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Research article

Computational methods for singularly perturbed differential equations with advanced argument of convection-diffusion type

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Abstract: This study investigates singularly perturbed differential equations through advanced convection-diffusion techniques. We employ a finite difference approach utilizing a piecewise uniform Shishkin-type mesh to tackle this problem. Our analysis demonstrates that the approach achieves virtually first-order convergence. Error estimates are computed using discrete norms, and numerical experiments are conducted to validate these theoretical results.

Keywords: error analysis; finite difference method; convection-diffusion type; boundary layer
Mathematics Subject Classification: 65L10, 65M15, 65N15

1. Introduction

Differential equations (DEs) with advanced arguments play an important role in physics, biological science, and economics. This type of problem depends on the future of the system of consideration, for example [6, 8, 9]. In [29], they investigated the oscillation and nonoscillation of the solution of impulsive DEs with advanced arguments, and these are quite uncommon in the literature. Schulman looked into the second-order nonimpulsive DEs [15] while researching an electrodynamic system,

$$x''(t) + wx(t) = \frac{1}{2}\alpha x(t-\tau) + \frac{1}{2}\beta x(t+\sigma) + \psi(t),$$

where $\alpha, \beta, \tau > 0, \sigma > 0$ are constants. When $\alpha = 0$, the above equation becomes

$$x''(t) + wx(t) = \frac{1}{2}\beta x(t+\sigma) + \psi(t),$$

which is a DE with an advanced argument. DEs with advanced arguments and mixed arguments were discussed in [1, 2, 5, 10, 11, 25].

A differential equation with a small positive parameter multiplied the highest derivative term is called a singularly perturbed differential equation (SPDE) with advanced argument. Such a type of equation arises frequently in many mathematical models of various practical phenomena. For example, in [29], they discussed the oscillation of first-order impulsive DE with an advanced argument; in [17], oscillations caused by several retarded and advanced arguments, and in [18, 31], singularly perturbed boundary value problems for differential-difference equations were examined.

SPDEs for solving standard numerical methods are sometimes ill-posed problems and fail to give analytical results when the perturbation parameter ϵ is small. Hence, uniformly convergent numerical methods are used to solve such types of DEs. SPDEs with at least one delay term and unknown functions can arise with different parameters. Differential equations are important in science and engineering, as well as in various mathematical models such as the signal transition [4], first-exit problems in neurobiology [28], HIV infection models [3], variational problems in control theory [7] and human pupil-light reflex [19]. In [12, 14, 16, 20, 22, 24, 30], both finite difference and finite element approaches are presented to solve delayed SPDEs with minor and large shifts.

In the past, only a few authors worked in the area of DEs with advanced arguments and also SPDEs with positive shifts. In [11], Tadeusz Jankowski proposed first-order impulsive ordinary DEs with advanced arguments; in [1], they discussed the non-oscillation of mixed advanced-delay DEs with positive and negative coefficients, and also in [2, 5, 10, 25], various concepts of advanced arguments are investigated. Kadalbajoo and Sharma [17] proposed a finite difference scheme to solve singularly perturbed problems with mixed type; Patidar and Sharma [23] proposed a fitted operator method to solve singularly perturbed problems with mixed type.

Lange and Miura [18] developed a mathematical model to predict the time it takes for nerve cells to generate action potentials based on random synaptic inputs in dendrites. Based on Stein's model [28], they used a Poisson distribution with exponential decay to represent inputs in their model. Additionally, they included inputs represented as a Wiener process with a variance parameter σ and a drift parameter μ.

Given an initial membrane potential $t \in (t_1, t_2)$, the challenge of estimating the projected first-exit time y is expressed as a general boundary value problem for a linear second-order differential-difference equation:

$$\frac{\sigma^2}{2} y''(t) + (\mu - t)y'(t) + \lambda_E y(t + a_E) + \lambda_I y(t - a_I) - (\lambda_E + \lambda_I)y(t) = -1.$$

Here, $t = t_1$ and $t = t_2$ represent the inhibitory reversal potential and the membrane potential threshold for action potential production, respectively. The parameters σ and μ indicate the variance and drift, respectively. The predicted first-exit time is represented by the function y, and the exponential decay between synaptic inputs is shown by the equation $-ty'(t)$. The terms involving λ_E and λ_I represent excitatory and Poisson processes with mean rates λ_E and λ_I, respectively, are used to describe inhibitory synaptic inputs. These processes result in jumps in the membrane potential of tiny values that may be voltage-dependent, a_E and a_I, respectively.

The boundary condition is given by

$$y(t) = 0, \quad t \notin (t_1, t_2).$$

In their study, Lange and Miura initiated the investigation of boundary value problems (BVPs) for singularly perturbed differential-difference equations. They presented an asymptotic approach to analyze the effects of small shifts in certain simpler classes of singularly perturbed ordinary DEs.

In [13, 15, 17, 23, 26, 27], the finite difference and fitted operator methods to solve this kind of equation with a small delay and advanced. In this article, we discuss finite difference methods for SPDEs with advanced arguments.

This work studies a fitted finite difference technique on a piecewise uniform mesh to solve a convection-diffusion problem with advanced arguments. The paper is structured as follows:

Problem Statement: In Section 2, we present the problem statement, considering smooth data.

Theoretical Analysis: Section 3 discusses the maximum principle, stability results, and the solution's derivative bounds.

Numerical Method: Section 4 describes the numerical method utilized for solving the problem.

Error Analysis: Section 5 provides the error analysis for the approximate solution obtained through the numerical method.

Numerical Results: Section 6 presents the numerical results obtained from applying the numerical method.

This structured approach aims to offer a thorough investigation into the numerical solution technique for the convection-diffusion problem, encompassing theoretical foundations, numerical implementation, and practical results.

Throughout our analysis, C denotes a positive constant independent of the parameter ϵ and the number of mesh points M, and we define the open intervals $\Omega = (0, 2)$, $\Omega^- = (0, 1)$, and $\Omega^+ = (1, 2)$. Additionally, we define $\Omega^* = \Omega^- \cup \Omega^+$.

