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Research article

A study of $*$-Ricci–Yamabe solitons on *LP*-Kenmotsu manifolds

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Abstract: In this study, we characterize *LP*-Kenmotsu manifolds admitting $*$-Ricci–Yamabe solitons ($*$-RYSs) and gradient $*$-Ricci–Yamabe solitons (gradient $*$-RYSs). It is shown that an *LP*-Kenmotsu manifold of dimension n admitting a $*$-Ricci–Yamabe soliton obeys Poisson's equation. We also determine the necessary and sufficient conditions under which the Laplace equation is satisfied by *LP*-Kenmotsu manifolds. Finally, by using a non-trivial example of an *LP*-Kenmotsu manifold, we verify some results of our paper.

Keywords: $*$-Ricci–Yamabe solitons; *LP*-Kenmotsu manifolds; Einstein manifolds; η-Einstein manifolds; Poisson's equation
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1. Introduction

Ricci flow is a technique widely used in differential geometry, geometric topology, and geometric analysis. The Yamabe flow that deforms the metric of a Riemannian manifold M is presented by the equations [1, 2]

$$\frac{\partial}{\partial t}g(t) + r(t)g(t) = 0, \qquad g(0) = g_0, \tag{1.1}$$

where $r(t)$ is the scalar curvature of M, and t indicates the time. For a 2-dimensional manifold, (1.1) is equivalent to the Ricci flow presented by

$$\frac{\partial}{\partial t}g(t) + 2S(g(t)) = 0, \tag{1.2}$$

where S is the Ricci tensor of M. However, in cases of dimension > 2, these flows do not coincide, since the Yamabe flow preserves the conformal class of $g(t)$ but the Ricci flow does not in general. The Ricci flow has been widely used for dealing with numerous significant problems, such as deformable surface registration in vision, parameterization in graphics, cancer detection in medical imaging, and manifold spline construction in geometric modeling. For more utilization in medical and engineering fields, see [3]. It has also been used in theoretical physics, particularly; in the study of the geometry of spacetime in the context of general relativity. In this area, it has been applied to understand the behavior of black holes and the large-scale structure of the universe.

The self-similar solutions of (1.1) and (1.2) are called the Ricci and Yamabe solitons, respectively [4, 5]. They are respectively expressed by the following equations:

$$\pounds_F g + 2S + 2\Lambda g = 0, \tag{1.3}$$

and

$$\pounds_F g + 2(\Lambda - r)g = 0, \tag{1.4}$$

where $\pounds_F$ is the Lie derivative operator along the smooth vector field F on M, $\Lambda \in \mathbb{R}$.

In 2010, Blair [6] defined the concept of $*$-Ricci tensor S^* in contact metric manifolds M as:

$$S^*(X, Y) = g(Q^* X, Y) = \text{Trace}\{\varphi \circ R(X, \varphi Y)\},$$

for any vector fields X and Y on M, here Q^* is the $*$-Ricci operator, R is the curvature tensor, and φ is a $(1, 1)$ tensor field. It is to be noted that the notion of the $*$-Ricci tensor on complex manifolds was introduced by Tachibana [7]. Later, Hamada [8] studied $*$-Ricci flat real hypersurfaces of complex space forms.

If we replace S with S^* in (1.3), then we recover the expression of $*$-Ricci soliton, proposed and defined by Kaimakamis and Panagiotidou [9] as follows:

Definition 1.1. *On a Riemannian* (*or a semi-Riemannian*) *M, the metric g is called a $*$-Ricci soliton; if*

$$\pounds_F g + 2S^* + 2\Lambda g = 0 \tag{1.5}$$

holds and $\Lambda \in \mathbb{R}$.

In 2019, a modern class of geometric flows, namely, the Ricci–Yamabe (RY) flow of type (ρ, σ) was established by Güler and Crasmareanu [10]; and is defined by

$$\frac{\partial}{\partial t}g(t) + 2\rho S(g(t)) + \sigma r(t)g(t) = 0, \quad g(0) = g_0$$

for $\rho, \sigma \in \mathbb{R}$.

The RY flow can be Riemannian, semi-Riemannian, or singular Riemannian due to the involvement of the scalars ρ and σ. This kind of different choice is useful in differential geometry and physics, especially in general relativity theory (i.e., a new bimetric approach to space–time geometry [11, 12]).

A Ricci–Yamabe soliton (RYS) is a solution of RY flow; if it depends only on one parameter group of diffeomorphism and scaling. A Riemannian manifold M is said to admit an RYS if

$$\pounds_F g + 2\rho S + (2\Lambda - \sigma r)g = 0, \tag{1.6}$$

where $\rho, \sigma \in \mathbb{R}$.

