AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 22500–22512.
DOI: 10.3934/math.20241094
Received: 23 May 2024
Revised: 05 July 2024
Accepted: 11 July 2024
Published: 19 July 2024

Research article

On stability of non-surjective (ε, s)-isometries of uniformly convex Banach spaces

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Abstract: In this paper, we established two results concerning non-surjective (ε, s)-isometries of uniformly convex Banach spaces, which extended some known results of Dolinar and Jung.

Keywords: stability; (ε, s)-isometry; linear isometry; uniformly convex space
Mathematics Subject Classification: 46B04, 46B20

1. Introduction

We first recall definitions of (ε, s)-isometry, ε-isometry, and isometry.

Definition 1.1. Let $(X, d_X), (Y, d_Y)$ be two metric spaces, $f : X \to Y$ be a mapping, and $\varepsilon, s \geq 0$. We say that f is an (ε, s)-isometry if

$$|d_Y(f(x), f(y)) - d_X(x, y)| \leq \varepsilon d_X(x, y)^s \quad \text{for all } \ x, y \in X;$$

f is said to be an ε-isometry if

$$|d_Y(f(x), f(y)) - d_X(x, y)| \leq \varepsilon \quad \text{for all } \ x, y \in X; \tag{1.1}$$

f is called an isometry if $\varepsilon = 0$ in (1.1).

The Mazur-Ulam theorem [1] (1932) is well-known for stating that every surjective isometry between two real normed spaces must be affine. This theorem provides the profound insight that a mapping f from a real normed space to another real normed space, which preserves distances and satisfies $f(0) = 0$, must be a linear isometry. Research on isometries and their extensions has been ongoing for more than 90 years since then. For more information on non-surjective isometries, please refer to the following references: Figiel [2], Godefroy and Kalton [3], Dutrieux and Lancien [4], and Cheng and Zhou [5].

In 1945, Hyers and Ulam [6] introduced the concept of ε-isometry. They raised a question: Given two Banach spaces E_1, E_2 and a positive constant β, can we find a surjective linear isometry $U : E_1 \to E_2$ that corresponds to every surjective ε-isometry $f : E_1 \to E_2$ with $f(0) = 0$, such that $\|f(x) - Ux\|$ is less than $\beta\varepsilon$? After extensive research by numerous mathematicians over fifty years (see, for instance, [7–9]), Omladič and Šemrl [10] eventually provided an affirmative answer to this question with the sharp estimate of $\beta = 2$. Inspired by Figiel theorem [2] and other important results on non-surjective ε-isometries (see [11,12]), Cheng, Dong, and Zhang [13] discovered a remarkable result known as the weak stability formula. It has received considerable attention from numerous researchers (see [5, 14–18]).

In 2000, Dolinar [19] investigated the stability of non-surjective (ε, s)-isometries of L_p spaces and Hilbert spaces when $0 \le s < 1$, and obtained the superstability of surjective (ε, s)-isometries between finite dimensional spaces when $s > 1$.

On the other hand, Cădariu and Radu [20] (2003) investigated the stability of Jensen's functional equation by using the fixed point approach; for related literature, see [21, 22]. Further, Jung [23] gave an interesting stability result for non-surjective perturbed isometries from a normed space to a Banach space in which the range space satisfies the parallelogram law. As an application, he obtained the stability results of non-surjective (ε, s)-isometries($s \neq 1$) of Hilbert spaces.

In this paper, we use a fixed point theorem to extend some results of (ε, s)-isometries, established by Dolinar [19] and Jung, from Hilbert spaces or L_p spaces to uniformly convex spaces with power type p. More precisely, we prove that if $f : E \to F$ is an (ε, s)-isometry with $f(0) = 0$, where E is a real normed space, F is a real uniformly convex space with power type p, and $s \neq 1$, then there exist a linear isometry $U : E \to F$ and a constant $M(\varepsilon, s, p) \ge 0$ with $\lim\limits_{\varepsilon\to 0} M(\varepsilon, s, p) = 0$ such that

$$\|f(u) - Uu\| \le M(\varepsilon, s, p) \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \quad \text{for all } u \in E.$$

2. Preliminaries

The modulus of convexity of a Banach space F is the function $\delta_F : [0, 2] \to [0, 1]$ defined by

$$\delta_F(\varepsilon) = \inf\{1 - \|\frac{u + v}{2}\| : \|u\| = \|v\| = 1, \|u - v\| \ge \varepsilon\}.$$

Definition 2.1. A Banach space F is called uniformly convex if $\delta_F(\varepsilon) > 0$ for all $0 < \varepsilon \le 2$. Let $p \ge 2$, and we say that a uniformly convex Banach space F has power type p if there is a constant $\alpha > 0$ so that $\delta_F(\varepsilon) \ge \alpha\varepsilon^p$ for all $0 < \varepsilon \le 2$.

