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Research article

The boundedness on BMO_{L_α} space of variation operators for semigroups related to the Laguerre operator

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Abstract: In this paper, we established a $T1$ criterion for the boundedness of Laguerre-Calderón-Zygmund operators on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ associated with Laguerre operators $L_\alpha(\alpha > -\frac{1}{2})$. As applications, we proved the boundedness on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ of variation operators for semigroups related to the Laguerre operator L_α.

Keywords: $T1$ criterion; BMO_{L_α} space; Laguerre operators; semigroups; variation operators
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1. Introduction

It is well-known that the $T1$ theorem plays a critical role in the analysis of L^2 boundedness of Calderón-Zygmund singular integral operators (see [1–3] and [4] (p. 590)). There are analogue $T1$ criterions of Calderón-Zygmund singular integral operators T for the endpoint boundedness, to be precise, T is bounded on $H^1(\mathbb{R}^n)$ if and only if T^*1=0, and bounded on $\mathrm{BMO}(\mathbb{R}^n)$ if and only if $T1$=0 (see e.g., [5]).

Betancor et al. [6] established a $T1$ criterion for Hermite-Calderón-Zygmund operators on $\mathrm{BMO}_H(\mathbb{R}^n)$ related to Hermite operator $H = -\Delta + |x|^2$. They then utilized this $T1$ criterion to demonstrate the boundedness on $\mathrm{BMO}_H(\mathbb{R}^n)$ of several singular integral operators associated with H, including Riesz transforms, maximal operators, Littlewood-Paley g-functions, and variation operators. Ma et al. [7] established an analogous $T1$ criterion for γ-Schrödinger-Calderón-Zygmund operators on Campanato space $\mathrm{BMO}_L^\alpha(\mathbb{R}^n)$ associated with the Schrödinger operator $L = -\Delta + V$ with nonnegative potential V satisfying the reverse Hölder inequality. As applications, they obtained regularity estimates for certain harmonic operators associated with L, including maximal operators, square functions, Laplace transform type multipliers, negative powers $L^{-\frac{\gamma}{2}}$, and Riesz transforms. More recently, Bui et al. [8] provided necessary and sufficient conditions in terms of $T1$ criteria for a generalize Calderón-Zygmund type operators to be bounded on $H^p_L(\mathbb{R}^n)$ and $\mathrm{BMO}_L(\mathbb{R}^n)$ with respect to the Schrödinger operator $L = -\Delta + V$ with nonnegative potential V that satisfies the reverse Hölder inequality. Their applications included proving the boundedness for certain singular operators associated with L, and their results not only recovered exiting results in [7], but also introduced new results.

Assume that $n \in \mathbb{N}, \alpha > -1$, the Laguerre function of Hermite type φ_α on $(0, \infty)$ is defined as

$$\varphi^\alpha_n(y) = \Big(\frac{\Gamma(n+1)}{\Gamma n + 1 + \alpha}\Big)^{1/2} e^{-y^2/2} y^\alpha L^\alpha_n(y^2)(2y)^{1/2},\ y \in (0, \infty),$$

where $L^\alpha_n(x)$ represents the Laguerre polynomial of degree n and order α, see [9]. It is well-known that for every $\alpha > -1$ the system $\{\varphi^\alpha_n\}^\infty_{n=0}$ forms an orthonormal basis of $L^2(0, \infty)$. Furthermore, these functions are eigenfunctions of the Laguerre differential operator

$$L_\alpha = \frac{1}{2}\Big(- \frac{d^2}{dy^2} + y^2 + \frac{\alpha^2 - 1/4}{y^2}\Big)$$

satisfying $L_\alpha\varphi^\alpha_n = (2n+\alpha+1)\varphi^\alpha_n$, and L_α can be extended to a positive self-adjoint operator on $L^2(0, \infty)$ by giving a suitable domain of definition, see [10]. Let $\alpha > -1/2$, the auxiliary function ρ_{L_α} related to Laguerre operator L_α is defined as

$$\rho_{L_\alpha}(x) = \frac{1}{8} \min{(x, 1/x)},\ x > 0. \tag{1.1}$$

Our aim of this paper is to study the boundedness of variation operators for semigroups related to Laguerre operator $L_\alpha(\alpha > -1/2)$ on $\mathrm{BMO}_{L_\alpha}(0, \infty)$. Inspired by [6], we first establish a simple $T1$ criterion of Laguerre-Calderón operators to be bounded on $\mathrm{BMO}_{L_\alpha}(0, \infty)$ related to the Laguerre operator, and then use this $T1$ criterion to obtain the boundedness of this variation operator on $\mathrm{BMO}_{L_\alpha}(0, \infty)$.

We now introduce the following Laguerre-Calderón-Zygmund operators associated with the Laguerre operator L_α for $\alpha > -1/2$.

Definition 1.1. Let T be a bounded linear opterator on $L^2(0, \infty)$ such that

$$Tf(x) = \int_0^\infty K(x, y)f(y)dy,\ \ f \in L^2_c(0, \infty)\ and\ a.e.\ x \notin supp(f).$$

We say that T is a Laguerre-Calderón-Zygmund operator if

(i) $|K(x, y)| \le \frac{C}{|x-y|} e^{-c|x-y|^2}$, for all $x, y \in (0, \infty)$ *and* $x \neq y$,

(ii) $|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \le C\frac{|y-z|}{|x-y|^2}$, when $|x - y| > 2|y - z|$.

