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Research article

Nondecreasing analytic radius for the a Kawahara-Korteweg-de-Vries equation

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Abstract: By using linear, bilinear, and trilinear estimates in Bourgain-type spaces and analytic spaces, the local well-posedness of the Cauchy problem for the a Kawahara-Korteweg-de-Vries equation

$$\partial_t u + \omega\partial_x^5 u + \nu\partial_x^3 u + \mu\partial_x u^2 + \lambda\partial_x u^3 + \mathfrak{d}(x)u = 0,$$

was established for analytic initial data u_0. Besides, based on the obtained local result, together with an analytic approximate conservation law, we prove that the global solutions exist. Furthermore, the analytic radius has a fixed positive lower bound uniformly for all time.

Keywords: analytic radius; Kawahara-Korteweg-de-Vries equation; damped effect; nonlinear equations
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1. Introduction and function spaces

The initial-boundary value problem

$$\partial_t u + a\partial_x u + \partial_{xxx} u + g(u)\partial_x u = 0 \tag{1.1}$$

includes the Korteweg-de-Vries equation when $g(u) = u$ and the modified Korteweg-de Vries equation if $g(u) = \mp u^2$, which describes the propagation of one-dimensional nonlinear waves in media with dispersion and without dissipation. Similar models have previously been considered [9, 16, 19, 21]. In [21], the authors considered the problem for the equation

$$\partial_t u + \partial_x u + \partial_{xxx} u + g(u)\partial_x u + b(x)u = 0. \tag{1.2}$$

It is assumed that the function b is non-negative in the space $L^2(0, L)$. It was also shown that problem (1.2) has a solution in a suitable space. In [19], similar results for problem (1.2) were obtained with stronger conditions being satisfied on b in [21]. In [16], it was shown that in the case $g(u) = u^4$, problem (1.2) has a solution in an appropriate space under some conditions for the initial and boundary data. In [9], authors considered questions on the existence and uniqueness of solutions and their decay at large times for the initial boundary value problem in the case of more general cases. In [10], it was analyzed whether the condition of smallness of the initial data for the Korteweg-de-Vries equation and the modified Korteweg-de-Vries equation with initial boundary conditions is necessary for the solutions to decrease to zero for large times. To answer this question, the authors found conditions under which these problems have stationary solutions $u = u(x)$.
On the other hand, the Cauchy problem for the Kawahara equation is given by

$$\partial_t u - \partial_{xxxxx} u + b\partial_{xxx} u + u\partial_x u = f(t, x). \tag{1.3}$$

Equation (1.3) was first derived in article [15] to describe long nonlinear waves in media with weak dispersion and was later called the Kawahara equation. It should be noted that in various physical models, the coefficient b can be positive, negative, or zero, see [13, 18]. The Kawahara equation is a generalization of the Korteweg-de-Vries equation

$$\partial_t u + \partial_{xxx} u + u\partial_x u = f,$$

to the case of a higher-order dispersion relation. The study of the Kawahara equation largely follows the study of the Korteweg-de-Vries equation, but has some special features. Equation (1.3) (for $f \equiv 0$) has two conservation laws

$$\int_{\mathbb{R}} u^2 dx = C, \quad \int_{\mathbb{R}} ((\partial_{xx} u)^2 + b(\partial_x u)^2 dx - \frac{1}{3} u^3) dx = C.$$

It was on the basis of these equalities that, in article [23], the result was obtained on the existence and uniqueness of a time-global solution to problem (1.3). A similar result was also established in [5]. From the article [11], the result on the existence of a global solution to the Kawahara equation for an irregular initial function was obtained, see [2–4, 8, 12]. Let $x \in \mathbb{R}$ and $t \geq 0$; in this article, we consider a Cauchy problem for a Kawahara-Korteweg-de-Vries equation in analytic spaces

$$\begin{cases} \partial_t u + \omega\partial_x^5 u + \nu\partial_x^3 u + \mu\partial_x u^2 + \lambda\partial_x u^3 = 0, \\ u(x, t = 0) = u_0(x). \end{cases} \tag{1.4}$$

Here, the coefficients $\omega \neq 0$, ν, λ, and μ are real constants. From a physical point of view, the Kawahara-Korteweg-de-Vries-type equation models magnetic-acoustic-waves in plasma and nonlinear water wave propagation in the long-wavelength region. In the absence of the last term in our model, it

becomes a higher-order nonlinear dispersive equation. When it comes to low regularity for the higher-order nonlinear dispersive equation, our strategy mainly relies on the standard contraction mapping principle, an approximate conservation law in the modified Gevrey space $H^{\sigma,1}$, linear estimates, and bilinear estimates; see [25, 26].
The Kawahara-Korteweg-de-Vries equation with a damping term is considered as

$$\begin{cases} \partial_t u + \omega\partial_x^5 u + \nu\partial_x^3 u + \mu\partial_x u^2 + \lambda\partial_x u^3 + \mathfrak{d}(x)u = 0, \\ u(x, t = 0) = u_0(x), \end{cases} \tag{1.5}$$

and $\mathfrak{d}(x)$ is the damping term.
We need to state assumptions on the damping term $\mathfrak{d}(x)$.

(A1) $\exists\delta > 0$ such that

$$\mathfrak{d}(x) \geq \delta > 0, \qquad \text{for all } x \in \mathbb{R},$$

this is to ensure the damping effect.

(A2) $\exists M_1, M_2 > 0$ such that

$$\|\partial_x^j \mathfrak{d}(x)\|_{L^\infty(\mathbb{R})} \leq M_1 M_2^j (j!), \qquad \text{for all } j \in \mathbb{N},$$

for the analyticity.

A closely related work [17] established a uniformly positive lower bound of analytic radius for the KdV equation with a damping term $|D|^\alpha u$. Although the damping effect is proposed in different setting, the result is the same. Note that the damping $|D|^\alpha u$ is stronger than $D(x)u$ for high frequencies $|\varepsilon| \geq 1$ but weaker for low frequencies $|\varepsilon| < 1$. Moreover, uniform positive lower bound still holds if the damping is replaced by log-damping $(log|D|)u$, or more generally $h(D)u$, where $h(D)$ is a Fourier multiplier; see [1].

Notation: We first introduce some notations and function spaces used in this article. We state the following operator

$$\Lambda^\rho = \cosh(\rho|D|), \tag{1.6}$$

$$\widehat{\Lambda^\rho u}^x(\zeta, t) = \cosh(\rho|\zeta|)\widehat{u}^x(\zeta, t), \tag{1.7}$$

$$\Delta = \partial_x\left[(\Lambda^\rho u)^2 - \Lambda^\rho u^2\right], \tag{1.8}$$

$$\Theta = \partial_x\left[(\Lambda^\rho u)^3 - \Lambda^\rho u^3\right], \tag{1.9}$$

$$\Gamma = \left[\mathfrak{d}\Lambda^\rho u - \Lambda^\rho(\mathfrak{d}u)\right]. \tag{1.10}$$

Here

$$V(x, t) = \cosh(\rho|D|)u(x, t),$$

where $u(x, t)$ is the solution to (1.5). Thus

$$u(x, t) = sech(\rho|D|)V(x, t).$$

A class of analytic functions suitable for our analysis is the analytic class $\mathcal{G}^\rho(\mathbb{R})$, which will be given by

$$\mathcal{G}^\rho(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|^2_{\mathcal{G}^\rho(\mathbb{R})} = \int_{\mathbb{R}} \exp(2\rho|\zeta|)|\mathcal{F}_x(f(\zeta))|^2 d\zeta < \infty \right\}. \tag{1.11}$$

In what follows, a nice choice of the analytic function space for our arguments is the modified analytic space $\mathcal{L}^{\rho}(\mathbb{R})$, $\rho \geq 0$, given by the norm

$$\mathcal{L}^{\rho}(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \|f\|^2_{\mathcal{L}^{\rho}(\mathbb{R})} = \int_{\mathbb{R}} \cosh^2(\rho|\zeta|)|\mathcal{F}_x(f(\zeta))|^2 d\zeta < \infty \right\}. \tag{1.12}$$

As a consequence of $\exp(\rho|\zeta|) \sim \cosh(\rho|\zeta|)$, the norms $\| \cdot \|^2_{\mathcal{L}^{\rho}(\mathbb{R})}$ and $\| \cdot \|^2_{\mathcal{G}^{\rho}(\mathbb{R})}$ are equivalent.