Remark 2.2. Pisier [24] showed that every uniformly convex Banach spaces can be renormed to admit power type p for some $2 \le p < +\infty$.

The main result in this section is based on the following inequality (2.1); we refer to [24].

Lemma 2.3. *(Pisier) A Banach space F has power type p if, and only if, there is a constant $\alpha \ge 1$ so that*

$$\left\|\frac{w_1 - w_2}{2}\right\| \le \alpha\Big(\frac{\|w_1\|^p + \|w_2\|^p}{2} - \left\|\frac{w_1 + w_2}{2}\right\|^p\Big)^{\frac{1}{p}} \quad \textit{for all } w_1, w_2 \in F. \tag{2.1}$$

We now recall a fixed point result in generalized metric space, which is essential in this article; one can refer to [25].

Definition 2.4. Let X be a nonempty set. A function $\varrho : X \times X \to [0, +\infty]$ is said to be a generalized metric on X if ϱ satisfies the following statements:
(1) $\varrho(u, v) = 0$ if, and only if, $u = v$;
(2) $\varrho(u, v) = \varrho(v, u)$ for all $u, v \in X$;
(3) $\varrho(u, w) \leq \varrho(u, v) + \varrho(v, w)$ for all $u, v, w \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

Definition 2.5. Let (X, ϱ) be a generalized metric space. We say that $T : X \to X$ is a strictly contractive operator with the constant λ if there is a constant λ such that $0 \leq \lambda < 1$ and

$$\varrho(Tu, Tv) \leq \lambda\varrho(u, v) \quad \text{for all } \ u, v \in X.$$

Lemma 2.6. *(Margolis-Diaz [25]) Let (X, ϱ) be a generalized complete metric space and let $T : X \to X$ be a strictly contractive operator with the constant λ. If there exists a nonnegative integer n so that $\varrho(T^{n+1}u, T^n u) < +\infty$ for some $u \in X$, then the following statements hold:*
(i) $\{T^n u\}$ converges to a fixed point u^* of T;
(ii) u^* is the unique fixed point of T in

$$Y = \{v \in X : \varrho(T^n u, v) < +\infty\};$$

(iii) If $v \in Y$, then

$$\varrho(v, u^*) \leq \frac{1}{1-\lambda}\varrho(Tv, v).$$

3. Stability of (ε, s)-isometries for $0 \leq s < 1$

We start with the following theorem:

Theorem 3.1. *Let E be a real normed space, F be a real uniformly convex space with power type p, and let $f : E \to F$ be an (ε, s)-isometry with $f(0) = 0$, where $0 \leq s < 1$. Then, there exist a linear isometry $U : E \to F$ and a constant $M(\varepsilon, s, p) \geq 0$ with $\lim_{\varepsilon\to 0} M(\varepsilon, s, p) = 0$ such that*

$$\|f(u) - Uu\| \leq M(\varepsilon, s, p) \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \quad \textit{for all } \ u \in E.$$

Proof. Let $X = \{g : E \to F |\, g(0) = 0\}$ and

$$d(g_1, g_2) = \inf\{C \in [0, +\infty] : \|g_1(u) - g_2(u)\| \leq C\varphi(u) \text{ for all } \ u \in E\}, \tag{3.1}$$

where $\varphi(u) = \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\}$. Then, (X, d) is a complete generalized metric space. Indeed, by definition of a space to have power type p, it is a Banach space, thus F is complete. This fact ensures that (X, d) is complete. We define the mapping $T : X \to X$ by

$$(Tg)(u) = \frac{1}{2}g(2u) \ \text{ and } \ (T^0 g)(u) = g(u), \ \ \forall g \in X, u \in E. \tag{3.2}$$