Obviously, Laplace operator $L = -\Delta$ and Hermite operator $H = -\Delta + |x|^2$ satisfy (i) (ii) above. For convenience, we also write as $T \in \mathrm{LCZO}$ if T is a Laguerre-Calderón-Zygmund operator. Note that each Laguerre-Calderón-Zygmund operator is also a classical Calderón-Zygmund operator, see [4].

Our first main result is the following $T1$ type theorem for Laguerre-Calderón-Zygmund operator T above to be bounded on $\mathrm{BMO}_{L_\alpha}(0, \infty)$ associated with Laguerre operator L_α, and the precise definition and properties of $\mathrm{BMO}_{L_\alpha}(0, \infty)$, we refer to Section 2.

Theorem 1.2. *Let T be a Laguerre-Calderón-Zygmund operator. Then T is a bounded operator on $BMO_{L_\alpha}(0,\infty)$ if and only if there exists constant $C > 0$ such that*

(i) $\frac{1}{|B(x,\rho_{L_\alpha}(x))|}\int_{B(x,\rho_{L_\alpha}(x))}|T1(y)|dy \le C,\ x \in (0,\infty)$*;*

(ii) $\left(1+\log\left(\frac{\rho_{L_\alpha}(x)}{r}\right)\right)\frac{1}{|B(x,r)|}\int_{B(x,r)}|T1(y)-(T1)_{B(x,r)}|dy \le C,\ x \in (0,\infty)$ *and* $0 < r \le \rho_{L_\alpha}(x)$*, where* ρ_{L_α} *defined in* (1.1).

Here $(T1)_{B(x,r)} = \frac{1}{|B(x,r)|}\int_{B(x,r)} T1(y)dy$.

Remark 1.3. Some further comments on Theorem 1.2:

(i) Theorem 1.2 also holds for vector valued setting, also see Remark 1.1 in [6].

(ii) Suppose that $T1$ is a bounded function in $(0,\infty)$. Then $T1$ satisfies the first condition of Theorem 1.2. The second condition is fulfilled whenever there exists $0 < \alpha \le 1$ such that $|T1(x) - T1(y)| \le C|x-y|^\alpha$ for $x, y \in (0,\infty)$. For example, if $\nabla T1 \in L^\infty(0,\infty)$, then condition (ii) holds.

As applications, we will use Theorem 1.2 to prove the boundedness on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ of several variation operators for the semigroups and Riesz transforms related to Laguerre operator L_α.

We now turn to variation operators. Let $(X,\mathcal{F},\mu)$ be a measure space, $\{T_t\}_{t>0}$ be an uniparametric family of bounded operators in $L^p(X)$ for $1 \le p < \infty$, and $\lim_{t\to 0^+} T_t f$ exists for a.e. $x \in X$. Over the past few years, many papers have focused on researching the speed of convergence of the limit above in terms of the boundedness properties of ρ-variation operators $\mathcal{V}_\rho(T_t)(\rho > 2)$ defined as

$$\mathcal{V}_\rho(T_t)(f)(x) = \sup_{t_j\searrow 0}\left(\sum_{j=1}^{\infty}|T_{t_j}f(x) - T_{t_{j+1}}f(x)|^\rho\right)^{\frac{1}{\rho}}, \tag{1.2}$$

where the supremum is taken over all the sequences of real numbers $\{t_j\}_{j\in\mathbb{N}}$ that decrease to zero. Gillespie and Torrea [11] studied the dimension free estimates of the oscillation for Riesz transforms. The L^p-theory of the variation operators related to the heat semigroup $\{W_t^H\}_{t>0}$ and Poisson semigroup $\{P_t^H\}_{t>0}$ generated by Hermite operator $H = -\Delta + |x|^2$, and the truncated integral operators of the Herimte-Riesz transform were studied in [12, 13]. Betancor et al. [6] studied the boundedness on $\mathrm{BMO}_H(\mathbb{R}^n)$ for the variation operators associated to the heat and Poisson semigroups generated by Hermite operator, and the case of truncated integral operators for Hermite-Riesz transform was also studied. Ping Li et al. [14] studied the variation operator associated with parabolic Hermite operator. They obtained the Lp boundedness of this variation operator, and the boundedness of the endpoint case is also considered. Y. Wen and X. Hou [15] studied the boundedness of variation related to commutators. For more papers about Riesz transform and variation operator, see [16–19]. Y. Ma et al. studied the oscillation of the Poisson semigroup associated with the parabolic Bessel operator, and they obtained the Lp boundedness of this oscillation operator, see [20]. Yali Xiao and Ping Li [21] studied the oscillation of the semigroups associated with discrete Laplacian, and they obtained the ℓ_p-boundedness of this oscillation operator, the endpoint case was also considered.

Here, we will consider the boundedness on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ for variation operators related to the heat semigroup $\{W_t^{L_\alpha}\}_{t>0}$ and Poisson semigroup $\{P_t^{L_\alpha}\}_{t>0}$ generated by the Laguerre operator L_α. We have the following theorem.