Remark 1.1. *If $\rho = 0$, the space $\mathcal{L}^0(\mathbb{R})$ is reduced to $L^2(\mathbb{R})$.*

We will need to recall the embedding property of analytic space. For all $0 < \rho' < \rho$, we have

$$\mathcal{L}^{\rho}(\mathbb{R}) \subset \mathcal{L}^{\rho'}(\mathbb{R}), \tag{1.13}$$

i.e.,

$$\|f\|_{\mathcal{L}^{\rho'}} \leq C_{\rho,\rho'} \|f\|_{\mathcal{L}^{\rho}}.$$

Let us consider the space that is a hybrid between the known analytic space and the space of Fourier restriction. For $b \in \mathbb{R}$ and $\rho > 0$, we define the spaces $\mathcal{Y}_{\rho,b}(\mathbb{R}^2)$ to be the Banach space equipped with

$$\|u\|^2_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \cosh^2(\rho|\zeta|)(1 + |\eta + \phi(\zeta)|)^{2b} |\mathcal{F}_{x,t}(u(\zeta,\eta))|^2 d\zeta d\eta,$$

where $\phi(\zeta) = \omega\zeta^5 - \nu\zeta^3$. For $\rho = 0$, $\mathcal{Y}_{\rho,b}(\mathbb{R}^2)$ coincides with the space $\mathcal{Y}_b(\mathbb{R}^2)$

$$\|u\|^2_{\mathcal{Y}_b(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |\eta + \phi(\zeta)|)^{2b} |\widehat{u}(\zeta,\eta)|^2 d\zeta d\eta.$$

For $T > 0$, the spaces $\mathcal{Y}^T_{\rho,b}(\mathbb{R}^2)$ denote the restricted analytic spaces given by

$$\|u\|_{\mathcal{Y}^T_{\rho,b}(\mathbb{R}^2)} = \inf \left\{ \|v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} : v = u \ on \ (0,T) \times \mathbb{R} \right\}. \tag{1.14}$$

We will show in the next lemma that $\mathcal{Y}_{\rho,b}(\mathbb{R}^2)$ is continuously embedded in $C([0,T], \mathcal{L}^{\rho}(\mathbb{R}))$, where $b > 1/2$.

Lemma 1.1. *Let $b > \frac{1}{2}$ and $\rho > 0$, Then, for all $u \in \mathcal{Y}_{\rho,b}(\mathbb{R}^2)$ we have*

$$|u|_{T,\rho} = \sup_{t\in[0,T]} \|u(\cdot,t)\|_{\mathcal{L}^{\rho}(\mathbb{R})} \lesssim_b \|u\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}.$$

2. Basic estimates

2.1. Linear estimates

By using the Duhamel's formula, the solution of (1.5) is given by

$$u(x,t) = S(t)u_0(x) - \int_0^t S(t-\tau)(N_1(x,\tau) + N_2(x,\tau) + +\mathfrak{d}(x)u(x,\tau))d\tau,$$

and the unit operator related to the corresponding linear equation is

$$S(t) = \mathcal{F}_x^{-1} e^{-it(\omega\zeta^5 - \nu\zeta^3)} \mathcal{F}_x,$$

where the nonlinear terms N_1, N_2 are given by $\mu\partial_x(u^2), \lambda\partial_x(u^3)$, respectively.
Using a cut-off function $\varpi \in C_0^\infty(\mathbb{R})$, where $\varpi = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$, supp$\varpi \subset [-1, 1]$ and $\varpi_T(t) = \varpi(\frac{t}{T})$ to localize it in a time variable. Let us define the operator Φu by

$$\Phi(u)(x,t) = \varpi_1(t)S(t)u_0(x) - \varpi_1(t)\int_0^t S(t-\tau)(N_1(x,\tau) + N_2(x,\tau) + \mathfrak{d}(x)u(x,\tau))d\tau. \tag{2.1}$$

We have to solve $\Phi(u) = u$ for estimating (2.1) in the next lemma.

Lemma 2.1. *[20] Let $\rho > 0$ and $\frac{1}{2} < b < 1$, we have*

$$\|\varpi_1(t)S(t)u_0(x)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \lesssim_{\varpi_1} \|u_0(x)\|_{\mathcal{L}^\rho(\mathbb{R})}. \tag{2.2}$$

$$\left\|\varpi_1(t)\int_0^t S(t-\tau)N(x,\tau)\mathrm{d}\tau\right\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \lesssim_{\varpi_1} \|N(x,t)\|_{\mathcal{Y}_{\rho,b-1}(\mathbb{R}^2)}. \tag{2.3}$$

If $\frac{1}{2} < b \le b' < 1$, then for any $T > 0$, we have

$$\|\varpi_T(t)N(x,t)\|_{\mathcal{Y}_{\rho,b-1}(\mathbb{R}^2)} \lesssim_{b,b'} T^{b'-b}\|N(x,t)\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}. \tag{2.4}$$

Lemma 2.2. *[22] If $-\frac{1}{2} < b < \frac{1}{2}$, then for every interval $I \subset [0, T]$*

$$\|\chi_I u\|_{\mathcal{Y}_{\rho,b}} \lesssim_b \|u\|_{\mathcal{Y}^T_{\rho,b}},$$

where χ_I denotes the characteristic function of I.

Lemma 2.3. *[24] For all $\rho > 0$, we have*

$$\left\|\chi_{[0,\delta]}(t)e^{-2\delta(T-t)}u\right\|_{\mathcal{Y}_{\rho,1-b}} \lesssim \|u\|_{\mathcal{Y}^T_{\rho,b}}. \tag{2.5}$$

Lemma 2.4. *[24] For all $\rho > 0, \mathfrak{d} \in \mathcal{A}^\rho(\mathbb{R}), u \in \mathcal{L}^\rho(\mathbb{R})$, we have*

$$\|\mathfrak{d}(x)u(x)\|_{\mathcal{L}^\rho(\mathbb{R})} \lesssim \|\mathfrak{d}(x)\|_{\mathcal{A}^\rho(\mathbb{R})}\|u(x)\|_{\mathcal{L}^\rho(\mathbb{R})}. \tag{2.6}$$

For all $T \in (0, 1], b' \le 0, b \ge 0$, we have

$$\|\mathfrak{d}(x)u(x,t)\|_{\mathcal{Y}^T_{\sigma,b'}} \lesssim \|\mathfrak{d}(x)\|_{\mathcal{A}^\rho(\mathbb{R})}\|u(x,t)\|_{\mathcal{Y}^T_{\sigma,b}}. \tag{2.7}$$

2.2. Trilinear and bilinear estimate

The following result provides the basic trilinear and bilinear estimation needed to prove the Theorems 3.1 and 4.1. The following lemma shows the required linear estimation.

Theorem 2.1. *[14] Let $b_1' > \frac{1}{2}$ be close enough to $\frac{1}{2}$ and $b_1 > \frac{1}{2}$. Then*

$$\left\|\partial_x \prod_{i=1}^{2} u_i\right\|_{\mathcal{Y}_{b_1'-1}(\mathbb{R}^2)} \lesssim_{b_1,b_1'} \prod_{i=1}^{2}\|u_i\|_{\mathcal{Y}_{b_1}(\mathbb{R}^2)}. \tag{2.8}$$

Remark 2.1. *Setting*

$$f_i(\zeta,\eta) = (1 + |\eta + \phi(\zeta)|)^{b_1}\widehat{u_i}(\zeta,\eta), \quad i = 1, 2,$$

the estimate of Theorem 2.1 can be rewritten as

$$\begin{aligned}
&\|\partial_x(u_1u_2)\|_{\mathcal{Y}_{b_1'-1}(\mathbb{R}^2)} \\
&= \left\| \frac{\zeta}{(1 + |\eta + \phi(\zeta)|)^{b_1'-1}} \widehat{u_1u_2}(\zeta,\eta) \right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\
&= C\left\| \frac{\zeta}{(1 + |\eta + \phi(\zeta)|)^{b_1'-1}} \int_{\mathbb{R}^2} \widehat{u_1}(\zeta_1,\eta_1)\widehat{u_2}(\zeta-\zeta_1,\eta-\eta_1)d\zeta_1 d\eta_1 \right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\
&= C\left\| \frac{\zeta}{(1 + |\eta + \phi(\zeta)|)^{b_1-1}} \int_{\mathbb{R}^2} \frac{\widehat{f_1}(\zeta_1,\eta_1)}{(1 + |\eta_1 + \phi(\zeta_1)|)^{b_1}} \frac{\widehat{f_2}(\zeta-\zeta_1,\eta-\eta_1)}{(1 + |\eta - \eta_1 + \phi(\zeta-\zeta_1)|)^{b_1}} \right. \\
&\quad \left. \cdot\widehat{u_2}(\zeta-\zeta_1,\eta-\eta_1)d\zeta_1 d\eta_1 \right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\
&\lesssim_{b_1,b_1'} \|f_1\|_{L^2_{\zeta,\eta}}\|f_2\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}.
\end{aligned} \tag{2.9}$$