We first show that T is a strictly contractive operator. Given $g_1, g_2 \in X$, let $C \in [0, +\infty]$ satisfy $C \geq d(g_1, g_2)$. It follows from (3.1) that

$$\|g_1(u) - g_2(u)\| \leq C\varphi(u) \text{ for all } u \in E.$$

This and (3.2) imply

$$\|(Tg_1)(u) - (Tg_2)(u)\| = \frac{1}{2}\|g_1(2u) - g_2(2u)\| \leq \frac{1}{2}C\varphi(2u) = \frac{1}{2}C \max\{\|2u\|^s, \|2u\|^{1-\frac{1-s}{p}}\}.$$

Since $0 \leq s < 1, p \geq 2$, we get $s < 1 - \frac{1-s}{p} < 1$, and then $2^s < 2^{1-\frac{1-s}{p}} < 2$. So,

$$\begin{aligned}\|(Tg_1)(u) - (Tg_2)(u)\| &\leq \frac{1}{2}C \cdot 2^{1-\frac{1-s}{p}} \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \\ &= \frac{1}{2}C \cdot 2\lambda \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \\ &= C\lambda\varphi(u),\end{aligned}$$

where $\lambda = \frac{2^{1-\frac{1-s}{p}}}{2} = 2^{\frac{s-1}{p}} < 1$. This entails that $d(Tg_1, Tg_2) \leq C\lambda$ by (3.1). Thus, $d(Tg_1, Tg_2) \leq \lambda d(g_1, g_2)$, i.e., T is a strictly contractive operator.

Next, we shall prove that

$$d(Tf, f) < +\infty.$$

Given $u \in E$, let us denote $w_1 = (Tf)(u) - \frac{1}{2}f(u), w_2 = \frac{1}{2}f(u)$ in (2.1) of Lemma 2.3. Then, there exists $\alpha \geq 1$ such that

$$\|(Tf)(u) - f(u)\|^p \leq 2^p\alpha\Big(\frac{\|(Tf)(u) - \frac{1}{2}f(u)\|^p + \|\frac{1}{2}f(u)\|^p}{2} - \|\frac{(Tf)(u)}{2}\|^p\Big).$$

According to the definition of T, we have

$$\|(Tf)(u) - \frac{1}{2}f(u)\|^p = \|\frac{1}{2}f(2u) - \frac{1}{2}f(u)\|^p = \frac{1}{2^p}\|f(2u) - f(u)\|^p,$$

and

$$\|\frac{Tf(u)}{2}\|^p = \frac{1}{2^p}\|\frac{f(2u)}{2}\|^p.$$

Then,

$$\|(Tf)(u) - f(u)\|^p \leq \alpha\Big(\frac{\|f(2u) - f(u)\|^p + \|f(u)\|^p}{2} - \|\frac{f(2u)}{2}\|^p\Big).$$

Since f is an (ε, s)-isometry, we get

$$\|f(2u) - f(u)\| \leq \|2u - u\| + \varepsilon\|2u - u\|^s = \|u\| + \varepsilon\|u\|^s,$$

and

$$\|f(u)\| \leq \|u\| + \varepsilon\|u\|^s.$$

Therefore,

$$\|(Tf)(u) - f(u)\|^p \leq \alpha\Big(\frac{(\|u\| + \varepsilon\|u\|^s)^p + (\|u\| + \varepsilon\|u\|^s)^p}{2} - \|\frac{f(2u)}{2}\|^p\Big)$$

$$= \alpha\Big((\|u\| + \varepsilon\|u\|^s)^p - \|\frac{f(2u)}{2}\|^p\Big). \tag{3.3}$$

We distinguish two cases:

i) If $\|u\| < \varepsilon\|u\|^s$, then (3.3) implies

$$\begin{aligned}\|(Tf)(u) - f(u)\| &\le \alpha^{\frac{1}{p}}\Big((\|u\| + \varepsilon\|u\|^s)^p - \|\frac{f(2u)}{2}\|^p\Big)^{\frac{1}{p}} \\ &\le \alpha^{\frac{1}{p}}(\|u\| + \varepsilon\|u\|^s) \\ &\le 2\alpha^{\frac{1}{p}}\varepsilon\|u\|^s. \end{aligned} \tag{3.4}$$

ii) If $\|u\| \ge \varepsilon\|u\|^s$, then

$$\|f(2u)\| \ge \|2u\| - \varepsilon\|2u\|^s = 2(\|u\| - \varepsilon 2^{s-1}\|u\|^s) \ge 0,$$

which implies

$$\|\frac{f(2u)}{2}\|^p \ge (\|u\| - \varepsilon 2^{s-1}\|u\|^s)^p.$$

Thus, by (3.3), we have

$$\begin{aligned}\|(Tf)(u) - f(u)\|^p &\le \alpha\Big((\|u\| + \varepsilon\|u\|^s)^p - (\|u\| - \varepsilon 2^{s-1}\|u\|^s)^p\Big) \\ &= \alpha\|u\|^p\Big((1 + \varepsilon\|u\|^{s-1})^p - (1 - \varepsilon 2^{s-1}\|u\|^{s-1})^p\Big).\end{aligned}$$