If $F = gradf$, $f \in C^\infty(M)$, then the RYS is called the gradient Ricci–Yamabe soliton (gradient RYS), and then (1.6) takes the form

$$\nabla^2 f + \rho S + (\Lambda - \frac{\sigma r}{2})g = 0, \tag{1.7}$$

where $\nabla^2 f$ indicates the Hessian of f. A RYS of types $(\rho, 0)$ and $(0, \sigma)$ is respectively known as ρ-Ricci soliton and σ-Yamabe soliton.

A manifold M is said to admit a $*$-Ricci–Yamabe soliton ($*$-RYS) if

$$\pounds_F g + 2\rho S^* + (2\Lambda - \sigma r^*)g = 0 \tag{1.8}$$

holds, where $\rho, \sigma, \Lambda \in \mathbb{R}$ and r^* is the $*$-scalar curvature tensor of M.

In a similar way to (1.7), the gradient $*$-Ricci–Yamabe soliton (gradient $*$-RYS) is defined by

$$\nabla^2 f + \rho S^* + (\Lambda - \frac{\sigma r^*}{2})g = 0. \tag{1.9}$$

A $*$-RYS is said to be shrinking if $\Lambda < 0$, steady if $\Lambda = 0$, or expanding if $\Lambda > 0$. A $*$-RYS is called a
(i) $*$-Yamabe soliton if $\rho = 0, \sigma = 1$,
(ii) $*$-Ricci soliton if $\rho = 1, \sigma = 0$,
(iii) $*$-Einstein soliton if $\rho = 1, \sigma = -1$,
(iv) $*$-ρ-Einstein soliton if $\rho = 1, \sigma = -2\rho$.

Note that the $*$-Ricci–Yamabe soliton is the generalization of the aforementioned cases (i)–(iv). Thus, the research on $*$-Ricci–Yamabe soliton is more significant and promising.

On the other hand, the Lorentzian manifold, which is one of the most important subclasses of pseudo-Riemannian manifolds, plays a key role in the development of the theory of relativity and cosmology [13]. In 1989, Matsumoto [14] proposed the notion of *LP*-Sasakian manifolds, while the same notion was independently studied by Mihai and Rosca [15] in 1992, and they contributed several important results on this manifold. Later, this manifold was studied by many researchers. Recently, in 2021, Haseeb and Prasad proposed and studied *LP*-Kenmotsu manifolds [16] as a subclass of Lorentzian paracontact manifolds.

Since the turn of the 21st century, the study of Ricci solitons and their generalizations has become highly significant due to their wide uses in various fields of science, engineering, computer science, medical etc. Here we are going to mention some works on Ricci solitons and their generalizations that were carried out by several authors, such as: the geometric properties of Einstein, Ricci and Yamabe solitons were studied by Blaga [17] in 2019; Deshmukh and Chen [18] find the sufficient conditions on the soliton vector field, where the metric of a Yamabe soliton is of constant scalar curvature; Chidananda et al. [19] have studied Yamabe and Riemann solitons in *LP*-Sasakian manifolds; the study of *LP*-Kenmotsu manifolds and ϵ-Kenmotsu manifolds admitting η-Ricci solitons have been carried out by Haseeb and Almusawa [20], and Haseeb and De [21]; in [22], the authors studied conformal Ricci soliton and conformal gradient Ricci solitons on Lorentz-Sasakian space forms; RYSs have been studied by Haseeb et al. [23], Singh and Khatri [24], Suh and De [25], Yoldas [26], Zhang et al. [27]. The study of Ricci solitons and their generalizations has been extended to $*$-Ricci solitons and their generalizations on various manifolds and has been explored by the authors: Dey [28], Dey et al. [29], Ghosh and Patra [30], Haseeb and Chaubey [31], Haseeb et al. [32], and Venkatesha et al. [33]. Recently, Azami et al. [34] investigated perfect fluid spacetimes and perfect fluid generalized Roberston–Walker spacetimes.

2. Preliminaries

A differentiable manifold M ($\dim M = n$) with the structure $(\varphi, \zeta, \eta, g)$ is named a Lorentzian almost paracontact metric manifold; in case φ: a $(1,1)$-tensor field, ζ: a contravariant vector field, η: a 1-form, and g: a Lorentzian metric g satisfy [13]

$$\eta(\xi) = -1, \tag{2.1}$$

$$\varphi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{2.3}$$

$$g(\varphi\cdot, \varphi\cdot) = g(\cdot, \cdot) + \eta(\cdot)\eta(\cdot), \tag{2.4}$$

$$g(\cdot, \xi) = \eta(\cdot). \tag{2.5}$$

We define the 2-form Φ on M as

$$\Phi(X, Y) = \Phi(Y, X) = g(X, \varphi Y), \tag{2.6}$$

for any $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on M. If

$$d\eta(X, Y) = \Phi(Y, X), \tag{2.7}$$

here d is an exterior derivative, then $(M, \varphi, \xi, \eta, g)$ is called a paracontact metric manifold.