Lemma 2.5. *Let $\rho > 0$ and $b_1' > \frac{1}{2}$ be close enough to $\frac{1}{2}$ and $b_1 > \frac{1}{2}$. Then*

$$\left\|\partial_x \prod_{i=1}^{2} u_i\right\|_{\mathcal{Y}_{\rho,b_1'-1}(\mathbb{R}^2)} \lesssim_{b_1,b_1'} \prod_{i=1}^{2} \|u_i\|_{\mathcal{Y}_{\rho,b_1}(\mathbb{R}^2)}. \tag{2.10}$$

Proof. By considering the operator Λ^ρ in (1.7), we have

$$\begin{aligned}
&\|\partial_x(u_1u_2)\|_{\mathcal{Y}_{\rho,b_1'-1}(\mathbb{R}^2)}\\
&= \left\|\cosh(\rho|\zeta|)(1+|\eta+\phi(\zeta)|)^{b_1'-1}\partial_x\widehat{(u_1u_2)}(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&\sim \left\|\cosh(\rho|\zeta|)(1+|\eta+\phi(\zeta)|)^{b_1'-1}\partial_x\widehat{(u_1u_2)}(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&= \left\|(1+|\eta+\phi(\zeta)|)^{b_1'-1}\zeta\cosh(\rho|\zeta|)\mathcal{F}_{x,t}\left(sech(\rho|D|)V_1(x,t)sech(\rho|D|)V_2(x,t)\right)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&\lesssim \left\|(1+|\eta+\phi(\zeta)|)^{b_1'-1}|\zeta|\int_{\mathbb{R}^2}\cdot\widehat{V_1^\rho}(\zeta-\zeta_1,\eta-\eta_1)\widehat{V_2^\rho}(\zeta_1,\eta_1)d\zeta_1d\eta_1\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&\lesssim \left\|(1+|\eta+\phi(\zeta)|)^{b_1'-1}|\zeta|\mathcal{F}_{x,t}(\cosh(\rho|D|)u_1\cosh(\rho|D|)u_2)(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&\sim \left\||D|(\Lambda^\rho u_1\Lambda^\rho u_2)\right\|_{\mathcal{Y}_{b_1'-1}(\mathbb{R}^2)},
\end{aligned}$$

where

$$\cosh(\rho|\zeta|)\left(sech(\rho|\zeta_1|)sech(\rho|\zeta-\zeta_1|)\right)\leq 2,$$

and

$$V_i^\rho(x,t)=\cosh(\rho|D|)u_i(x,t),\ i=1,2,$$

with u being the solution to (1.5). Thus

$$u_i(x,t)=sech(\rho|D|)V_i(x,t).$$

Now, by using Theorem 2.1, we get

$$\begin{aligned}
\||D|(\Lambda^\rho u_1\Lambda^\rho u_2)\|_{Xb_1'-1(\mathbb{R}^2)} &\lesssim_{b_1,b_1'} \|\Lambda^\rho u_1\|_{\mathcal{Y}_{b_1}(\mathbb{R}^2)}\|\Lambda^\rho u_2\|_{\mathcal{Y}_{b_1}(\mathbb{R}^2)}\\
&= \|u_1\|_{\mathcal{Y}_{\rho,b_1}(\mathbb{R}^2)}\|u_2\|_{\mathcal{Y}_{\rho,b_1}(\mathbb{R}^2)}.
\end{aligned}$$

□

The next result states the needed trilinear estimate.

Lemma 2.6. *[14] Let $b>\frac{1}{2}$ and $\frac{1}{2}<b'-1<\frac{7}{10}$. Then*

$$\left\|\partial_x\prod_{i=1}^{3}u_i\right\|_{\mathcal{Y}_{b'-1}(\mathbb{R}^2)}\lesssim_{b,b'}\prod_{i=1}^{3}\|u_i\|_{\mathcal{Y}_b(\mathbb{R}^2)}.$$

Remark 2.2. *Setting*

$$g_i(\zeta,\eta) = (1+|\eta+\phi(\zeta)|)^b \widehat{u_i}(\zeta,\eta), \quad i=1,2,3,$$

the estimate of Theorem 2.6 can be rewritten as

$$\begin{aligned}
&\|\partial_x(u_1u_2u_3)\|_{\mathcal{Y}_{b'-1}(\mathbb{R}^2)}\\
&= \left\| \frac{\zeta}{(1+|\eta+\phi(\zeta)|)^{b'-1}} \widehat{u_1u_2u_3}(\zeta,\eta) \right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&= C\left\| \frac{\zeta}{(1+|\eta+\phi(\zeta)|)^{b'-1}} \int_{\mathbb{R}^4} \widehat{u_1}(\zeta_1,\eta_1)\widehat{u_2}(\zeta_2,\eta_2)\widehat{u_3}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&= C\left\| \frac{\zeta}{(1+|\eta+\phi(\zeta)|)^{b'-1}} \int_{\mathbb{R}^4} \frac{\widehat{g_1}(\zeta_1,\eta_1)}{(1+|\eta_1+\phi(\zeta_1)|)^b} \frac{\widehat{g_2}(\zeta_2,\eta_2)}{(1+|\eta_2+\phi(\zeta_2)|)^b} \right.\\
&\quad \left. \cdot \frac{\widehat{g_3}(\zeta-\zeta_1-\zeta_2,\eta-\eta_1-\eta_2)}{(1+|\eta-\eta_1-\eta_2+\phi(\zeta-\zeta_1-\zeta_2)|)^b} d\zeta_1 d\zeta_2 d\eta_1 d\eta_2 \right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}\\
&\lesssim_{b,b'} \|g_1\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \|g_2\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \|g_3\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)}.
\end{aligned}$$

Lemma 2.7. *Let* $\rho > 0$, $b > \frac{1}{2}$ *and* $\frac{1}{2} < b' - 1 < \frac{7}{10}$. *Then*

$$\left\| \partial_x \prod_{i=1}^{3} u_i \right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)} \lesssim_{b,b'} \prod_{i=1}^{3} \| u_i \|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} .$$

Proof. The proof is similar to that of Lemma 2.5. □

2.3. Energy estimates

Lemma 2.8. *Given* $b = \frac{1}{2} + \epsilon$, *such that for all* $T > 0$ *and* $u \in \mathcal{Y}_{\rho,b}(\mathbb{R}^2)$, *we have*

$$\|\chi_{[0,T]}(\cdot)\Delta\|_{\mathcal{Y}_{b-1}(\mathbb{R}^2)} \lesssim \rho^2 \|u\|^2_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}, \tag{2.11}$$

$$\|\chi_{[0,T]}(\cdot)\Theta\|_{\mathcal{Y}_{b-1}(\mathbb{R}^2)} \lesssim \rho^2 \|u\|^3_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}. \tag{2.12}$$

Proof. By symmetry, we may assume $|\zeta_1| \geq |\zeta_2| \geq \cdots \geq |\zeta_p|$. By Lemma 3 in [7] ,

$$\begin{aligned}
\left| 1 - \cosh(\rho|\zeta|) \prod_{j=1}^{p} \operatorname{sech}\left(\rho|\zeta_j|\right) \right| &\leq 2^p \sum_{j\neq k=1}^{p} |\rho\zeta_j|\,|\rho\zeta_k| \\
&\leq p^2 2^p \rho^2\,|\zeta_1|\,|\zeta_2|\,.
\end{aligned} \tag{2.13}$$