Note that, for $0 \le r \le 1$ and $p \ge 1$, we have

$$(1 + r)^p \le 1 + (2^p - 1)r,$$

and

$$(1 - r)^p \ge 1 - rp.$$

Then,

$$\begin{aligned}\|Tf(u) - f(u)\|^p &\le \alpha\|u\|^p\Big(1 + (2^p - 1)\varepsilon\|u\|^{s-1} - (1 - \varepsilon 2^{s-1}p\|u\|^{s-1})\Big) \\ &= \alpha\|u\|^p\Big((2^p - 1 + 2^{s-1}p)\varepsilon\|u\|^{s-1}\Big) \\ &= (2^p - 1 + 2^{s-1}p)\alpha\varepsilon\|u\|^{p+s-1},\end{aligned}$$

and so

$$\begin{aligned}\|(Tf)(u) - f(u)\| &\le (2^p - 1 + 2^{s-1}p)^{\frac{1}{p}}\alpha^{\frac{1}{p}}\varepsilon^{\frac{1}{p}}\|u\|^{1+\frac{s-1}{p}} \\ &= M_1(\varepsilon, s, p)\|u\|^{1-\frac{1-s}{p}},\end{aligned} \tag{3.5}$$

where $M_1(\varepsilon, s, p) = (2^p - 1 + 2^{s-1}p)^{\frac{1}{p}}\alpha^{\frac{1}{p}}\varepsilon^{\frac{1}{p}} \ge 0$. Therefore, combining (3.4) and (3.5), we obtain that

$$\|(Tf)(u) - f(u)\| \le M_2(\varepsilon, s, p)\max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\},$$

where $M_2(\varepsilon, s, p) = \max\{M_1(\varepsilon, s, p), 2\alpha^{\frac{1}{p}}\varepsilon\}$. It follows from the definition of d that

$$d(Tf, f) \leq M_2(\varepsilon, s, p) < +\infty.$$

Then, by Lemma 2.6, we obtain that $\{T^n f\}$ converges to a fixed point U of T and $U \in X$. So,

$$Uu = \lim_{n\to\infty}(T^n f)(u) = \lim_{n\to\infty} \frac{f(2^n u)}{2^n} \quad \text{for all } u \in E.$$

Note that $U : E \to F$ is an isometry. Indeed,

$$\begin{aligned}\big|\|Uu - Uv\| - \|u - v\|\big| &= \lim_{n\to\infty} \frac{\big|\|f(2^n u) - f(2^n v)\| - \|2^n u - 2^n v\|\big|}{2^n} \\ &\leq \lim_{n\to\infty} \frac{\varepsilon\|2^n u - 2^n v\|^s}{2^n} \\ &= 0.\end{aligned}$$

Baker [26] showed that an isometry from a normed space into a strictly convex normed space, which maps the origin to the origin, is a linear transformation. Since F is a strictly convex space and $U0 = 0$, we obtain that U is a linear isometry. From (ii) in Lemma 2.6, we get

$$U \in Y = \{g \in X : d(T^0 f, g) < \infty\} = \{g \in X : d(f, g) < \infty\}.$$

Clearly, $f \in Y$. It follows from (iii) in Lemma 2.6 that

$$d(f, U) \leq \frac{1}{1-\lambda} d(Tf, f) \leq \frac{M_2(\varepsilon, s, p)}{1-\lambda},$$

where $\lambda = 2^{\frac{s-1}{p}}$. Let $M(\varepsilon, s, p) = \frac{M_2(\varepsilon,s,p)}{1-\lambda}$, then $\lim_{\varepsilon\to 0} M(\varepsilon, s, p) = 0$.

