AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 22393–22413.
DOI: 10.3934/math.20241089
Received: 26 April 2024
Revised: 29 June 2024
Accepted: 09 July 2024
Published: 18 July 2024

Research article

Several fixed-point theorems for generalized Ćirić-type contraction in G_b-metric spaces

Yunpeng Zhao[1,2], **Fei He**[1,*] **and Shumin Lu**[1]

[1] School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China
[2] Shanxi Vocational University of Engineering Science and Technology, Jinzhong 030619, China

* **Correspondence:** Email: hefei@imu.edu.cn.

Abstract: In the framework of G_b-metric spaces, we introduce the concept of a generalized Ćirić-type contraction and obtain several fixed-point theorems for this contraction. First, we present a significant lemma, which is used to ensure that the Picard sequence is a Cauchy sequence. Using this lemma, we establish three fixed-point theorems satisfying different conditions. Second, we construct new examples to illustrate our results. As applications, we deduce the famous Ćirić fixed-point theorem in terms of b-metric spaces using our results. In addition, we obtain Reich-type contraction fixed-point theorems in such a space using the aforementioned lemma. Our results improve and complement many recent findings. In particular, we substantially enlarge the range of the contraction constant in our results. Finally, we consider the existence and uniqueness of solutions for integral equation applying our new results.

Keywords: generalized Ćirić-type contraction; Reich-type contraction; fixed point; G_b-metric spaces
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

In 1989, Bakhtin [1] presented a kind of generalized metric space, that was named b-metric space. In 1998, Czerwik [2] reintroduced this notion from the constant $s = 2$ to a constant $s \geq 1$. After this pioneering work, many types of fixed-point theorems were presented by different authors (for example, refer to [3–9]). In 2006, Mustafa and Sims [10] gave the notion of G-metric, a function $d : X \times X \times X \to [0, \infty)$ satisfying certain conditions. Based on G-metric spaces, a large number of fixed-point theorems were proposed (for example, see [11–16]). In 2014, A. Aghajani et al. [17] defined a new type of metric using the concepts of G-metric and b-metric, which was called G_b-metric.

On the other hand, in 1974, Ćirić [18] established the Ćirić-type fixed point theorem in metric spaces. From then on, fixed-point results for many kinds of Ćirić-type contractions in metric and

b-metric spaces have been studied (for example, see [19–23]). Also, in 2008, Mustafa et al. [11] proved some fixed-point results for Reich-type and Ćirić-type contractions in complete G-metric space. In 2017, A. H. Ansari et al. [24] proved a new approach for some common fixed-point results in complete G_p-metric spaces defined by partial metric spaces and G-metric spaces. More recently, in 2019, Min Liang et al. [25] established some new theorems for various cyclic contractions in G_b-metric spaces. In 2019, Hassen Aydi et al. [26] investigated the unique fixed point of Ćirić-type contractions with seven metrics in G_b-metric space with coefficient $s \geq 1$, in which the range of contraction constant is $[0, \frac{1}{2s})$. In 2021, Vishal Gupat et al. [27] investigated a kind of Reich-type fixed-point theorem where the range of contraction constant is $[0, \frac{1}{2s})$ in G_b-metric space.

Inspired by the above literature, in this work, we introduce the generalized Ćirić-type contraction in G_b-metric spaces and investigate some Ćirić-type and Reich-type fixed-point theorems. The generalized Ćirić-type contraction introduced in this paper involves ten metrics, while previous literature only considered, at most, seven metrics. Our new theorems can generalize and improve the related results in [26,27]. In particular, the range of contraction constant is enlarged from $[0, \frac{1}{2s})$ to $[0, \frac{1}{s})$ for Ćirić-type contraction, and the range of contraction constant is extended from $[0, \frac{1}{2s})$ to $[0, \frac{1}{\max\{2,s\}})$ for Reich-type fixed-point theorems. Moreover, a new example is given to illustrate our results, where G_b-metric is discontinuous. Using our results, we can obtain the famous Ćirić fixed-point theorem in metric spaces. Finally, we show the existence of a solution for the integral equation formulated in G_b-metric spaces.

Throughout this paper, we denote by $\mathbb{R}, \mathbb{N}_0$, and $\mathbb{N}$ the sets of real numbers, natural numbers, and positive integer numbers, respectively.

2. Preliminaries

First, we recall some concepts that are going to be used later.

A. Aghajani et al. [17] defined a new notion of G_b-metric using the concepts of G-metric and b-metric as follows.

Definition 2.1. *[17]* Let X be a nonempty set and $s \geq 1$ a given real number. Suppose that a mapping G_b: $X \times X \times X \to [0, \infty)$ satisfies:

(G_b1) $G_b(x, y, z) = 0$ if $x = y = z$,

(G_b2) $0 < G_b(x, x, y)$; for all $x, y \in X$ with $x \neq y$,

(G_b3) $G_b(x, x, y) \leq G_b(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,

(G_b4) $G_b(x, y, z) = G_b(x, z, y) = G_b(y, z, x) = ...$, (symmetry in all three variables),

(G_b5) $G_b(x, y, z) \leq s(G_b(x, a, a) + G_b(a, y, z))$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G_b is called a generalized b-metric, and the pair (X, G_b) is a generalized b-metric space or G_b-metric space with coefficient s.

We can construct an example of G_b-metric space by b-metric space.

Example 2.2. Let (X, d) be a b-metric space with $s \geq 1$. Define $G_b : X^3 \to [0, \infty)$ by

$$G_b(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\} \text{ for all } x, y, z \in X.$$

Then, (X, G_b) is a G_b-metric space with identical coefficient $s \geq 1$.

Proof. It is not difficult to check that G_b satisfies (G_b1)–(G_b4). Next, we prove that G_b satisfies the condition (G_b5).

In fact, for all $x, y, z \in X$, and $s \geq 1$, we find

$$\begin{aligned} G_b(x,y,z) &= \max\{d(x,y), d(y,z), d(x,z)\} \\ &\leq \max\{s[d(x,a)+d(a,y)], s[d(x,a)+d(y,z)], s[d(x,a)+d(a,z)]\} \\ &= s[d(x,a)+\max\{d(a,y), d(y,z), d(a,z)\}] \\ &= s[G_b(x,a,a)+G_b(a,y,z)]. \end{aligned}$$

Thus G_b satisfies (G_b5), and then (X, G_b) is a G_b-metric space with identical coefficient $s \geq 1$.

Definition 2.3. *[17]* Let (X, G_b) be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) G_b-Cauchy sequence, or simply Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer n_0 such that, for all $m, n, l \geq n_0$, $G_b(x_n, x_m, x_l) < \epsilon$.

(2) G_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer n_0 such that, for all $m, n \geq n_0$, $G_b(x_n, x_m, x) < \epsilon$.

Definition 2.4. *[17]* A G_b-metric space (X, G_b) is called G_b-complete, or simply complete, if every G_b-Cauchy sequence is G_b-convergent in X.

Proposition 2.5. *[17]* Let (X, G_b) be a G_b-metric space, then the following are equivalent:

(1) the sequence $\{x_n\}$ is G_b-Cauchy.

(2) for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq n_0$.

Proposition 2.6. *[17]* Let (X, G_b) be a G_b-metric space, then the following are equivalent:

(1) $\{x_n\}$ is G_b-convergent to x.

(2) $G_b(x_n, x_n, x) \to 0$ as $n \to \infty$.

(3) $G_b(x_n, x, x) \to 0$ as $n \to \infty$.

The following is a new example of a G_b-metric space, which is completely different from the known examples in [25–27].

Example 2.7. Let $X = [0, \infty)$ and define

$$G_b(x,y,z) = \begin{cases} |x-y|+|y-z|+|x-z|, & xyz \neq 0, \\ 2(|x-y|+|y-z|+|x-z|), & xyz = 0. \end{cases}$$

Then the following holds:

(1) (X, G_b) is a G_b-metric space with coefficient $s = 2$.

(2) (X, G_b) is G_b-complete.

(3) G_b is discontinuous on X.

