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Research article

Solvability of fractional differential system with parameters and singular nonlinear terms

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Abstract: In this article, we consider the parametric high-order fractional system with integral boundary value conditions involving derivatives of order *p*-*q*. With the aid of the fixed-point theorem, an exact interval from the existence to the solution of the system will be obtained, under the condition that the nonlinearities of the system may have singularities. Finally, we provide an instance to show the practicality of the primary outcomes.

Keywords: positive solution; parameter; fractional differential system; singular
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1. Introduction

The singularity phenomenon exists in a large number of physical models and biological processes, for instance, viscoelasticity, aerodynamics, hydrodynamics, rheology, and infectious diseases. In nonlinear elastic fracture mechanics, there is a singular relationship between the range q and the stress near the crack tip; the stress shows that the power singularity is $q^{-\frac{1}{2}}$, where q is the range determined from the crack tip [1]. As is well known, fractional differential equations have significant advantages in describing local limitations and long-term, large-scale physical phenomena. Westerland [2] depicts the transmission of electromagnetic waves in the following model:

$$\vartheta\varsigma\frac{\partial^2\mathcal{B}(u,s)}{\partial u^2}+\vartheta\varsigma\chi\mathcal{D}_s^{\nu}\mathcal{B}(u,s)+\frac{\partial^2\mathcal{B}(u,s)}{\partial s^2}=0,$$

here ϑ,ς,χ are constants, and $\mathcal{D}_s^{\nu}\mathcal{B}(u,s)=\frac{\partial^{\nu}\mathcal{B}(u,s)}{\partial t^{\nu}}$ is a fractional derivative. The authors in article [3] constructed a fractional Maxwell model

$$\Upsilon+\lambda^{\rho}\frac{d^{\rho}\sigma}{dt^{\rho}}=E\lambda^{\varrho}\frac{d^{\varrho}\varepsilon}{dt^{\varrho}}$$

with fractional parameters ρ and ϱ satisfying $0 < \rho \le \varrho < 1$.

This work treats the singular fractional order system:

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha_1} w(t) + \lambda_1 f_1(t, w(t), v(t)) = 0, \\ \mathcal{D}_{0^+}^{\alpha_2} v(t) + \lambda_2 f_2(t, w(t), v(t)) = 0, \ 0 < t < 1, \\ w(0) = w'(0) = \cdots w^{(n_1-2)} = 0, \ \mathcal{D}_{0^+}^{p_1} w(1) = \mu_1 \int_0^{\eta_1} h_1(s) \mathcal{D}_{0^+}^{q_1} w(s) ds, \\ v(0) = v'(0) = \cdots v^{(n_2-2)} = 0, \ \mathcal{D}_{0^+}^{p_2} v(1) = \mu_2 \int_0^{\eta_2} h_2(s) \mathcal{D}_{0^+}^{q_2} v(s) ds, \end{cases} \tag{1.1}$$

where $\lambda_i > 0$ $(i = 1, 2)$ is a parameter, and $\mathcal{D}_{0^+}^{\alpha_i}$ is the Riemann-Liouville derivative. $n_i - 1 < \alpha_i \le n_i$, $n_i \ge 3$, $1 \le p_i \le n_i - 2$, $0 \le q_i \le p_i$, $\mu_i > 0$, $0 < \eta_i \le 1$, $h \in L^1[0, 1]$ is nonnegative, $\Lambda_i = \Gamma(\alpha_i)/\Gamma(\alpha_i - p_i)(1 - \mu_i \int_0^{\eta_i} h_1(s) s^{\alpha_i - q_i - 1} ds) > 0$, $f_i : (0, 1) \times [0, +\infty)^2 \setminus \{(0, 0)\} \to [0, +\infty)$ is continuous, $f_i(t, x, y)$ may have singularity at $t = 0,\ 1$ and $(x, y) = (0, 0)$.

Fractional calculus has attracted widespread attention from scholars in various fields. Additionally, a large number of theories and arguments have been concentrated on that can be utilized to characterize various differential equations, such as operator theory [4–6], space theory [7–10], variational methods [11–13], fixed point theorems [14–17], etc. For instance, employing the fixed index theory, Xu et al. [18] talked over the fractional boundary value problem (BVP):

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} u(t) + h(t) f(t, u(t)) = 0, \ 0 < t < 1, \ n - 1 < \alpha \le n, \ n > 3, \\ u^k(0) = 0, \ 0 \le k \le n - 2, \ \left[\mathcal{D}_{0^+}^{\beta} u\right]_{t=1} = 0, 1 \le \beta \le n - 2, \end{cases} \tag{1.2}$$

where $\mathcal{D}_{0^+}^{\alpha}$ is the Riemann-Liouville derivative. $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$, $h \in C(0, 1) \cap L(0, 1)$, h is nonnegative. By adopting the famous Krasnosel'skii fixed-point theorem, the author in [19] also researched BVP (1.2) with $f : [0, 1] \times [0, +\infty) \to [0, +\infty))$ being continuous and $h \equiv 1$.

Henderson and Luca [20] focus on the following equation:

$$\mathcal{D}_{0^+}^{\alpha} \mathrm{u}(t) + \overline{\mu} f(t, \mathrm{u}(t)) = 0, \ 0 < t < 1, \ n - 1 < \alpha \le n, \ n \ge 3, \tag{1.3}$$

with multi-point boundary value conditions of fractional order

$$\mathrm{u}^{\mathrm{j}}(0) = 0, \ \mathrm{j} = 0, 1, 2, ..., n - 2, \ \mathcal{D}_{0^+}^{p} \mathrm{u}(1) = \sum_{i=1}^{m} a_i \mathcal{D}_{0^+}^{q} \mathrm{u}(\xi_i), \tag{1.4}$$

where $\overline{\mu} > 0$ is a parameter and $\mathcal{D}_{0^+}^{\alpha}$ is the Riemann-Liouville derivative. $0 < \xi_1 < \cdots < \xi_m < 1$, $1 \le p \le n - 2$, $0 \le q \le p$, the sign of f can be changed, and it can be singular when $t = 0,\ 1$. Existence results for at least one positive solution are given in [20] according to Krasnosel'skii fixed-point theorem.

Wang and Jiang [21] also studied Eq (1.3) with the parameter $\overline{\mu} = 1$, and the boundary condition is

$$\mathrm{u}^{\mathrm{j}}(0) = 0, \ \mathrm{j} = 0, 1, 2, ..., n - 2, \ \mathcal{D}_{0^+}^{p} \mathrm{u}(1) = \overline{\mu} \int_0^{\eta} h(s) \mathcal{D}_{0^+}^{q} \mathrm{u}(s) ds. \tag{1.5}$$

They determined that there exist two positive solutions on the basis of the Leray-Schauder nonlinear alternative and the fixed-point theory of cone tension and compression.

Motivated by the aforementioned works, in this article, we consider the system (1.1), under the argument of fixed-point theory, whose existence will be gained. What is more, we obtain the exact interval in which the positive solution exists under the confinement of the singularity in nonlinear term. The remaining parts of this article are arranged as below: In Section 2, we put forward some preliminary and necessary lemmas. The proof of system (1.1) will be constructed under the fixed-point theory in Section 3. In the following section, an example for demonstrating the applicability of the primary conclusions will be given. In Section 5, we draw conclusions from this article.