Consequently, by (3.1), we have

$$\|f(u) - Uu\| \leq M(\varepsilon, s, p)\varphi(u) = M(\varepsilon, s, p)\max\{\|x\|^s, \|x\|^{1-\frac{1-s}{p}}\} \quad \text{for all } u \in E.$$

The proof is completed. □

Note that Hilbert spaces have power type 2 and the power type of L_p spaces can be characterized by the following two cases: For $p > 2$, the power type is p; for $1 < p \leq 2$, the power type is 2 (see [27], on page 69). Then, by Theorem 3.1, we have the following corollaries which were obtained by Dolinar.

Corollary 3.2. *[19, Proposition 2] Let H be a Hilbert space, and let $f : E \to H$ be an (ε, s)-isometry with $f(0) = 0$, where $0 \leq s < 1$. Then, there exist a linear isometry $U : E \to H$ and a constant $M(\varepsilon, s) \geq 0$ such that $\lim_{\varepsilon\to 0} M(\varepsilon, s) = 0$ and*

$$\|f(u) - Uu\| \leq M(\varepsilon, s)\max\{\|u\|^s, \|u\|^{\frac{1+s}{2}}\} \quad \text{for all } u \in E.$$

Corollary 3.3. *[19, Proposition 3] Let (Ω, Σ, μ) be a measure space, and let $f : E \to L_p(\Omega, \Sigma, \mu)(1 < p < \infty)$ be an (ε, s)-isometry with $f(0) = 0$, where $0 \leq s < 1$. Then, there exist a linear isometry $U : E \to L_p(\Omega, \Sigma, \mu)$ and a constant $M(\varepsilon, s, p') \geq 0$ such that $\lim_{\varepsilon\to 0} M(\varepsilon, s, p') = 0$ and*

$$\|f(u) - Uu\| \leq M(\varepsilon, s, p')\max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p'}}\} \quad \text{for all } u \in E,$$

where p' denotes the power type of the space $L_p(\Omega, \Sigma, \mu)$.

Remark 3.4. Jung obtained the stability of (ε, s)-isometries into Hilbert spaces for $0 \leq s < 1$, where one has to restrict ε to $1 < \varepsilon \leq 2^{1-s}$ (see [23] (Corollary 3.2)). Note that our results no longer impose any additional requirements on ε.

4. Stability of (ε, s)-isometries for $s > 1$

Theorem 4.1. *Let E be a real normed space, F be a real uniformly convex space with power type p, and let $f : E \to F$ be an (ε, s)-isometry with $f(0) = 0$, where $s > 1$. Then, there exist a linear isometry $U : E \to F$ and a constant $\widetilde{M}(\varepsilon, s, p) \geq 0$ with $\lim_{\varepsilon \to 0} \widetilde{M}(\varepsilon, s, p) = 0$ such that*

$$\|f(u) - Uu\| \leq \widetilde{M}(\varepsilon, s, p) \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \quad \text{for all} \ \ u \in E.$$

Proof. Let $X = \{g : E \to F | \, g(0) = 0\}$ and

$$d(g_1, g_2) = \inf\{C \in [0, +\infty] : \|g_1(u) - g_2(u)\| \leq C\varphi(u) \text{ for all } \ u \in E\}, \tag{4.1}$$

where $\varphi(u) = \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\}$. Then, (X, d) is a complete generalized metric space. Indeed, by definition of a space to have power type p, it is a Banach space, thus F is complete. This fact ensures that (X, d) is complete. We define the mapping $\widetilde{T} : X \to X$ by

$$(\widetilde{T}g)(u) = 2g(\frac{1}{2}u) \ \text{ and } \ (\widetilde{T}^0 g)(u) = g(u), \ \ \forall g \in X, u \in E. \tag{4.2}$$

We first show that $\widetilde{T}$ is a strictly contractive operator. Given $g_1, g_2 \in X$, let $C \in [0, +\infty]$ satisfy $C \geq d(g_1, g_2)$. It follows from (4.1) that

$$\|g_1(u) - g_2(u)\| \leq C\varphi(u) \text{ for all } \ u \in E.$$

This and (4.2) imply

$$\|(\widetilde{T}g_1)(u) - (\widetilde{T}g_2)(u)\| = 2\|g_1(\frac{1}{2}u) - g_2(\frac{1}{2}u)\| \leq 2C\varphi(\frac{1}{2}u) = 2C \max\{\|\frac{1}{2}u\|^s, \|\frac{1}{2}u\|^{1-\frac{1-s}{p}}\}.$$