Proof. (1) It is not difficult to obtain that G_b satisfies (G_b1), (G_b2), (G_b4). Now, we verify that G_b satisfies (G_b3), i.e., $G_b(x,x,y) \leq G_b(x,y,z)$, for all $x, y, z \in X$ with $z \neq y$. We consider the following two cases:

Case 1. Assume that $xyz \neq 0$, then

$$G_b(x,x,y) = 2|x-y| \leq |x-y|+|y-z|+|x-z| = G_b(x,y,z).$$

Case 2. Suppose that $xyz = 0$, then at least one of x, y, and z must be 0.

If $x = 0, y = 0$, then we have $G_b(x,x,y) = 0$, and obviously, (G_b3) holds.

If $x = 0, y \neq 0$ or $x \neq 0, y = 0$, then

$$G_b(x,x,y) = 4|x-y| \leq 2(|x-y|+|y-z|+|x-z|) = G_b(x,y,z).$$

If $x \neq 0, y \neq 0$, then $z = 0$, and

$$G_b(x, x, y) = 2|x - y| \leq 2(|x - y| + |y - z| + |x - z|) = G_b(x, y, z).$$

Hence, (G_b3) holds.

Next, we verify that G_b satisfies (G_b5), i.e., $G_b(x, y, z) \leqslant 2(G_b(x, a, a) + G_b(a, y, z))$, for all $x, y, z, a \in X$. We consider the following two cases:

Case 1. Assume that $xyz \neq 0$.

If $a \neq 0$, then we get

$$\begin{aligned} G_b(x, y, z) &= |x - y| + |y - z| + |x - z| \leq 2|x - a| + |a - y| + |y - z| + |a - z| \\ &= G_b(x, a, a) + G_b(a, y, z) \leq 2(G_b(x, a, a) + G_b(a, y, z)). \end{aligned}$$

If $a = 0$, then we have that

$$\begin{aligned} &G_b(x, a, a) = 4|x - a| = 4|x|, \\ &G_b(a, y, z) = 2(|a - y| + |y - z| + |a - z|) = 2(|y| + |y - z| + |z|). \end{aligned}$$

Hence,

$$\begin{aligned} G_b(x, y, z) &= |x - y| + |y - z| + |x - z| \leq |x| + |y| + |y - z| + |x| + |z| \\ &\leq 4|x| + 2|y| + |y - z| + 2|z| \leq 4|x| + 2|y| + 2|y - z| + 2|z| \\ &= G_b(x, a, a) + G_b(a, y, z)) \leq 2(G_b(x, a, a) + G_b(a, y, z)). \end{aligned}$$

Case 2. Assume that $xyz = 0$.

If $a = 0$, then we know

$$\begin{aligned} G_b(x, y, z) &= 2(|x - y| + |y - z| + |x - z|) \leq 2|x| + 2|y| + 2|y - z| + 2|x| + 2|z|, \\ &= 4|x| + 2(|y| + |y - z| + +|z|) = G_b(x, a, a) + G_b(a, y, z) \\ &\leq 2(G_b(x, a, a) + G_b(a, y, z)). \end{aligned}$$

If $a \neq 0$, and $x \neq 0$, then at least one of y and z must be 0. Hence, $G_b(x, a, a) = 2|x - a|$, $G_b(a, y, z) = 2(|y - a| + |y - z| + |a - z|)$, and

$$\begin{aligned} G_b(x, y, z) &= 2(|x - y| + |y - z| + |x - z|) \\ &\leq 2(|x - a| + |y - a|) + 2|y - z| + 2(|x - a| + |a - z|) \\ &= 2G_b(x, a, a) + G_b(a, y, z) \leq 2(G_b(x, a, a) + G_b(a, y, z)). \end{aligned}$$

If $a \neq 0$, and $x = 0$, then $G_b(x, a, a) = 4|x - a|$. If $y \neq 0$, and $z \neq 0$, then we know $G_b(a, y, z) = |a - y| + |y - z| + |a - z|$, and

$$\begin{aligned} G_b(x, y, z) &= 2(|x - y| + |y - z| + |x - z|) \\ &\leq 2(|x - a| + |y - a|) + 2|y - z| + 2(|x - a| + |a - z|) \\ &= G_b(x, a, a) + 2G_b(a, y, z) \leq 2(G_b(x, a, a) + G_b(a, y, z)). \end{aligned}$$

If at least one of y and z must be 0, then $G_b(a, y, z) = 2(|a - y| + |y - z| + |a - z|)$, and

$$\begin{aligned} G_b(x, y, z) &= 2(|x - y| + |y - z| + |x - z|) \\ &\leq 2(|x - a| + |y - a|) + 2|y - z| + 2(|x - a| + |a - z|) \\ &= G_b(x, a, a) + G_b(a, y, z) \leq 2(G_b(x, a, a) + G_b(a, y, z)). \end{aligned}$$

Therefore, (X, G_b) is a G_b-metric space with coefficient $s = 2$.

(2) We prove that (X, G_b) is G_b-complete. Let $\{z_n\} \subset X$ be G_b-Cauchy sequence, i.e., $\lim_{m,n\to\infty} G_b(z_n, z_m, z_m) = 0$, that is, $|z_n - z_m| \to 0$ as $m, n \to \infty$. Thus, $\{z_n\}$ is a Cauchy sequence in $[0, \infty)$. Since $[0, \infty)$ with ordinary metric is complete, hence there exists $z_0 \in [0, \infty)$ such that $|z_n - z_0| \to 0$ as $n \to \infty$, thus, $G_b(z_n, z_0, z_0) \to 0$ as $n \to \infty$.

(3) We show that G_b is discontinuous on X. Indeed, let $x_n = \frac{1}{n}$, $n \in \mathbb{N}$ and $a = 1$, we have $x_n \to 0$ as $n \to \infty$ and $x_n \neq 0$ for any $n \in \mathbb{N}$. Hence,

$$\lim_{n\to\infty} G_b(x_n, x_n, a) = \lim_{n\to\infty} 2|1 - \frac{1}{n}| = 2 \neq 4 = G_b(0, 0, a).$$

3. Main results

Now we give the notion of generalized Ćirić-type contraction in G_b-metric spaces.

Definition 3.1. Let (X, G_b) be a G_b-metric space with coefficient $s \geq 1$, $T : X \to X$ is a map. Then, the map T is called a generalized Ćirić-type contraction, if there exists $\lambda \in [0, 1)$ such that for all $x, y, z \in X$,

$$\begin{aligned} G_b(Tx, Ty, Tz) \leq \lambda \max\{&G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), \\ &G_b(z, Tz, Tz), G_b(x, Ty, Ty), G_b(y, Tz, Tz), G_b(z, Tx, Tx) \\ &G_b(x, Tz, Tz), G_b(y, Tx, Tx), G_b(z, Ty, Ty)\}. \end{aligned} \tag{3.1}$$

And λ is called the contraction constant of T.

The following lemma is crucial and will be used to prove our main results.

Lemma 3.2. Let (X, G_b) be a G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map and $z_0 \in X$. Let $\{z_n\}_{n\in\mathbb{N}_0}$ be a sequence defined by $z_n = Tz_{n-1} = T^n z_0$ for all $n \in \mathbb{N}$. If T is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, 1)$, then $\{z_n\}_{n\in\mathbb{N}_0}$ is a G_b-Cauchy sequence.

Proof. Denote a set $\{(m, n) : m \in \mathbb{N}_0, n \in \mathbb{N} \text{ and } m < n\}$ by D.

Define $P : D \to [0, \infty)$ by

$$P(m, n) = \max\{G_b(z_i, z_j, z_j) : m \leq i, j \leq n\}.$$

We will prove it in four steps.