2. Preliminaries and lemmas

Please refer to [22, 23] for the related definitions and lemmas of fractional derivative and integral. We put in several lemmas for the rest of the paper. The following Lemmas 2.1 and 2.2 have been proved in [20, 24, 25].

Lemma 2.1. *Assume that* $\Lambda_i \neq 0$, $\mathrm{y}_i \in C[0,1]$ $(i = 1, 2)$, *the boundary value problems*

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha_i}\overline{z}(t) + \mathrm{y}_i(t) = 0, \ 0 < t < 1, \\ \overline{z}(0) = \overline{z}'(0) = \cdots \overline{z}^{(n_i-2)} = 0, \ \mathcal{D}_{0^+}^{p_i}\overline{z}(1) = \mu_i \displaystyle\int_0^{\eta_i} h_i(s)\mathcal{D}_{0^+}^{q_i}\overline{z}(s)ds, \end{cases}$$

has the representation

$$\overline{z}(t) = \int_0^1 \mathcal{G}_i(t,s)\mathrm{y}_i(s)ds,$$

in which

$$\mathcal{G}_i(t,s) = \mathcal{G}_{i1}(t,s) + \mathcal{G}_{i2}(t,s), \ i = 1, 2, \tag{2.1}$$

$$\mathcal{G}_{i1}(t,s) = \frac{1}{\Gamma(\alpha_i)}\begin{cases} t^{\alpha_i-1}(1-s)^{\alpha_i-p_i-1} - (t-s)^{\alpha_i-p_i-1}, & 0 \le s \le t \le 1, \\ t^{\alpha_i-1}(1-s)^{\alpha_i-p_i-1}, & 0 \le t \le s \le 1, \end{cases}$$

$$\mathcal{G}_{i2}(t,s) = \frac{\mu_i t^{\alpha_i-1}}{\Lambda_i}\int_0^{\eta_i} h_i(t)\mathcal{H}_i(t,s)dt,$$

$$\mathcal{H}_i(t,s) = \frac{1}{\Gamma(\alpha_i - q_i)}\begin{cases} t^{\alpha_i-q_i-1}(1-s)^{\alpha_i-p_i-1} - (t-s)^{\alpha_i-p_i-1}, & 0 \le s \le t \le 1, \\ t^{\alpha_i-q_i-1}(1-s)^{\alpha_i-p_i-1}, & 0 \le t \le s \le 1. \end{cases}$$

□

Lemma 2.2. $\mathcal{G}_i(t,s)$ $(i = 1, 2)$ *labeled (2.1) has the following properties:*

(1) $\mathcal{G}_i(t,s) \in C([0,1] \times [0,1], [0,+\infty))$.

(2) $\mathcal{G}_i(t,s) \le \xi_i(s), \ 0 \le t, s \le 1,$

$$\xi_i(s) = \bar{h}_i(s) + \frac{\mu_i}{\Lambda_i}\int_0^{\eta_i} h_i(t)\mathcal{H}_i(t,s)dt, \ \bar{h}_i(s) = \frac{(1-s)^{\alpha_i-p_i-1}(1-(1-s)^{p_i})}{\Gamma(\alpha_i)}.$$

(3) $\widehat{\omega}(t)\xi_i(s) \le \mathcal{G}_i(t,s) \le \sigma\bar{\omega}(t), \ 0 \le t, s \le 1,$

$$\widehat{\omega}(t) = \min\{t^{\alpha_1-1}, t^{\alpha_2-1}\}, \ \bar{\omega}(t) = \max\{t^{\alpha_1-1}, t^{\alpha_2-1}\},$$

$$\sigma = \max\left\{\frac{1}{\Gamma(\alpha_1)} + \mu_1 \int_0^{\eta_1} h_1(t)t^{\alpha_1-q_1-1}dt/\Lambda_1\Gamma(\alpha_1 - q_1)\right.$$
$$\left.\frac{1}{\Gamma(\alpha_2)} + \mu_2 \int_0^{\eta_2} h_2(t)t^{\alpha_2-q_2-1}dt/\Lambda_2\Gamma(\alpha_2 - q_2)\right\}.$$

□

Denote the Banach space $X = C[0,1] \times C[0,1]$ with norms:

$$\|(w,v)\| = \|w\| + \|v\|,\ \|w\| = \max_{0\le t\le 1}|w(t)|,\ \|v\| = \max_{0\le t\le 1}|v(t)|.$$

Let

$$\mathcal{K} = \{(w,v) \in X : w(t) \ge \widehat{\omega}(t)\|w\|,\ v(t) \ge \widehat{\omega}(t)\|v\|,\ 0 \le t \le 1\},$$

then $\mathcal{K} \subset X$ is a positive cone. For any real number $0 < \overline{r} < \overline{R}$, let

$$\mathcal{K}_{[\overline{r},\overline{R}]} = \{(w,v) \in \mathcal{K} : \overline{r} \le \|(w,v)\| \le \overline{R}\},\ \mathcal{K}_{\overline{r}} = \{(w,v) \in \mathcal{K} : \|(w,v)\| < \overline{r}\}.$$

For $i = 1,2$, the assumption through the work must hold.

($\boldsymbol{H}_1$) $f_i : (0,1) \times [0,+\infty)^2 \setminus \{(0,0)\} \to [0,+\infty)$ *is continuous and*

$$|f_i(t,x,y)| \le \mathsf{b}_i(t)F_i(t,x,y),\ (t,x,y) \in (0,1) \times [0,+\infty)^2 \setminus \{(0,0)\},$$

$\mathsf{b}_i \in C(0,1)$, b_i *is singular at* $t = 0$ *and/or* $t = 1$, $\mathsf{b}_i(t) \not\equiv 0$ *on* $[0,+\infty)$, $F_i : [0,1] \times [0,+\infty)^2 \setminus \{(0,0)\} \to [0,+\infty)$ *is continuous,*

$$\lim_{\widetilde{m}\to+\infty} \sup_{(w,v)\in\mathcal{K}_{[r_1^*,r_2^*]}} \int_{H(\widetilde{m})} \xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds = 0,\ 0 < r_1^* < r_2^* < +\infty,$$

where $H(\widetilde{m}) = \left[0,\frac{1}{\widetilde{m}}\right] \cup \left[\frac{\widetilde{m}-1}{\widetilde{m}},1\right]$.

($\boldsymbol{H}_2$) $\int_0^1 \xi_i(s)\mathsf{b}_i(s)ds < +\infty$.