Definition 2.1. *A Lorentzian almost paracontact manifold M is called a Lorentzian para-Kenmotsu* (*in brief, LP-Kenmotsu*) *manifold if*

$$(\nabla_X\varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X, \tag{2.8}$$

for any X and Y on M [16, 23, 35].

In the case of an LP-Kenmotsu manifold, we have

$$\nabla_X\xi = -X - \eta(X)\xi, \tag{2.9}$$

$$(\nabla_X\eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.10}$$

where ∇ indicates the Levi–Civita connection with respect to g.

Furthermore, in an LP-Kenmotsu manifold of dimension n (in brief, $(LP\text{-}K)_n$), the following relations hold [16]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.14}$$

$$S(X, \xi) = (n-1)\eta(X), \quad S(\xi, \xi) = -(n-1), \tag{2.15}$$

$$Q\xi = (n-1)\xi, \tag{2.16}$$

for any $X, Y, Z \in \chi(M)$.

Lemma 2.1. *[36] In an $(LP\text{-}K)_n$, we have*

$$(\nabla_X Q)\xi = QX - (n-1)X, \tag{2.17}$$

$$(\nabla_\xi Q)X = 2QX - 2(n-1)X, \tag{2.18}$$

for any X on $(LP\text{-}K)_n$.

Lemma 2.2. *[37] In an $(LP\text{-}K)_n$, we have*

$$S^*(Y, Z) = S(Y, Z) - ng(Y, Z) - \eta(Y)\eta(Z) + ag(Y, \varphi Z), \tag{2.19}$$

$$r^* = r - n^2 + 1 + a^2, \tag{2.20}$$

for any Y, Z on $(LP\text{-}K)_n$, and a is the trace of φ.

Lemma 2.3. *In an $(LP\text{-}K)_n$, the eigenvalue of the $*$-Ricci operator Q^* corresponding to the eigenvector ξ is zero, i.e., $Q^*\xi = 0$.*

Proof. From (2.19), we have

$$Q^*Y = QY - nY - \eta(Y)\xi + a\varphi Y, \tag{2.21}$$

which, by putting $Y = \xi$; and using (2.1), (2.3), and (2.16) gives $Q^*\xi = 0$.

Lemma 2.4. *The $*$-Ricci operator Q^* in an $(LP\text{-}K)_n$ satisfies the following identities:*

$$(\nabla_X Q^*)\xi = QX - nX - \eta(X)\xi + a\varphi X, \tag{2.22}$$

$$(\nabla_\xi Q^*)X = 2QX - 2(n-1)X + \xi(a)\varphi X, \tag{2.23}$$

for any X on $(LP\text{-}K)_n$.

Proof. By the covariant differentiation of $Q^*\xi = 0$ with respect to X and using (2.9), (2.21) and $Q^*\xi = 0$, we obtain (2.22). Next, differentiating (2.21) covariantly with respect to ξ and using (2.8)–(2.10), (2.18), and (2.21), we obtain (2.23).

Lemma 2.5. *In an $(LP\text{-}K)_n$, we have [23]*

$$\xi(r) = 2(r - n(n-1)), \tag{2.24}$$

$$X(r) = -2(r - n(n-1))\eta(X), \tag{2.25}$$

$$\eta(\nabla_\xi Dr) = 4(r - n(n-1)), \tag{2.26}$$

for any X on M, and Dr stands for the gradient of r.

Remark 2.1. *From the equation (2.24), it is observed that if r of an $(LP\text{-}K)_n$ is constant, then $r = n(n-1)$.*

3. $*$-RYS on $(LP\text{-}K)_n$

In this section, we first prove the following result:

Theorem 3.1. *In an $(LP\text{-}K)_n$ admitting a $*$-RYS, the scalar curvature r of the manifold satisfies the Poisson's equation $\Delta r = \Psi$, where $\Psi = \frac{4(n-1)\Lambda}{\sigma} + 2r(n-3) + (n-1)\{h - 2a^2 - 2(n^2 - 4n + 1)\}$, $\sigma \neq 0$.*