Step 1. We prove that

$$P(m + 1, n) \leq \lambda P(m, n) \text{ for any } (m, n) \in D \text{ with } n > m + 1. \tag{3.2}$$

Let $(m, n) \in D$ with $n > m + 1$ be given. For any $i, j \in [m + 1, n]$ and $i, j \in \mathbb{N}$, since T is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, 1)$, we have that

$$G_b(z_i, z_j, z_j) = G_b(Tz_{i-1}, Tz_{j-1}, Tz_{j-1})$$

$$\begin{aligned}
&\le \lambda \max\{G_b(z_{i-1}, z_{j-1}, z_{j-1}), G_b(z_{i-1}, z_i, z_i), G_b(z_{j-1}, z_j, z_j),\\
&\quad G_b(z_{j-1}, z_j, z_j), G_b(z_{i-1}, z_j, z_j), G_b(z_{j-1}, z_j, z_j), G_b(z_{j-1}, z_i, z_i),\\
&\quad G_b(z_{i-1}, z_j, z_j), G_b(z_{j-1}, z_i, z_i), G_b(z_{j-1}, z_j, z_j)\}\\
&\le \lambda P(m, n),
\end{aligned}$$

which deduces $P(m + 1, n) \le \lambda P(m, n) < P(m, n)$.

Step 2. We verify that

$$P(m, n) = \max\{G_b(z_m, z_{p_1}, z_{p_1}), G_b(z_{p_2}, z_m, z_m) : m < p_1, p_2 \le n\}. \tag{3.3}$$

Let $(m, n) \in D$ be given. If $n - m = 1$, then $n = m + 1$ and hence

$$\begin{aligned}
P(m, n) &= \max\{G_b(z_m, z_n, z_n), G_b(z_m, z_m, z_n)\}\\
&= \max\{G_b(z_m, z_{p_1}, z_{p_1}), G_b(z_{p_2}, z_m, z_m) : m < p_1, p_2 \le n\}.
\end{aligned}$$

We now suppose that $n - m > 1$, then $n > m + 1$. For any $i, j \in [m + 1, n]$ and $i, j \in \mathbb{N}$, by (3.2), we obtain $G_b(z_i, z_j, z_j) \le P(m + 1, n) \le \lambda P(m, n) < P(m, n)$. Thus,

$$P(m, n) = \max\{G_b(z_m, z_{p_1}, z_{p_1}), G_b(z_{p_2}, z_m, z_m) : m < p_1, p_2 \le n\}.$$

Step 3. We shall show that there exists $M > 0$ such that

$$P(0, n) \le M \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Since $0 \le \lambda < 1$, there exists $q \in \mathbb{N}$ such that $\lambda^q < \frac{1}{s}$.

If $P(0, n) \le P(0, q)$ for all $n \in \mathbb{N}$, then the conclusion holds. Otherwise, if $P(0, n_q) > P(0, q)$ for some $n_q \in \mathbb{N}$, by (3.3), we consider two cases:

On the one hand, there exists an integer $p_1 \le n_q$ and $p_1 > q$ such that

$$\begin{aligned}
P(0, n_q) = G_b(z_0, z_{p_1}, z_{p_1}) &\le sG_b(z_0, z_q, z_q) + sG_b(z_q, z_{p_1}, z_{p_1})\\
&\le sG_b(z_0, z_q, z_q) + sP(q, p_1)\\
&\le sG_b(z_0, z_q, z_q) + sP(q, n_q)\\
&\le sG_b(z_0, z_q, z_q) + s\lambda P(q - 1, n_q)\\
&\le ...\\
&\le sG_b(z_0, z_q, z_q) + s\lambda^q P(0, n_q),
\end{aligned}$$

hence, we have $P(0, n_q) \le \frac{s}{1-s\lambda^q} G_b(z_0, z_q, z_q)$.

On the other hand, there exists an integer $p_2 \le n_q$ and $p_2 > q$ such that

$$P(0, n_q) = G_b(z_0, z_0, z_{p_2}).$$

Using a similar technique, we have $P(0, n_q) \le \frac{s}{1-s\lambda^q} G_b(z_0, z_0, z_q)$. Let

$$M =: \max\{P(0, q), \frac{s}{1 - s\lambda^q} G_b(z_0, z_q, z_q), \frac{s}{1 - s\lambda^q} G_b(z_0, z_0, z_q)\}.$$

Therefore, we have $P(0, n) \leq M$ for all $n \in \mathbb{N}$.
Step 4. We shall prove that $\{z_n\}$ is a G_b-Cauchy sequence.

For any $m, n \in \mathbb{N}$ and $m < n$, by applying (3.2), we have

$$\begin{aligned} G_b(z_m, z_m, z_n) &\leq P(m, n) \\ &\leq \lambda P(m-1, n) \\ &\leq ... \leq \lambda^m P(0, n) \\ &\leq \lambda^m \cdot M. \end{aligned} \tag{3.5}$$

Therefore, by (3.5) and $0 \leq \lambda < 1$, we find that $G_b(z_m, z_m, z_n) \to 0$ as $m, n \to \infty$, which shows that $\{z_n\}$ is a G_b-Cauchy sequence.

The following corollary is an immediate consequence of Lemma 3.2.

Corollary 3.3. *(see [26, Theorem 3])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. Suppose the mapping $T : X \to X$ satisfies

$$G_b(Tx, Ty, Tz) \leq \lambda G_b(x, y, z)$$

for all $x, y, z \in X$, where $\lambda \in [0, 1)$ is a given constant. Then, T has a unique fixed point (say u) in X and, for $x \in X$, the Picard sequence $\{T^n x\}_{n \in \mathbb{N}_0}$ converges to u.

Proof. Obviously, T is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, 1)$. Then, by Lemma 3.2, for $x \in X$, we get that $\{T^n x\}_{n \in \mathbb{N}_0}$ is a G_b-Cauchy sequence. Since (X, G_b) is G_b-complete, there exists $u \in X$ satisfying $T^n x \to u (n \to \infty)$.

On the other hand, we find

$$G_b(T^{n+1}x, Tu, Tu) = G_b(T(T^n x), Tu, Tu) \leq \lambda G_b((T^n x, u, u) \to 0 (n \to \infty).$$

So, $T^{n+1}x \to Tu (n \to \infty)$. Thus, u is a fixed point of T.

Next, we shall verify that u is the unique fixed point of T. If v is another fixed point of T, then $G_b(u, v, v) = G_b(Tu, Tv, Tv) \leq \lambda G_b(u, v, v) < G_b(u, v, v)$. Thus, we conclude that $u = v$. Hence, T has a unique fixed point (say u) in X. Obviously, the Picard sequence $\{T^n x\}_{n \in \mathbb{N}_0}$ converges to u.

Next, we establish and prove main results of this section.

Theorem 3.4. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. If $T : X \to X$ is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, \frac{1}{s})$. Then there exists a unique fixed point of T.

Proof. Let $x_0 \in X$ be an arbitrary point and $\{x_n\}_{n \in \mathbb{N}}$ be a sequence defined by $x_n = Tx_{n-1} = T^n x$ for all $n \in \mathbb{N}$.

Since T is a generalized Ćirić-type contraction and $\lambda \in [0, \frac{1}{s}) \subset [0, 1)$, by Lemma 3.2, we get that $\{x_n\}_{n \in \mathbb{N}_0}$ is a G_b-Cauchy sequence. Since (X, G_b) is G_b-complete, there exists $v \in X$ satisfying $\lim_{n \to \infty} G_b(x_n, v, v) = 0$. Using (3.1), we have

$$\begin{aligned} G_b(x_{n+1}, Tv, Tv) \leq \lambda \max\{&G_b(x_n, v, v), G_b(x_n, x_{n+1}, x_{n+1}), G_b(v, Tv, Tv), \\ &G_b(v, Tv, Tv), G_b(x_n, Tv, Tv), G_b(v, Tv, Tv), G_b(v, x_{n+1}, x_{n+1}), \\ &G_b(x_n, Tv, Tv), G_b(v, x_{n+1}, x_{n+1}), G_b(v, Tv, Tv_0)\} \\ = \lambda \max\{&G_b(x_n, v, v), G_b(x_n, x_{n+1}, x_{n+1}), G_b(v, Tv, Tv), \end{aligned} \tag{3.6}$$

$$G_b(x_n, Tv, Tv), G_b(v, x_{n+1}, x_{n+1})\}.$$

Now we consider three cases:
Case 1. If

$$\begin{aligned}\max\{G_b(x_n, v, v), &G_b(x_n, x_{n+1}, x_{n+1}), G_b(v, Tv, Tv), G_b(x_n, Tv, Tv), G_b(v, x_{n+1}, x_{n+1})\}\\ &= \max\{G_b(x_n, v, v), G_b(x_n, x_{n+1}, x_{n+1}), G_b(v, x_{n+1}, x_{n+1})\},\end{aligned}$$

then by $\{x_n\}$ G_b-convergent to x and (3,6), we have that

$$\lim_{n\to\infty} G_b(x_{n+1}, Tv, Tv) = 0.$$

Hence, $Tv = v$.
Case 2. If

$$\begin{aligned}\max\{G_b(x_n, v, v), G_b(x_n, x_{n+1}, x_{n+1}), G_b(v, Tv, Tv), G_b(x_n, Tv, Tv), G_b(v, x_{n+1}, x_{n+1})\}\\ = G_b(v, Tv, Tv),\end{aligned}$$

then using Definition 2.1 (G_b5), we obtain that

$$G_b(v, Tv, Tv) \le s[G_b(v, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tv, Tv)].$$