Theorem 1.4. *Let $\rho > 2$. Denote by $\{T_t\}_{t>0}$ any of the uniparametric families of operators $\{W_t^{L_\alpha}\}_{t>0}$ or $\{P_t^{L_\alpha}\}_{t>0}$. Then the variation operators $\mathcal{V}_\rho(T_t)$ are bounded from $BMO_{L_\alpha}(0,\infty)$ into itself.*

Betancor et al. [22] studied the transference between Laguerre and Hermite settings and obtained some new properties of the Laguerre operators. Dziubański [23] studied the Hardy space $H^1_{L_\alpha}(0,\infty)$ related to the Laguerre operator L_α for $\alpha > -1/2$, and the author utilized the maximal function related to the heat-diffusion semigroup generated by L_α and atomic decompositions to characterize this Hardy space. In the sequel, Betancor et al. [24] characterized Hardy space associated with certain Laguerre expansions using the Laguerre-Riesz transform. Betancor et al. [25] studied the L^p-boundedness of the area Littlewood-Paley g-functions associated with Hermite and Laguerre operators. Dziubański et al. [26] studied BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. Cha and Liu [27] studied $\mathrm{BMO}_{L_\alpha}(0,\infty)$ related to L_α for $\alpha > -1/2$, which is identified as the dual space of $H^1_{L_\alpha}(0,\infty)$ associated with L_α. They characterized the $\mathrm{BMO}_{L_\alpha}(0,\infty)$ by Carleson measures related to appropriate square functions, and obtained the boundedness on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ of fractional integral operator and the Riesz transform related to L_α. For more references about BMO space, see [28–32].

The outline of this paper is as follows. In Section 2, we mostly introduce some definitions and lemmas needed in the process of proof. In Section 3, we prove Theorem 1.2 by combining the properties of $T1$. In Section 4, we are devoted to establishing the boundedness of variational operators on $\mathrm{BMO}_{L_\alpha}(0,\infty)$, that is, we give the proof of Theorem 1.4.

Throughout this paper, C and c always denote suitable positive constants, though they are not necessarily the same in each occurrence. We will repeatedly apply the inequality $t^\alpha e^{-\beta t} \le C, \alpha \ge 0, \beta > 0$.

2. Preliminaries

In this section, we first introduce the definition of $\mathrm{BMO}_{L_\alpha}(0,\infty)$ related to the Laguerre operator L_α for $\alpha > -1/2$, and then give some properties that are used frequently later, see e.g., [27].

Let $\alpha > -\frac{1}{2}$, $B_s(y)$ be any ball in $(0,\infty)$ with the center y and the radius s. A locally integrable function f on $(0,\infty)$ belongs to $\mathrm{BMO}_{L_\alpha}(0,\infty)$ if there exists $C > 0$ independent of s and y such that

(i) $\frac{1}{|B_s(y)|}\int_{B_s(y)} |f(x) - f_{B_s(y)}|dx \le C,$ for $s < \rho_{L_\alpha(y)}$;

(ii) $\frac{1}{|B_s(y)|}\int_{B_s(y)} |f(x)|dx \le C,$ for $s \ge \rho_{L_\alpha(y)}$,

where $f_{B_s(y)} = \frac{1}{|B_s(y)|}\int_{B_s(y)} f(x)dx$ and the critical radii ρ_{L_α} defined in (1.1). Let $\|f\|_{BMO_{L_\alpha}}$ denote the smallest C in the two inequalities above.

Lemma 2.1. *([27, Lemma 1]) Suppose that $x_0 = 1, x_j = x_{j+1} + \rho_{L_\alpha(x_{j-1})}$ for $j > 1$, and $x_j = x_{j+1} - \rho_{L_\alpha(x_{j+1})}$ for $j < 1$. Let $B = \{B_k\}_{k=-\infty}^{\infty}$, where $B_k = \{x \in (0,\infty) : |x - x_k| < \rho_{L_\alpha(x_k)}\}$. Then we have*

(i) $\cup_{k=-\infty}^{\infty} B_k = (0,\infty)$;

(ii) For every $k \in \mathbb{Z}$, $B_k \cap B_j = \emptyset$ provided that $j \notin \{k-1, k, k+1\}$;

(iii) For any $y_0 \in (0,\infty)$, at most three balls in B have nonempty intersection with $B(y_0, \rho_{L_\alpha(y_0)})$.

It is not hard to check that for every $B_R(x) \subseteq (0,\infty)$ with $R > \rho_{L_\alpha(y_0)}$, there exists $C > 0$ such that

$$|B_R(x)| \leq \sum_{B_k \in B, B_k \cap B_R(x) \neq \emptyset} |B_k| \leq C|B_R(x)|.$$

Lemma 2.2. *([26], p.341) Let $\alpha > -\frac{1}{2}$. An operator V defined on $BMO_{L_\alpha}(0,\infty)$ is bounded from $BMO_{L_\alpha}(0,\infty)$ into itself if there exists $C > 0$ such that for every $f \in BMO_{L_\alpha}$ and $k \in \mathbb{N}$,*

(i) $\frac{1}{|B_k|}\int_{B_k} |Vf(x)|dx \leq C\|f\|_{BMO_{L_\alpha}}$;

(ii) $\|Vf\|_{BMO_{L_\alpha}} \leq C\|f\|_{BMO_{L_\alpha}}$.

Here $\|f\|_{BMO_{L_\alpha}} = \sup_{B_k} \frac{1}{|B_k^*|}\int_{B_k^*} |f(x) - f_{B_k^*}|dy$. *For a ball B, B^* denotes the ball with the same center than B and twice radius.*

3. Proof of Theorem 1.2

In this section, we are devoting to proving Theorem 1.2. Before proving Theorem 1.2 we first introduce the definition of Tf for $f \in \mathrm{BMO}_{L_\alpha}(0,\infty)$.