Proof. Let the metric of an $(LP\text{-}K)_n$ be a $*$-RYS, then in view of (2.19), (1.8) takes the form

$$(\pounds_F g)(X, Y) = -2\rho S(X, Y) + 2\{\rho n - \Lambda + \frac{\sigma r^*}{2}\}g(X, Y) + 2\rho\eta(X)\eta(Y) - 2a\rho g(X, \varphi Y), \tag{3.1}$$

for any X, Y on M.
Taking the covariant derivative of (3.1) with respect to Z, we find

$$\begin{aligned}(\nabla_Z \pounds_F g)(X, Y) &= -2\rho(\nabla_Z S)(X, Y) + \sigma(Zr^*)g(X, Y) - 2\rho\{g(X, Z)\eta(Y) + g(Y, Z)\eta(X)\} \\ &+ 2a\rho\{g(\varphi Z, Y)\eta(Y) + g(\varphi Z, X)\eta(Y)\} - 4\rho\eta(X)\eta(Y)\eta(Z).\end{aligned} \tag{3.2}$$

As g is parallel with respect to ∇, then the following formula [38]

$$(\pounds_F \nabla_X g - \nabla_X \pounds_F g - \nabla_{[F,X]} g)(Y, Z) = -g((\pounds_F \nabla)(X, Y), Z) - g((\pounds_F \nabla)(X, Z), Y)$$

turns to

$$(\nabla_X \pounds_F g)(Y, Z) = g((\pounds_F \nabla)(X, Y), Z) + g((\pounds_F \nabla)(X, Z), Y).$$

Due to the symmetric property of $\pounds_V \nabla$, we have

$$2g((\pounds_F \nabla)(X, Y), Z) = (\nabla_X \pounds_F g)(Y, Z) + (\nabla_Y \pounds_F g)(X, Z) - (\nabla_Z \pounds_F g)(X, Y),$$

which, by using (3.2), becomes

$$\begin{aligned}g((\pounds_F \nabla)(X, Y), Z) &= \rho\{(\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)\} \\ &+\frac{\sigma}{2}\{(Xr^*)g(Y, Z) + (Yr^*)g(X, Z) - (Zr^*)g(X, Y)\} \\ &-2\rho\{g(X, Y) + \eta(X)\eta(Y)\}\eta(Z) + 2a\rho g(\varphi X, Y)\eta(Z).\end{aligned}$$

By putting $Y = \xi$ and using (2.3), (2.1), (2.5), (2.17), and (2.18), the preceding equation gives

$$(\pounds_F \nabla)(Y, \xi) = -2\rho\{QY - (n-1)Y\} + \frac{\sigma}{2}\{(Yr^*)\xi + (\xi r^*)Y - (Dr^*)\eta(Y)\}. \tag{3.3}$$

By using the relation (2.20) in (3.3), we have

$$\begin{aligned}(\pounds_F \nabla)(Y, \xi) &= -2\rho\{QY - (n-1)Y\} + \frac{\sigma}{2}\{g(Dr, Y)\xi + 2(r - n(n-1))Y - (Dr)\eta(Y)\} \\ &+\frac{\sigma}{2}\{(Ya^2)\xi + (\xi a^2)Y - (Da^2)\eta(Y)\}.\end{aligned} \tag{3.4}$$

The covariant derivative of (3.4) with respect to X leads to

$$(\nabla_X \pounds_F \nabla)(Y, \xi) = (\pounds_F \nabla)(Y, X) - 2\rho\{QY - (n-1)Y\}\eta(X) - 2\rho(\nabla_X Q)Y$$

$$
\begin{aligned}
&+\sigma(r-n(n-1))\{\eta(Y)X-\eta(X)Y\}+\frac{\sigma}{2}(\xi a^2)\eta(X)Y \\
&+\frac{\sigma}{2}g(\nabla_X Dr, Y)\xi-\frac{\sigma}{2}(\nabla_X Dr)\eta(Y)+\frac{\sigma}{2}(Dr)g(X,Y) \\
&+\frac{\sigma}{2}\{-(Ya^2)X+\nabla_X(\xi a^2)Y-\nabla_X(Da^2)\eta(Y)+(Da^2)g(X,Y)\},
\end{aligned}
\tag{3.5}
$$

where (2.1), (2.4), and (3.4) are used.
Again, in [38], we have

$$
(\pounds_F R)(X,Y)Z=(\nabla_X \pounds_F \nabla)(Y,Z)-(\nabla_Y \pounds_F \nabla)(X,Z),
$$

which, by setting $Z=\xi$ and using (3.5), becomes

$$
\begin{aligned}
(\pounds_F R)(X,Y)\xi &= 2\rho\{\eta(Y)QX-(n-1)\eta(Y)X-\eta(X)QY+(n-1)\eta(X)Y\} \\
&-2\rho\{(\nabla_X Q)Y-(\nabla_Y Q)X\}+\frac{\sigma}{2}(\xi a^2)\{\eta(X)Y-\eta(Y)X\} \\
&+\frac{\sigma}{2}\{g(\nabla_X Dr, Y)\xi-g(\nabla_Y Dr, X)\xi-(\nabla_X Dr)\eta(Y)+(\nabla_Y Dr)\eta(X)\} \\
&+2\sigma(r-n(n-1))\{\eta(Y)X-\eta(X)Y\} \\
&+\frac{\sigma}{2}\{-(Ya^2)X+\nabla_X(\xi a^2)Y-\nabla_X(Da^2)\eta(Y) \\
&+(Xa^2)Y-\nabla_Y(\xi a^2)X+\nabla_Y(Da^2)\eta(X)\}.
\end{aligned}
\tag{3.6}
$$