Remark 2.1. ($\mathbf{H}_1$) can be inferred from ($\mathbf{H}_2$). As a matter of fact, by ($\mathbf{H}_1$), for $0 < r_1 < r_2 < +\infty$, choose $(w(t),v(t)) \equiv \left(\frac{r_1}{2},\frac{r_1}{2}\right) \in \mathcal{K}_{[r_1,r_2]}$, for any fixed $\widetilde{m} > 0$, we have

$$\int_{H(\widetilde{m})} \xi_i(s)\mathsf{b}_i(s)F_i\left(s,\frac{r_1}{2},\frac{r_1}{2}\right)ds < +\infty,\ i = 1,2.$$

Since $F_i : [0,1] \times [0,+\infty)^2 \setminus \{(0,0)\} \to [0,+\infty)$ is continuous, we have

$$\int_{H(\widetilde{m})} \xi_i(s)\mathsf{b}_i(s)ds < +\infty.$$

As $\mathsf{b}_i,\ \xi_i : \left[\frac{1}{\widetilde{m}},\frac{\widetilde{m}-1}{\widetilde{m}}\right] \to [0,+\infty)$ is continuous, we see that

$$\int_{\frac{1}{\widetilde{m}}}^{\frac{\widetilde{m}-1}{\widetilde{m}}} \xi_i(s)\mathsf{b}_i(s)ds < +\infty,$$

and so

$$\int_0^1 \xi_i(s)\mathsf{b}_i(s)ds = \int_{H(\widetilde{m})} \xi_i(s)\mathsf{b}_i(s)ds + \int_{\frac{1}{\widetilde{m}}}^{\frac{\widetilde{m}-1}{\widetilde{m}}} \xi_i(s)\mathsf{b}_i(s)ds < +\infty,\ i = 1,2.$$

According to conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, for any $(w,v) \in \mathcal{K} \setminus \{(0,0)\}$, define $\mathcal{T} : \mathcal{K} \setminus \{(0,0)\} \to X$,

$$\mathcal{T}(w,v)(t) = (\mathcal{T}_1(w,v)(t), \mathcal{T}_2(w,v)(t)),\ 0 \le t \le 1, \tag{2.2}$$

$$\mathcal{T}_i(w,v)(t) = \lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s, w(s), v(s))ds,\ i = 1,2.$$

It can be declared that $\mathcal{T}(w,v)$ is well defined. Actually, for any fixed $(w_0,v_0) \in \mathcal{K} \setminus \{(0,0)\}$, there exists $r > 0$, such that $\|(w_0,v_0)\| = r$. In $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we will show that

$$\mathcal{T}_i(w_0,v_0)(t) = \lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s, w_0(s), v_0(s))ds < +\infty,\ 0 \le t \le 1,\ i = 1,2. \tag{2.3}$$

By $(\mathbf{H}_1)$, there exists $\widetilde{l} \in \mathrm{N}$ (N represents the set of natural numbers) such that

$$\sup_{(w,v)\in\partial\mathcal{K}_r} \lambda_i \int_{H(\widetilde{l})} \xi_i(s)\mathsf{b}_i(s)F_i(s, w(s), v(s))ds < 1,\ i = 1,2.$$

For each $(w,v) \in \partial\mathcal{K}_r$, $t \in \left[\frac{1}{\widetilde{l}}, \frac{\widetilde{l}-1}{\widetilde{l}}\right]$,

$$\omega^* r \le w(t) + v(t) \le r,\ \omega^* = \min\left\{\widehat{\omega}(t) : t \in \left[\frac{1}{\widetilde{l}}, \frac{\widetilde{l}-1}{\widetilde{l}}\right]\right\}.$$

Let

$$M = \max\left\{g_i(t,x,y) : \frac{1}{\widetilde{l}} \le t \le \frac{\widetilde{l}-1}{\widetilde{l}},\ \omega^* r \le x + y \le r,\ x \ge 0,\ y \ge 0,\ i = 1,2\right\}.$$

So, by $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ and Lemma 2.2, for $0 \le t \le 1,\ i = 1,2$, we have

$$\begin{aligned}
&\mathcal{T}_i(w_0,v_0)(t)\\
\le& \sup_{(w,v)\in\partial\mathcal{K}_r} \lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s, w(s), v(s))ds\\
\le& \sup_{(w,v)\in\partial\mathcal{K}_r} \lambda_i \int_0^1 \xi_i(s)\mathsf{b}_i(s)F_i(s, w(s), v(s))ds\\
\le& \sup_{(w,v)\in\partial\mathcal{K}_r} \lambda_i \int_{H(\widetilde{l})} \xi_i(s)\mathsf{b}_i(s)F_i(s, w(s), v(s))ds\\
&+ \sup_{(w,v)\in\partial\mathcal{K}_r} \lambda_i \int_{\frac{1}{\widetilde{l}}}^{\frac{\widetilde{l}-1}{\widetilde{l}}} \xi_i(s)\mathsf{b}_i(s)F_i(s, w(s), v(s))ds\\
\le& 1 + M\lambda_i \int_0^1 \xi_i(s)\mathsf{b}_i(s)ds < +\infty.
\end{aligned}$$

Then, we know (2.3) holds. In combination with the continuity of $\mathcal{G}_i(t,s)$, $\mathcal{T}_i(w_0,v_0) \in C[0,1]$, $\mathcal{T}_i : \mathcal{K} \setminus \{(0,0)\} \to C[0,1]$ is well defined. Therefore, $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : \mathcal{K} \setminus \{(0,0)\} \to X$ is well defined. The fixed point of operator $\mathcal{T}$ in $\mathcal{K} \setminus \{(0,0)\}$ is the solution of the system (1.1).

Lemma 2.3. *Assume that* $(\boldsymbol{H}_1)$ $(\boldsymbol{H}_2)$ *hold. Then* $\mathcal{T} : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$ *is completely continuous.*

Proof. Firstly, we illustrate $\mathcal{T}(\mathcal{K}_{[r_1,r_2]}) \subseteq \mathcal{K}$. For $i = 1,2$, $(w,v) \in \mathcal{K}_{[r_1,r_2]}$, $0 \le t \le 1$, as for the proof of (2.3), we know

$$\lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s,w(s),v(s))ds \le \lambda_i \int_0^1 \xi_i(s) f_i(s,w(s),v(s))ds < +\infty.$$

So, by $(\mathbf{H}_2)$ and Lemma 2.2,

$$\begin{aligned}\|\mathcal{T}_i(w,v)\| = \max_{0\le t\le 1} |T_i(w,v)(t)| &= \max_{0\le t\le 1} \left| \lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s,w(s),v(s))ds \right| \\ &\le \lambda_i \int_0^1 \xi_i(s) f_i(s,w(s),v(s))ds < +\infty.\end{aligned}$$

By $(\mathbf{H}_2)$ and Lemma 2.2, for any $(w,v) \in \mathcal{K}_{[r_1,r_2]}$, $0 \le t \le 1$,

$$\begin{aligned}\mathcal{T}_i(w,v)(t) =&\lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s,w(s),v(s))ds \\ \ge&\lambda_i \int_0^1 \omega(t)\xi_i(s) f_i(s,w(s),v(s))ds.\end{aligned}$$

Then $\mathcal{T}_i(w,v)(t) \ge \widehat{\omega}(t)\|\mathcal{T}_i(w,v)\|$, $T(\mathcal{K}_{[r_1,r_2]}) \subseteq \mathcal{K}$.