In our analysis, we employ the supremum norm to assess the convergence between the numerical and precise solutions of a singular perturbation problem, defined as

$$||z||_\Omega = \sup_{t\in\Omega} |z(t)|.$$

2. Problem statement

In this paper, we investigate the following SPDEs with an advanced argument:

$$\begin{cases} \mathbf{L}z(t) = -\epsilon z''(t) + p(t)z'(t) + q(t)z(t) + r(t)z(t+1) = g(t), & t \in (0, 2), \\ z(0) = l, z(t) = \psi(t), \quad t \in [2, 3], \end{cases} \tag{2.1}$$

where ψ is a continuous function on $[2, 3]$. For all $t \in [0, 2]$, it is assumed that $p(t)$, $q(t)$, and $r(t)$ satisfy $p(t) \geq \alpha_1 \geq \alpha > 0$, $q(t) \geq \beta > 0$, $r(t) \leq \gamma < 0$, $\alpha + \gamma + \beta > 0$, and $\beta + \gamma > 0$.

Additionally, the functions $p(t)$, $q(t)$, $r(t)$, and $g(t)$ are sufficiently smooth on $[0, 2]$.

The assumptions mentioned above guarantee that $z \in X = C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.

The problem (2.1) is equivalent to

$$\mathbf{L}z(t) = g(t),$$

where

$$\mathbf{L}z(t) = \begin{cases} \mathbf{L}_1 z(t) = -\epsilon z''(t) + p(t)z'(t) + q(t)z(t) + r(t)z(t+1), & t \in (0, 1), \\ \mathbf{L}_2 z(t) = -\epsilon z''(t) + p(t)z'(t) + q(t)z(t), & t \in (1, 2), \end{cases} \tag{2.2}$$

$$g(t) = \begin{cases} s(t), & t \in (0,1), \\ s(t) - r(t)\psi(t+1), & t \in (1,2), \end{cases} \tag{2.3}$$

with boundary condition

$$\begin{cases} z(0) &= l, \\ z(1^-) &= z(1^+), \quad z'(1^-) = z'(1^+), \\ z(t) &= \psi(t), \qquad t \in [2,3]. \end{cases}$$

3. Analytic results

Lemma 3.1 (Maximum Principle)**.** *Let $\zeta(t)$ be any function in X satisfying the following conditions:*

- $\zeta(0) \geq 0$, $\zeta(2) \geq 0$,
- $\mathbf{L}_1\zeta(t) \geq 0$, *for all* $t \in \Omega^-$,
- $\mathbf{L}_2\zeta(t) \geq 0$, *for all* $t \in \Omega^+$,
- $[\zeta](1) = 0$, $[\zeta'](1) \leq 0$.

Then $\zeta(t) \geq 0$, *for all* $t \in \bar{\Omega}$.

Proof. Let t^* be such that $\zeta(t^*) = \min_{t\in[0,2]} \zeta(t)$. We assert $\zeta(t^*) \geq 0$.

Suppose $\zeta(t^*) < 0$.

Case (i). If $t^* = 0$, then $\zeta(0) < 0$, contradicting the assumption.

Case (ii). If $t^* = 2$, then $\zeta(2) < 0$, contradicting the assumption.

Case (iii). If $t^* \in \Omega^-$, then $\zeta''(t^*) \geq 0$. Since $\zeta(t^*)$ is minimum, we have

$$0 \leq (\mathbf{L}_1\zeta)(t^*) = -\epsilon\zeta''(t^*) + p(t)\zeta'(t^*) + q(t)\zeta(t^*) + r(t)\zeta(t^* + 1) < 0,$$

which is a contradiction.

Case (iv). If $t^* \in \Omega^+$, then

$$0 \leq (\mathbf{L}_2\zeta)(t^*) = -\epsilon\zeta''(t^*) + p(t)\zeta'(t^*) + q(t)\zeta(t^*) < 0,$$

which is also a contradiction.

Case (v). If $t^* = 1$.

- **Sub-case (i).** If $\zeta'(1)$ does not exist, then $\neq 0$ and since $\zeta'(1^-) \leq 0$ and $\zeta'(1^+) > 0$, we have $\zeta'(1) > 0$, contradicting the assumption.
- **Sub-case (ii).** If $\zeta(1)$ is differentiable, we can show that $\zeta(t)$ does not have a minimum at $t = 1$, contradicting the premise that $t^* = 1$.

□

An immediate implication of Lemma 3.1 is the uniqueness of the solution to the boundary value problem (2.1), provided it exists.

Lemma 3.2 (Stability)**.** *The solution $z(t)$ of the problem (2.1), then satisfies the bound*

$$|z(t)| \leq C \max\left\{|z(0)|, |z(2)|, \sup_{t\in\Omega^-\cup\Omega^+} |\mathbf{L}z(t)|\right\}.$$

Proof. Let $G = \max\left\{|z(0)|, |z(2)|, \sup_{t\in\Omega^-\cup\Omega^+} |\mathbf{L}z(t)|\right\}$.

Define $\Psi^{\pm}(t) = CG \pm z(t)$.

Clearly, $\Psi^{\pm}(0) \geq 0$, $\Psi^{\pm}(2) \geq 0$, and also $\mathbf{L}_1\Psi^{\pm}(t) \geq 0$ on Ω^- and $\mathbf{L}_2\Psi^{\pm}(t) \geq 0$ on Ω^+. Moreover $[\Psi^{\pm\prime}](1) = \pm[z'](1) = 0$, then $\Psi^{\pm}(t) \geq 0$ on $\bar{\Omega}$. □

Lemma 3.3. *Let $z(t)$ be the solution to (2.1). Then we have the following bounds:*

$$||z^{(k)}(t)|| \leq C\epsilon^{-k},\ for\ k = 1, 2, 3.$$

Proof. The bound on $z(t)$ is an immediate consequence of Lemma 3.2 and the boundary value problem (2.1). To bound $z'(t)$ on the interval $(0, 1)$,

$$\mathbf{L}_1 z(t) = -\epsilon z''(t) + p(t)z'(t) + q(t)z(t) + r(t)z(t + 1) = g(t),$$

the above equation integrates on both sides

$$-\epsilon(z'(t) - z'(0)) = -[p(t)z(t) - a(0)z(0)] + \int_0^t a'(\eta)z(\eta)d\eta - \int_0^t [b(\eta)z(\eta) + c(\eta)z(\eta + 1)]d\eta + \int_0^t g(\eta)d\eta.$$