Now, by putting $Y=\xi$ in (3.6) and using (2.5), (2.1), (2.17), and (2.18), we have

$$
\begin{aligned}
(\pounds_F R)(X,\xi)\xi &= \frac{\sigma}{2}\{(\xi a^2)-4(r-n(n-1))\}(X+\eta(X)\xi) \\
&+ \frac{\sigma}{2}\{\eta(\nabla_X Dr)\xi-g(\nabla_\xi Dr, X)\xi+(\nabla_X Dr)+(\nabla_\xi Dr)\eta(X)\} \\
&+ \frac{\sigma}{2}\{-(\xi a^2)X+\nabla_X(\xi a^2)\xi+\nabla_X(Da^2)+(Xa^2)\xi-\nabla_\xi(\xi a^2)X+\nabla_\xi(Da^2)\eta(X)\}.
\end{aligned}
\tag{3.7}
$$

For the h constant, we assume that $Da^2=h\xi$, and hence we deduce the following values:

$$
(i)\ \xi a^2=-h,\ (ii)\ \nabla_X(Da^2)=-hX-h\eta(X)\xi,\ (iii)\ X(a^2)=h\eta(X). \tag{3.8}
$$

In light of (3.8), (3.7) reduces to

$$
\begin{aligned}
(\pounds_F R)(X,\xi)\xi &= -\frac{\sigma}{2}\{h+4(r-n(n-1))\}(X+\eta(X)\xi) \\
&+\frac{\sigma}{2}\{\eta(\nabla_X Dr)\xi-g(\nabla_\xi Dr, X)\xi+(\nabla_X Dr)+(\nabla_\xi Dr)\eta(X)\}.
\end{aligned}
\tag{3.9}
$$

By contracting (3.9) over X, we lead to

$$
(\pounds_F S)(\xi,\xi) = -\frac{h\sigma(n-1)}{2}-2\sigma(n-2)(r-n(n-1))+\frac{\sigma}{2}\Delta r, \tag{3.10}
$$

where (2.26) is used and Δ symbolizes the Laplacian operator of g.
The Lie derivative of (2.15) along F gives

$$
(\pounds_F S)(\xi,\xi)=-2(n-1)\eta(\pounds_F \xi). \tag{3.11}
$$

By setting $X = Y = \xi$ in (3.1) and using (2.3), (2.1), (2.5), and (2.16), we have

$$(\pounds_F g)(\xi, \xi) = 2\Lambda - \sigma r^*. \tag{3.12}$$

The Lie derivative of $1 + g(\xi, \xi) = 0$ gives

$$(\pounds_F g)(\xi, \xi) = -2\eta(\pounds_F \xi). \tag{3.13}$$

Now, combining (3.10)–(3.13), we deduce

$$\Delta r = \Psi, \tag{3.14}$$

where $\Psi = \frac{4(n-1)\Lambda}{\sigma} + 2r(n-3) + (n-1)\{h - 2a^2 - 2(n^2 - 4n + 1)\},\ \sigma \neq 0.$

For the smooth functions θ and Ψ, an $(LP\text{-}K)_n$ satisfies Poisson's equation if $\theta = \Psi$ holds. In the case; where $\theta = 0$, Poisson's equation transforms into Laplace's equation. This completes the proof of our theorem.

A function $\upsilon \in C^\infty(M)$ is said to be subharmonic if $\Delta\upsilon \geq 0$, harmonic if $\Delta\upsilon = 0$, and superharmonic if $\Delta\upsilon \leq 0$. Thus, from (3.14), we state the following corollaries:

Corollary 3.1. *In an $(LP\text{-}K)_n$ admitting a $*$-RYS, we have*

The values of scalar curvature (r)	*Harmonicity cases*
$\geq \frac{(n-1)}{(n-3)}\{a^2 + (n^2 - 4n + 1) - \frac{h}{2} - \frac{2\Lambda}{\sigma}\}$	*subharmonic*
$= \frac{(n-1)}{(n-3)}\{a^2 + (n^2 - 4n + 1) - \frac{h}{2} - \frac{2\Lambda}{\sigma}\}$	*harmonic*
$\leq \frac{(n-1)}{(n-3)}\{a^2 + (n^2 - 4n + 1) - \frac{h}{2} - \frac{2\Lambda}{\sigma}\}$	*superharmonic*