Since $s > 1, p \geq 2$, we get $s > 1 - \frac{1-s}{p} > 1$, and then $(\frac{1}{2})^s < (\frac{1}{2})^{1-\frac{1-s}{p}} < 1$. So,

$$\begin{aligned} \|(\widetilde{T}g_1)(u) - (\widetilde{T}g_2)(u)\| &\leq 2C \cdot (\frac{1}{2})^{1-\frac{1-s}{p}} \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \\ &= 2C \cdot \frac{1}{2}\widetilde{\lambda} \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \\ &= C\widetilde{\lambda}\varphi(u), \end{aligned}$$

where $\widetilde{\lambda} = \frac{(\frac{1}{2})^{1-\frac{1-s}{p}}}{\frac{1}{2}} = (\frac{1}{2})^{\frac{s-1}{p}} < 1$. This entails that $d(\widetilde{T}g_1, \widetilde{T}g_2) \leq C\widetilde{\lambda}$ by (4.1). Thus, $d(\widetilde{T}g_1, \widetilde{T}g_2) \leq \widetilde{\lambda}d(g_1, g_2)$, i.e., $\widetilde{T}$ is a strictly contractive operator.

Next, we shall prove that

$$d(\widetilde{T}f, f) < +\infty.$$

Given $u \in E$, let us denote $w_1 = f(u) - \frac{1}{2}(\widetilde{T}f)(u), w_2 = \frac{1}{2}(\widetilde{T}f)(u)$ in (2.1) of Lemma 2.3. Then, there exists $\alpha \geq 1$ such that

$$\|f(u) - (\widetilde{T}f)(u)\|^p \leq 2^p\alpha\Big(\frac{\|f(u) - \frac{1}{2}(\widetilde{T}f)(u)\|^p + \|\frac{1}{2}(\widetilde{T}f)(u)\|^p}{2} - \|\frac{f(u)}{2}\|^p\Big).$$

According to the definition of T, we have

$$\|f(u) - \frac{1}{2}(\widetilde{T}f)(u)\|^p = \|f(u) - f(\frac{1}{2}u)\|^p,$$

and

$$\|\frac{1}{2}(\widetilde{T}f)(u)\|^p = \|f(\frac{1}{2}u)\|^p.$$

Then,

$$\|f(u) - (\widetilde{T}f)(u)\|^p \leq 2^p\alpha\Big(\frac{\|f(u) - f(\frac{1}{2}u)\|^p + \|f(\frac{1}{2}u)\|^p}{2} - \|\frac{f(u)}{2}\|^p\Big).$$

Since f is an (ε, s)-isometry, we get

$$\|f(u) - f(\frac{1}{2}u)\| \leq \|u - \frac{1}{2}u\| + \varepsilon\|u - \frac{1}{2}u\|^s = \|\frac{1}{2}u\| + \varepsilon\|\frac{1}{2}u\|^s,$$

and

$$\|f(\frac{1}{2}u)\| \leq \|\frac{1}{2}u\| + \varepsilon\|\frac{1}{2}u\|^s.$$

Therefore,

$$\begin{aligned}\|f(u) - (\widetilde{T}f)(u)\|^p &\leq 2^p\alpha\Big(\frac{(\|\frac{1}{2}u\| + \varepsilon\|\frac{1}{2}u\|^s)^p + (\|\frac{1}{2}u\| + \varepsilon\|\frac{1}{2}u\|^s)^p}{2} - \|\frac{f(u)}{2}\|^p\Big)\\ &= 2^p\alpha\Big((\|\frac{1}{2}u\| + \varepsilon\|\frac{1}{2}u\|^s)^p - \|\frac{f(u)}{2}\|^p\Big)\\ &= \alpha\Big((\|u\| + \varepsilon\|u\|^s)^p - \|f(u)\|^p\Big). \qquad (4.3)\end{aligned}$$

We distinguish two cases:

i) If $\|u\| < \varepsilon\|u\|^s$, then (4.3) implies

$$\begin{aligned}\|f(u) - (\widetilde{T}f)(u)\| &\leq \alpha^{\frac{1}{p}}\Big((\|u\| + \varepsilon\|u\|^s)^p - \|f(u)\|^p\Big)^{\frac{1}{p}}\\ &\leq \alpha^{\frac{1}{p}}(\|u\| + \varepsilon\|u\|^s)\\ &\leq 2\alpha^{\frac{1}{p}}\varepsilon\|u\|^s. \qquad (4.4)\end{aligned}$$

ii) If $\|u\| \geq \varepsilon\|u\|^s$, then

$$\|f(u)\| \geq \|u\| - \varepsilon\|u\|^s \geq 0.$$

Thus, by (4.3), we have

$$\begin{aligned}\|f(u) - (\widetilde{T}f)(u)\|^p &\leq \alpha\Big((\|u\| + \varepsilon\|u\|^s)^p - (\|u\| - \varepsilon\|u\|^s)^p\Big)\\ &= \alpha\|u\|^p\Big((1 + \varepsilon\|u\|^{s-1})^p - (1 - \varepsilon\|u\|^{s-1})^p\Big).\end{aligned}$$