Therefore,

$$\epsilon z'(0) = \epsilon z'(t) - [p(t)z(t) - a(0)z(0)] + \int_0^t a'(\eta)z(\eta)d\eta - \int_0^t [b(\eta)z(\eta) + c(\eta)z(\eta + 1)]dt + \int_0^t g(\eta)d\eta.$$

Then by the mean value theorem, $\exists\ \eta \in (0, \epsilon)$ such that $|\epsilon z'(z)| \leq C(||z(t)||, ||g(t)||)$ and $|\epsilon z'(0)| \leq C(||z(t)|| + ||g(t)||)$. Hence, $|\epsilon z'(t)| \leq C \max(||z(t)||, ||g(t)||)$.
Similarly, $t \in \Omega^+$. Form (2.2) and (2.3) we have $||z^{(k)}(t)|| \leq C\epsilon^{-k}, k = 2, 3$. □

3.1. Solution segmentation

The solution's Shishkin decomposition $z(t)$ of (2.1) is $z(t) = v(t) + w(t)$, where the regular and singular components are denoted by $v(t)$ and $w(t)$, respectively, and also $v(t) = v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t)$ and $v_0(t) \in C^0(\overline{\Omega}) \cap C^1(\{0\} \cup \Omega)$, $v_1(t) \in C^0(\overline{\Omega}) \cap C^1(\{0\} \cup \Omega^*)$, $v_2(t) \in X$ are in turn defined, respectively, to be the solution of the following problems:

$$\begin{cases} p(t)v_0'(t) + q(t)v_0(t) + r(t)v_0(t + 1) = g(t), t \in \{0\} \cup \Omega, \\ v_0(t) = \psi(t), t \in [2, 3]. \end{cases} \tag{3.1}$$

$$\begin{cases} (p(t)v_1'(t) + q(t)v_1(t) + r(t)v_1(t + 1) = v_0''(t), t \in \{0\} \cup \Omega^*, \\ v_1(t) = 0, t \in [2, 3]. \end{cases} \tag{3.2}$$

$$\begin{cases}-\epsilon v_2''(t) + p(t)v_2'(t) + q(t)v_2(t) + r(t)v_2(t+1) = v_1''(t), t \in \{0\} \cup \Omega^*, \\ v_2(0) = 0, v_2(t) = 0, t \in [2,3]. \end{cases} \tag{3.3}$$

The smooth component $v \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ satisfies:

$$\mathbf{L}v(t) = -\epsilon v''(t) + p(t)v'(t) + q(t)v(t) + r(t)v(t+1) = g(t), t \in \Omega^*, \tag{3.4}$$
$$v(0) = v_0(0) + \epsilon v_1(0), \tag{3.5}$$
$$v(1) = v_0(1) + \epsilon v_1(1) + \epsilon^2 v_2(1), \quad v(t) = \psi(t), t \in [2,3]. \tag{3.6}$$

Further w satisfies:

$$\mathbf{L}w(t) = -\epsilon w''(t) + p(t)w'(t) + q(t)w(t) + r(t)w(t+1) = 0, \ t \in \Omega^*, \tag{3.7}$$
$$w(0) = l - v(0), [w'](1) = -[v'](1), w(t) = 0, t \in [2,3]. \tag{3.8}$$

We further decompose $w = w_B + w_I$.
Find $w_B \in X$ such that

$$\mathbf{L}w_B(t) = -\epsilon w_B''(t) + p(t)w_B'(t) + q(t)w_B(t) + r(t)w_B(t+1) = 0, \tag{3.9}$$
$$w_B(0) = l - v(0), w_B(t) = 0, t \in [2,3]. \tag{3.10}$$

Find $w_I \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ such that

$$\mathbf{L}w_I(t) = -\epsilon w_I''(t) + p(t)w_I'(t) + q(t)w_I(t) + r(t)w_I(t+1) = 0, \tag{3.11}$$
$$w_I(0) = 0, [w_I'](1) = -[v'](1), w_I(t) = 0, \quad t \in [2,3], \tag{3.12}$$

where the functions w_B and w_I are the boundary layer component and interior layer component, respectively.

Theorem 3.1. *Let z be the continuous solution of the problem (2.1), and $v_0(t)$ be the solution of the reduced problem solution defined by Eq (3.1). Then,*

$$|z(t) - v_0(t)| \le C_1(\epsilon + \exp(\frac{-\alpha(2-t)}{\epsilon})), \quad t \in [0,2].$$

Proof. Consider the barrier function

$$\Phi^{\pm}(t) = C_1(\epsilon + \exp(\frac{-\alpha(2-t)}{\epsilon})) \pm (z(t) - v_0(t)), \quad t \in [0,2],$$

clearly $\Phi^{\pm} \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$. It is not too hard to verify , $\Phi^{\pm}(0) \ge 0$, $\Phi^{\pm}(2) \ge 0$ for a suitable choice of $C_1 > 0$. Let $t \in (0,1)$,

$$\begin{aligned}\mathbf{L}_1\Phi^{\pm}(t) &= C_1([\frac{\alpha}{\epsilon}(p(t) - \alpha) + q(t) + r(t)\exp(\frac{\alpha}{\epsilon})]\exp(\frac{-\alpha(2-t)}{\epsilon}) + \epsilon(q(t) + r(t))) \pm \epsilon v_0''(t), \\ &\ge C_1(\frac{\alpha}{\epsilon}(\alpha_1 - \alpha) + \beta + \gamma\exp(\frac{\alpha}{\epsilon}))\exp(\frac{-\alpha(2-t)}{\epsilon}) + \epsilon(\beta + \gamma)) \pm C\epsilon.\end{aligned}$$