Corollary 3.2. *In an $(LP\text{-}K)_n$ admitting a $*$-RYS, the scalar curvature r of the manifold satisfies the Laplace equation if and only if*

$$r = -\frac{2(n-1)\Lambda}{(n-3)\sigma} - \frac{(n-1)}{(n-3)}\{\frac{h}{2} - a^2 - (n^2 - 4n + 1)\}, \quad \sigma \neq 0. \tag{3.15}$$

Let an $(LP\text{-}K)_n$ admit a $*$-RYS, and if the scalar curvature r of the manifold satisfies Laplace's equation, then (3.15) holds. If this value of r is constant, then by virtue of Remark 2.1, we find $\Lambda = -\frac{\sigma}{2}(n + \frac{h}{2} - a^2 - 1)$. Thus, we have:

Corollary 3.3. *In an $(LP\text{-}K)_n$ admitting a $*$-RYS, we have*

Condition	*Values of* σ	*Values of* Λ	*Conditions for the $*$-RYS to be shrinking, steady, or expanding*
	(i) $\sigma > 0$	(i) $\Lambda < 0$	(i) *shrinking*
$n-1+\frac{h}{2} > a^2$	(ii) $\sigma = 0$	(ii) $\Lambda = 0$	(ii) *steady*
	(iii) $\sigma < 0$	(iii) $\Lambda > 0$	(iii) *expanding*
$n-1+\frac{h}{2} = a^2$	$\sigma > 0, = 0$ *or* < 0	$\Lambda = 0$	*steady*
	(i) $\sigma > 0$	(i) $\Lambda > 0$	(i) *shrinking*
$n-1+\frac{h}{2} < a^2$	(ii) $\sigma = 0$	(ii) $\Lambda = 0$	(ii) *steady*
	(iii) $\sigma < 0$	(iii) $\Lambda < 0$	(iii) *expanding*

4. Gradient $*$-RYS on $(LP\text{-}K)_n$

This section explores the properties of gradient $*$-RYS on $(LP\text{-}K)_n$.

Let M be an $(LP\text{-}K)_n$ with g as a gradient $*$-RYS. Then (1.9) can be written as

$$\nabla_X Df + \rho Q^* X + \left(\Lambda - \frac{\sigma r^*}{2}\right) X = 0, \tag{4.1}$$

for all X on $(LP\text{-}K)_n$, where D indicates the gradient operator of g.
The covariant differentiation of (4.1) along Y leads to

$$\nabla_Y \nabla_X Df = -\rho\{(\nabla_Y Q^*)X + Q^*(\nabla_Y X)\} + \sigma \frac{Y(r^*)}{2} X - \left(\Lambda - \frac{\sigma r^*}{2}\right) \nabla_Y X. \tag{4.2}$$

By virtue of (4.2) and the curvature identity $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we find

$$R(X,Y)Df = \rho\{(\nabla_Y Q^*)X - (\nabla_X Q^*)Y\} + \frac{\sigma}{2}\{X(r^*)Y - Y(r^*)X\}. \tag{4.3}$$

By contracting (4.3) along X, we have

$$S(Y, Df) = \{\frac{\rho - (n-1)\sigma}{2}\} Y(r^*). \tag{4.4}$$

From (2.15), we have

$$S(\xi, Df) = (n-1) g(\xi, Df). \tag{4.5}$$

Thus, from (4.4) and (4.5), it follows that

$$g(\xi, Df) = \frac{\rho - (n-1)\sigma}{2(n-1)} g(\xi, Dr^*). \tag{4.6}$$

Now, we divide our study into two cases, as follows:

Case I. Let $\rho = (n-1)\sigma$. For this case, we prove the following theorem:

Theorem 4.1. *Let an $(LP\text{-}K)_n$ admit a gradient $*$-RYS and $\rho = (n-1)\sigma$. Then, $(LP\text{-}K)_n$ possesses a constant scalar curvature r^*.*

Proof. Let $\rho = (n-1)\sigma$. Then, from (4.6), we obtain

$$g(\xi, Df) = 0. \tag{4.7}$$

The covariant derivative of (4.7) along X gives

$$X(f) + \left(\Lambda - \frac{\sigma r^*}{2}\right) \eta(X) = 0, \tag{4.8}$$

where (2.9), (2.19), (4.1), and (4.7) are used.
By setting $X = \xi$ in (4.8), and then using (4.7) and (2.1), we infer that

$$\Lambda = \frac{\sigma r^*}{2}, \tag{4.9}$$

which informs us that r^* is constant. This completes the proof.