Taking the Fourier transform we have

$$\mathcal{F}_x\left[(\Lambda^\rho u)^2 - \Lambda^\rho u^2\right](\zeta) = \int_{\zeta=\zeta_1+\zeta_2} \left[1 - \cosh(\rho|\zeta|) \prod_{j=1}^{2} \operatorname{sech}\left(\rho|\zeta_j|\right) \right] \prod_{j=1}^{2} \widehat{V^\rho}\left(\zeta_j\right) d\zeta_1 d\zeta_2,$$

and

$$\mathcal{F}_x\Big[(\Lambda^\rho u)^3 - \Lambda^\rho u^3\Big](\zeta) = \int_{\zeta=\zeta_1+\zeta_2+\zeta_3}\left[1-\cosh(\rho|\zeta|)\prod_{j=1}^{3}\operatorname{sech}\Big(\rho|\zeta_j|\Big)\right]\prod_{j=1}^{3}\widehat{V^\rho}\Big(\zeta_j\Big)\,d\zeta_1 d\zeta_2 d\zeta_3.$$

Now, set $w_\rho = \mathcal{F}_x^{-1}(|\widehat{V^\rho}|)$. Then, applying (2.13)

$$\begin{aligned}\left|\mathcal{F}_x\Big[(\Lambda^\rho u)^2 - \Lambda^\rho u^2\Big](\zeta)\right| &\leq p^2 2^p \rho^2 \int_{\zeta=\zeta_1+\zeta_2} |\zeta_1||\zeta_2|\left|\widehat{V^\rho}(\zeta_1)\right|\left|\widehat{V^\rho}(\zeta_2)\right| d\zeta_1 d\zeta_2 \\ &= p^2 2^p \rho^2 \int_{\zeta=\zeta_1+\zeta_2} |\zeta_1|\,|\zeta_2|\,\widehat{w_\rho}(\zeta_1)\,\widehat{w_\rho}(\zeta_2)\,d\zeta_1 d\zeta_2 \\ &= p^2 2^p \rho^2 \mathcal{F}_x\left(\Big(|D|w_\rho\Big)^2\right)(\zeta),\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{F}_x\Big[(\Lambda^\rho u)^3 - \Lambda^\rho u^3\Big](\zeta)\right| &\leq p^2 2^p \rho^2 \int_{\zeta=\zeta_1+\zeta_2+\zeta_3} |\zeta_1|\,|\zeta_2|\left|\widehat{V^\rho}(\zeta_1)\right|\left|\widehat{V^\rho}(\zeta_2)\right|\left|\widehat{V^\rho}(\zeta_3)\right| d\zeta_1 d\zeta_2 d\zeta_3 \\ &= p^2 2^p \rho^2 \int_{\zeta=\zeta_1+\zeta_2+\zeta_3} |\zeta_1|\,|\zeta_2|\,\widehat{w_\rho}(\zeta_1)\,\widehat{w_\rho}(\zeta_2)\,\widehat{w_\rho}(\zeta_3)\,d\zeta_1 d\zeta_2 d\zeta_3 \\ &= p^2 2^p \rho^2 \mathcal{F}_x\left(\Big(|D|w_\rho\Big)^2\cdot w_\rho\right)(\zeta).\end{aligned}$$

Therefore, using Plancherel identity, Hölder inequality, and Sobolev embedding, we obtain

$$\begin{aligned}|\chi_{[0,T]}(\cdot)\Delta| &\lesssim \rho^2 \left\|\Big(|D|w_\rho\Big)^2\right\|_{\mathcal{Y}_{b'-1}} \\ &\lesssim \rho^2 \left\|w_\rho\right\|_{\mathcal{Y}_b}^2 \\ &\sim \rho^2 \|V^\rho\|_{\mathcal{Y}_b}^2,\end{aligned}$$

and

$$\begin{aligned}|\chi_{[0,T]}(\cdot)\Theta| &\lesssim \rho^2 \left\|\Big(|D|w_\rho\Big)^2 w_\rho\right\|_{\mathcal{Y}_{b'-1}} \\ &\lesssim \rho^2 \left\|w_\rho\right\|_{\mathcal{Y}_b}^3 \\ &\sim \rho^2 \|V^\rho\|_{\mathcal{Y}_b}^3.\end{aligned}$$

□

Lemma 2.9. *([24]) Let $d \in \mathcal{A}^{\rho_0}$ with $\rho_0 > 0$. Then, for all $0 < \rho \leq \rho_0$ we have*

$$\|\Gamma\|_{L^2(\mathbb{R})} \leq 2\frac{\rho}{\rho_0}\|d\|_{\mathcal{A}^{\rho_0}(\mathbb{R})}\|u\|_{\mathcal{L}^\rho(\mathbb{R})} + 2\|\mathfrak{d}\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}. \tag{2.14}$$

3. Local well-posedness

Based on the above estimations, using the contraction argument, we prove the local well-posedness. Without the damping term, Coclite and Ruvo [6] established the well-posedness of the classical solutions of (1.4). We are going to extend this result and prove the local well-posedness in $\mathcal{L}^\rho$ of the Cauchy problem (1.4) with a weakly damping in (1.5).

Theorem 3.1. *(Local well-posedness in $\mathcal{L}^\rho(\mathbb{R})$) Let $\rho > 0$ and $u_0 \in \mathcal{L}^\rho$. Then, $\exists b > \frac{1}{2}$, which is near enough to $\frac{1}{2}$ and $T > 0$ depending only on ρ, s, and u_0, such that (1.5) is locally well-posed in $C([0,T], \mathcal{L}^\rho(\mathbb{R}))$. Besides, the solution u satisfies*

$$\|u(t)\|_{\mathcal{L}^\rho} \leq c\|u\|_{\mathcal{Y}_{\rho,b}} \leq 2C\|u_0\|_{\mathcal{L}^\rho},\ t \in [0,T], \tag{3.1}$$

where

$$T = \frac{c_0}{(1 + \|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})} + \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + \|u_0\|_{\mathcal{L}^\rho(\mathbb{R})})^{\frac{1}{b'-b}}}, \tag{3.2}$$

for certain constants

$$c_0 = (16C^3 2^{b'-b} \max\{|\mu|, |\lambda|\})^{-\frac{1}{b'-b}} > 0, \quad a > 1, \quad C > 0.$$

3.1. Existence of solution

Let $\mathbb{B}$ be a closed ball. The standard Banach contraction maping of the functions $\mathcal{Y}_{\sigma,\rho,s,b}(\mathbb{R}^2)$ in $\mathbb{B}$ will be used to prove the local well-posedness in the analytic Gevrey spaces. For this end, we will need the integral operator

$$\begin{aligned} \Phi(u)(x,t) &= \varpi_1(t)S(t)u_0(x) - \varpi_1(t)\int_0^t S(t-\tau)(\mathfrak{d}(x)u(x,\tau)) \\ &+ \partial_x(\mu u^2(x,\tau) + \lambda u^3(x,\tau)))d\tau, \end{aligned} \tag{3.3}$$

to deal with the nonlinear estimates. Additionally, we introduce a cut-off in

$$\mu\partial_x u^2, \lambda\partial_x u^3, \mathfrak{d}(x),$$

and consider

$$\begin{aligned} \Phi(u)(x,t) = \varpi_1(t)S(t)u_0(x) - \varpi_1(t)\int_0^t S(t-\tau)(\varpi_{2T}(\tau)\mathfrak{d}(x)u(x,\tau) \\ + \partial_x(\mu\varpi_{2T}(\tau)u^2(x,\tau) + \lambda\varpi_{2T}(\tau)u^3(x,\tau)))d\tau. \end{aligned} \tag{3.4}$$

Since $\varpi_{2T} = 1$ on support of ϖ_T, the Eq (3.4) is identical with (3.3).