Note that, for $0 \le r \le 1$ and $p \ge 1$, we have

$$(1+r)^p \le 1 + (2^p - 1)r,$$

and

$$(1-r)^p \ge 1 - rp.$$

Then,

$$\begin{aligned}\|f(u) - (\widetilde{T}f)(u)\|^p &\le \alpha\|u\|^p\Big(1 + (2^p-1)\varepsilon\|u\|^{s-1} - (1 - \varepsilon p\|u\|^{s-1})\Big)\\ &= \alpha\|u\|^p\Big((2^p - 1 + p)\varepsilon\|u\|^{s-1}\Big)\\ &= (2^p - 1 + p)\alpha\varepsilon\|u\|^{p+s-1},\end{aligned}$$

and so

$$\begin{aligned}\|f(u) - (\widetilde{T}f)(u)\| &\le (2^p - 1 + p)^{\frac{1}{p}}\alpha^{\frac{1}{p}}\varepsilon^{\frac{1}{p}}\|u\|^{1+\frac{s-1}{p}}\\ &= \widetilde{M_1}(\varepsilon, s, p)\|u\|^{1-\frac{1-s}{p}},\end{aligned} \tag{4.5}$$

where $\widetilde{M_1}(\varepsilon, s, p) = (2^p - 1 + p)^{\frac{1}{p}}\alpha^{\frac{1}{p}}\varepsilon^{\frac{1}{p}} \ge 0$. Therefore, combining (4.4) and (4.5), we obtain that

$$\|f(u) - (\widetilde{T}f)(u)\| \le \widetilde{M_2}(\varepsilon, s, p)\max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\},$$

where $\widetilde{M_2}(\varepsilon, s, p) = \max\{\widetilde{M_1}(\varepsilon, s, p), 2\alpha^{\frac{1}{p}}\varepsilon\}$. It follows from the definition of d that

$$d(\widetilde{T}f, f) \le \widetilde{M_2}(\varepsilon, s, p) < +\infty.$$

Then, by Lemma 2.6, we obtain that $\{\widetilde{T}^n f\}$ converges to a fixed point U of $\widetilde{T}$ and $U \in X$. So,

$$Uu = \lim_{n\to\infty}(\widetilde{T}^n f)(u) = \lim_{n\to\infty} 2^n f(\frac{u}{2^n}) \quad \text{for all } \ u \in E.$$

Note that $U : E \to F$ is an isometry. Indeed,

$$\begin{aligned}\Big|\|Uu - Uv\| - \|u - v\|\Big| &= \lim_{n\to\infty} 2^n \left|\|f(\frac{u}{2^n}) - f(\frac{v}{2^n})\| - \|\frac{u}{2^n} - \frac{v}{2^n}\|\right|\\ &\le \lim_{n\to\infty} 2^n\varepsilon\|\frac{u-v}{2^n}\|^s\\ &= 0.\end{aligned}$$

Baker [26] showed that an isometry from a normed space into a strictly convex normed space, which maps the origin to the origin, is a linear transformation. Since F is a strictly convex space and $U0 = 0$, we obtain that U is a linear isometry. From (ii) in Lemma 2.6, we get

$$U \in Y = \{g \in X : d(\widetilde{T}^0 f, g) < \infty\} = \{g \in X : d(f, g) < \infty\}.$$

Clearly, $f \in Y$. It follows from (iii) in Lemma 2.6 that

$$d(f, U) \le \frac{1}{1-\widetilde{\lambda}}d(\widetilde{T}f, f) \le \frac{\widetilde{M_2}(\varepsilon, s, p)}{1-\widetilde{\lambda}},$$

where $\widetilde{\lambda} = 2^{\frac{s-1}{p}}$. Let $\widetilde{M}(\varepsilon, s, p) = \frac{\widetilde{M_2}(\varepsilon,s,p)}{1-\widetilde{\lambda}}$, then $\lim_{\varepsilon\to 0} \widetilde{M}(\varepsilon, s, p) = 0$.