Applying (3.6) and $\lambda \in [0, \frac{1}{s})$, we have $G_b(x_{n+1}, Tv, Tv) \le \frac{\lambda s}{1-\lambda s} G_b(v, x_{n+1}, x_{n+1})$, thus,

$$\lim_{n\to\infty} G_b(x_{n+1}, Tv, Tv) = 0.$$

Hence, $Tv = v$.
Case 3. If

$$\begin{aligned}\max\{G_b(x_n, v, v), G_b(x_n, x_{n+1}, x_{n+1}), G_b(v, Tv, Tv), G_b(x_n, Tv, Tv), G_b(v, x_{n+1}, x_{n+1})\}\\ = G_b(x_{n+1}, Tv, Tv),\end{aligned}$$

then as in the proof of case 2, we also get

$$\lim_{n\to\infty} G_b(x_{n+1}, Tv, Tv) = 0.$$

Therefore, we have $Tv = v$.

Finally, we shall show that T has the unique fixed point v in X.

If u is another fixed point of T, then

$$\begin{aligned}G_b(u, v, v) = G_b(Tu, Tv, Tv) \le \lambda \max\{&G_b(u, v, v), G_b(u, Tu, Tu), G_b(v, Tv, Tv),\\ &G_b(v, Tv, Tv), G_b(u, Tv, Tv), G_b(v, Tv, Tv), G_b(v, Tu, Tu),\\ &G_b(u, Tv, Tv), G_b(v, Tu, Tu), G_b(v, Tv, Tv)\}\\ = \lambda \max\{&G_b(u, v, v), G_b(v, u, u)\}.\end{aligned}$$

Similarly, $G_b(u,u,v) \le \lambda \max\{G_b(u,u,v), G_b(v,v,u)\}$. Thus,

$$\max\{G_b(u,u,v), G_b(v,v,u)\} \le \lambda \max\{G_b(u,u,v), G_b(v,v,u)\}.$$

We conclude that $u = v$. Hence, T has a unique fixed point in X. The proof is completed.

Note that T and G_b are not required to be continuous in the above theorem; now, we construct the following example to illustrate Theorem 3.4.

Example 3.5. Let $X = [0,4]$ and define

$$G_b(x,y,z) = \begin{cases} |x-y| + |y-z| + |x-z|, & xyz \ne 0, \\ 2(|x-y| + |y-z| + |x-z|), & xyz = 0. \end{cases}$$

As in the proof of Example 2.7, we know that (X, G_b) is a G_b-complete G_b-metric space with coefficient $s = 2$. Also, G_b is discontinuous. Now we consider $T : X \to X$ defined by

$$Tx = \begin{cases} \frac{1}{3}x, & x \ne 4, \\ 1, & x = 4. \end{cases}$$

Then, the following holds:

(1) T is not continuous on X;

(2) T is a generalized Ćirić-type contraction with contraction constant $\lambda = \frac{1}{3} < \frac{1}{s}$;

(3) T has a unique fixed point in X.

Proof. (1) We shall prove that T is not continuous at $x = 4$. In fact, $4 - \frac{1}{n} \to 4$ as $n \to \infty$. However, $T(4 - \frac{1}{n}) \to \frac{4}{3} \ne T4 = 1 (n \to \infty)$. Hence, T is not continuous on X.

(2) We claim that $G_b(Tx, Ty, Tz) \le \frac{1}{3} M(x,y,z)$ for any $x, y, z \in X$, where

$$\begin{aligned} M(x,y,z) = \max\{&G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), \\ &G_b(z,Tz,Tz), G_b(x,Ty,Ty), G_b(y,Tz,Tz), G_b(z,Tx,Tx) \\ &G_b(x,Tz,Tz), G_b(y,Tx,Tx), G_b(z,Ty,Ty)\} \text{ for any } x,y,z \in X. \end{aligned} \tag{3.7}$$

We consider the following three cases.

Case 1. Suppose that $x = 0$.

If $y, z \in [0,4)$, then we have

$$\begin{aligned} G_b(Tx,Ty,Tz) &= G_b(\frac{1}{3}x, \frac{1}{3}y, \frac{1}{3}z) = \frac{2}{3}(|x-y| + |y-z| + |x-z|) \\ &= \frac{1}{3} G_b(x,y,z) \le \frac{1}{3} M(x,y,z). \end{aligned}$$

If $y = z = 4$, then we have

$$G_b(Tx,Ty,Tz) = G_b(0,1,1) = 4, G_b(x,y,z) = G_b(0,4,4) = 16.$$

Thus, $G_b(Tx,Ty,Tz) \le \frac{1}{3} M(x,y,z)$.

If $z = 4, y \in [0, 4)$ or $y = 4, z \in [0, 4)$, without loss of generality, now we consider the former, i.e., $z = 4, y \in [0, 4)$, then

$$G_b(Tx, Ty, Tz) = G_b(0, 1, \frac{1}{3}z) = 2(1 + \frac{1}{3}z + |\frac{1}{3}z - 1|).$$

Suppose that $z \in [3, 4)$, then

$$G_b(Tx, Ty, Tz) = 2(1 + \frac{1}{3}z + \frac{1}{3}z - 1) = \frac{4}{3}z < \frac{16}{3},$$

$$G_b(x, y, z) = G_b(0, 4, z) = 2(4 + |z - 4| + |z|) = 16.$$

Therefore,

$$G_b(Tx, Ty, Tz) < \frac{1}{3}M(x, y, z).$$

Suppose that $z \in [0, 3)$, then

$$G_b(Tx, Ty, Tz) = 2(1 + \frac{1}{3}z + 1 - \frac{1}{3}z) = 4, G_b(x, y, z) = 16.$$

Therefore, $G_b(Tx, Ty, Tz) \leq \frac{1}{3}M(x, y, z)$.

Case 2. Suppose that $x \in (0, 4)$.

If $y, z \in (0, 4)$ then

$$\begin{aligned} G_b(Tx, Ty, Tz) &= \frac{1}{3}(|x - y| + |y - z| + |x - z|) \\ &= \frac{1}{3}G_b(x, y, z) \leq \frac{1}{3}M(x, y, z). \end{aligned}$$

If $y = z = 4$, then $G_b(Tx, Ty, Tz) = G_b(\frac{1}{3}x, 1, 1) = 2|\frac{1}{3}x - 1|$. Suppose that $x \in (0, 3)$, we get

$$G_b(Tx, Ty, Tz) = 2(1 - \frac{1}{3}x) < 2, G_b(z, Tz, Tz) = G_b(4, 1, 1) = 6.$$

Consequently, $G_b(Tx, Ty, Tz) \leq \frac{1}{3}M(x, y, z)$. Suppose that $x \in [3, 4)$, then

$$G_b(Tx, Ty, Tz) = 2(\frac{1}{3}x - 1) < 2(\frac{4}{3} - 1) = \frac{2}{3}.$$

Therefore, $G_b(Tx, Ty, Tz) \leq \frac{1}{3}M(x, y, z)$.

If $y = z = 0$, then

$$G_b(Tx, Ty, Tz) = G_b(\frac{1}{3}x, 0, 0) = \frac{4}{3}x, G_b(x, y, z) = G_b(x, 0, 0) = 4x.$$

Thus, $G_b(Tx, Ty, Tz) \leq \frac{1}{3}M(x, y, z)$.