Next, we explain $\mathcal{T} : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$ is a continuous operator. Let $(w_n,v_n), (w,v) \in \mathcal{K}_{[r_1,r_2]}$, such that $\|(w_n,v_n) - (w,v)\| \to 0$ $(n \to +\infty)$, we will prove that $\|\mathcal{T}(w_n,v_n) - \mathcal{T}(w,v)\| \to 0$ $(n \to +\infty)$. By $(\mathbf{H}_1)$, for any $\varepsilon > 0$, there exists $\ell \in \mathrm{N}$ satisfying

$$\sup_{(u,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{H(\ell)} \xi_i(s) \mathrm{b}_i(s) F_i(s,w(s),v(s))ds < \frac{\varepsilon}{4},\ i = 1,2. \tag{2.4}$$

For each $(w,v) \in \mathcal{K}_{[r_1,r_2]}$, $t \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right]$, we have

$$\omega' r_1 \le w(t) + v(t) \le r_2,\ \omega' = \min\left\{\widehat{\omega}(t) : t \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right]\right\}.$$

For $t \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right]$, $\omega' r_1 \le w(t) + v(t) \le r_2$, $w(t)$, $v(t)$ meet one of the following three cases:

(1) $\frac{\omega' r_1}{2} \le w(t) \le r_2$, $\frac{\omega' r_1}{2} \le v(t) \le r_2$.

(2) $\frac{\omega' r_1}{2} \le w(t) \le r_2$, $0 \le v(t) \le \frac{\omega' r_1}{2}$.

(3) $0 \le w(t) \le \frac{\omega' r_1}{2}$, $\frac{\omega' r_1}{2} \le v(t) \le r_2$.

Since for $(t,x,y) \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right] \times \left[\frac{\omega' r_1}{2}, r_2\right] \times \left[\frac{\omega' r_1}{2}, r_2\right]$ or $(t,x,y) \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right] \times \left[\frac{\omega' r_1}{2}, r_2\right] \times \left[0, \frac{\omega' r_1}{2}\right]$ or $(t,x,y) \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right] \times \left[0, \frac{\omega' r_1}{2}\right] \times \left[\frac{\omega' r_1}{2}, r_2\right]$, $f_i(t,x,y)$ is uniformly continuous, we have

$$\lim_{n\to+\infty} |f_i(s,w_n(s),v_n(s)) - f_i(s,w(s),v(s))| = 0,\ i = 1,2$$

holds uniformly on $s \in \left[\frac{1}{\ell}, \frac{\ell-1}{\ell}\right]$. Then the Lebesgue-dominated convergence theorem yields that

$$\lambda_i \int_{\frac{1}{\ell}}^{\frac{\ell-1}{\ell}} \xi_i(s)|f_i(s, w_n(s), v_n(s)) - f_i(s, w(s), v(s))|ds \to 0, \ \ n \to +\infty, \ i = 1, 2.$$

So, for the above $\varepsilon > 0$, there exists a sufficiently large $N_0 (N_0 \in \mathrm{N})$, when $n > N_0$,

$$\lambda_i \int_{\frac{1}{\ell}}^{\frac{\ell-1}{\ell}} \xi_i(s)|f_i(s, w_n(s), v_n(s)) - f_i(s, w(s), v(s))|ds < \frac{\varepsilon}{2}, \ i = 1, 2. \tag{2.5}$$

It follows from (2.4) and (2.5) that, when $n > N_0$,

$$\begin{aligned}
&\|\mathcal{T}_i(w_n, v_n) - \mathcal{T}_i(w, v)\| \\
\leq &\lambda_i \int_0^1 |\mathcal{G}_i(t, s) f_i(s, w_n(s), v_n(s)) - \mathcal{G}_i(t, s) f_i(s, w(s), v(s))|ds \\
\leq &\lambda_i \int_{\frac{1}{\ell}}^{\frac{\ell-1}{\ell}} \xi_i(s)|f_i(s, w_n(s), v_n(s)) - f_i(s, w(s), v(s))|ds \\
&+ \sup_{(u,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{H(\ell)} \xi_i(s)\mathrm{b}_i(s)|\mathsf{F}_i(s, w_n(s), v_n(s)) + \mathsf{F}_i(s, w(s), v(s))|ds \\
<&\varepsilon, \ i = 1, 2.
\end{aligned}$$

That is, $\mathcal{T} : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$ is a continuous operator.

Then, we show $\mathcal{T} : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$ is compact. Let $\mathcal{A} \subset \mathcal{K}_{[r_1,r_2]}$ be any bounded set. For $(w, v) \in \mathcal{A}$, we have $r_1 \leq \|(w, v)\| \leq r_2$. By $(\mathbf{H}_1)$, there exists $\jmath \in \mathrm{N}$ satisfying

$$\sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{H(\jmath)} \xi_i(s)\mathrm{b}_i(s)\mathsf{F}_i(s, w(s), v(s))ds < 1, \ i = 1, 2.$$

For each $(w, v) \in \mathcal{K}_{[r_1,r_2]}$, $t \in \left[\frac{1}{\jmath}, \frac{\jmath-1}{\jmath}\right]$, we can see

$$\omega'' r_1 \leq w(t) + v(t) \leq r_2, \ \omega'' = \min\left\{\widehat{\omega}(t) : t \in \left[\frac{1}{\jmath}, \frac{\jmath-1}{\jmath}\right]\right\}.$$

Let

$$M'' = \max\left\{\mathsf{F}_i(t, x, y) : \frac{1}{\jmath} \leq t \leq \frac{\jmath-1}{\jmath}, \ \omega'' r_1 \leq x + y \leq r_2, \ x \geq 0, \ y \geq 0, \ i = 1, 2\right\}.$$

So, by $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we have

$$\begin{aligned}
&\sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s,w(s),v(s))ds \\
\leq &\sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_0^1 \xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds \\
\leq &\sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{H(J)} \xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds \\
&+ \sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{\frac{1}{J}}^{\frac{J-1}{J}} \xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds \\
\leq &\, 1 + M''\lambda_i \int_0^1 \xi_i(s)\mathsf{b}_i(s)ds < +\infty,\ i = 1,2.
\end{aligned}$$

Therefore, for any $(w,v) \in \mathcal{A}$, $0 \leq t \leq 1$, we have

$$\begin{aligned}
&\lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s,w(s),v(s))ds \\
\leq &\sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_0^1 \mathcal{G}_i(t,s) f_i(s,w(s),v(s))ds \\
\leq &1 + M''\lambda_i \int_0^1 \xi_i(s)\mathsf{b}_i(s)ds < +\infty,\ i = 1,2.
\end{aligned}$$

So, $\mathcal{T}(\mathcal{A})$ is bounded in X.