Definition of Tf for $f \in$BMO$_{L_\alpha}(0,\infty)$. Assume that $f \in L^2(0,\infty)$. For every $R > 0$, let $B_R = B(0,R)$, then we can write

$$Tf = T(f\chi_{B_R}) + T(f\chi_{B_R^c}) = T(f\chi_{B_R}) + \lim_{n\to\infty} T(f\chi_{B_R^c \cap B_n}), \tag{3.1}$$

where the limit is understood in $L^2(0,\infty)$. The identity (3.1) suggests to define the operator T on $\mathrm{MBO}_{L_\alpha}(0,\infty)$ as follows. Suppose that $f \in \mathrm{BMO}_{L_\alpha}(0,\infty)$ and $R > 1$. For every $x \in B_R$, by using the condition (i) in Definition 1.1, it follows that

$$\begin{aligned}
\int_{B_R^c} |K(x,y)||f(y)|dy &\leq C\sum_{j=1}^{\infty}\int_{2^jR<|y|<2^{j+1}R} \frac{e^{-c|x-y|^2}}{|x-y|}|f(y)|dy \\
&\leq C\sum_{j=1}^{\infty}\frac{1}{(2^jR)^2}\int_{2^jR<|y|<2^{j+1}R} |f(y)|dy \\
&\leq C\sum_{j=1}^{\infty}\frac{1}{(2^jR)^2}\int_{|y|<2^{j+1}R} |f(y)|dy \\
&\leq \frac{C}{R}\|f\|_{BMO_{L_\alpha}}.
\end{aligned}$$

Moreover, if $R < r$, it holds that

$$\begin{aligned}
T(f\chi_{B_r}) - T(f\chi_{B_R}) = T(f\chi_{B_r\setminus B_R})(x) &= \int_{B_r\setminus B_R} K(x,y)f(y)dy \\
&= \int_{B_{R^c}} K(x,y)f(y)dy - \int_{B_r^c} K(x,y)f(y)dy,\ a.e.\ x \in B_R.
\end{aligned}$$

For $R > 1$, we define

$$\widetilde{T}f = T(f\chi_{B_R}) + \int_{B_R^c} K(x,y)f(y)dy,\ a.e.\ x \in B_R. \tag{3.2}$$

Note that the definition of $\widetilde{T}f$ in (3.2) is consistent in the choice of $R > 1$ in the sense that if $r > R > 1$ then the definition using B_r coincides almost everywhere in B_R with the one just given.

In order to prove our main conclusion, we derive an expression of $\widetilde{T}f$. We write for $B = B(x_0, r_0)$,

$$f = (f - f_B)\chi_{B^*} + (f - f_B)\chi_{(B^*)^c} + f_B =: f_1 + f_2 + f_3. \tag{3.3}$$

Choosing $R > 0$ such that $B^* \subset B_R$, it follows from (3.3) that

$$\begin{aligned}
\widetilde{T}f(x) =& T(f\chi_{B_R})(x) + \int_{B_R^c} K(x,y)f(y)dy \\
=& T((f - f_B)\chi_{B^*})(x) + T((f - f_B)\chi_{B_R - B^*})(x) + f_B T(\chi_{B_R})(x) \\
&+ \int_{B_R^c} K(x,y)(f(y) - f_B)dy + f_B \int_{B_R^c} K(x,y)dy \\
=& T((f - f_B)\chi_{B^*})(x) + \int_{(B^*)^c} K(x,y)(f(y) - f_B)dy + f_B\widetilde{T}1(x) \\
=:& I_1 + I_2 + I_3,\ a.e.\ x \in B^*.
\end{aligned} \tag{3.4}$$

Proof of Theorem 1.2. We prove that conditions (i) and (ii) of Theorem 1.2 can deduce that T is a bounded operator on $\mathrm{BMO}_{L_\alpha}(0,\infty)$.

We first compute the first term I_1. Notice that T is bounded on $L^2(0,\infty)$, thanks to Hölder's inequality and John-Nirenberg inequality we have

$$\begin{aligned}
\frac{1}{|B_k|}\int_{B_k} I_1 dx &\le C\Big(\frac{1}{|B_k|}\int_{B_k} |T((f - f_B)\chi_{B_k^*})(x)|^2 dx\Big)^{\frac{1}{2}} \\
&\le C\Big(\frac{1}{|B_k|}\int_{B_k^*} |(f(x) - f(_{B_k}))|^2 dx\Big)^{\frac{1}{2}} \le C\|f\|_{BMO_{L_\alpha}}.
\end{aligned}$$

For the second term I_2. It holds that

$$\begin{aligned}
|I_2| &\le \int_{(B_k^*)^c} |K(x,y)||f(y) - f_{B_k}|dy \\
&\le \sum_{j=1}^{\infty} \int_{2^j r_k < |y| < 2^{j+1} r_k} \frac{e^{-|x-y|^2}}{|x-y|} |f(y) - f_{B_k}|dy \\
&\le \sum_{j=1}^{\infty} \int_{|y| < 2^{j+1} r_k} \frac{1}{|x-y|^2} |f(y) - f_{B_k}|dy \\
&\le \frac{1}{r}\|f\|_{BMO_{L_\alpha}}.
\end{aligned}$$

Hence, we obtain that

$$\frac{1}{|B_k|}\int_{B_k} |I_2|dx \le \|f\|_{BMO_{L_\alpha}}. \tag{3.5}$$

Finally, according to condition (i) of Theorem 1.2 and Corollary 5 in [27], then it follows that

$$\begin{aligned}
\frac{1}{|B_k|}\int_{B_k} |I_3|dx &= \frac{1}{|B_k|}\int_{B_k} |f_{B_k}\widetilde{T}1(x)|dx \\
&= |f_{B_k}|\frac{1}{|B_k|}\int_{B_k} |\widetilde{T}1(x)|dx \le C\|f\|_{BMO_{L_\alpha}}.
\end{aligned} \tag{3.6}$$

Combining with all computations above, it immediately obtain that $\widetilde{T}$ satisfies the condition (i) of Lemma 2.2 that does not depent on K.