If $y \in (0, 4), z = 4$, then we get

$$G_b(Tx, Ty, Tz) = G_b(\frac{1}{3}x, \frac{1}{3}y, 1) = \frac{1}{3}|x - y| + |\frac{1}{3}x - 1| + |\frac{1}{3}y - 1|.$$

Suppose that $x \in (0,3], y \in (0,3]$. Without loss of generality, now we assume $x \le y$, then

$$G_b(Tx, Ty, Tz) = \frac{1}{3}(y-x) + 1 - \frac{1}{3}x + 1 - \frac{1}{3}y = 2 - \frac{2}{3}x < 2,$$
$$G_b(z, Tz, Tz) = G_b(4,1,1) = 6.$$

Thus, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$.

Suppose that $x \in (0,3), y \in [3,4)$, then we get

$$G_b(Tx, Ty, Tz) = \frac{1}{3}(y-x) + 1 - \frac{1}{3}x + \frac{1}{3}y - 1 = \frac{2}{3}(y-x) < \frac{2}{3}(4-x) = \frac{1}{3}(8-2x),$$

$$G_b(x,y,z) = G_b(x,y,4) = 4 - x + 4 - y + y - x = 8 - 2x.$$

Hence, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$.

Suppose that $x \in [3,4), y \in [3,4)$. Without loss of generality, now we suppose $x \le y$, then we know

$$G_b(Tx, Ty, Tz) = \frac{1}{3}(y-x) + \frac{1}{3}x - 1 + \frac{1}{3}y - 1 = \frac{2}{3}y - 2 < \frac{2}{3} \times 4 - 2 = \frac{2}{3},$$

$$G_b(z, Tz, Tz) = G_b(4,1,1) = 6.$$

Therefore, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$.

If $y = 0, z = 4$, then we have

$$G_b(Tx, Ty, Tz) = G_b(\frac{1}{3}x, 0, 1) = 2(1 + \frac{1}{3}x + |\frac{1}{3}x - 1|).$$

Suppose that $x \in (0,3]$, then we have

$$G_b(Tx, Ty, Tz) = 2(1 + \frac{1}{3}x + (1 - \frac{1}{3}x)) = 4, G_b(z, Ty, Ty) = G_b(4,0,0) = 16.$$

Hence, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$. Suppose that $x \in (3,4]$, then we have

$$G_b(Tx, Ty, Tz) = 2(1 + \frac{1}{3}x + \frac{1}{3}x - 1) = \frac{4}{3}x \le \frac{16}{3}, G_b(z, Ty, Ty) = G_b(4,0,0) = 16.$$

Therefore, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$.

If $y = 0, z \in (0,4)$, then we get

$$G_b(Tx, Ty, Tz) = G_b(\frac{1}{3}x, 0, \frac{1}{3}z) = 2(\frac{1}{3}x + \frac{1}{3}z + \frac{1}{3}|x-z|) = \frac{1}{3}G_b(x,0,z).$$

Thus, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$.

Case 3. Suppose that $x = 4$.

If $y \in (0,4)$, then we prove that $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$ from a similar argument in Case 2.

If $y = 0$, then we conclude that $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$ as in the proof of Case 1.

If $y = z = 4$, then $G_b(Tx, Ty, Tz) = 0$, apparently, $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$.

Therefore, we have that $G_b(Tx, Ty, Tz) \le \frac{1}{3}M(x,y,z)$ for any $x, y, z \in X$.

(3) Clearly, $x = 0$ is the unique fixed point for T.

If T is continuous, we have the following theorem.

Theorem 3.6. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping, and T is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, 1)$. If T is continuous, then there exists a unique fixed point of T.

Proof. Let $x_0 \in X$ be an arbitrary point and $\{x_n\}_{n\in\mathbb{N}_0}$ a sequence defined by $x_n = Tx_{n-1} = T^n x$ for all $n \in \mathbb{N}$. Then, by Lemma 3.2, we get that $\{x_n\}_{n\in\mathbb{N}}$ is a G_b-Cauchy sequence. Since (X, G_b) is complete, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$.

Applying the continuity of T, we have $v = \lim_{n\to\infty} z_n = \lim_{n\to\infty} Tz_{n-1} = Tv$. By the same method as Theorem 3.3, we can obtain that v is the unique fixed point for T.

We give an example in which T is continuous to illustrate Theorem 3.6.

Example 3.7. We consider G_b-metric space in Example 2.7, let $X = [0, \infty)$ and define

$$G_b(x, y, z) = \begin{cases} |x - y| + |y - z| + |x - z|, & xyz \neq 0, \\ 2(|x - y| + |y - z| + |x - z|), & xyz = 0. \end{cases}$$

Let $T : X \to X$ be a map defined by $Tx = \frac{2}{3}x$ for any $x \in X$. Then, the following holds:

(1) T is a generalized Ćirić-type contraction with contraction constant $\lambda = \frac{2}{3} > \frac{1}{s}$;

(2) T is continuous on X;

(3) T has a unique fixed point in X.

Proof. (1) We will prove that $G_b(Tx, Ty, Tz) \leq \frac{2}{3}M(x, y, z)$ for any $x, y, z \in X$, with the definition of $M(x, y, z)$ in (3.7). We consider the following two cases:

Case 1. If $xyz \neq 0$, then $Tx \cdot Ty \cdot Tz \neq 0$.

$$\begin{aligned} G_b(Tx, Ty, Tz) &= G_b(\frac{2}{3}x, \frac{2}{3}y, \frac{2}{3}z) \\ &= \frac{2}{3}(|x - y| + |y - z| + |x - z|) \\ &= \frac{2}{3}G_b(x, y, z) \leq \frac{2}{3}M(x, y, z). \end{aligned}$$

Case 2. If $xyz = 0$, then $Tx \cdot Ty \cdot Tz = 0$.

$$\begin{aligned} G_b(Tx, Ty, Tz) &= G_b(\frac{2}{3}x, \frac{2}{3}y, \frac{2}{3}z) \\ &= \frac{4}{3}(|x - y| + |y - z| + |x - z|) \\ &= \frac{2}{3}G_b(x, y, z) \leq \frac{2}{3}M(x, y, z). \end{aligned}$$

The proof is completed.

(2) For any $x \in X$ and sequence $\{x_n\} \subset X$ converging to x, we shall prove that $\{Tx_n\}$ converges to Tx. We consider the following two cases:

Case 1. Assume that $x > 0$, by $G_b(x_n, x, x) \to 0$ as $n \to \infty$, then there exists a positive integer n_0 such that $x_n > 0$ for any integer $n \geq n_0$. Thus

$$\lim_{n\to\infty} G_b(Tx_n, Tx, Tx) = \lim_{n\to\infty} \frac{4}{3}|x_n - x| = \frac{2}{3}\lim_{n\to\infty} G_b(x_n, x, x) = 0$$

Case 2. If $x = 0$, then

$$\lim_{n\to\infty} G_b(Tx_n, Tx, Tx) = \lim_{n\to\infty} \frac{8}{3}|x_n - x| = \frac{2}{3}\lim_{n\to\infty} G_b(x_n, x, x) = 0.$$

From the above two cases, we show that T is continuous on X;

(3) Clearly, $x = 0$ is the unique fixed point for T.

If G_b is continuous, then we have the following result.

Theorem 3.8. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, 1)$. If G_b is continuous, then there exists a unique fixed point of T.

Proof. Let $x_0 \in X$ be an arbitrary point and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by $x_n = Tx_{n-1} = T^n x$ for all $n \in \mathbb{N}$. Then, by Lemma 3.2, we get that $\{x_n\}_{n\in\mathbb{N}_0}$ is a G_b-Cauchy sequence. Since (X, G_b) is complete, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$.

Applying the continuity of G_b, taking the limit as n tends to ∞ on both sides of (3.6), we have that $G_b(v, Tv, Tv) \leq \lambda \max G_b(v, Tv, Tv)$, hence, $Tv = v$.

By the same method as Theorem 3.3, we can obtain that v is the unique fixed point for T.

Using Theorems 3.4, we can obtain Ćirić fixed-point theorems in b-metric space.