Lemma 3.1. *Let $\rho > 0$ and $b > \frac{1}{2}$. Then, for all $u_0 \in \mathcal{L}^\rho(\mathbb{R})$ and $0 < T < 1$, with some constant $C > 0$, we have*

$$\begin{aligned} \|\Phi(u)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} &\leq C\|u_0\|_{\mathcal{L}^\rho(\mathbb{R}^2)} + C\max\{|\mu|, |\lambda|\}(2T)^{b'-b} \\ &\times\Big(\|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})}\|u\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} + \|u\|^2_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} + \|u\|^3_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\Big), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} &\leq C\max\{|\mu|, |\lambda|\}(2T)^{b'-b} \\ &\times\|\mathfrak{d} + u^2 + v^2 + uv + u + v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\|u - v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}. \end{aligned} \tag{3.6}$$

Proof. To begin by the estimate (3.5), it follows

$$\begin{aligned}
&\|\Phi(u)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\\
&\leq \|\varpi_1(t)S(t)u_0\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\\
&+\left\|\varpi_1(t)\int_0^t S(t-\tau)(\varpi_{2T}(\tau)\mathfrak{d}(x)u(\tau)+\partial_x(\mu\varpi_{2T}(\tau)u^2(\tau)+\lambda\varpi_{2T}(\tau)u^3(\tau)))d\tau\right\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\\
&\leq C\|u_0\|_{\mathcal{L}^\rho(\mathbb{R}^2)}+C\max\{|\mu|,|\lambda|\}\left\|\varpi_{2T}(t)(\mathfrak{d}u+\partial_x(u^2+u^3))\right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}\\
&\leq C\|u_0\|_{\mathcal{L}^\rho(\mathbb{R}^2)}+C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\left\|\mathfrak{d}u+\partial_x(u^2+u^3)\right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}\\
&\leq C\|u_0\|_{\mathcal{L}^\rho(\mathbb{R}^2)}+C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}(\|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})}\|u\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}+\|u\|^2_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}+\|u\|^3_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}).
\end{aligned}$$

Next, for (3.6), we have

$$\begin{aligned}
&\|\Phi(u)-\Phi(v)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\\
&=\left\|\varpi_1(t)\int_0^t S(t-\tau)\varpi_{2T}(\tau)\left(\mathfrak{d}(u-v)+\mu\partial_x(u^2-v^2)+\lambda\partial_x(u^3-v^3)\right)(\tau)d\tau\right\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\\
&\leq C\max\{|\mu|,|\lambda|\}\left\|\varpi_{2T}(\tau)(\mathfrak{d}(u-v)+\partial_x\left((u^2-v^2)+(u^3-v^3)\right)\right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}\\
&\leq C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\left\|\mathfrak{d}(u-v)+\partial_x\left((u^2-v^2)+(u^3-v^3)\right)\right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}\\
&=C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\left\|\partial_x\left(\mathfrak{d}(u-v)+(u+v)(u-v)+(u^2+v^2+uv)(u-v)\right)\right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}\\
&=C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\left\|\partial_x\left((\mathfrak{d}+(u+v)+(u^2+v^2+uv))(u-v)\right)\right\|_{\mathcal{Y}_{\rho,b'-1}(\mathbb{R}^2)}\\
&\leq C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\|\mathfrak{d}+u^2+v^2+uv+u+v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\|u-v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}.
\end{aligned}$$

Then, the proof is now completed. □

It will be shown that the map $\Phi:\mathbb{B}(0,r)\to\mathbb{B}(0,r)$ is a contraction, where

$$\mathbb{B}(0,r)=\{u\in\mathcal{Y}_{\rho,b}(\mathbb{R}^2);\|u\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\leq r\}\text{ with }r=2C\|u_0\|_{\mathcal{L}^\rho(\mathbb{R})}.$$

Proposition 3.1. *Let $\rho>0$ and $b>\frac{1}{2}$. Then, for all $u_0\in\mathcal{L}^\rho(\mathbb{R})$, such that*

$$T=\frac{c_0}{(1+\|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})}+\|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})}+\|u_0\|_{\mathcal{L}^\rho(\mathbb{R})})^{\frac{1}{b'-b}}},\tag{3.7}$$

then, the map Φ is a contraction on the ball $\mathbb{B}(0,r)$ to $\mathbb{B}(0,r)$.

Proof. By Lemma 3.1, $\forall u \in \mathbb{B}(0,r)$, we have

$$\begin{aligned}\|\Phi(u)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} &\leq C\|u_0\|_{\mathcal{L}^\rho(\mathbb{R}^2)} + C\max\{|\mu|,|\lambda|\}(2T)^{b'-b} \\ &\times\Big(\|\eth\|_{\mathcal{A}^\rho(\mathbb{R})}\|u\|_{\mathcal{L}^\rho(\mathbb{R})} + \|u\|^2_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} + \|u\|^3_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\Big) \\ &\leq \frac{r}{2} + C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}(\|\eth\|_{\mathcal{A}^\rho(\mathbb{R})}r + r^2 + r^3).\end{aligned}$$

If we take

$$c_0 = (16C^3 2^{b'-b}\max\{|\mu|,|\lambda\})^{-\frac{1}{b'-b}},$$

then, for T given as in (3.7), we get

$$T \leq \frac{1}{(4C\max\{|\mu|,|\lambda\}2^{b'-b}(\|\eth\|_{\mathcal{A}^\rho(\mathbb{R})} + r^2 + r))^{\frac{1}{b'-b}}},$$

and hence

$$\|\ \Phi(u)\ \|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \leq r, \quad \forall u \in \mathbb{B}(0,r).$$

Then, $\Phi : \mathbb{B}(0,r) \to \mathbb{B}(0,r)$ is a contraction, since

$$\begin{aligned}&\|\Phi(u) - \Phi(v)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \\ &\leq C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\|\eth + u^2 + v^2 + uv + u + v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\|u-v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \\ &\leq C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}\left(\|\eth\|_{\mathcal{A}^\rho(\mathbb{R})} + 3r^2 + 2r\right)\|u-v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \\ &\leq C\max\{|\mu|,|\lambda|\}(2T)^{b'-b}3\left(\|\eth\|_{\mathcal{A}^\rho(\mathbb{R})} + r^2 + r\right)\|u-v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \\ &\leq \frac{3}{4}\ \|\ u - v\ \|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}\ .\end{aligned}$$

This completes the proof. □

3.2. The uniqueness

We shall prove the uniqueness of the solution in $C([0,T],\mathcal{L}^\rho(\mathbb{R}))$.

Lemma 3.2. *Let u and v be two solutions of (1.5) in $C([0,T],\mathcal{L}^\rho(\mathbb{R}))$ with $u(\cdot,t=0) = v(\cdot,t=0)$ in $\mathcal{L}^\rho(\mathbb{R})$, where $\rho > 0$. Then, $u = v$.*

Proof. Putting $w = u - v$, we see that w solves

$$\partial_t w + \omega\partial_x^5 w + \nu\partial_x^3 w + \eth w + \mu\partial_x w(u+v) + \lambda\partial_x w(u^2 + v^2 + uv) = 0, \tag{3.8}$$

with $w(\cdot,t=0) = 0$. By multiplication of (3.8) with w and after integration with respect to x, we find

$$w\partial_t w + \omega w\partial_x^5 w + \nu w\partial_x^3 w + \eth w + \mu w\partial_x w(u+v) + \lambda w\partial_x w(u^2 + v^2 + uv) = 0.$$

Then, by simple calculation, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R})} &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} w^2 dx = \int_{\mathbb{R}} w\partial_t w dx \\ &= \int_{\mathbb{R}} \eth w^2 dx - \mu \int_{\mathbb{R}} w\partial_x w(u+v)dx \\ &-\lambda \int_{\mathbb{R}} w\partial_x w(u^2+v^2+uv)dx,\end{aligned} \tag{3.9}$$

since

$$\int_{\mathbb{R}} w\partial_x^5 w dx = \int_{\mathbb{R}} w\partial_x^3 w dx = 0.$$

By (3.9), we have

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}} \eth w^2 dx - \mu \int_{\mathbb{R}} w\partial_x w(u+v)dx \\ &-\lambda \int_{\mathbb{R}} w\partial_x w(u^2+v^2+uv)dx, \\ &= -2\int_{\mathbb{R}} \eth w^2(t) dx - 2\mu \int_{\mathbb{R}} w\partial_x \left[fw\right] dx \\ &-2\lambda \int_{\mathbb{R}} w\partial_x \left[gw\right] dx,\end{aligned}$$

where $f = u + v$ and $g = u^2 + v^2 + uv$. With integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}} \eth w^2 dx - \mu \int_{\mathbb{R}} \partial_x f w^2 dx \\ &- \lambda \int_{\mathbb{R}} \partial_x g w^2 dx.\end{aligned}$$

Then,

$$\left|\frac{d}{dt}\|w(t,\cdot)\|^2_{L^2(\mathbb{R})}\right| \leq \max\{|\mu|,|\lambda|\}(\|\eth\|_{L^\infty(\mathbb{R})} + \|\partial_x f\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x g\|_{L^\infty([0,T]\times\mathbb{R})})\|w(t)\|^2_{L^2(\mathbb{R})}. \tag{3.10}$$

Since $u, v \in C([0,T], \mathcal{L}^\rho(\mathbb{R}))$ we have that u and v are continuous in t on the compact set $[0,T]$ and are $\mathcal{L}^\rho(\mathbb{R})$ in x. Thus, we can conclude that

$$\max\{|\mu|,|\lambda|\}(\|\eth\|_{L^\infty(\mathbb{R})} + \|\partial_x f\|_{L^\infty([0,T]\times\mathbb{R})} + \|\partial_x g\|_{L^\infty([0,T]\times\mathbb{R})}) \leq c < \infty. \tag{3.11}$$

Therefore, from (3.10) and (3.11), we obtain the differential inequality

$$\left|\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R})}\right| \leq c\|w(t)\|^2_{L^2(\mathbb{R})}, \quad t \in [0,T].$$

Then,

$$\|w(t)\|^2_{L^2(\mathbb{R})} \leq e^c\|w(0)\|^2_{L^2(\mathbb{R})}, \; t \in [0,T]. \tag{3.12}$$

Since $\|w(0)\|^2_{L^2(\mathbb{R})} = 0$, from (3.12) we find that $w(t) = 0$, $0 \leq t \leq T$ or $u = v$.