Next we prove that $\widetilde{T}$ satisfies the condition (ii) of Lemma 2.2. Let $B = B(x_0, r_0) \subseteq B_k^*$ where $x_0 \in (0,\infty)$ and $r_0 > 0$. We divide into $r_0 > \rho_{L_\alpha}(x_0)$ and $r_0 \le \rho_{L_\alpha}(x_0)$ two cases to show our conclusion.

If $r_0 > \rho_{L_\alpha}(x_0)$. Notice that $\rho_{L_\alpha}(x_0) \sim \rho_{L_\alpha}(x_k) \sim r_0$. Hence proceeding as above we obtain that

$$\frac{1}{|B|}\int_B |\widetilde{T}f(x) - (\widetilde{T}f)_B|dx \le \frac{2}{|B|}\int_B |\widetilde{T}f(x)|dx \le C\|f\|_{BMO_{L_\alpha}}.$$

By the definition of T, it follows that

$$T1(x) = \int_0^\infty K(x,y)dy = T(\chi_{B_k^{**}})(x) + \int_{(B_k^{**})^c} K(x,y)dy,\ \ x \in B_k^*.$$

Applying condition (i) of Theorem 1.2 and Hölder' s inequality, it follows that

$$\begin{aligned}\frac{1}{|B|}\int_B |T1(x)|dx &\le \frac{1}{|B_k|}\int_{B_k}|T1(x)|dx + \frac{1}{|B_k^*|}\int_{B_k^*\setminus B_k}|T1(x)|dx\\ &\le C + \Big(\frac{1}{|B_k^*|}\int_{B_k^*}|T(\chi_{B_k^{**}})(x)|^2dx\Big)^{\frac12} + \frac{1}{|B_k^*|}\int_{B_k^*\setminus B_k}\int_{(B_k^{**})^c}|K(x,y)|dydx\\ &\le C + C\|T(\chi_{B_k^{**}})\|_{L^2} + \frac{1}{|B_k^*|}\int_{(B_k^{**})^c}\int_{B_k^*\setminus B_k}\frac{e^{-|x-y|^2}}{|x-y|}dxdy \le C.\end{aligned}$$

If $r_0 \le \rho_{L_\alpha}(x_0)$. By using (3.3) and (3.4) we have

$$\begin{aligned}\frac{1}{|B|}\int_B |\widetilde{T}f(x) - (\widetilde{T}f)_B|dx \le& \frac{1}{|B|}\int_B\frac{1}{|B|}\int_B |Tf_1(x) - Tf_1(z)|dzdx\\ &+ \frac{1}{|B|}\int_B\frac{1}{|B|}\int_B\Big|\int_{(B^*)^c}(K(x,y)-K(z,y))f_2(y)dy\Big|dzdx\\ &+ \frac{1}{|B|}\int_B|\widetilde{T}f_3(x) - (\widetilde{T}f_3)_B|dx\\ =:& J_1 + J_2 + J_3,\ \ x,z \in B.\end{aligned}$$

For the first term J_1, noting that T is bounded on $L^2(0,\infty)$, applying Hölder' s inequality we have

$$J_1 \le \frac{2}{|B|}\int_B |Tf_1(x)|dx \le C\Big(\frac{1}{|B|}\int_{B^*}|f(x) - f_B|^2dx\Big)^{\frac12} \le C\|f\|_{BMO_{L_\alpha}}.$$

For the second term J_2, by using the same argument with J_1, it holds that

$$J_2 \le C\frac{1}{|B|}\int_B\frac{1}{|B|}\int_B \|f\|_{BMO_{L_\alpha}}dzdx \le C\|f\|_{BMO_{L_\alpha}}.$$

Now we estimate J_3. Thanks to Corollary 5 in [27], it follows that

$$\begin{aligned}J_3 &\le \frac{|f_B|}{|B|}\int_B |\widetilde{T}1(x) - (\widetilde{T}1)_B|dx\\ &\le C\|f\|_{BMO_{L_\alpha}}\Big(1 + \log\frac{\rho_{L_\alpha(x)}}{r}\Big)\frac{1}{|B|}\int_B|\widetilde{T}1(x) - (\widetilde{T}1)_B|dx\\ &\le C\|f\|_{BMO_{L_\alpha}}.\end{aligned}$$

Hence, we obtain that for all $B \in B_k^*$,

$$\frac{1}{|B|}\int_B |\widetilde{T}f(x) - (\widetilde{T}f)_B|dx \le C\|f\|_{BMO_{L_\alpha}}.$$

Hence, $\widetilde{T}$ satisfies the condition (ii) of Lemma 2.2. So by Lemma 2.2 we immediately get that T is bounded operator form BMO_{L_α} into itself.