Consequently, by (4.1), we have

$$\|f(u) - Uu\| \le \widetilde{M}(\varepsilon, s, p)\varphi(u) = \widetilde{M}(\varepsilon, s, p) \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p}}\} \quad \text{for all } u \in E.$$

The proof is completed. □

The following results are obtained, similar to Corollaries 3.2 and 3.3.

Corollary 4.2. *Let H be a Hilbert space, and let $f : E \to H$ be an (ε, s)-isometry with $f(0) = 0$, where $s > 1$. Then, there exist a linear isometry $U : E \to H$ and a constant $\widetilde{M}(\varepsilon, s) \ge 0$ such that $\lim_{\varepsilon\to 0} \widetilde{M}(\varepsilon, s) = 0$ and*

$$\|f(u) - Uu\| \le \widetilde{M}(\varepsilon, s) \max\{\|u\|^s, \|u\|^{\frac{1+s}{2}}\} \quad \textit{for all } u \in E.$$

Corollary 4.3. *Let (Ω, Σ, μ) be a measure space, and let $f : E \to L_p(\Omega, \Sigma, \mu)(1 < p < \infty)$ be an (ε, s)-isometry with $f(0) = 0$, where $s > 1$. Then, there exist a linear isometry $U : E \to L_p(\Omega, \Sigma, \mu)$ and a constant $\widetilde{M}(\varepsilon, s, p') \ge 0$ such that $\lim_{\varepsilon\to 0} \widetilde{M}(\varepsilon, s, p') = 0$ and*

$$\|f(u) - Uu\| \le \widetilde{M}(\varepsilon, s, p') \max\{\|u\|^s, \|u\|^{1-\frac{1-s}{p'}}\} \quad \textit{for all } u \in E,$$

where p' denotes the power type of the space $L_p(\Omega, \Sigma, \mu)$.

Remark 4.4. Let H be a Hilbert space, $s > 1$, $0 < \varepsilon_1 \le 1$, and $1 < \varepsilon_2 \le 2^{s-1}$. Jung obtained the stability of the mapping $f : E \to H$ defined by

$$\Big| \|f(u) - f(v)\| - \|u - v\| \Big| \le \begin{cases} \varepsilon_1 \|u - v\|^s, & \|x\| < 1, \\ \varepsilon_2 \|u - v\|^s, & \|x\| \ge 1. \end{cases}$$

For more details, see [23] (Corollary 3.4). Note that our results no longer impose any additional requirements on ε for any (ε, s)-isometry.

5. Discussion

In this article, we use a fixed theorem to extend two results for the stability of non-surjective (ε, s)-isometries from Hilbert spaces or L_p spaces to uniformly convex Banach spaces. Note that our results no longer impose any additional requirements on ε for any (ε, s)-isometry. For more recent related literature on the stability of perturbed isometries and functional equations in Banach spaces, see, for example, [28–30].

6. Conclusions

Dolinar [19] studied the stability of non-surjective (ε, s)-isometries of Hilbert spaces and $L_p(1 < p < \infty)$ spaces, here $0 \le s < 1$. In 2006, Jung [23] used a fixed point theorem to establish the stability of a class of perturbed isometries of Banach spaces that satisfy the parallelogram law. In particular, he give two results for (ε, s)-isometries of Hilbert spaces where $s \neq 1$. In this paper, we extend the above results for the stability of (ε, s)-isometries from Hilbert spaces or L_p spaces to uniformly convex spaces. In addition, Dolinar [19] also proved that there exists a non-surjective $(\varepsilon, 1)$-isometries which is not stable.

As for future work, we suggest the following:

(1). If $f : X \to Y$ is an (ε, s)-isometry, does there exist an isometry from X to Y?

(2). Are the findings of this study applicable in the case where Y is a Bochner space?

Author contributions

Yuqi Sun: Writing-original draft, Writing-review and editing; Xiaoyu Wang, Jing Dong and Jiahong Lv: Writing-original draft. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Supported by National Natural Science Foundation of China, No. 12301163, Fund Project for Central Leading Local Science and Technology Development, No. 2022ZY0194 and Research Program of Science at Universities of Inner Mongolia Autonomous Region, No. NJZY22345.

Conflict of interest

The authors declare no conflicts of interest.

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