□

3.3. Continuous dependence of the initial data

The next lemma will be useful.

Lemma 3.3. *Let $\rho > 0$ and $b > \frac{1}{2}$. Then, for all $u_0, v_0 \in \mathcal{L}^\rho(\mathbb{R})$, if u and v are two solutions to (1.5) corresponding to initial data u_0 and v_0. We have*

$$|u-v|_{T,\rho} \leq 4C_0C\|u_0 - v_0\|_{\mathcal{L}^\rho(\mathbb{R})}. \tag{3.13}$$

Proof. We have by Lemma 1.1

$$\begin{aligned} |u-v|_{T,\rho} &= \sup_{t\in[0,T]} \|u(\cdot,t) - v(\cdot,t)\|_{G^\rho(\mathbb{R})} \\ &\leq C_0\|u-v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \\ &= C_0\|\Phi(u) - \Phi(v)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}. \end{aligned}$$

Taking $u, v \in \mathbb{B}(0,r)$ and

$$T \leq \frac{1}{(4C\max\{|\mu|,|\lambda|\}2^{b'-b}(\|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})} + r^2 + r))^{\frac{1}{b'-b}}},$$

we have

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \leq C\|u_0 - v_0\|_{\mathcal{L}^\rho(\mathbb{R})} + \frac{3}{4}\|u-v\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)}.$$

Thus

$$\|\Phi(u) - \Phi(v)\|_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} \leq 4C\|u_0 - v_0\|_{\mathcal{L}^\rho(\mathbb{R})},$$

then

$$|u-v|_{T,\rho} \leq 4C_0C\|u_0 - v_0\|_{\mathcal{L}^\rho(\mathbb{R})}.$$

□

This completes the proof of Theorem 3.1.

4. Lower bound for radius of spatial analyticity

We need to state the property of the space $\mathcal{L}^\rho(\mathbb{R})$. For $\rho > 0$, it is not hard to show that if the function $f \in \mathcal{L}^\rho(\mathbb{R})$, then it is the restriction to the real line of a holomorphic function $f(x+iy)$ in the strip

$$S_\rho = \{x + iy \in \mathbb{C}, |y| < \rho\}.$$

The $\rho > 0$ is the uniform radius of spatial analyticity for f.
The next Paley-Wiener theorem provides an alternative description of $\mathcal{L}^\rho$.

Paley-Wiener Theorem. The function $f \in \mathcal{L}^\rho$ if and only if $f(x)$ is the restriction to the real line of a holomorphic function $f(x+iy)$ in the strip

$$S_\rho = \{x + iy \in \mathbb{C}, |y| < \rho\},$$

and satisfies the bound

$$\sup_{|y|<\rho} \|f(x+iy)\|_{L_x^2} < \infty.$$

From the point of view of Paley-Wiener theory, it is natural to take the initial data in $\mathcal{L}^{\rho}(\mathbb{R})$ and get a better understanding of the behavior of the solution as we try to scale it globally in time. It means that given $u_0 \in \mathcal{L}^{\rho}(\mathbb{R})$ for some initial radius $\rho > 0$, we want to estimate the behavior of the radius of analyticity $\rho(T)$ as time T grows.

In Section 3, we proved the local well-posedness in $\mathcal{L}^{\rho}(\mathbb{R})$ with $\rho > 0$, i.e., the local solution is analytic in x. Now, we use the obtained local result together with a Gevrey approximate conservation law to be extended for all time. Furthermore, we obtain explicit lower bounds on the radius of spatial analyticity $r(t)$ at any time $t \geq 0$, which is given by $r(t) \geq ct^{-1}$, where $c > 0$, based on an approximate conservation law in the modified Gevrey space; see [27].

Theorem 4.1. *Assume that (A1), (A2) hold and $u_0 \in \mathcal{L}^{\rho_0}$ for some $\rho_0 > 0$. Then, the initial value problem (1.5) has a global solution $u(t) \in \mathcal{L}^{\rho(t)}$ with*

$$\rho(t) \geq \rho_1 > 0, \quad \textit{moreover} \quad \|u(t)\|_{\mathcal{L}^{\rho_1}} \leq Ce^{-\frac{\delta t}{2}}, \quad \textit{for all } t \geq 0,$$

where ρ_1 depends on $\|u(t)\|_{\mathcal{L}^{\rho_0}}, \rho_0, \mathfrak{d}(x)$ and C depends only on $\mathfrak{d}(x), \|u(t)\|_{\mathcal{L}^{\rho_0}}$.

The method used here for proving lower bounds on the radius of analyticity was introduced in [22] in the context of the 1D Dirac-Klein-Gordon equations. It was also applied to the non-periodic KdV equation in [7]. Ming Wang [24] found that, for the damped KdV equation, the analytic radius has a fixed positive lower bound uniformly for all time.

4.1. Approximate conservation law

Recall that

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(x,t)dx + 2\int_{\mathbb{R}} \mathfrak{d}(x)u^2(x,t)dx = 0. \tag{4.1}$$

If $\mathfrak{d}(x) \geq \delta > 0$ on $\mathbb{R}$, then by Gronwall inequality, we deduce from (4.1) that

$$\|u(t)\|_{L^2}^2 \leq e^{-2\delta t}\|u_0\|_{L^2}^2. \tag{4.2}$$

We will establish an approximate conservation law for a solution to (1.5) based on the conservation $L^2(\mathbb{R})$ norm of solutions.

Theorem 4.2. *Let $0 < T < T_1 < 1$, T_1 be as in Theorem 3.1, $\exists b = \frac{1}{2} + \epsilon$, such that $\forall \rho > 0$ and any solution $u \in \mathcal{Y}_{\rho,b}^T(\mathbb{R}^2)$ to the problem (1.5) on $[0,T]$, we have*

$$\|u(T)\|_{\mathcal{L}^{\rho}(\mathbb{R})}^2 \leq e^{-2\delta T}\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})}^2 + C_1\rho^2\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})}^3 + C_2\rho^2\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})}^4$$

$$+ C_3\rho\|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})}\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})}^2 + C_4\|\mathfrak{d}\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^2(\mathbb{R})}\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})}.$$

Proof. Let $V^{\rho}(t,x) = \cosh(\rho|D|)u(t,x)$. Since u is real-valued, we also have V^{ρ} real-valued; applying the exponential $\cosh(\rho|D|)$ to Eq (1.5), it is easily seen that we obtain

$$\partial_t V^\rho + \omega\partial_x^5 V^\rho + \nu\partial_x^3 V^\rho + \mu\Lambda^\rho(\partial_x u^2) + \lambda\Lambda^\rho(\partial_x u^3) + \Lambda^\rho(\mathfrak{d}u) = 0,$$

which is equivalent to

$$\begin{aligned}&\partial_t V^\rho + \omega\partial_x^5 V^\rho + \nu\partial_x^3 V^\rho + 2\mu V^\rho\partial_x V^\rho + 3\lambda(V^\rho)^2\partial_x V^\rho + \mathfrak{d}V^\rho\\ &= \mu\partial_x((V^\rho)^2 - \Lambda^\rho u^2) + \lambda\partial_x((V^\rho)^3 - \Lambda^\rho u^3) + \mathfrak{d}\Lambda^\rho u - \Lambda^\rho(\mathfrak{d}u).\end{aligned}$$