Corollary 4.1. *Let an $(LP\text{-}K)_n$ admit a gradient $*$-RYS and $\rho = (n-1)\sigma$. Then, we have:*

Values of σ	*Values of r^**	*Values of Λ*	*Conditions for the $*$-RYS to be shrinking, steady, or expanding*
	(i) $r^* > 0$	(i) $\Lambda > 0$	(i) *expanding*
$\sigma > 0$	(ii) $r^* = 0$	(ii) $\Lambda = 0$	(ii) *steady*
	(iii) $r^* < 0$	(iii) $\Lambda < 0$	(iii) *shrinking*
$\sigma = 0$	$r^* > 0, = 0$ *or* < 0	$\Lambda = 0$	*steady*
	(i) $r^* > 0$	(i) $\Lambda < 0$	(i) *shrinking*
$\sigma < 0$	(ii) $r^* = 0$	(ii) $\Lambda = 0$	(ii) *steady*
	(iii) $r^* < 0$	(iii) $\Lambda > 0$	(iii) *expanding*

Now, from (4.8) and (4.9), we conclude that

$$X(f) = 0. \tag{4.10}$$

This indicates that the gradient function f of the gradient $*$-RYS is constant on $(LP\text{-}K)_n$. Thus, we have

Corollary 4.2. *Let an $(LP\text{-}K)_n$ admit a gradient $*$-RYS and $\rho = (n-1)\sigma$. Then the function f is constant, and hence the gradient $*$-RYS is trivial. Moreover, $(LP\text{-}K)_n$ is an Einstein manifold.*

Case II. Let $\rho \neq (n-1)\sigma$. For this case, we prove the following theorem:

Theorem 4.2. *Let an $(LP\text{-}K)_n$ admit a gradient $*$-RYS and $\rho \neq (n-1)\sigma$. Then the gradient of the potential function f is pointwise collinear with ξ.*

Proof. Replacing $X = \xi$ in (4.3), then using (2.12) and Lemma 2.5, we have

$$R(\xi, Y)Df = \rho\{-QY + (n-2)Y - \eta(Y)\xi + a\varphi Y - \xi(a)\varphi Y\} + \frac{\sigma}{2}\{\xi(r^*)Y - Y(r^*)\xi\}.$$

Also from (2.12), we have

$$R(\xi, Y)Df = g(Y, Df)\xi - \eta(Df)Y = Y(f)\xi - \xi(f)Y.$$

By equating the last two equations, we have

$$\begin{aligned} Y(f)\xi - \xi(f)Y \quad &= \quad \rho\{-QY + (n-2)Y - \eta(Y)\xi + a\varphi Y - \xi(a)\varphi Y\} \\ &\quad + \frac{\sigma}{2}\{\xi(r^*)Y - Y(r^*)\xi\}. \end{aligned} \tag{4.11}$$

By contracting Y in (4.11), we have

$$\xi(f) = \frac{\rho}{n-1}\{r - n(n-2) - 1 - a(a - \xi(a))\} - \frac{\sigma}{2}\xi(r^*). \tag{4.12}$$

Now, from (2.20), (2.24), (2.25), and (3.8), we easily find

$$X(r^*) = -2(r - n(n-1))\eta(X) + X(a^2). \tag{4.13}$$

This implies

$$\xi(r^*) = -h + 2(r - n(n-1)), \tag{4.14}$$

where $g(\xi, Da^2) = -h$.
By using (4.14) in (4.12), we obtain

$$\xi(f) = \frac{\rho}{n-1}\{r - n(n-2) - 1 - a(a - \xi(a))\} - \sigma(r - n(n-1)) - \frac{h\sigma}{2}. \tag{4.15}$$

From (4.11) and (4.15), it follows that

$$\begin{aligned} Y(f)\xi &= \frac{\rho}{n-1}\{r - n(n-2) - 1 - a(a - \xi(a))\}Y \\ &\quad -\rho\{QY - (n-2)Y + \eta(Y)\xi - a\varphi Y + \xi(a)\varphi Y\} \\ &\quad +\sigma(r - n(n-1))\eta(Y)\xi - \frac{\sigma}{2}Y(a^2)\xi. \end{aligned} \tag{4.16}$$

The inner product of (4.16) with ξ and using (2.2), (2.1), (2.5), and (2.15) gives

$$Y(f) = -\frac{\rho}{n-1}\{r - n(n-2) - 1 - a(a - \xi(a))\}\eta(Y) + \sigma(r - n(n-1))\eta(Y) - \frac{h\sigma}{2}\eta(Y), \tag{4.17}$$

where we assumed $Da^2 = h\xi$.
From (4.17), we conclude that $Df = k\xi$, where k is a smooth function given by

$$k = -\frac{\rho}{n-1}\{r - n(n-2) - 1 - a(a - \xi(a))\} + \sigma(r - n(n-1)) - \frac{h\sigma}{2}.$$

This completes the proof.