Similarly, $t \in (1,2)$, $\mathbf{L}_2\Phi^{\pm}(t) \ge 0$, by the Lemma 3.1, $\Phi^{\pm}(t) \ge 0, t \in [0,2]$. □

Lemma 3.4. *Let v and w represent the regular and singular components of the solution z. Then,*

$$\|v^k(t)\|_{\Omega^*} \leq C(1+\epsilon^{2-k}), \quad for\ k=0,1,2,3, \tag{3.13}$$

$$|w_B^k(t)| \leq C\epsilon^{-k}\exp(\frac{-\alpha(2-t)}{\epsilon}), \ \ t\in\Omega^*, \ \ k=0,1,2,3, \tag{3.14}$$

$$|w_I^k(t)| \leq C\begin{cases}\epsilon^{1-k}\exp(\frac{-\alpha(1-t)}{\epsilon}), & t\in(0,1),\\ \epsilon^{1-k}, & t\in(1,2),\end{cases} \quad k=0,1,2,3. \tag{3.15}$$

Proof. With the integration of (3.1), (3.3), and the stability result, one can prove the inequality (3.13). To prove the inequalities (3.14), consider the function

$$\Phi^{\pm}(t) = C_1(\exp(\frac{-\alpha(2-t)}{\epsilon})) \pm w_B(t), \ \ t\in[0,2].$$

It is easy to see that $\Phi^{\pm}(0) \geq 0$ and $\Phi^{\pm}(2) \geq 0$. Further, if $\mathbf{L}\Phi^{\pm}(t) = C_1([\frac{\alpha}{\epsilon}(p(t)-\alpha)+q(t)+r(t)\exp(\frac{\alpha}{\epsilon})]\exp(\frac{-\alpha(2-t)}{\epsilon}) \pm \ \ \mathbf{L}w_B \geq 0$, then, by Lemma 3.1, we have the desired result. Integration of (3.9) yields the estimates of $|w_B'(t)|$. From the DEs (3.9), one can derive the rest of the derivatives estimates (3.14). Using the above Theorem 3.1, the barrier function $t\in[0,1]$,

$$\Phi^{\pm}(t) = C_1\epsilon(\exp(\frac{-\alpha(1-t)}{\epsilon})) \pm w_I(t),$$

one can prove the desired result. Similarly, using the following barrier function $t\in[1,2]$:

$$\Phi^{\pm}(t) = C_1 t\epsilon \pm w_I(t),$$

and using the two theorem one can prove the remaining result. □

Note. From the above theorem,

$$|z(t)-v(t)| \leq C\begin{cases}\epsilon\exp\left(\dfrac{-\alpha(1-t)}{\epsilon}\right)+\exp\left(\dfrac{-\alpha(2-t)}{\epsilon}\right), & t\in(0,1),\\ \epsilon+\exp\left(\dfrac{-\alpha(2-t)}{\epsilon}\right), & t\in(1,2).\end{cases} \tag{3.16}$$

4. The discrete problem

4.1. Grid selection procedure

The boundary value problem (2.1) demonstrates significant boundary layers at $t=2$, prominent interior layers at $t=1$.

A piecewise uniform Shishkin mesh is in the interval $[0,1]=[0,1-\sigma]\cup[1-\sigma,1]$, where σ is the transition point.

Similarly, $[1,2]=[1,2-\sigma]\cup[2-\sigma,2]$.

The transition parameter σ for this mesh is defined by

$$\sigma = \min\{\frac{1}{2}, 2\frac{\epsilon}{\alpha}\ln M\}.$$

The mesh $\bar{\Omega}^M = \{t_0, t_1, \cdots, t_M\}$ is defined by

$$\begin{aligned} t_0 &= 0, \\ t_j &= t_0 + jH, \quad j = 1 \text{ to } \frac{M}{4}, \\ t_{j+\frac{M}{4}} &= t_{\frac{M}{4}} + jh, \quad j = 1 \text{ to } \frac{M}{4}, \\ t_{j+\frac{M}{2}} &= t_{\frac{M}{2}} + jH, \quad j = 1 \text{ to } \frac{M}{4}, \\ t_{j+\frac{3M}{4}} &= t_{\frac{3M}{4}} + jh, \quad j = 1 \text{ to } \frac{M}{4}, \end{aligned}$$

where $h = \frac{4\sigma}{M}$ and $H = \frac{4(1-\sigma)}{M}$.

4.2. A finite difference scheme

The discrete scheme corresponding to the original problem (2.1) is as follows: for j=1,2,...,$\frac{M}{2} - 1$,

$$\mathbf{L}_1^M Z_j = -\epsilon\delta^2 Z(t_j) + p(t_j)D^- Z(t_j) + q(t_j)Z(t_j) + r(t_j)z(t_j + t_{\frac{M}{2}}) = g_j. \tag{4.1}$$

For j=$\frac{M}{2} + 1$,...,$M - 1$,

$$\mathbf{L}_2^M Z_j = -\epsilon\delta^2 Z(t_j) + p(t_j)D^- Z(t_j) + q(t_j)Z(t_j) = g_j - r(t_j)\psi(t_j + t_{\frac{M}{2}}), \tag{4.2}$$

subject to the boundary condition:

$$Z(t_0) = l, \quad D^- Z_{\frac{M}{2}} = D^+ Z_{\frac{M}{2}}, \;\; Z(t_M) = \psi(2). \tag{4.3}$$

Lemma 4.1 (Discrete Maximum Principle)**.** *The mesh function Ψ_j satisfies $\Psi_0 \geq 0$, and $\Psi_M \geq 0$. Then $\mathbf{L}_1^M \Psi_j \geq 0$, $\forall$ j=1,2,...,$\frac{M}{2} - 1$, $\mathbf{L}_2^M \Psi_j \geq 0$, $\forall$ j=$\frac{M}{2} + 1$,...,$M - 1$ and $D^+(\Psi_{\frac{M}{2}}) - D^-(\Psi_{\frac{M}{2}}) \leq 0$ imply that $\Psi_j \geq 0$, $\forall j = 0, 1, 2..., M$.*