Finally, we explain that $\mathcal{T}_i(\mathcal{A})$ is equicontinuous. By $(\mathbf{H}_1)$, for any $\varepsilon > 0$, there exists $\hbar \in \mathrm{N}$, we have

$$\sup_{(w,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{H(\hbar)} \xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds < \frac{\varepsilon}{4},\ i = 1,2.$$

Let

$$M_0 = \max\left\{F_i(t,x,y) : \frac{1}{\hbar} \leq t \leq \frac{\hbar-1}{\hbar},\ \omega_0 r_1 \leq x+y \leq r_2,\ x \geq 0,\ y \geq 0,\ i = 1,2\right\},$$

$\omega_0 = \min\left\{\widehat{\omega}(t) : t \in \left[\frac{1}{\hbar}, \frac{\hbar-1}{\hbar}\right]\right\}$. The uniform continuity of $\mathcal{G}_i(t,s)$ on $[0,1]\times[0,1]$ means, for the above $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta_0$, $s \in \left[\frac{1}{\hbar}, \frac{\hbar-1}{\hbar}\right]$, we have

$$|\mathcal{G}_i(t_1,s) - \mathcal{G}_i(t_2,s)| < \frac{\varepsilon}{2}\left(M_0\lambda_i \int_{\frac{1}{\hbar}}^{\frac{\hbar-1}{\hbar}} \mathsf{b}_i(s)ds\right)^{-1},\ i = 1,2.$$

Thus, when $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta_0$, for any $(w,v) \in \mathcal{A}$,

$$\begin{aligned}
&|T_i(w,v)(t_1) - T_i(w,v)(t_2)| \\
\le&\lambda_i \int_{\frac{1}{\hbar}}^{\frac{\hbar-1}{\hbar}} |\mathcal{G}_i(t_1,s) - \mathcal{G}_i(t_2,s)| f_i(s,w(s),v(s))ds \\
&+ \sup_{(u,v)\in\mathcal{K}_{[r_1,r_2]}} \lambda_i \int_{H(\hbar)} |\mathcal{G}_i(t_1,s) - \mathcal{G}_i(t_2,s)| \mathsf{b}_i(s) \mathsf{F}_i(s,w(s),v(s))ds \\
\le&\frac{\varepsilon}{2} + 2M_0\lambda_i \int_{\frac{1}{\hbar}}^{\frac{\hbar-1}{\hbar}} \xi_i(s)\mathsf{b}_i(s)ds < \varepsilon,\ i = 1,2.
\end{aligned}$$

This means that $T_i(\mathcal{A})$ is equicontinuous. By the Arzela-Ascoli theorem, $T_i(\mathcal{A})$ is a relatively compact set. So $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2) : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$ is compact. Combining with the continuity of $T : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$, $\mathcal{T} : \mathcal{K}_{[r_1,r_2]} \to \mathcal{K}$ is a completely continuous operator. □

The following Lemmas 2.4 and 2.5 can be used to explain the existence of the fixed point, that is, the existence of positive solutions to the system (1.1).

Lemma 2.4. [26] *Let $\mathcal{P}$ be a positive cone in a Banach space E, Ω_1 and Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $\mathtt{A} : \mathcal{P} \cap \overline{\Omega}_2 \backslash \Omega_1 \to \mathcal{P}$ is a completely continuous operator. If the following conditions are satisfied:*

$\|\mathtt{A}x\| \le \|x\|$, $\forall x \in \mathcal{P} \cap \partial\Omega_1$, $\quad \|\mathtt{A}x\| \ge \|x\|$, $\forall x \in \mathcal{P} \cap \partial\Omega_2$,

or

$\|\mathtt{A}x\| \ge \|x\|$, $\forall x \in \mathcal{P} \cap \partial\Omega_1$, $\quad \|\mathtt{A}x\| \le \|x\|$, $\forall x \in \mathcal{P} \cap \partial\Omega_2$,

then $\mathtt{A}$ has at least one fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.5. [27] *Let $\mathcal{P}$ be a positive cone in a real Banach space E. Denote $\mathcal{P}_r = \{x \in \mathcal{P} : \|x\| < r\}$, $\overline{\mathcal{P}}_{r,R} = \{x \in \mathcal{P} : r \le \|x\| \le R\}$, $0 < r < R < +\infty$. Let $\mathtt{A} : \overline{\mathcal{P}}_{r,R} \to \mathcal{P}$ be a completely continuous operator. If the following conditions are satisfied:*

(1) $\|\mathtt{A}x\| \le \|x\|$, $\forall x \in \partial\mathcal{P}_R$,

(2) *There exists a $x_0 \in \partial\mathcal{P}_1$, such that $x \ne \mathtt{A}x + mx_0$, $\forall x \in \partial\mathcal{P}_r$, $m > 0$,*

then $\mathtt{A}$ has fixed points in $\overline{\mathcal{P}}_{r,R}$.

Remark 2.2. If (1) and (2) are satisfied for $x \in \partial\mathcal{P}_r$ and $x \in \partial\mathcal{P}_R$, respectively, then Lemma 2.6 still holds.

3. Main results

Theorem 3.1. *Assume that $(\boldsymbol{H}_1)$ $(\boldsymbol{H}_2)$ $(\boldsymbol{H}_3)$ hold.*

$(\boldsymbol{H}_3)$

$$0 \le \mathsf{F}_i^\infty = \limsup_{\substack{x+y\to+\infty \\ (x,y)\ne(0,0)}} \max_{t\in[0,1]} \frac{\mathsf{F}_i(t,x,y)}{x+y} < L_i,$$

$$0 < l_i < f_{i0} = \liminf_{\substack{x+y\to0^+ \\ (x,y)\ne(0,0)}} \min_{t\in[a,b]\subset(0,1)} \frac{|f_i(t,x,y)|}{x+y} \le +\infty, i = 1,2,$$

where $L_i = \left(4\int_0^1 \xi_i(s)\mathsf{b}_i(s)ds\right)^{-1}$, $l_i = \left(2\overline{\omega}^2\int_a^b \xi_i(s)ds\right)^{-1}$. $\overline{\omega} = \min_{t\in[a,b]}\widehat{\omega}(t)$. *For any*

$$\lambda_1 \in \left(\frac{l_1}{f_{10}}, \frac{L_1}{F_1^\infty}\right),\ \lambda_2 \in \left(\frac{l_2}{f_{20}}, \frac{L_2}{F_2^\infty}\right), \tag{3.1}$$

system (1.1) has at least one positive solution.

Proof. For $i = 1, 2$, let λ_i satisfy (3.1) and let $\varepsilon_0 > 0$ be chosen such that $L_i - \varepsilon_0 > 0$ and $\lambda_i g_i^\infty \leq L_i - \varepsilon_0$. From the first inequality in ($\mathbf{H}_3$), it can be inferred that there exists $\overline{r} > 0$, making

$$F_i(t, x, y) \leq \frac{1}{\lambda_i}(L_i - \varepsilon_0)(x + y),\ x + y > \overline{r},\ t \in [0, 1]. \tag{3.2}$$

Let

$$\overline{M} = \max_{i=1,2}\ \sup_{(w,v)\in\partial\mathcal{K}_{\overline{r}}} \lambda_i \int_0^1 \xi_i(s)\mathsf{b}_i(s)F_i(s, w(s), v(s))ds.$$

As for the proof of (2.3), we obtain $\overline{M} < +\infty$. Take

$$R_1 > \max\left\{\overline{r},\ \frac{4\overline{M}L_1}{1+\varepsilon_0},\ \frac{4\overline{M}L_2}{1+\varepsilon_0}\right\},\ \mathcal{K}_{R_1} = \{(w, v) \in \mathcal{K} : \|(w, v)\| < R_1\}.$$

For any $(w, v) \in \partial\mathcal{K}_{R_1}$, let

$$D(w, v) = \{t \in [0, 1] : w(t) + v(t) > \overline{r}\}.$$

So, for any $t \in D(w, v)$, we receive

$$\overline{r} < w(t) + v(t) \leq \|(w, v)\| = R_1.$$

Thus, by (3.2),

$$F_i(t, w(t), v(t)) \leq \frac{1}{\lambda_i}(L_i - \varepsilon_0)(w(t) + v(t)),\ (w, v) \in \partial\mathcal{K}_{R_1},\ t \in D(w, v). \tag{3.3}$$