5. Example

We consider the 3-dimensional manifold $M = \{(u_1, u_2, u_3) \in \mathbb{R}^3, u_3 > 0\}$, where (u_1, u_2, u_3) are the standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, and e_3 be the vector fields on $\mathcal{M}$ given by

$$e_1 = e^{u_3}\frac{\partial}{\partial u_1}, \quad e_2 = e^{u_3}\frac{\partial}{\partial u_2}, \quad e_3 = e^{u_3}\frac{\partial}{\partial u_3} = \xi,$$

which are linearly independent at each point of M.

Define a Lorentzian metric g on M such that

$$g(e_1, e_1) = g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1.$$

Let η be the 1-form on M defined by $\eta(X) = g(X, e_3) = g(X, \xi)$ for all X on M; and let φ be the (1,1)-tensor field on M is defined as

$$\varphi e_1 = -e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0.$$

By applying the linearity of φ and g, we have

$$\begin{cases} \eta(\xi) = g(\xi, \xi) = -1, \quad \varphi^2 X = X + \eta(X)\xi, \quad \eta(\varphi X) = 0, \\ g(X, \xi) = \eta(X), \quad g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \end{cases} \tag{5.1}$$

for all X, Y on M .

Let ∇ be the Levi–Civita connection with respect to the Lorentzian metric g. Thus, we have

$$[e_1, e_2] = [e_2, e_1] = 0, \quad [e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2.$$

With the help of Koszul's formula, we easily calculate

$$\nabla_{e_i} e_j = \begin{cases} -e_3, & 1 \leq i = j \leq 2, \\ -e_i, & 1 \leq i \leq 2, \ j = 3, \\ 0, & otherwise. \end{cases} \tag{5.2}$$

One can also easily verify that

$$\nabla_X \xi = -X - \eta(X)\xi \quad \text{and} \quad (\nabla_X \varphi)Y = -g(\varphi X, Y)\xi - \eta(Y)\varphi X.$$

Hence, M is a Lorentzian para-Kenmotsu manifold of dimension 3. By using (5.2), we obtain

$$R(e_1, e_2)e_1 = -e_2, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_2, e_3)e_1 = 0,$$

$$R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_2 = 0, \quad R(e_2, e_3)e_2 = e_3,$$

$$R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_3)e_3 = e_2.$$

Now, with the help of the above components of the curvature tensor, it follows that

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{5.3}$$

Thus, the manifold is of constant curvature.

From (5.3), we get $S(Y, Z) = 2g(Y, Z)$. This implies that

$$QY = 2Y. \tag{5.4}$$

Now, by putting $Y = \xi$ in (2.21), then using (5.1) and (5.4), we obtain $Q^*\xi = 0$. This proves Lemma 2.3. Next, by contracting $S(Y, Z) = 2g(Y, Z)$, we find $r = 6$. Since r is constant, therefore, (2.24) leads to $\xi(r) = 0 \implies r = 6$, where $n = 3$. This proves Remark 2.1.

6. Conclusions

In the present study, we obtain certain important results on $(LP\text{-}K)_n$ admitting a $*$-RYS and a gradient $*$-RYS. First, we prove that the scalar curvature r of $(LP\text{-}K)_n$ admitting a $*$-RYS satisfies Poission's equation, and we discuss the conditions for the $*$-RYS to be shrinking, steady, and expanding. Furthermore, we deal with the study of gradient $*$-RYS on $(LP\text{-}K)_n$ in two cases: (i) $\rho = (n-1)\sigma$; in this case, we showed that the gradient function f of the gradient $*$-RYS is constant, and hence the gradient $*$-RYS is trivial. Moreover, $(LP\text{-}K)_n$ is an Einstein manifold; (ii) $\rho \neq (n-1)\sigma$, in this case, we proved that if an $(LP\text{-}K)_n$ admits a gradient $*$-RYS, then the gradient of the potential function f is pointwise collinear with ξ.

Author contributions

Abdul Haseeb: Conceptualization, investigation, methodology, writing–original draft; Fatemah Mofarreh: Investigation, methodology, writing–review & editing; Sudhakar Kumar Chaubey: Conceptualization, methodology, writing–review & editing; Rajendra Prasad: Conceptualization, investigation, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

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The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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