Now, we prove the necessity of the Theorem 1.2. Assume that T is bounded from BMO_{L_α} into itself. Since the function $f(x) = 1$, $x \in (0,\infty)$, belongs to BMO_{L_α}, and $\widetilde{T}1 \in \mathrm{BMO}_{L_\alpha}$. Then, condition (i) holds in Theorem 1.2, and there exists $C > 0$ such that for every ball B,

$$\frac{1}{|B|}\int_B |\widetilde{T}1(y) - (\widetilde{T}1)_B|dy \le C.$$

Let $x_0 \in (0,\infty)$ and $0 < r_0 < \rho_{L_\alpha(x_0)}$, we define

$$f(\cdot, r_0, x_0) =: \chi_{[0,r_0]}(|x - x_0|)\log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big) + \chi_{[r_0,\rho_{L_\alpha(x_0)}]}(|x - x_0|)\log\Big(\frac{\rho_{L_\alpha(x_0)}}{|x - x_0|}\Big). \tag{3.7}$$

Similar to the proof of Lemma 2.1 in [6], we get that $f(\cdot, r_0, x_0)$ belongs to BMO_{L_α}. By the same argument with J_3 above, it holds that

$$\log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big)\frac{1}{|B|}\int_B |\widetilde{T}1(y) - (\widetilde{T}1)_B|dy \le C.$$

Then, condition (ii) in Theorem 1.2 holds.

Hence, the proof of Theorem 1.2 is completed.

Corollary 3.1. *Let g be a measurable function on $(0,\infty)$. We define the multiplier operator $T_g(f) = fg, f \in BMO_{L_\alpha}$, then T_g is bounded on BMO_{L_α} if and only if*

(i) $g \in L^\infty(0,\infty)$;

(ii) there exists $C > 0$ such that

$$\log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big)\frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)} |g(y) - g_{B(x_0,r_0)}|dy \le C,$$

for every $x_0 \in (0,\infty)$ and every ball $B(x_0,r_0)$ with $0 < r_0 < \frac{\rho_{L_\alpha(x_0)}}{2}$.

Proof. Let g be a measurable function on $(0,\infty)$ satisfying the conditions (i) and (ii) in Corollary 3.1. By the same argument with the proof of Theorem 1.2, we konw that g defines pointwise multiplier in BMO_{L_α}, and the kernel of operator $T = T_g$ is zero.

Suppose that g is a pointwise multiplier in BMO_{L_α}. Note that the function $f(\cdot, r_0, x_0)$ defined in (3.7) belongs to BMO_{L_α}, for any ball $B = B(x_0, r_0)$ with $0 < r_0 < \frac{\rho_{L_\alpha(x_0)}}{2}$, applying Corollary 5 in [27] it holds that

$$\begin{aligned}\log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big)\frac{1}{|B|}\int_B |g(x)|dx &= \frac{1}{|B|}\int_B |f(x)g(x)|dx \\ &\le \frac{1}{|B|}\int_B |(fg)(x) - (fg)_B|dx + (fg)_B \\ &\le C\|f\|_{BMO_{L_\alpha}} + \log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big)\|fg\|_{BMO_{L_\alpha}} \\ &\le C\log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big)\|f\|_{BMO_{L_\alpha}}.\end{aligned}$$

Hence, $|g|_B \le C$ where C is independent of B. Thus we obtain that g is bounded on BMO_{L_α}.

On the other hand, if $x_0 \in (0,\infty)$ and $0 < r_0 < \frac{\rho_{L_\alpha(x_0)}}{2}$, by the boundedness on BMO_{L_α} of T_g we have

$$\begin{aligned}&\log\Big(\frac{\rho_{L_\alpha(x_0)}}{r_0}\Big)\frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)}|g(x)-g_{B(x_0,r_0)}|dx\\ \le&\frac{1}{|B(x_0,r_0)|}\int_{B(x_0,r_0)}|g(x)f(x,r_0,x_0)-(gf(x,r_0,x_0))_{B(x_0,r_0)}|dx\\ \le&\|gf(\cdot,r_0,x_0)\|_{BMO_{L_\alpha}}\\ \le&C\|f(\cdot,r_0,x_0)\|_{BMO_{L_\alpha}}.\end{aligned}$$

Here, the constants $C > 0$ appearing in this proof do not depend on $x_0 \in (0,\infty)$ and $0 < r_0 < \rho_{L_\alpha(x_0)}$. □

Remark 3.1. For example, if g is a Lipschitz function, then the condition (ii) in Corollary 3.1 is fulfilled.

Let T be a Laguerre-Calderón-Zygmund operator, then $T1$ defines a pointwise multiplier in BMO_{L_α} space. By Corollary 3.1 and Theorem 1.2, we immediately obtain that T is a bounded operator on BMO_{L_α} sapce.

4. Proof of Theorem 1.4

In this section, we are devoted to proving Theorem 1.4. We first introduce some definitions and properties of the heat-diffusion semigroup generated by Laguerre operator L_α for $\alpha > -\frac{1}{2}$, see e.g., [23,27].

For $f \in L^2(0,\infty)$, the heat-diffusion semigroup $\{W_t^{L_\alpha}\}_{t>0}$ generated by L_α is given by

$$W_t^{L_\alpha}f(x) \equiv e^{-tL_\alpha}f(x) = \int_0^\infty W_t^\alpha(x,y)f(y)dy,\ \ x\in(0,\infty),\ t>0, \tag{4.1}$$

with the kernel

$$W_t^\alpha(x,y) = \Big(\frac{2e^{-t}}{1-e^{-2t}}\Big)^{\frac{1}{2}}\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)^{\frac{1}{2}}I_\alpha\Big(\frac{2xye^{-t}}{1-e^{-2t}}\Big)e^{-\frac{1}{2}\frac{1+e^{-2t}}{1-e^{-2t}}(x^2+y^2)}, \tag{4.2}$$

where I_α is the modified Bessel function of the first kind and order α. The heat semigroup $\{W_t^{L_\alpha}\}_{t>0}$ is contractive in $L^p(0,\infty)$ for $1 \le p < \infty$, and selfadjoint in $L^2(0,\infty)$ but it is not Markovian. Moreover, for every $f \in L^p(0,\infty)$ with $1 \le p < \infty$, then $\lim\limits_{t\to 0^+} W_t^{L_\alpha}f(x) = f(x)$ in $L^p(0,\infty)$ and a.e. $x \in (0,\infty)$.