For any $(w, v) \in \partial\mathcal{K}_{R_1}$, let

$$w_1(t) = \min\left\{\frac{w(t) + v(t)}{2},\ \frac{\overline{r}}{2}\right\},\ v_1(t) = \min\left\{\frac{w(t) + v(t)}{2},\ \frac{\overline{r}}{2}\right\},$$

then we have

$$\|(w_1, v_1)\| = \max_{t\in[0,1]}|w_1(t)| + \max_{t\in[0,1]}|v_1(t)| = \overline{r},\ (w_1, v_1) \in \partial\mathcal{K}_{\overline{r}}.$$

So for any $(w,v)\in\partial\mathcal{K}_{R_1}$, by (3.3),

$$\begin{aligned}&\|\mathcal{T}_i(w,v)\|\\ =&\max_{t\in[0,1]}\lambda_i\left|\int_0^1\mathcal{G}_i(t,s)f_i(s,w(s),v(s))ds\right|\\ \le&\lambda_i\int_0^1\xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds\\ \le&\lambda_i\int_{D(w,v)}\xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds\\ &+\lambda_i\int_{[0,1]\setminus D(w,v)}\xi_i(s)\mathsf{b}_i(s)F_i(s,w(s),v(s))ds\\ \le&\frac{1}{\lambda_i}(L_i-\varepsilon_0)\lambda_i\int_0^1\xi_i(s)\mathsf{b}_i(s)(w(s)+v(s))ds\\ &+\lambda_i\int_0^1\xi_i(s)\mathsf{b}_i(s)F_i(s,w_1(s),v_1(s))ds\\ \le&\frac{(L_i-\varepsilon_0)R_1}{4L_i}+\overline{M}<\frac{R_1}{2}=\frac{\|w\|+\|v\|}{2}.\end{aligned}$$

Thus

$$\|\mathcal{T}(w,v)\|=\|\mathcal{T}_1(w,v)\|+\|\mathcal{T}_2(w,v)\|\le\|w\|+\|v\|=\|(w,v)\|,\ (w,v)\in\partial\mathcal{K}_{R_1}. \tag{3.4}$$

For $i=1,2$, let λ_i satisfy (3.1) and let $\overline{\varepsilon}>0$ be chosen so that $l_i+\overline{\varepsilon}\le\lambda_i f_{i0}$. By the second inequality of $(\mathbf{H}_3)$, there exists $R_2<R_1$ meeting

$$|f_i(t,x,y)|\ge\frac{1}{\lambda_i}(l_i+\overline{\varepsilon})(x+y),\ \ 0<x+y\le R_2,\ t\in[a,b]. \tag{3.5}$$

Choose $\mathcal{K}_{R_2}=\{(w,v)\in\mathcal{K}:\|(w,v)\|<R_2\}$. For any $(w,v)\in\partial\mathcal{K}_{R_2}$,

$$R_2\ge w(t)+v(t)\ge\widehat{\omega}(t)\|w\|+\widehat{\omega}(t)\|v\|\ge\overline{\omega}R_2>0,\ t\in[a,b].$$

Combining with (3.5), we gain

$$|f_i(t,w(t),v(t))|\ge\frac{1}{\lambda_i}(l_i+\overline{\varepsilon})(w(t)+v(t)),\ (w,v)\in\partial\mathcal{K}_{R_2},\ t\in[a,b]. \tag{3.6}$$

Therefore, for any $(w,v)\in\partial\mathcal{K}_{R_2}$, by (3.6),

$$\begin{aligned}&\mathcal{T}_i(w,v)(t)\\ =&\lambda_i\int_0^1\mathcal{G}_i(t,s)f_i(s,w(s),v(s))ds\\ \ge&\lambda_i\int_a^b\mathcal{G}_i(t,s)f_i(s,w(s),v(s))ds\\ \ge&\frac{1}{\lambda_i}(l_i+\overline{\varepsilon})\lambda_i\int_a^b\widehat{\omega}(t)\xi_i(s)(w(s)+v(s))ds\\ \ge&\frac{1}{\lambda_i}(l_i+\overline{\varepsilon})\lambda_i\overline{\omega}\int_a^b\xi_i(s)(w(s)+v(s))ds\\ >&\frac{R_2}{2}=\frac{\|w\|+\|v\|}{2}.\end{aligned}$$

Thus

$$\|\mathcal{T}(w,v)\| = \|\mathcal{T}_1(w,v)\| + \|\mathcal{T}_2(w,v)\| \geq \|w\| + \|v\| = \|(w,v)\|,\ (w,v) \in \partial\mathcal{K}_{R_2}. \tag{3.7}$$

As can be seen by (3.4), (3.7), and Lemmas 2.3 and 2.4, $\mathcal{T}$ has a fixed point (w,v) with $0 < R_2 \leq \|(w,v)\| \leq R_1$. Then system (1.1) has a positive solution (w,v). □

Remark 3.1. From Theorem 3.1, the superlinear and sublinear conditions of $f_i(t,x,y)$ and $F_i(t,x,y)$ are not necessary. In reality, Theorem 3.1 still applies to the following situations:

(1) $F_i^\infty < L_i$, $f_{i0} = +\infty$, $\lambda_i \in \left(0, \frac{L_i}{F_i^\infty}\right)$.

(2) $F_i^\infty = 0$, $f_{i0} = +\infty$, $\lambda_i \in (0, +\infty)$.

(3) $F_i^\infty = 0$, $f_{i0} > l_i > 0$, $\lambda_i \in \left(\frac{l_i}{f_{i0}}, +\infty\right)$.

Remark 3.2. For $i = 1,2$, since $0 < \frac{l_i}{f_{i0}} < 1$, $\frac{L_i}{F_i^\infty} > 1$, we have $1 \in \left(\frac{l_i}{f_{i0}}, \frac{L_i}{F_i^\infty}\right)$, so when $\lambda_1 = \lambda_2 = 1$, Theorem 3.1 is true.

Theorem 3.2. *Assume that* $(\boldsymbol{H}_1)$ $(\boldsymbol{H}_2)$ $(\boldsymbol{H}_4)$ *hold.*

$(\boldsymbol{H}_4)$

$$0 \leq F_i^0 = \limsup_{\substack{x+y\to 0^+\\(x,y)\neq(0,0)}} \max_{t\in[0,1]} \frac{F_i(t,x,y)}{x+y} < L_i,$$

$$0 < l_i' < f_{i\infty} = \liminf_{\substack{x+y\to +\infty\\(x,y)\neq(0,0)}} \min_{t\in[a,b]\subset(0,1)} \frac{|f_i(t,x,y)|}{x+y} \leq +\infty,\ i = 1,2,$$

the definitions of L_i *is the same in Theorem 3.1:* $l_i' = \left(2\min_{t\in[a,b]} \int_a^b \mathcal{G}_i(t,s)ds\right)^{-1}$. *For any*

$$\lambda_1 \in \left(\frac{l_1'}{f_{1\infty}}, \frac{L_1}{F_1^0}\right),\ \lambda_2 \in \left(\frac{l_2'}{f_{2\infty}}, \frac{L_2}{F_2^0}\right),$$

system (1.1) has at least one positive solution.