Corollary 3.9. *(see [19, Theorem 3])* Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \to X$ be a map such that for some $\lambda \in [0, \frac{1}{s})$ and all $x, y \in X$

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then, T has a unique fixed point.

Proof. Let $G_b(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$, for all $x, y, z \in X$. From Example 2.4, we know that (X, G_b) is a G_b-metric space. Furthermore, it is obvious that (X, G_b) is G_b-complete due to completeness of (X, d). Hence,

$$\begin{aligned}
G_b(Tx, Ty, Tz) = \max\{d(Tx, Ty), d(Ty, Tz), d(Tx, Tz)\}& \\
\leq \lambda \max\{&d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(y, z), d(y, Ty), \\
&d(z, Tz), d(y, Tz), d(z, Ty), d(x, z), d(x, Tx), d(z, Tz), d(x, Tz), d(z, Tx)\} \\
= \lambda \max\{&G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), \\
&G_b(z, Tz, Tz), G_b(x, Ty, Ty), G_b(y, Tz, Tz), G_b(z, Tx, Tx), \\
&G_b(x, Tz, Tz), G_b(y, Tx, Tx), G_b(z, Ty, Ty)\}
\end{aligned}$$

for all $x, y, z \in X$. Thus, T is a generalized Ćirić-type contraction with contraction constant $\lambda \in [0, 1)$. By Theorem 3.4, T has a unique fixed point in (X, G_b).

Therefore, T has a unique fixed point in (X, d).

Remark 3.10. Let $s = 1$ in Corollary 3.9; we can obtain the famous fixed-point theorem (so-called Ćirić fixed-point theorem [18]) in the setting of metric spaces.

Using Theorem 3.6, we can obtain the following result in b-metric space as in the proof of Corollary 3.9.

Corollary 3.11. *(see [19, Theorem 3])* Let (X, d) be a complete b-metric space with $s \geq 1$. Let $T : X \to X$ be a map and T be continuous such that for some $\lambda \in [0, 1)$ and all $x, y \in X$

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then, T has a unique fixed point.

The following corollary can be immediately deduced from Theorem 3.4, and the range of contraction constant in Theorem 8 in [26] is enlarged from $[0, \frac{1}{2s})$ to $[0, \frac{1}{s})$.

Corollary 3.12. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map. If there exists $\lambda \in [0, \frac{1}{s})$ such that

$$\begin{aligned} G_b(Tx, Ty, Tz) \leq \lambda \max\{&G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), \\ &G_b(z, Tz, Tz), G_b(x, Ty, Ty), G_b(y, Tz, Tz), G_b(z, Tx, Tx)\} \end{aligned} \tag{3.8}$$

for all $x, y, z \in X$, then there exists a unique fixed point of T.

Remark 3.13. It is not difficult to prove that mapping T in Example 3.5 satisfies (3.8). The contraction constant of T is $\lambda = \frac{1}{3}$. However, the contraction constant λ does not satisfy the range of contraction constant in Theorem 8 in [26]. That is, $\lambda = \frac{1}{3} \notin [0, \frac{1}{2s}) = [0, \frac{1}{4})$. Also, by calculating the corresponding values of $x = 3, y = 0, z = 4$, we can know the above fact. Our results are used widely.

Theorem 3.4 can also deduce the following two corollaries, which are results in [27].

Corollary 3.14. *(see [27, Theorem 3.3])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map. If there exists $\lambda \in [0, \frac{1}{s})$ such that

$$G_b(Tx, Ty, Tz) \leq \lambda \max\{G_b(x, Tx, Tx), G_b(y, Ty, Ty), G_b(z, Tz, Tz)\}$$

for all $x, y, z \in X$, then there exists a unique fixed point of T.

Corollary 3.15. *(see [27, Theorem 3.5])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map. If there exists $\lambda \in [0, \frac{1}{s})$ such that

$$G_b(Tx, Ty, Tz) \leq \lambda \max\{G_b(x, Ty, Ty), G_b(y, Tx, Tx), G_b(y, Ty, Ty)\}$$

for all $x, y, z \in X$, then there exists a unique fixed point of T.

Applying the same method of Theorems 3.4 and 3.6, we have the following results.

Theorem 3.16. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map. If there exists $\lambda \in [0, \frac{1}{s})$ such that

$$\begin{aligned} G_b(Tx, Ty, Tz) \leq \lambda \max\{&G_b(x, y, z), G_b(x, x, Tx), G_b(y, y, Ty), \\ &G_b(z, z, Tz), G_b(x, x, Ty), G_b(y, y, Tz), G_b(z, z, Tx), \\ &G_b(x, x, Tz), G_b(y, y, Tx), G_b(z, z, Ty)\} \end{aligned}$$

for all $x, y, z \in X$, then there exists a unique fixed point of T.

Theorem 3.17. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map and T is continuous. If there exists $\lambda \in [0, 1)$ such that

$$G_b(Tx, Ty, Tz) \leq \lambda \max\{G_b(x, y, z), G_b(x, x, Tx), G_b(y, y, Ty),$$

$$G_b(z,z,Tz), G_b(x,x,Ty), G_b(y,y,Tz), G_b(z,z,Tx),$$
$$G_b(x,x,Tz), G_b(y,y,Tx), G_b(z,z,Ty)\}$$

for all $x,y,z \in X$, then there exists a unique fixed point of T.

The following corollaries are immediate consequences of Theorems 3.16 and 3.17.

Corollary 3.18. *(see [27, Theorem 3.3])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map. If there exists $\lambda \in [0, \frac{1}{s})$ such that

$$G_b(Tx, Ty, Tz) \leq \lambda \max\{G_b(x,x,Tx), G_b(y,y,Tx), G_b(z,z,Tz))\}$$

for all $x,y,z \in X$, then there exists a unique fixed point of T.

Corollary 3.19. *(see [27, Theorem 3.5])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. $T : X \to X$ is a map. If there exists $\lambda \in [0, \frac{1}{s})$ such that

$$G_b(Tx, Ty, Tz) \leq \lambda \max\{G_b(x,x,Ty), G_b(y,y,Tx), G_b(y,y,Ty))\}$$

for all $x,y,z \in X$, then there exists a unique fixed point of T.

In G_b-metric spaces, Reich-type contraction fixed-point theorems have been studied in [26,27]. Using Lemma 3.2, we can obtain the following Reich-type fixed-point results in the context of G_b-metric spaces.

Theorem 3.20. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$\begin{aligned} G_b(Tx, Ty, Tz) \leq{}& a_1 G_b(x,y,z) + a_2 G_b(x,Tx,Tx) + a_3 G_b(y,Ty,Ty) \\ &+ a_4 G_b(z,Tz,Tz) + a_5 G_b(x,Ty,Ty) + a_6 G_b(y,Tz,Tz) + a_7 G_b(z,Tx,Tx) \\ &+ a_8 G_b(x,Tz,Tz) + a_9 G_b(y,Tx,Tx) + a_{10} G_b(z,Ty,Ty) \end{aligned}$$

for all $x,y,z \in X$, where $0 \leq \sum_{i=1}^{10} a_i < 10$ and $s(a_3 + a_4 + a_5 + a_6 + a_8 + a_{10}) < 1$. Then, there exists a unique fixed point of T.