We use Δ, Θ and Γ, and obtain

$$\partial_t V^\rho + \omega\partial_x^5 V^\rho + \nu\partial_x^3 V^\rho + 2\mu V^\rho\partial_x V^\rho + 3\lambda(V^\rho)^2\partial_x V^\rho + \mathfrak{d}V^\rho = \mu\Delta + \lambda\Theta + \Gamma. \tag{4.3}$$

Multiplying (4.3) by V^ρ and integrating in space, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} V^\rho\partial_t V^\rho dx + \omega\int_{\mathbb{R}} V^\rho\partial_x^5 V^\rho dx + \nu\int_{\mathbb{R}} V^\rho\partial_x^3 V^\rho dx + 2\mu\int_{\mathbb{R}}(V^\rho)^2\partial_x V^\rho dx+\\ &3\lambda\int_{\mathbb{R}}(V^\rho)^3\partial_x V^\rho dx + \int_{\mathbb{R}} V^\rho\mathfrak{d}V^\rho dx = \mu\int_{\mathbb{R}} V^\rho\Delta dx + \lambda\int_{\mathbb{R}} V^\rho\Theta dx + \int_{\mathbb{R}} V^\rho\Gamma dx.\end{aligned} \tag{4.4}$$

The integration by parts is justified, since we may assume that $V^\rho(t, x)$ decays to zero as $|x| \longrightarrow \infty$, and the same holds for all spatial derivatives. Thus, (4.4) can be rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(V^\rho)^2 dx + \frac{\omega}{2}\int_{\mathbb{R}}\partial_x(\partial_x^2 V^\rho\partial_x^2 V^\rho)dx + \frac{\nu}{2}\int_{\mathbb{R}}\partial_x(\partial_x V^\rho\partial_x V^\rho)dx + \frac{2\mu}{3}\int_{\mathbb{R}}\partial_x(V^\rho)^3 dx+\\ &\frac{3\lambda}{4}\int_{\mathbb{R}}\partial_x(V^\rho)^4 dx + \int_{\mathbb{R}}\mathfrak{d}(x)(V^\rho)^2 dx = \mu\int_{\mathbb{R}} V^\rho\Delta dx + \lambda\int_{\mathbb{R}} V^\rho\Theta dx + \int_{\mathbb{R}} V^\rho\Gamma dx,\end{aligned} \tag{4.5}$$

and the second, third, forth, and fifth terms on the left side vanish

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}(V^\rho)^2 dx + \int_{\mathbb{R}}\mathfrak{d}(x)(V^\rho)^2(x,t)dx = \mu\int_{\mathbb{R}} V^\rho\Delta dx + \lambda\int_{\mathbb{R}} V^\rho\Theta dx + \int_{\mathbb{R}} V^\rho\Gamma dx.$$

Since $\mathfrak{d}(x) \geq \delta > 0$ for all $x \in \mathbb{R}$, we get

$$\frac{d}{dt}\int_{\mathbb{R}}(V^\rho)^2 dx + 2\delta\int_{\mathbb{R}}(V^\rho)^2(x,t)dx \leq 2\mu\int_{\mathbb{R}} V^\rho\Delta dx + 2\lambda\int_{\mathbb{R}} V^\rho\Theta dx + 2\int_{\mathbb{R}} V^\rho\Gamma dx.$$

Applying Gronwall lemma , we have

$$\begin{aligned}\|u(T)\|_{\mathcal{L}^\rho}^2 \leq e^{-2\delta T}\|u(0)\|_{\mathcal{L}^\rho}^2 &+ 2|\mu|\left|\int_{\mathbb{R}^2}\chi_{[0,T]}(t)e^{-2\delta(T-\eta)}V^\rho\Delta\; dxd\eta\right|\\ &+ 2|\lambda|\left|\int_{\mathbb{R}^2}\chi_{[0,T]}(t)e^{-2\delta(T-\eta)}V^\rho\Theta\; dxd\eta\right|\\ &+ 2\left|\int_0^T\int_{\mathbb{R}}\chi_{[0,T]}(t)e^{-2\delta(T-\eta)}V^\rho\Gamma\; dxd\eta\right|\end{aligned}$$

$$\leq e^{-2\delta T}\|u(0)\|^2_{\mathcal{L}^\rho} + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

Recalling that V^ρ is real-valued, it follows from Plancherel's identity and Cauchy-Schwarz inequality that

$$\begin{aligned}\mathcal{I}_1 &= 2|\mu|\left|\int_{\mathbb{R}^2}(\chi_{[0,T]}(\cdot)e^{-2\delta(T-\eta)}V^\rho)(\zeta,\eta)\overline{(\chi_{[0,T]}(\cdot)\Delta)}(\zeta,\eta)d\zeta d\eta\right| \\ &= 2|\mu|\left|\int_{\mathbb{R}^2}(\chi_{[0,T]}(\cdot)\widehat{e^{-2\delta}(T-\eta)}V^\rho)(\zeta,\eta)\overline{(\widehat{\chi_{[0,T]}(\cdot)}\Delta)}(\zeta,\eta)d\zeta d\eta\right|.\end{aligned}$$

Then, Hölder's inequality yields

$$\begin{aligned}\mathcal{I}_1 &\leq 2|\mu|\left\|(1+|\eta+\phi(\zeta)|)^{1-b}(\chi_{[0,T]}\widehat{e^{-2\delta(T-\eta)}}V^\rho)(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\ &\times\left\|(1+|\eta+\phi(\zeta)|)^{b-1}(\widehat{\chi_{[0,T]}}\Delta)(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\ &\leq 2|\mu|\left\|\chi_{[0,T]}(\cdot)e^{-2\delta(T-\eta)}V^\rho\right\|_{\mathcal{Y}_{1-b}(\mathbb{R}^2)}\left\|\chi_{[0,T]}(\cdot)\Delta\right\|_{\mathcal{Y}_{b-1}(\mathbb{R}^2)},\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}\mathcal{I}_2 &= 2|\lambda|\left|\int_{\mathbb{R}^2}(\chi_{[0,T]}(\cdot)e^{-2\delta(T-\eta)}V^\rho)(\zeta,\eta)\overline{(\chi_{[0,T]}(\cdot)\Theta)}(\zeta,\eta)d\zeta d\eta\right| \\ &= 2|\lambda|\left|\int_{\mathbb{R}^2}(\chi_{[0,T]}(\cdot)\widehat{e^{-2\delta}(T-\eta)}V^\rho)(\zeta,\eta)\overline{(\widehat{\chi_{[0,T]}(\cdot)}\Theta)}(\zeta,\eta)d\zeta d\eta\right|.\end{aligned}$$

Then, Hölder's inequality yields

$$\begin{aligned}\mathcal{I}_2 &\leq 2|\lambda|\left\|(1+|\eta+\phi(\zeta)|)^{1-b}(\chi_{[0,T]}\widehat{e^{-2\delta(T-\eta)}}V^\rho)(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\ &\times\left\|(1+|\eta+\phi(\zeta)|)^{b-1}(\widehat{\chi_{[0,T]}}\Theta)(\zeta,\eta)\right\|_{L^2_{\zeta,\eta}(\mathbb{R}^2)} \\ &\leq 2|\lambda|\left\|\chi_{[0,T]}(\cdot)e^{-2\delta(T-\eta)}V^\rho\right\|_{\mathcal{Y}_{1-b}(\mathbb{R}^2)}\left\|\chi_{[0,T]}(\cdot)\Theta\right\|_{\mathcal{Y}_{b-1}(\mathbb{R}^2)},\end{aligned}\tag{4.7}$$

we have both $-\frac{1}{2} < b-1 < \frac{1}{2}$ and $\frac{1}{2} < 1-b < \frac{1}{2}$. Therefore, one can use Lemmas 2.3 and 2.2 to obtain

$$\begin{aligned}\left\|\chi_{[0,T]}(\cdot)e^{-2\delta(T-\eta)}V^\rho\right\|_{\mathcal{Y}_{1-b}(\mathbb{R}^2)} &= \left\|\chi_{[0,T]}(\cdot)e^{-2\delta(T-\eta)}u\right\|_{\mathcal{Y}_{\rho,1-b}(\mathbb{R}^2)} \\ &\lesssim \left\|u\right\|_{\mathcal{Y}^T_{\rho,1-b}(\mathbb{R}^2)},\end{aligned}$$

since we have $1 - b < b$, we can conclude from it and Lemma 2.8 that

$$\mathcal{I}_1 + \mathcal{I}_2 \lesssim \rho^2 \|u\|^3_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)} + \rho^2 \|u\|^4_{\mathcal{Y}_{\rho,b}(\mathbb{R}^2)},$$

by using the condition (3.1) we conclude that

$$\mathcal{I}_1 + \mathcal{I}_2 \lesssim \rho^2 \|u_0\|^3_{\mathcal{L}^\rho(\mathbb{R})} + \rho^2 \|u_0\|^4_{\mathcal{L}^\rho(\mathbb{R})}.$$

$$\mathcal{I}_3 \leq C_3 \rho \|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})} \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + C_4 \|\mathfrak{d}\|_{L^\infty(\mathbb{R})} \|u_0\|_{L^2(\mathbb{R})} \|u_0\|_{\mathcal{L}^\rho(\mathbb{R})}.$$