Proof. For $i = 1,2$, for any $\lambda_i \in \left(\frac{l_i'}{f_{i\infty}}, \frac{L_i}{F_i^0}\right)$, there exists $\varepsilon' > 0$ so that

$$\frac{l_i'}{f_{i\infty} - \varepsilon'} \leq \lambda_i \leq \frac{L_i}{F_i^0 + \varepsilon'},\ f_{i\infty} - \varepsilon' > 0.$$

According to the first inequality of $(\mathbf{H}_4)$, there exists $r > 0$,

$$F_i(t,x,y) \leq (F_i^0 + \varepsilon')(x+y),\ \ 0 < x+y \leq r,\ t \in [0,1]. \tag{3.8}$$

Set

$$\mathcal{K}_{r_1} = \{(w,v) \in \mathcal{K} : \|(w,v)\| < r_1\},\ (r_1 \leq r).$$

From $(\mathbf{H}_1)$, there exists m' ($m' \in \mathrm{N}$),

$$\sup_{(w,v)\in\partial\mathcal{K}_{r_1}} \lambda_i \int_{H(m')} \xi_i(s)\mathrm{b}_i(s)F_i(s,w(s),v(s))ds < \frac{r_1}{4}. \tag{3.9}$$

For any $(w, v) \in \partial\mathcal{K}_{r_1}$, we can acquire

$$0 < w' r_1 \leq w(t) + v(t) \leq r_1 \leq r,\ t \in \left[\frac{1}{m'}, \frac{m'-1}{m'}\right],$$

where $\omega' = \min\left\{\widehat{\omega}(t) : t \in \left[\frac{1}{m'}, \frac{m'-1}{m'}\right]\right\}$. From (3.8), we know

$$F_i(t, w(t), v(t)) \leq (F_i^0 + \varepsilon_i')(w(t) + v(t)),\ (u, v) \in \partial\mathcal{K}_{r_1},\ t \in \left[\frac{1}{m'}, \frac{m'-1}{m'}\right]. \tag{3.10}$$

Equations (3.9) and (3.10) can be introduced so that, for any $(u, v) \in \partial\mathcal{K}_{r_1}$,

$$\begin{aligned}
&\|\mathcal{T}_i(w, v)\| \\
&= \max_{t\in[0,1]} \lambda_i \left| \int_0^1 \mathcal{G}_i(t, s) f_i(s, w(s), v(s)) ds \right| \\
&\leq \sup_{(w,v)\in\partial\mathcal{K}_{r_1}} \lambda_i \int_{H(m')} \xi_i(s) \mathsf{b}_i(s) F_i(s, w(s), v(s)) ds \\
&\quad + \lambda_i \int_{\frac{1}{m'}}^{\frac{m'-1}{m'}} \xi_i(s) \mathsf{b}_i(s) F_i(s, w(s), v(s)) ds \\
&\leq \frac{r_1}{4} + (g^0 + \varepsilon')\lambda_i \int_{\frac{1}{m'}}^{\frac{m'-1}{m'}} \xi_i(s) \mathsf{b}_i(s)(w(s) + v(s)) ds \\
&\leq \frac{r_1}{2} = \frac{\|w\| + \|v\|}{2}.
\end{aligned}$$

Thus

$$\|\mathcal{T}(w, v)\| = \|\mathcal{T}_1(w, v)\| + \|\mathcal{T}_2(w, v)\| \leq \|w\| + \|v\| = \|(w, v)\|,\ (w, v) \in \partial\mathcal{K}_{r_1}. \tag{3.11}$$

For $i = 1, 2$, for the above $\varepsilon' > 0$, on the basis of the second inequality of $(\mathbf{H}_5)$,

$$r_0 > \overline{\omega} r_1 > 0,\ \overline{\omega} = \min_{t\in[a,b]} \widehat{\omega}(t)$$

satisfying

$$|f_i(t, x, y)| \geq (f_{i\infty} - \varepsilon')(x + y),\quad x + y \geq r_0,\ t \in [a, b]. \tag{3.12}$$

Let

$$r_2 = r_0/\overline{\omega} > r_1,\ \mathcal{K}_{r_2} = \{(w, v) \in \mathcal{K} : \|(w, v)\| < r_2\},\ (u_0.v_0) = \left(\frac{1}{2}, \frac{1}{2}\right) \in \partial\mathcal{K}_1.$$

Then, we demonstrate

$$(w, v) \neq \mathcal{T}(u, v) + \mu(w_0.v_0), \forall (u, v) \in \partial\mathcal{K}_{r_2}, \forall \mu > 0. \tag{3.13}$$

Otherwise, there exists $(w_2, v_2) \in \partial\mathcal{K}_{r_2}$ and $\mu_2 > 0$ such that

$$(w_2, v_2) = \mathcal{T}(w_2, v_2) + \mu_2(w_0, v_0).$$

Owing to the fact that

$$w_2(t) + v_2(t) \geq \overline{\omega}\|w_2\| + \overline{\omega}\|v_2\| = \overline{\omega}r_2 = r_0,\ a \leq t \leq b,$$

by (3.12), we can find that

$$|f_i(t, w_2(t), v_2(t))| \geq (f_{i\infty} - \varepsilon')(w_2(t) + v_2(t)),\ (w_2, v_2) \in \partial\mathcal{K}_{r_2},\ a \leq t \leq b. \tag{3.14}$$

Suppose

$$\xi = \min\{w_2(t) + v_2(t) :\ a \leq t \leq b\}. \tag{3.15}$$

Then $w_2(t) + v_2(t) \geq \xi > 0$, $a \leq t \leq b$. Therefore, for any $a \leq t \leq b$, by (3.14),

$$\begin{aligned}
&w_2(t) + v_2(t) \\
=&\mathcal{T}_1(w_2, v_2) + \mathcal{T}_2(w_2, v_2) + \mu_2(w_0 + v_0) \\
=&\lambda_1 \int_0^1 \mathcal{G}_1(t, s) f_1(s, w_2(s), v_2(s))ds \\
&+ \lambda_2 \int_0^1 \mathcal{G}_2(t, s) f_2(s, w_2(s), v_2(s))ds + \mu_2(w_0 + v_0) \\
\geq&\lambda_1 \int_a^b \mathcal{G}_1(t, s) f_1(s, w_2(s), v_2(s))ds \\
&+ \lambda_2 \int_a^b \mathcal{G}_2(t, s) f_2(s, w_2(s), v_2(s))ds + \mu_2 \\
\geq& \min_{s\in[a,b]}(w_2(s) + v_2(s))(f_{1\infty} - \varepsilon')\lambda_1 \min_{t\in[a,b]} \int_a^b \mathcal{G}_1(t, s)ds \\
&+ \min_{s\in[a,b]}(w_2(s) + v_2(s))(f_{2\infty} - \varepsilon')\lambda_2 \min_{t\in[a,b]} \int_a^b \mathcal{G}_2(t, s)ds + \mu_2 \\
\geq&\xi + \mu_2 > \xi.
\end{aligned}$$

Thus

$$w_2(t) + v_2(t) \geq \xi + \mu_2,\ t \in [a, b]. \tag{3.16}$$

Obviously, (3.16) and (3.15) yield contradiction, thus (3.13) holds. It follows from (3.11), (3.13), and Lemmas 2.3 and 2.5 that $\mathcal{T}$ has a fixed point (w, v) with $0 < r_1 \leq \|(w, v)\| \leq r_2$. Then system (1.1) has a positive solution (w, v). □

Remark 3.3. Similar to Remark 3.1, for $i = 1, 2$, Theorem 3.2 holds to the situations:

(1) $f_{i\infty} = +\infty, F_i^0 < L_i, \lambda_i \in \left(0, \frac{L_i}{F_i^0}\right)$.