Proof. Since $0 \leq \sum_{i=1}^{10} a_i < 1$, then T is a generalized Ćirić-type contraction and contraction constant $\lambda \in [0,1)$. By Lemma 3.2, for $x \in X$, we get that $\{T^n x\}_{n\in\mathbb{N}_0}$ is a G_b-Cauchy sequence. Because (X, G_b) is G_b-complete, there exists $u \in X$ satisfying $T^n x \to u (n \to \infty)$. Next, we will prove u is a fixed point.

$$\begin{aligned} &G_b(x_{n+1}, fu, fu) \\ &\leq a_1 G_b(x_n,u,u) + a_2 G_b(x_n, x_{n+1}, x_{n+1}) + a_3 G_b(u, fu, fu) \\ &\quad + a_4 G_b(u,fu,fu) + a_5 G_b(x_n, fu, fu) + a_6 G_b(u,fu,fu) + a_7 G_b(u, x_{n+1}, x_{n+1}) \\ &\quad + a_8 G_b(x_n, fu, fu) + a_9 G_b(u, x_{n+1}, x_{n+1}) + a_{10} G_b(u,fu,fu) \\ &= a_1 G_b(x_n,u,u) + a_2 G_b(x_n, x_{n+1}, x_{n+1}) + (a_3 + a_4 + a_6 + a_{10}) G_b(u,fu,fu) \\ &\quad + (a_5 + a_8) G_b(x_n, fu, fu) + a_7 G_b(u, x_{n+1}, x_{n+1}) + a_9 G_b(u, x_{n+1}, x_{n+1}) \\ &\leq a_1 G_b(x_n,u,u) + a_2 G_b(x_n, x_{n+1}, x_{n+1}) \\ &\quad + (a_3 + a_4 + a_6 + a_{10}) s[G_b(u, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, fu, fu)] \\ &\quad + (a_5 + a_8) s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, fu, fu)] \end{aligned}$$

$$+ a_7 G_b(u, x_{n+1}, x_{n+1}) + a_9 G_b(u, x_{n+1}, x_{n+1}).$$

Therefore,

$$\begin{aligned}
[1 - s(a_3 + a_4 + a_6 + a_{10}) - s(a_5 + a_8)]G_b(x_{n+1}, fu, fu) \leq\ & a_1 G_b(x_n, u, u) \\
& + a_2 G_b(x_n, x_{n+1}, x_{n+1}) + (a_3 + a_4 + a_6 + a_{10}) s[G_b(u, x_{n+1}, x_{n+1}) \\
& + (a_5 + a_8) s[G_b(x_n, x_{n+1}, x_{n+1} + (a_7 + a_9) G_b(u, x_{n+1}, x_{n+1}).
\end{aligned}$$

As $n \to \infty$, we know $[1 - s(a_3 + a_4 + a_6 + a_{10}) - s(a_5 + a_8)]G_b(x_{n+1}, fu, fu) \leq 0$. Since $s(a_3 + a_4 + a_5 + a_6 + a_8 + a_{10}) < 1$, so $x_{n+1} \to fu$, thus, $u = fu$.

Finally, we shall show that T has the unique fixed point u in X. If v is another fixed point of T, then

$$\begin{aligned}
G_b(u, v, v) = G_b(fu, fv, fv) &\leq a_1 G_b(u, v, v) + a_2 G_b(u, fu, fu) + a_3 G_b(v, fv, fv) \\
&\quad + a_4 G_b(v, fv, fv) + a_5 G_b(u, fv, fv) + a_6 G_b(v, fv, fv) + a_7 G_b(v, fu, fu) \\
&\quad + a_8 G_b(u, fv, fv) + a_9 G_b(v, fu, fu) + a_{10} G_b(v, fv, fv) \\
&= (a_1 + a_5 + a_8) G_b(u, v, v) + a_7 + a_9) G_b(u, u, v) \\
&\leq (a_1 + a_5 + a_8 + a_7 + a_9) \max\{G_b(u, v, v), G_b(u, u, v)\} \\
&\leq (a_1 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) \max\{G_b(u, v, v), G_b(u, u, v)\}.
\end{aligned}$$

Similarly, we get

$$G_b(u, u, v) \leq (a_1 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) \max\{G_b(u, v, v), G_b(u, u, v)\}.$$

Hence,

$$\begin{aligned}
\max\{G_b(u, v, v), G_b(u, u, v)\} \leq\ & (a_1 + a_5 + a_6 + a_7 \\
& + a_8 + a_9 + a_{10}) \max\{G_b(u, v, v), G_b(u, u, v)\}.
\end{aligned}$$

Thus, $\max\{G_b(u, v, v), G_b(u, u, v)\} = 0$, we have $u = v$.

By Theorem 3.20 the following results immediately follows.

Corollary 3.21. *(refer to [26, Theorem 7])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$\begin{aligned}
G_b(Tx, Ty, Tz) \leq\ & a_1 G_b(x, y, z) + a_2 G_b(x, Tx, Tx) + a_3 G_b(y, Ty, Ty) \\
& + a_4 G_b(z, Tz, Tz) + a_5 G_b(x, Ty, Ty) + a_6 G_b(y, Tz, Tz) + a_7 G_b(z, Tx, Tx)
\end{aligned}$$

for all $x, y, z \in X$, where $0 \leq \sum_{i=1}^{7} a_i < 1$ and $s(a_3 + a_4 + a_5 + a_6) < 1$. Then, there exists a unique fixed point of T.

Remark 3.22. Corollary 3.22 generalizes Theorem 7 in [26], and the contraction condition is relaxed from $0 \leq a_1 + a_2 + a_3 + a_4 + 2sa_5 + a_6 + a_7 < 1$ to $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 < 1$.

Corollary 3.23. *(refer to [27, Theorem 3.1])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$G_b(Tx, Ty, Tz) \leq a_1 G_b(x, y, z) + a_2 G_b(x, Tx, Tx) + a_3 G_b(y, Ty, Ty) + a_4 G_b(z, Tz, Tz),$$

for all $x, y, z \in X$, where $0 \leq \sum_{i=1}^{4} a_i < 1$ and $s(a_3 + a_4) < 1$. Then, there exists a unique fixed point of T.

Proof. Take $a_8 = a_9 = a_{10} = 0$ in Theorem 3.20.

Corollary 3.24. *(refer to [27, Theorem 3.9])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$G_b(Tx, Ty, Ty) \leq \lambda(G_b(x, Ty, Ty) + G_b(y, Tx, Tx))$$

for all $x, y, z \in X$, where $0 \leq \lambda < \frac{1}{\max\{2,s\}}$. Then, there exists a unique fixed point of T.

Proof. The assertion follows if we take $a_1 = a_2 = a_3 = a_4 = a_7 = a_8 = a_{10} = 0$, $a_5 = a_9 = \lambda$ and $z = y$ in Theorem 3.20.

Remark 3.25. Corollary 3.24 improves Theorem 3.9 in [27], and the range of contraction constant is relaxed from $[0, \frac{1}{2s})$ to $[0, \frac{1}{\max\{2,s\}})$.

Corollary 3.26. *(refer to [26, Corollary 2])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$ and mapping $T : X \to X$. If there exists $0 \leq \lambda < \frac{1}{2s}$ such that

$$G_b(Tx, Ty, Tz) \leq \lambda(G_b(x, Ty, Ty) + G_b(y, Tz, Tz) + G_b(z, Tx, Tx)),$$

for all $x, y, z \in X$, then there exists a unique fixed point of T.

Proof. Take $a_5 = a_6 = a_7 = \lambda$ and $a_1 = a_2 = a_3 = a_4 = a_8 = a_9 = a_{10} = 0$ in Theorem 3.20.

Remark 3.27. Corollary 3.26 generalizes Corollary 2 in [26]. The range of contraction constant is extended from $[0, \frac{1}{2s+1})$ to $[0, \frac{1}{2s})$.

By the same method of Theorem 3.20, we have the following theorem.

Theorem 3.28. Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$\begin{aligned} G_b(Tx, Ty, Tz) \leq a_1 G_b(x, y, z) &+ a_2 G_b(x, x, Tx) + a_3 G_b(y, y, Ty) \\ &+ a_4 G_b(z, z, Tz) + a_5 G_b(x, x, Ty) + a_6 G_b(y, y, Tz) + a_7 G_b(z, z, Tx) \\ &+ a_8 G_b(x, x, Tz) + a_9 G_b(y, y, Tx) + a_{10} G_b(z, z, Ty) \end{aligned}$$

for all $x, y, z \in X$, where $0 \leq \sum_{i=1}^{10} a_i < 10$ and $s(a_3 + a_4 + a_5 + a_6 + a_8 + a_{10}) < 1$. Then, there exists a unique fixed point of T.

Corollary 3.29. *(refer to [27, Theorem 3.1])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$G_b(Tx, Ty, Tz) \leq a_1 G_b(x, y, z) + a_2 G_b(x, x, Tx) + a_3 G_b(y, y, Ty) + a_4 G_b(z, z, Tz),$$

for all $x, y, z \in X$, where $0 \leq \sum_{i=1}^{4} a_i < 1$ and $s(a_3 + a_4) < 1$. Then, there exists a unique fixed point of T.