Therefore, we conclude that

$$\begin{aligned}\|u(T)\|^2_{\mathcal{L}^\rho} &\leq e^{-2\delta T} \|u(0)\|^2_{\mathcal{L}^\rho} + \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \\ &\leq e^{-2\delta T} \|u_0\|^2_{\mathcal{L}^\rho} + C_1 \rho^\kappa \|u_0\|^3_{\mathcal{L}^\rho(\mathbb{R})} + C_2 \rho^\varrho \|u_0\|^4_{\mathcal{L}^\rho(\mathbb{R})} \\ &\quad + C_3 \rho \|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})} \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + C_4 \|\mathfrak{d}\|_{L^\infty(\mathbb{R})} \|u_0\|_{L^2(\mathbb{R})} \|u_0\|_{\mathcal{L}^\rho(\mathbb{R})}.\end{aligned}$$

□

4.2. *Global existence and analytic radius lower bound*

According to assumption (A2), we have $\|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})} < \infty$, and we can assume that $u_0 \in \mathcal{L}^{\rho_0}(\mathbb{R})$ with $\rho_0 < M_2^{-1}$. Then, we show that $d(x) \in \mathcal{A}^{\rho_0}(\mathbb{R})$. According to Theorem 3.1, there exists a unique solution

$$C\left([0, T], \mathcal{L}^\rho(\mathbb{R})\right),$$

of the initial value problem (1.5) with life span

$$T = \frac{c_0}{(1 + \|\mathfrak{d}\|_{\mathcal{A}^\rho(\mathbb{R})} + \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + \|u_0\|_{\mathcal{L}^\rho(\mathbb{R})})^{\frac{1}{b'-b}}}.$$

Moreover, by Theorem 4.2, we have the estimate

$$\begin{aligned}\|u(T)\|^2_{\mathcal{L}^\rho(\mathbb{R})} &\leq e^{-2\delta T} \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + C_1 \rho^\kappa \|u_0\|^3_{\mathcal{L}^\rho(\mathbb{R})} + C_2 \rho^\varrho \|u_0\|^4_{\mathcal{L}^\rho(\mathbb{R})} \\ &\quad + C_3 \rho \|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})} \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + C_4 \|\mathfrak{d}\|_{L^\infty(\mathbb{R})} \|u_0\|_{L^2(\mathbb{R})} \|u_0\|_{\mathcal{L}^\rho(\mathbb{R})}.\end{aligned}$$

Now, using the inequality

$$C_4 \|\mathfrak{d}\|_{L^\infty(\mathbb{R})} \|u_0\|_{L^2(\mathbb{R})} \|u_0\|_{\mathcal{L}^\rho(\mathbb{R})} \leq \frac{1 - e^{-2\delta T}}{2} \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + C_5 \|u_0\|^2_{L^2(\mathbb{R})},$$

we deduce that

$$\|u(T)\|^2_{\mathcal{L}^\rho(\mathbb{R})} \leq \frac{1 + e^{-2\delta T}}{2} \|u_0\|^2_{\mathcal{L}^\rho(\mathbb{R})} + C_1 \rho^\kappa \|u_0\|^3_{\mathcal{L}^\rho(\mathbb{R})} + C_2 \rho^\varrho \|u_0\|^4_{\mathcal{L}^\rho(\mathbb{R})}$$

$$+ C_3\rho\|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})}\|u_0\|^2_{\mathcal{L}^{\rho}(\mathbb{R})} + C_5\|u_0\|^2_{L^2(\mathbb{R})}.$$

Now choose

$$\rho = \rho_1 = \min\left\{\rho_0, \frac{1-e^{-2\delta T}}{6C_3\|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})}}, \left(\frac{1-e^{-2\delta T}}{6C_1\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})}}\right)^{\frac{1}{\kappa}}, \left(\frac{1-e^{-2\delta T}}{6C_2\|u_0\|^2_{\mathcal{L}^{\rho}(\mathbb{R})}}\right)^{\frac{1}{\varrho}}\right\}, \tag{4.8}$$

so that

$$C_1\rho^{\kappa}\|u_0\|_{\mathcal{L}^{\rho}(\mathbb{R})} + C_2\rho^{\varrho}\|u_0\|^2_{\mathcal{L}^{\rho}(\mathbb{R})} + C_3\rho\|\mathfrak{d}\|_{\mathcal{A}^{\rho_0}(\mathbb{R})} \leq \frac{1-e^{-2\delta T}}{2}.$$

Then,

$$\|u(T)\|^2_{\mathcal{L}^{\rho_1}(\mathbb{R})} \leq \|u_0\|^2_{\mathcal{L}^{\rho_1}(\mathbb{R})} + C_5\|u_0\|^2_{L^2(\mathbb{R})}. \tag{4.9}$$

Now, the solution on $[nT, (n+1)T]$ proceeds as above

$$\|u(nT)\|^2_{\mathcal{L}^{\rho_1}(\mathbb{R})} \leq \|u_0\|^2_{\mathcal{L}^{\rho_1}(\mathbb{R})} + C_5\sum_{i=1}^{n} e^{-2(i-1)\delta T}\|u_0\|^2_{L^2(\mathbb{R})}, \quad \forall n \geq 1. \tag{4.10}$$

we find that

$$\|u(t)\|^2_{\mathcal{L}^{\rho_1}(\mathbb{R})} \leq \|u_0\|^2_{\mathcal{L}^{\rho_1}(\mathbb{R})} + C_6\|u_0\|^2_{L^2(\mathbb{R})}, \quad \forall t \geq 0 \tag{4.11}$$

where $C_6 > 0$ is independent of k. Thanks to (4.11), we have

$$\|u(t)\|_{\mathcal{L}^{\rho_1}(\mathbb{R})} \leq C_7, \quad \forall t \geq 0. \tag{4.12}$$

Recalling the exponential decay $\|u(t)\|^2_{L^2} \leq e^{-2\delta t}\|u_0\|^2_{L^2}$, we deduce from (4.12) that

$$\|u(t)\|_{\mathcal{L}^{\frac{\rho_1}{2}}(\mathbb{R})} \leq \|u(t)\|^{\frac{1}{2}}_{\mathcal{L}^{\rho_1}(\mathbb{R})}\|u(t)\|^{\frac{1}{2}}_{L^2(\mathbb{R})} \leq \sqrt{C_7}e^{\frac{-\delta t}{2}}\|u_0\|^{\frac{1}{2}}_{L^2(\mathbb{R})}, \quad \forall t \geq 0.$$

This completes the proof of Theorem 4.1.

5. Conclusions

The local well-posedness for the Kawahara-Korteweg-de-Vries equation with a weakly damping (1.5) in modified analytic space $\mathcal{L}^{\rho}(\mathbb{R})$, $\rho \geq 0$ defined in (1.12) with $\|f\|^2\mathcal{L}^{\rho}(\mathbb{R}) \sim \|f\|^2\mathcal{G}^{\rho}(\mathbb{R})$ is discussed. The local well-posedness is showed by using the Banach contraction mapping principle together with the bilinear and trilinear estimates in the space of Fourier restriction $\mathcal{Y}_{\rho,b}(\mathbb{R}^2)$. The local result with the approximate conservation law

$$\frac{d}{dt}\int_{\mathbb{R}} u^2(x,t)dx + 2\int_{\mathbb{R}} \mathfrak{d}(x)u^2(x,t)dx,$$

is extended to be global in time. Besides, the analytic radius lower bound is proved.

Author contributions

Aissa Boukarou: Writing—original draft preparation; Mohamed Bouye, Abdelkader Moumen: Writing—review and editing; Khaled Zennir: Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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