(2) $f_{i\infty} = +\infty, F_i^0 = 0, \lambda_i \in (0, +\infty)$.

(3) $f_{i\infty} > l_i', F_i^0 = 0, \lambda_i \in \left(\frac{l_i'}{f_{i\infty}}, +\infty\right)$.

Remark 3.4. Since $0 < \frac{l_i'}{f_{i\infty}} < 1$, $\frac{L_i}{F_i^0} > 1$, we have $1 \in \left(\frac{l_i'}{f_{i\infty}}, \frac{L_i}{F_i^0}\right)$, so when $\lambda_1 = \lambda_2 = 1$, Theorem 3.2 also holds.

4. An example

Example 4.1. *Take account of the fractional system*

$$\begin{cases} \mathcal{D}_{0^+}^{\frac{5}{2}} w(t) + f_1(t, w(t), v(t)) = 0, \\ \mathcal{D}_{0^+}^{\frac{7}{2}} v(t) + f_2(t, w(t), v(t)) = 0, \ 0 < t < 1, \\ w(0) = w'(0) = 0, \ w'(1) = \dfrac{1}{3} \displaystyle\int_0^{\frac{1}{3}} t^{-\frac{1}{4}} \mathcal{D}_{0^+}^{\frac{1}{2}} w(t) dt, \\ v(0) = v'(0) = v''(0) = 0, \ \mathcal{D}_{0^+}^{\frac{5}{4}} v(1) = \dfrac{1}{2} \displaystyle\int_0^{\frac{4}{5}} t^{-\frac{3}{4}} \mathcal{D}_{0^+}^{\frac{5}{4}} v(t) dt, \end{cases} \tag{4.1}$$

where

$$f_1(t, x, y) = 3 + t \sin \frac{1}{x + y} + |\ln(x + y)|,$$

$$f_2(t, x, y) = 12 - t + \sqrt{x + y} + |\ln(x + y)|.$$

Obviously,

$$\alpha_1 = \frac{5}{2}, \ \alpha_2 = \frac{7}{2}, \ \mathsf{b}_i(t) = 1, \ F_i(t, x, y) = f_i(t, x, y), \ i = 1, 2.$$

By computation

$$\Gamma(\alpha_1) = 1.3294, \ \Gamma(\alpha_2) = 3.3237, \ \Lambda_1 = 0.7714, \ \Lambda_2 = 0.8485, \ \xi_1 \approx 1.1843, \ \xi_2 \approx 0.8901.$$

For $\widehat{\omega}(t) = t^{\frac{7}{2}}$, define a cone

$$\mathcal{K} = \left\{ (w, v) \in X : w(t) \geq t^{\frac{7}{2}} \|w\|, \ v(t) \geq t^{\frac{7}{2}} \|v\|, \ 0 \leq t \leq 1 \right\}.$$

For any $0 < r_1^* < r_2^* < +\infty$ and $(w, v) \in \mathcal{K}_{[r_1^*, r_2^*]}$, $\widehat{\omega}(t) r_1^* \leq w(t) + v(t) \leq r_2^*$. So

$$\begin{aligned} &F_1(t, w(t), v(t)) \leq 4 + |\ln r_1^*| + |\ln r_2^*| + |\ln \widehat{\omega}(t)|, \\ &F_2(t, w(t), v(t)) \leq 12 + \sqrt{r_2^*} + |\ln r_1^*| + |\ln r_2^*| + |\ln \widehat{\omega}(t)|. \end{aligned} \tag{4.2}$$

$$\int_0^1 |\ln \widehat{\omega}(t)| dt < 4. \tag{4.3}$$

The absolute continuity of the integral gives

$$\lim_{\widetilde{m} \to +\infty} \int_{H(\widetilde{m})} |\ln \widehat{\omega}(t)| dt = 0.$$

By (4.2) (4.3), we obtain

$$
\begin{aligned}
0 &\le \sup_{(w,v)\in\mathcal{K}_{[r_1^*,r_2^*]}} \int_{H(\widetilde{m})} \xi_1(s)\mathsf{b}_1(s)F_1(s,w(s),v(s))ds \\
&\le 1.1843 \int_{H(\widetilde{m})} (4+|\ln r_1^*|+|\ln r_2^*|+|\ln\widehat{\omega}(s)|)ds \\
&< 1.1843(4+|\ln r_1^*|+|\ln r_2^*|)\int_{H(\widetilde{m})} ds + 2\int_{H(\widetilde{m})} |\ln\widehat{\omega}(s)|ds \\
&= \frac{2.3686\left(4+|\ln r_1^*|+|\ln r_2^*|\right)}{\widetilde{m}} + \int_{H(\widetilde{m})} |\ln\widehat{\omega}(s)|ds.
\end{aligned}
$$

By

$$\lim_{\widetilde{m}\to+\infty}\frac{1}{\widetilde{m}}=0,\ \lim_{\widetilde{m}\to+\infty}\int_{H(\widetilde{m})}|\ln\widehat{\omega}(t)|dt=0,$$

we know

$$\lim_{\widetilde{m}\to+\infty}\frac{2.3686(4+|\ln r_1^*|+|\ln r_2^*|)}{\widetilde{m}}+\int_{H(\widetilde{m})}|\ln\widehat{\omega}(s)|ds=0,$$

so, we have

$$\lim_{\widetilde{m}\to+\infty}\sup_{(w,v)\in\mathcal{K}_{[r_1^*,r_2^*]}}\int_{H(\widetilde{m})}\xi_1(s)\mathsf{b}_1(s)F_1(s,w(s),v(s))ds=0. \tag{4.4}$$

Similar to (4.4), we also have

$$\lim_{\widetilde{m}\to+\infty}\sup_{(w,v)\in\mathcal{K}_{[r_1^*,r_2^*]}}\int_{H(\widetilde{m})}\xi_1(s)\mathsf{b}_2(s)F_2(s,w(s),v(s))ds=0.$$

These imply that condition ($\mathbf{H}_1$) holds.

By calculation, $F_1^\infty=F_2^\infty=0$, and $f_{10}=f_{20}=+\infty$. Therefore, by Theorem 3.1, for each $\lambda_i\in(0,+\infty)$ ($i=1,2$), we get that system (4.1) has at least one positive solution.

5. Conclusions

In this article, we investigate a singular fractional differential system involving integral boundary value conditions. The existence and explicit interval can be acquired under the argument of fixed-point theory. Since f_i ($i=1,2$) is the abstract function, in the actual world, there are quantities of functions that satisfy the requirements of this article, which proves the effectiveness and feasibility of these theorems.

Author contributions

Ying Wang: Writing original draft, methodology, proof of conclusions; Linmin Guo: Methodology, proof of conclusions; Yumei Zi: Validation, writing review, editing; Jing Li: Validation, writing review, editing. All the authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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