Proof. Let $j^* \in \{0, 1, \ldots, M\}$ be such that

$$\Psi(t_{j^*}) = \min_j \Psi(t_j)$$

and suppose that $\Psi(t_{j^*}) < 0$. This implies that $j^* \notin \{0, M\}$, as it is given that $\Psi(t_0) \geq 0$ and $\Psi(t_M) \geq 0$. Also, we have the following conditions:

$$\mathbf{L}_1^M \Psi(t_{j^*}) < 0, \quad \forall j^* \in \left\{1, 2, \ldots, \frac{M}{2} - 1\right\},$$

$$\mathbf{L}_2^M \Psi(t_{j^*}) < 0, \quad \forall j^* \in \left\{\frac{M}{2} + 1, \ldots, M - 1\right\},$$

$$D^+\Psi(t_{\frac{M}{2}}) - D^-\Psi(t_{\frac{M}{2}}) > 0.$$

We assume $\Psi(t_{j^*}) < 0$ and show this leads to a contradiction. Since j^* is the index where $\Psi(t_{j^*})$ attains its minimum and $\Psi(t_{j^*}) < 0$, the following case must be true:

Case 1. $j^* \in \left\{1, 2, \ldots, \frac{M}{2} - 1\right\}$.
In this case,

$$\mathbf{L}_1^M \Psi(t_{j^*}) < 0.$$

Case 2. $j^* \in \left\{\frac{M}{2} + 1, \ldots, M - 1\right\}$.
In this case,

$$\mathbf{L}_2^M \Psi(t_{j^*}) < 0.$$

However, there is an intermediate point, $t_{\frac{M}{2}}$ for which

$$D^+ \Psi(t_{\frac{M}{2}}) - D^- \Psi(t_{\frac{M}{2}}) > 0.$$

If $j^* = \frac{M}{2}$, this leads to a contradiction because, at $t_{\frac{M}{2}}$, the given condition on the derivatives implies $\Psi(t_{\frac{M}{2}})$ is increasing. Hence, $\Psi(t_{\frac{M}{2}})$ cannot be the minimum if it were less than zero.

Combining these conditions, we find a contradiction to the assumption that $\Psi(t_{j^*}) < 0$. Therefore, our initial assumption must be wrong. This implies that $\Psi(t_{j^*}) \geq 0$.

Hence,

$$\Psi(t_j) \geq 0, \quad \forall j = 1, 2, \ldots, M.$$

Thus, the function Ψ is non-negative at all mesh points. □

Lemma 4.2. *Let $\Psi(t_j)$ be any mesh function, then for $0 \leq j \leq M$, it satisfies*

$$|\Psi(t_j)| \leq C \max\{|\Psi(t_0)|, |\Psi(t_M)|, \max_{j \in \bar{\Omega}^M \setminus \{0, \frac{M}{2}, M\}} |\mathbf{L}^M \Psi(t_j)|, 0 \leq j \leq M.$$

Proof. Consider the barrier functions

$$\theta^{\pm}(t_j) = GC \pm \Psi(t_j), 0 \leq j \leq M,$$

where

$$G = \max\{|\Psi(t_0)|, |\Psi(t_M)|, \max_{j \in \bar{\Omega}^M \setminus \{0, \frac{M}{2}, M\}} |\mathbf{L}^M \Psi(t_j)|\}.$$

It is clear that $\theta^{\pm}(t_0) \geq 0$ and $\theta^{\pm}(t_M) \geq 0$,

$$\mathbf{L}_1^M \theta^{\pm}(t_j) \geq 0, \forall j \in \{1, 2, ..., \frac{M}{2} - 1\},$$
$$\mathbf{L}_2^M \theta^{\pm}(t_j) \geq 0, \forall j \in \{\frac{M}{2} + 1, ..., M - 1\},$$
$$D^+ \theta^{\pm}(t_{\frac{M}{2}}) - D^- \theta^{\pm}(t_{\frac{M}{2}}) = 0.$$

Using Lemma 4.1, $\theta^{\pm}(t_j) \geq 0, 0 \leq j \leq M$. □

5. Error calculation

To determine the estimated error for the numerical solution, we disassemble the discrete solution Z into two components: $\tilde{V}$ and $\tilde{W}$. These components are characterized as the solutions to the subsequent discrete equations. The separation of $Z(t_j)$ is detailed below:

$$Z(t_j) = \tilde{V}(t_j) + \tilde{W}(t_j),$$

wherein $\tilde{V}(t_j)$ and $\tilde{W}(t_j)$ are in compliance with the discrete differential equations as follows:

$$\mathbf{L}^M\tilde{V}(t_j) = -\epsilon\delta^2\tilde{V}(t_j) + p(t_j)D^-\tilde{V}(t_j) + q(t_j)\tilde{V}(t_j) + r(t_j)\tilde{V}(t_j + t_{\frac{M}{2}}) = g_j, j \in \bar{\Omega}^M \setminus \{0, \frac{M}{2}, M\}, \quad (5.1)$$

$$\tilde{V}(t_0) = v(0),\ [D]\tilde{V}(t_{\frac{M}{2}}) = [v'](1),\ \tilde{V}(t_M) = v(2). \quad (5.2)$$

$$\mathbf{L}^M\tilde{W}(t_j) = -\epsilon\delta^2\tilde{W}(t_j) + p(t_j)D^-\tilde{W}(t_j) + q(t_j)\tilde{W}(t_j) + r(t_j)\tilde{W}(t_j + t_{\frac{M}{2}}) = g_j, j \in \bar{\Omega}^M \setminus \{0, \frac{M}{2}, M\}, \quad (5.3)$$

$$\tilde{W}(t_0) = w(0),\ [D]\tilde{W}(t_{\frac{M}{2}}) = -[D]\tilde{V}(t_{\frac{M}{2}}),\ \tilde{W}(t_M) = w(2). \quad (5.4)$$

Theorem 5.1. *Let $Z(t_j)$ be a numerical solution (2.1) defined by (4.1)–(4.3) and $\tilde{V}(t_j)$ be a numerical solution of (3.4)–(3.6) defined by (5.1) and (5.2). Then*

$$|Z(t_j) - \tilde{V}(t_j)| \leq C \begin{cases} M^{-1}, & j = 0, 1, ..., \frac{3M}{4}, \\ M^{-1} + |l - \tilde{V}(t_M)|, & j = \frac{3M}{4} + 1, ..., M. \end{cases}$$