By Bochner's subordination formula, the Poisson semigroup $\{P_t^{L_\alpha}\}_{t>0}$ associated with L_α is given by

$$P_t^{L_\alpha}f(x) \equiv e^{-t\sqrt{L_\alpha}}f(x) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty e^{-\frac{t^2}{4u}L_\alpha}f(x)e^{-u}\frac{du}{u^{\frac{1}{2}}},\ \ t>0. \tag{4.3}$$

Assume now that for $f \in \mathrm{BMO}_{L_\alpha}$, it is clearly that for every $t > 0$ and $x \in (0,\infty)$ the integral

$$W_t^\alpha f(x) \equiv \int_0^\infty W_t^\alpha(x,y)f(y)dy$$

is absolutely convergent. Hence, for $f \in \mathrm{BMO}_{L_\alpha}$ we define $W_t^{L_\alpha}f$ and $P_t^{L_\alpha}f$ by (4.1) and (4.3), respectively.

Let $r = e^{-2t}$. Thanks to (4.2) we have

$$W_t^{\alpha}(x,y) = H(r,x,y)\Phi(r,x,y)\Psi_{\alpha}(r,x,y), \tag{4.4}$$

where $H(r,x,y) = \frac{(1+r)^{\frac{1}{2}}}{(1-r)^{\frac{1}{2}}} e^{-\frac{1}{2}\frac{1+r}{1-r}|x-y|^2}$, $\Phi(r,x,y) = \frac{\sqrt{2}r^{\frac{1}{4}}}{(1+r)^{\frac{1}{2}}} e^{-\frac{1-r}{(1+\sqrt{r})^2}xy}$, and

$$\Psi_{\alpha}(r,x,y) = \Big(\frac{2r^{\frac{1}{2}}xy}{1-r}\Big)^{\frac{1}{2}} e^{-\frac{2r^{\frac{1}{2}}xy}{1-r}} I_{\alpha}\Big(\frac{2r^{\frac{1}{2}}xy}{1-r}\Big).$$

For every $x, y \in (0,\infty)$ and $t > 1$, we have

$$W_t^{\alpha}(x,y) \leq Ce^{-ct}e^{-c\frac{|x-y|^2}{t}}\bigg(\frac{2xye^{-t}}{1-e^{-2t}}\bigg)^{\frac{1}{2}+\alpha} e^{-\frac{2xye^{-t}}{1-e^{-2t}}} \leq Ce^{-ct}e^{-c|x-y|^2}. \tag{4.5}$$

For every $x, y \in (0,\infty)$ and $0 < t \leq 1$, we have

$$W_t^{\alpha}(x,y) \leq Ct^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}e^{-txy}\frac{1}{\sqrt{2\pi}}e^{\frac{2xye^{-t}}{1-e^{-2t}}}e^{-\frac{2xye^{-t}}{1-e^{-2t}}}(1+O(\frac{1}{z})) \leq Ct^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}e^{-txy}. \tag{4.6}$$

Hence, for every $x, y \in (0,\infty)$ and $t > 0$, it follows that

$$W_t^{\alpha}(x,y) \leq Ct^{-\frac{1}{2}}e^{-c\frac{|x-y|^2}{t}}e^{-txy}\chi_{(0,1]}(t) + Ce^{-ct}e^{-c|x-y|^2}\chi_{(1,\infty)}(t). \tag{4.7}$$

We also use frequently the following properties of Bessel function I_{α}:

$$I_{\alpha}(z) \sim z^{\alpha}, \ z \to 0; \tag{4.8}$$

$$z^{\frac{1}{2}}I_{\alpha}(z) = \frac{1}{\sqrt{2\pi}}e^{z}(1+O(\frac{1}{z})), \ z \to \infty; \tag{4.9}$$

$$\frac{d}{dz}\Big(z^{-\alpha}I_{\alpha}(z)\Big) = z^{-\alpha}I_{\alpha+1}(z), \ z \in (0,\infty). \tag{4.10}$$

Now, we begin to show Theorem 1.4. Here, we prove only Theorem 1.4 for the heat semigroup $\{W_t^{L_{\alpha}}\}_{t>0}$, since the case of Poisson semigroup $\{W_t^{L_{\alpha}}\}_{t>0}$ proceeds identically. We will use Theorem 1.2 in a vector-valued setting(see Remark 1.3) to obtain our results.

