $p \ge 2$ in [9, 12] doesn't hold there. Let $0 < p \le 1$. By computation, it is easy to verify that Assumption 3 holds, that is, for any $x \in \mathbb{R}$,

$$(1 + |x|^2)(2xf(x,t) + \|g(x,t)\|^2) - (2-p)\left|xg(x,t)\right|^2 \le K(t)(1 + |x|^2)^2.$$

Then, by Theorem 3.1 we can conclude that the SDE (4.1) has a unique global solution $X(t)$ on $t \ge 0$ with the boundedness of the pth moment on $[0, T]$, that is,

$$\sup_{t\in[0,T]} \mathbb{E}\big[|X(t)|^p\big] \le C, \qquad \forall T \in [0,\infty).$$

5. Conclusions

The current focus lies in the existence and uniqueness of solutions for stochastic differential equations with locally one-sided Lipschitz condition, and we can obtain the pth moment boundedness. In future research, we are going to study the stability of the solution, furthermore, we shall investigate an implicit numerical scheme for these equations under a local one-sided Lipschitz condition.

Author contributions

Fangfang Shen and Huaqin Peng: Conceptualization, Methodology, Investigation, Writing-original draft, Writing-review and editing. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this paper.

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