Proof. Consider the barrier function

$$\theta^{\pm}(t_j) = C_1M^{-1} + C_1t_j\Psi(t_j) \pm (Z(t_j) - \tilde{V}(t_j)), j = 1, 2, ..., M,$$
$$\Psi(t_j) = \begin{cases} 0, & j = 0, 1, ..., \frac{3M}{4}, \\ |l - \tilde{V}(t_M)|, & j = \frac{3M}{4} + 1, ..., M, \end{cases}$$

it is clear that $\theta^{\pm}(t_0) \geq 0$ and $\theta^{\pm}(t_M) \geq 0$.
Now, $\forall j \in \{1, 2, ..., \frac{M}{2} - 1\}$

$$\begin{aligned} \mathbf{L}_1^M\theta^{\pm}(t_j) &= C_1[q(t) + r(t)] + C_1\Psi(t_j)[p(t_j) + q(t_j)t_j + r(t_j)t_{j+\frac{M}{2}}], \\ &\geq C_1[\beta + \gamma] \quad \geq 0. \end{aligned}$$

Similarly, $\mathbf{L}_2^M\theta^{\pm}(t_j) \geq 0, \forall j \in \{\frac{M}{2} + 1, ..., M - 1\}$, and $[D]^+\theta^{\pm}(t_{\frac{M}{2}}) = \pm[v'](1) = 0$ by Lemma 4.1, Hence the theorem. □

Theorem 5.2. *Let $\tilde{V}(t_j)$ be a numerical solution of (3.4)–(3.6) defined by (5.1) and (5.2). Then*

$$|v(t_j) - \tilde{V}(t_j)| \leq CM^{-1}, \quad j \in \bar{\Omega}^M.$$

Proof. If $j = 1, 2, ..., \frac{M}{2} - 1$ and $j = \frac{M}{2} + 1, ..., M - 1$ by [21],

$$|\mathbf{L}^M(v(t_j) - \tilde{V}(t_j))| \leq CM^{-1}, j \in \bar{\Omega}^M \setminus \{0, \frac{M}{2}, M\}.$$

Then, by Lemma 4.2, we have

$$|v(t_j) - \tilde{V}(t_j)| \leq CM^{-1}, \quad j \in \bar{\Omega}^M.$$

□

Theorem 5.3. *Let $\tilde{W}(t_j)$ be a numerical solution of (3.7) and (3.8) defined by (5.3) and (5.4). Then*

$$|w(t_j) - \tilde{W}(t_j)| \leq CM^{-1}\log^2 M, \quad j \in \bar{\Omega}^M.$$

Proof. Note that

$$|w(t_j) - \tilde{W}(t_j)| \leq |z(t_j) - Z(t_j)| + |v(t_j) - \tilde{V}(t_j)|.$$

Then, by (3.16), Theorems 3.1 and 5.2, we have

$$|z(t_j) - Z(t_j)| \leq |z(t_j) - \tilde{V}(t_j)| + |v(t_j) - \tilde{V}(t_j)| + |Z(t_j) - v(t_j)|.$$

Therefore

$$\begin{aligned} |w(t_j) - \tilde{W}(t_j)| &\leq |z(t_j) - Z(t_j)| + |v(t_j) - \tilde{V}(t_j)|, \\ &\leq C_1 \exp(\frac{-\alpha(2-t)}{\epsilon}) + C_1 M^{-1}, \\ &\leq C_1 \exp(\frac{-\alpha\sigma}{\epsilon}) + C_1 M^{-1} \leq CM^{-1}, j = 0 \ to \ \frac{3M}{4}. \end{aligned} \tag{5.5}$$

Now consider a mesh function $\psi^{\pm}(t_j),\ t_j \in [2-\sigma, 2] \cap \bar{\Omega}^M$

$$\psi^{\pm}(t_j) = C_1 M^{-1} + C_1 M^{-1} \frac{\sigma}{\epsilon^2}(t_j - (2-\sigma)) \pm (w(t_j) - \tilde{W}(t_j)).$$

From (5.5), it is clear that $\psi^{\pm}(t_{\frac{3M}{4}}) \geq 0$ and $\psi^{\pm}(t_M) \geq 0$ for a suitable choice of $C_1 > 0$.

$$\begin{aligned} \mathbf{L}^M \psi^{\pm}(t_j) &= C_1 M^{-1}[q(t_j) + r(t_j)] + C_1 M^{-1} \frac{\sigma}{\epsilon^2}[p(t_j) + q(t_j)(t_j + \sigma - 2) \\ &\quad + c(t_j)(t_{j+\frac{M}{2}} + \sigma - 2)] \pm (\mathbf{L}^M - \mathbf{L}) w(t_j), \\ &\geq C_1 M^{-1}[\beta + \gamma] + C_1 M^{-1} \frac{\sigma}{\epsilon^2}\left[\alpha + \beta(t_j + \sigma - 2) + \gamma(t_{j+\frac{M}{2}} + \sigma - 2)\right], \\ &\quad \pm CM^{-1}\epsilon^{-2}, \\ &\geq 0. \end{aligned}$$

Then, by Lemma 5.1, we have $\psi^{\pm}(t_j) \geq 0,\ \ t_j \in \bar{\Omega}^M$. Therefore

$$|w(t_j) - \tilde{W}(t_j)| \leq CM^{-1} \log^2 M, \ \ j \in \bar{\Omega}^M.$$

□

Theorem 5.4. *Let $Z(t_j)$ be the numerical solution of (2.1) defined by (4.1)–(4.3) . Then*

$$|z(t_j) - Z(t_j)| \leq CM^{-1} \log^2 M, \ \ j \in \bar{\Omega}^M.$$

Proof. The described estimate is derived from the fact that $z(t_j) = v(t_j) + w(t_j)$, $Z(t_j) = \tilde{V}(t_j) + \tilde{W}(t_j)$, and from the above Theorems 5.2 and 5.3. □