We first consider the Banach space E_{ρ} which is defined as follows. A complex function g defined in $[0,\infty)$ is in $E_{\rho}, \rho > 2$, then

$$\|g\|_{E_{\rho}} =: \sup_{t_j \searrow 0}\Big(\sum_{j=1}^{\infty}|g(t_j) - g(t_{j+1})|^{\rho}\Big)^{\frac{1}{\rho}} < \infty.$$

Applying the kernel $W_t^{\alpha}(x,y)$ in (4.2), we can get the kernel of variation operator $\mathcal{V}_{\rho}(W_t^{L_{\alpha}})$. It is clearly

$$\mathcal{V}_{\rho}(W_t^{\alpha})f(x) = \|W_t^{\alpha}f(x)\|_{E_{\rho}}, \ x \in (0,\infty).$$

It is well know that variation operator $\mathcal{V}_{\rho}(W_t^{\alpha})$ is bounded from $L^2(0,\infty)$ into itself. In order to prove that $\mathcal{V}_{\rho}(W_t^{\alpha})$ is bounded form $\mathrm{BMO}_{L_{\alpha}}$ into itself, by using Theorem 1.2 in a vector-valued setting (see Remark 1.3), it is enough to prove that the operator Υ_{ρ} defined by

$$\Upsilon_{\rho}(f) = (W_t^{\alpha}f)_{t>0}, \ f \in BMO_{L_{\alpha}},$$

is bounded from $\mathrm{BMO}_{L_{\alpha}}(0,\infty)$ into $\mathrm{BMO}_{L_{\alpha}}((0,\infty);E_{\rho})$. Finally, in order to use Theorem 1.2 in a vector-valued setting, we need to check that the kernel of $\mathcal{V}_{\rho}(W_t^{\alpha})$ satisfies the following proposition.

Proposition 4.1. *Let $\rho > 2$. There exist constants C such that*

(i) $\|W_t^\alpha(x,y)\|_{E_\rho} \le \frac{C}{|x-y|}e^{-|x-y|^2}$, $x, y \in (0,\infty)$, $x \neq y$;

(ii) $\|\nabla_x W_t^\alpha(x,y)\|_{E_\rho} \le \frac{C}{|x-y|^2}$, $x, y \in (0,\infty)$, $x \neq y$;

(iii) $\Upsilon_\rho(1) \in L^\infty((0,\infty); E_\rho)$ *and* $\nabla\Upsilon_\rho(1) \in L^\infty((0,\infty); E_\rho)$.

Proof. (i) Suppose that $\{t_j\}_{j>0} \subset (0,1)$ is a decreasing sequence and $\lim_{j\to\infty} t_j = 0$. By (4.4) and (4.8)–(4.10), then we have for $x, y \in (0,\infty)$, $x \neq y$,

$$\begin{aligned}\Big(\sum_{j=1}^{\infty}|W_{t_j}^\alpha(x,y) - W_{t_{j+1}}^\alpha(x,y)|^\rho\Big)^{\frac{1}{\rho}} &\le \sum_{j=1}^{\infty}|W_{t_j}^\alpha(x,y) - W_{t_{j+1}}^\alpha(x,y)| \\ &\le \int_0^\infty |\partial_t W_t^\alpha(x,y)|dt \\ &\le C\int_0^1 e^{-c\frac{|x-y|^2}{t}}dt + C\int_1^\infty e^{-ct}e^{-c|x-y|^2}dt \\ &\le C\frac{1}{|x-y|}e^{-|x-y|^2}.\end{aligned}$$

(ii) Similar to the proof of Proposition 4.4 in [6], we obtain the desired conclusion.

(iii) By (4.4)–(4.10), we can obtain that $\mathcal{V}_\rho(W_t^\alpha)(1)(x) \le C$. Hence $\Upsilon_\rho(1) \in L^\infty((0,\infty); E_\rho)$. Similarly, we get that for $x \in (0,\infty)$,

$$\begin{aligned}&\sum_{j=1}^{\infty}\Big|\nabla\Big(W_{t_j}^\alpha 1(x)) - W_{t_{j+1}}^\alpha 1(x)\Big)\Big| \\ \le& C\sum_{j=1}^{\infty}\Big|\Big((4t_j - t_j^3)e^{-c(4t_j-t_j^3)|x|^2} - (4t_{j+1} - t_{j+1}^3)e^{-c(4t_{j+1}-t_{j+1}^3)|x|^2}\Big)|x|\chi_{(0,1)}(t)\Big| \\ \le& C\int_0^1\Big||x|\partial_t(4t-t^3)e^{-c(4t-t^3)|x|^2}\Big|dt \\ \le& C\int_0^1\Big|\frac{4-3t^2}{(4t-t^3)^{\frac{1}{2}}}(4t-t^3)^{\frac{1}{2}}|x|e^{-c(4t-t^3)|x|^2} - \frac{4-3t^2}{(4t-t^3)^{\frac{1}{2}}}((4t-t^3)|x|^2)^{\frac{3}{2}}e^{-c(4t-t^3)|x|^2}\Big|dt \le C.\end{aligned}$$

As a result, $\nabla\Upsilon_\rho(1) \in L^\infty((0,\infty); E_\rho)$.

Hence, by using Proposition 4.1 and Theorem 1.2 in a vector-valued setting (see Remark 1.3), we finished the proof of Theorem 1.4. □

5. Conclusions

In this work, we first established a $T1$ criterion of the boundedness on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ of Laguerre-Calderón-Zygmund operators associated with the Laguerre operators $L_\alpha(\alpha > -\frac{1}{2})$, and then used this $T1$ criterion to prove the boundedness on $\mathrm{BMO}_{L_\alpha}(0,\infty)$ of variation operators for semigroups related to the Laguerre operator L_α.

Author contributions

Fan Chen: Formal Analysis, commenting; Houwei Du: Formal Analysis, commenting; Jinglan Jia: Writing-original draft, commenting; Ping Li: Writing-original draft, funding acquisition, commenting; Zhu Wen: Writing-original draft, commenting. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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