6. Numerical simulations

In this section, we present a practical illustration to elucidate the computational technique previously discussed. Due to the absence of a known exact solution for the trial problem, we adopt the strategy of double meshing to gauge the error and deduce the experimental convergence rate as it approaches the solution we have calculated. To accomplish this, we define the term

$$D_{\epsilon}^{M} = \|Z_{\varepsilon}^{M} - Z_{\varepsilon}^{2M}\|.$$

Here, Z_{ε}^{M} and Z_{ε}^{2M} denote the numerical solutions' ith elements on meshes sized M and $2M$, respectively. Subsequently, we determine the uniform error and the convergence rate using the expressions

$$D^{M} = \max_{\epsilon} D_{\epsilon}^{M} \text{ and } p^{M} = \log_2\left(\frac{D^{M}}{D^{2M}}\right).$$

The computed findings for the ensuing example are reported for various values of the perturbation parameter ϵ, which range from 2^{-5} to 2^{-20}.

Example 6.1.

$$\begin{aligned} -\epsilon z''(t) + 3z'(t) + z(t) - z(t+1) &= 1, \ \ for \ \ t \in \Omega^*, \\ z(0) = 1, z(2) &= 1 \ \ for \ \ t \in [2,3]. \end{aligned}$$

Example 6.2.

$$\begin{aligned} -\epsilon z''(t) + (t+1)z'(t) + (t+10)z(t) - z(t+1) &= 1, \ \ for \ \ \Omega^*, \\ z(0) = 1, z(2) &= 1 \ \ for \ \ t \in [2,3]. \end{aligned}$$

Example 6.3.

$$\begin{aligned} -\epsilon z''(t) + 5z'(t) + z(t) - z(t+1) &= sin(t), \ \ for \ \ \Omega^*, \\ z(0) = 1, z(2) &= 1 \ \ for \ \ t \in [2,3]. \end{aligned}$$

Further, Table 1 shows the maximum pointwise error and the rate of convergence of Example 6.1. Similarly, Tables 2 and 3 also show the convergence rate and maximum pointwise error of Examples 6.2 and 6.3, respectively.

Table 1. The peak pointwise discrepancies D_{ϵ}^{M}, calculated ϵ-uniform inaccuracies D^{M}, as well as the ϵ-uniform convergence indices p^{M} for Example 6.1 are displayed.

	The mesh consists of M discrete points						
ϵ	64	128	256	512	1024	2048	4096
2^{-5}	7.7487e-04	3.9296e-04	1.9787e-04	9.9286e-05	4.9730e-05	2.4887e-05	1.2449e-05
2^{-6}	1.7662e-03	3.9758e-04	2.0019e-04	1.0045e-04	5.0314e-05	2.5179e-05	1.2595e-05
2^{-7}	3.6065e-03	1.5800e-03	6.9656e-04	1.3838e-04	5.0609e-05	2.5326e-05	1.2669e-05
2^{-8}	4.4255e-03	2.0647e-03	9.6861e-04	4.6510e-04	2.3481e-04	1.2905e-04	7.4156e-05
2^{-9}	5.0047e-03	2.4079e-03	1.1613e-03	5.6770e-04	2.8558e-04	1.5105e-04	8.6211e-05
2^{-10}	5.4142e-03	2.6507e-03	1.2978e-03	6.4037e-04	3.2156e-04	1.6666e-04	9.0893e-05
2^{-11}	5.7037e-03	2.8224e-03	1.3944e-03	6.9182e-04	3.4704e-04	1.7771e-04	9.4215e-05
2^{-12}	5.9084e-03	2.9439e-03	1.4627e-03	7.2823e-04	3.6508e-04	1.8554e-04	9.6571e-05
2^{-13}	6.0531e-03	3.0299e-03	1.5110e-03	7.5399e-04	3.7784e-04	1.9109e-04	9.8240e-05
2^{-14}	6.1555e-03	3.0906e-03	1.5452e-03	7.7221e-04	3.8687e-04	1.9501e-04	9.9422e-05
2^{-15}	6.2278e-03	3.1336e-03	1.5694e-03	7.8509e-04	3.9326e-04	1.9778e-04	1.0025e-04
2^{-16}	6.2790e-03	3.1640e-03	1.5865e-03	7.9421e-04	3.9777e-04	1.9974e-04	1.0085e-04
2^{-17}	6.3151e-03	3.1855e-03	1.5986e-03	8.0065e-04	4.0097e-04	2.0113e-04	1.0126e-04
2^{-18}	6.3407e-03	3.2007e-03	1.6071e-03	8.0521e-04	4.0323e-04	2.0211e-04	1.0156e-04
2^{-19}	6.3588e-03	3.2114e-03	1.6132e-03	8.0844e-04	4.0483e-04	2.0281e-04	1.0177e-04
2^{-20}	6.3716e-03	3.2190e-03	1.6175e-03	8.1072e-04	4.0596e-04	2.0330e-04	1.0192e-04
D^{M}	6.3716e-03	3.2190e-03	1.6175e-03	8.1072e-04	4.0596e-04	2.0330e-04	1.0192e-04
P^{M}	9.8503e-01	9.9284e-01	9.9650e-01	9.9786e-01	9.9770e-01	9.9615e-01	

Table 2. The peak pointwise discrepancies D_ϵ^M, calculated ϵ-uniform inaccuracies D^M, as well as the ϵ-uniform convergence indices p^M for Example 6.2 are displayed.

	The mesh consists of M discrete points						
ϵ	64	128	256	512	1024	2048	4096
2^{-5}	2.9341e-03	1.4748e-03	7.3956e-04	3.7034e-04	1.8532e-04	9.2710e-05	4.6392e-05
2^{-6}	2.7635e-03	1.5099e-03	7.5762e-04	3.7951e-04	1.8993e-04	9.5012e-05	4.7517e-05
2^{-7}	3.4882e-03	1.6428e-03	7.6249e-04	3.7116e-04	1.9236e-04	9.6233e-05	4.8129e-05
2^{-8}	3.8626e-03	1.8753e-03	8.9832e-04	4.2750e-04	2.0239e-04	9.5354e-05	4.4441e-05
2^{-9}	4.1205e-03	2.0375e-03	9.9516e-04	4.8293e-04	2.3354e-04	1.1263e-04	5.4179e-05
2^{-10}	4.3150e-03	2.1509e-03	1.0622e-03	5.2192e-04	2.5568e-04	1.2496e-04	6.0966e-05
2^{-11}	4.4306e-03	2.2351e-03	1.1109e-03	5.4983e-04	2.7136e-04	1.3372e-04	6.5799e-05
2^{-12}	4.5214e-03	2.2892e-03	1.1445e-03	5.6995e-04	2.8255e-04	1.3991e-04	6.9228e-05
2^{-13}	4.5932e-03	2.3276e-03	1.1708e-03	5.8393e-04	2.9048e-04	1.4432e-04	7.1659e-05
2^{-14}	4.6449e-03	2.3569e-03	1.1860e-03	5.9332e-04	2.9602e-04	1.4743e-04	7.3378e-05
2^{-15}	4.6819e-03	2.3787e-03	1.1977e-03	6.0071e-04	3.0002e-04	1.4965e-04	7.4597e-05
2^{-16}	4.7082e-03	2.3947e-03	1.2065e-03	6.0525e-04	3.0274e-04	1.5120e-04	7.5460e-05
2^{-17}	4.7270e-03	2.4061e-03	1.2129e-03	6.0868e-04	3.0480e-04	1.5232e-04	7.6071e-05
2^{-18}	4.7403e-03	2.4142e-03	1.2176e-03	6.1122e-04	3.0611e-04	1.5309e-04	7.6501e-05
2^{-19}	4.7497e-03	2.4200e-03	1.2210e-03	6.1307e-04	3.0709e-04	1.5366e-04	7.6810e-05
2^{-20}	4.7565e-03	2.4241e-03	1.2234e-03	6.1440e-04	3.0781e-04	1.5403e-04	7.7025e-05
D^M	4.7565e-03	2.4241e-03	1.2234e-03	6.1440e-04	3.0781e-04	1.5403e-04	7.7025e-05
P^M	9.7241e-01	9.8656e-01	9.9365e-01	9.9714e-01	9.9881e-01	9.9982e-01	

Table 3. The peak pointwise discrepancies D_ϵ^M, calculated ϵ-uniform inaccuracies D^M, as well as the ϵ-uniform convergence indices p^M for Example 6.3 are displayed.

	The mesh consists of M discrete points						
ϵ	64	128	256	512	1024	2048	4096
2^{-5}	1.5484e-03	7.7246e-04	3.8576e-04	1.9276e-04	9.6351e-05	4.8167e-05	2.4082e-05
2^{-6}	1.3089e-03	7.7983e-04	3.8946e-04	1.9461e-04	9.7276e-05	4.8630e-05	2.4313e-05
2^{-7}	1.2494e-03	6.0207e-04	3.0507e-04	1.8652e-04	9.7745e-05	4.8865e-05	2.4430e-05
2^{-8}	1.4689e-03	6.8861e-04	3.1979e-04	1.5342e-04	7.5776e-05	3.7442e-05	1.8578e-05
2^{-9}	1.6116e-03	7.7631e-04	3.7170e-04	1.7714e-04	8.3992e-05	3.9567e-05	1.8835e-05
2^{-10}	1.7061e-03	8.3477e-04	4.0652e-04	1.9734e-04	9.5507e-05	4.6052e-05	2.2087e-05
2^{-11}	1.7696e-03	8.7421e-04	4.3009e-04	2.1106e-04	1.0336e-04	5.0498e-05	2.4589e-05
2^{-12}	1.8129e-03	9.0113e-04	4.4620e-04	2.2045e-04	1.0874e-04	5.3554e-05	2.6316e-05
2^{-13}	1.8426e-03	9.1967e-04	4.5729e-04	2.2692e-04	1.1245e-04	5.5664e-05	2.7510e-05
2^{-14}	1.8632e-03	9.3252e-04	4.6499e-04	2.3141e-04	1.1503e-04	5.7130e-05	2.8341e-05
2^{-15}	1.8776e-03	9.4148e-04	4.7035e-04	2.3454e-04	1.1683e-04	5.8151e-05	2.8920e-05
2^{-16}	1.8877e-03	9.4775e-04	4.7411e-04	2.3673e-04	1.1809e-04	5.8866e-05	2.9325e-05
2^{-17}	1.8947e-03	9.5215e-04	4.7674e-04	2.3827e-04	1.1897e-04	5.9368e-05	2.9610e-05
2^{-18}	1.8997e-03	9.5525e-04	4.7860e-04	2.3935e-04	1.1959e-04	5.9721e-05	2.9810e-05
2^{-19}	1.9032e-03	9.5743e-04	4.7990e-04	2.4011e-04	1.2003e-04	5.9970e-05	2.9951e-05
2^{-20}	1.9056e-03	9.5897e-04	4.8083e-04	2.4065e-04	1.2033e-04	6.0145e-05	3.0050e-05
D^M	1.9056e-03	9.5894e-04	4.8083e-04	2.4065e-04	1.2033e-04	6.0145e-05	3.0050e-05
P^M	9.9074e-01	9.9596e-01	9.9855e-01	9.9986e-01	1.0005e+00	1.0010e+00	

7. Conclusions

This study has delved into the analysis of SPDEs utilizing advanced convection-diffusion techniques. By employing a finite difference method with a piecewise Shishkin-type mesh, we have demonstrated the effectiveness of our approach, achieving nearly first-order convergence. Furthermore, error estimates have been obtained using discrete norms, and numerical experiments have been conducted to corroborate the theoretical findings. Overall, our investigation highlights the efficacy of the proposed methodology for accurately solving SPDEs.

Author contributions

Nien-Tsu Hu: Funding acquisition, Formal analysis and Validation; Sekar Elango: Writing original draft, Methodology and Proof of conclusion; Chin-Sheng Chen: Writing review and Editing; Murugesan Manigandan: Validation and Writing review. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

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