Corollary 3.30. *(refer to [27, Theorem 3.9])* Let (X, G_b) be a G_b-complete G_b-metric space with coefficient $s \geq 1$. The mapping $T : X \to X$ satisfies the following conditions:

$$G_b(Tx, Ty, Ty) \leq \lambda(G_b(x, x, Ty) + G_b(y, y, Tx)),$$

for all $x, y, z \in X$, $0 \leq \lambda < \frac{1}{\max\{2,s\}}$. Then there exists a unique fixed point of T.

Proof. The assertion follows if we take $a_1 = a_2 = a_3 = a_4 = a_7 = a_8 = a_{10} = 0$, $a_5 = a_9 = \lambda$ and $z = y$ in Theorem 3.28.

Remark 3.31. Corollary 3.30 improves Theorem 3.9 in [27], and the range of contraction constant is enlarged from $[0, \frac{1}{2s})$ to $[0, \frac{1}{\max\{2,s\}})$.

4. Application to existence of solutions to the integral equation

In this section, by using Theorem 3.8, we discuss the existence of solutions of the following integral equations:

$$x(t) = h(t) + \mu \int_a^b G(t,v) f(v, x(v)) dv, \tag{4.1}$$

where $h : [a,b] \to \mathbb{R}$, $G : [a,b] \times [a,b] \to \mathbb{R}$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Let $\mathcal{X} = C[a,b]$, and define $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ by $d(x,y) = \|x-y\|^2$ for all $x, y \in \mathcal{X}$, where $\|x-y\| = \max\limits_{a\le t\le b} |x(t) - y(t)|$. Apparently, $(\mathcal{X}, d)$ is a complete b-metric space with coefficient $s = 2$.

Let $G_b(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\}$ for all $x, y, z \in \mathcal{X}$, then $(\mathcal{X}, G_b)$ is a G_b-metric space with identical constant $s = 2$ by Example 2.2, and G_b is continuous.

Consider the self-mapping $T : \mathcal{X} \to \mathcal{X}$ defined by

$$Tx(t) = h(t) + \mu \int_a^b G(t,v) f(v, x(v)) dv,$$

Clearly, $x(t)$ is a solution of (4.1) if and only if x is a fixed point of T. Suppose the following conditions are satisfied:

(1) $\mu^2 < 1$;

(2) $\max\limits_{a\le t\le b} \int_a^b G^2(t,v) dv \le \frac{1}{b-a}$;

(3) $\max\limits_{a\le t\le b} |f(t,x) - f(t,y)|^2 \le d(x,y)$. Hence, we have

$$\begin{aligned} |Tx - Ty|^2 =& \mu^2 \left| \int_a^b G(t,v)[f(v,x(v)) - f(v,y(v))dv] \right|^2 \\ \le& \mu^2 \left| \left[\int_a^b G^2(t,v)dv\right]^{\frac{1}{2}} \cdot \left[\int_a^b |f(v,x(v)) - f(v,y(v)|^2 dv\right]^{\frac{1}{2}} \right|^2 \\ =& \mu^2 \int_a^b G^2(t,v)dv \cdot \int_a^b |f(v,x(v)) - f(v,y(v)|^2 dv \\ \le& \mu^2 \frac{1}{b-a} \cdot (b-a) d(x,y) = \mu^2 d(x,y), \end{aligned}$$

then, we get $d(Tx,Ty) \le \mu^2 d(x,y)$, for all $x,y \in \mathcal{X}$. Similarly, $d(Ty,Tz) \le \mu^2 d(y,z)$ and $d(Tx,Tz) \le \mu^2 d(x,z)$, for all $x,y,z \in \mathcal{X}$.

Therefore, $G_b(Tx,Ty,Tz) \le \mu^2 G_b(x,y,z) \le \mu^2 M(x,y,z)$, for all $x,y,z \in \mathcal{X}$. The conditions of Theorem 3.8 are satisfied, so T has a unique fixed point in $\mathcal{X}$, and then (4.1) has a unique solution $x(t) \in \mathcal{X}$.

5. Conclusions

In this paper, we present the notion of generalized Ćirić-type contraction in G_b-metric space. Using a significant lemma, we derive a generalized Ćirić-type fixed-point theorem. Satisfyingly, we can deduce the famous Ćirić fixed-point theorem in metric space using our results. Moreover, we construct new and interesting examples to illustrate our results. In addition, we also obtain several Ćirić and Reich-types fixed-point theorems. As an application, we show the existence of a solution to the integral equation in G_b-metric space.

Author contributions

All authors contributed equal to the job. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are thankful to the referees for their valuable comments and suggestions to improve this paper. The research supported by the National Natural Science Foundation of China (12061050) and the Natural Science Foundation of Inner Mongolia (2020MS01004).

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, *Funct. Anal.*, **30** (1989), 26–37.
2. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Sem. Math. Fis. Univ. Modena*, **46** (1998), 263–276.
3. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1** (1993), 5–11.
4. N. V. Dung, V. T. L. Hang, On relaxations of contraction constants and Caristi's theorem in b-metric spaces, *J. Fixed Point Theory Appl.*, **18** (2016), 267–284. https://doi.org/10.1007/s11784-015-0273-9
5. J. Wu, F. He, S. Li, A fixed point theorem for generalized Ćirić-type contraction in Kaleva-Seikkala's type fuzzy b-metric spaces, *Axioms*, **12** (2023), 616. https://doi.org/10.3390/axioms12070616
6. P. Wang, F. He, X. Liu, Answers to questions on Kannan's fixed point theorem in strong b-metric spaces, *AIMS Math.*, **9** (2024), 3671–3684. https://doi.org/10.3934/math.2024180
7. K. M. Pankaj, S. Shweta, S. K. Banerjee, Some fixed point theorems in b-metric space, *Turkish J. Anal. Number Theory*, **2** (2014), 19–22. https://doi.org/10.12691/tjant-2-1-5

8. T. Suzuki, Basic inequality on a b-metric space and its applications, *J. Inequal. Appl.*, **2017** (2017), 256. https://doi.org/10.1186/s13660-017-1528-3

9. H. Afshari, H. Aydi, Erdal Karapınar, On generalized α-ψ-Geraghty contractions on b-metric spaces, *Georgian Math. J.*, **27** (2020), 9–21. https://doi.org/10.1515/gmj-2017-0063

10. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex A.*, **7** (2006), 289–297.

11. Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory Appl.*, **2008** (2008), 189870. https://doi.org/10.1155/2008/189870

12. Z. Mustafa, B. Sims, Fixed point theorems for contractive mappings in complete metric spaces, *Fixed point theory Appl.*, **2009** (2009), 917175. https://doi.org/10.1155/2009/917175

13. T. V. An, N. V. Dung, V. T. L. Hang, A new approach to fixed point theorems on G-metric spaces, *Topology Appl.*, **160** (2013), 1486–1493. https://doi.org/10.1016/j.topol.2013.05.027

14. H. Aydi, M. Postolache, W. Shatanawi, Coupled fixed point results for(ψ, ϕ)-weakly contractive mappings in ordered G-metric spaces, *Comput. Math. Appl.*, **63** (2012), 298–309. https://doi.org/10.1016/j.camwa.2011.11.022

15. M. Abbas, A. Hussain, B. Popovic, S. Radenovic, Istratescu-Suzuki-Ćirić type fixed points results in the framework of G-metric spaces, *J. Nonlinear Sci. Appl.*, **9** (2016), 6077–6095. https://doi.org/10.22436/jnsa.009.12.15

16. M. Jaradat, Z. Mustafa, S. U. Khan, M. Arshad, J. Ahmad, Some fixed point results on G-metric and G_b-metric spaces, *Demonstr. Math.*, **50** (2017), 190–207. https://doi.org/10.1515/dema-2017-0018

17. A. Aghajani, M. Abbas, J. R. Roshan, Common fixed point of generallized weak contractive mappings in partially ordered G_b-metric spaces, *Filomat*, **28** (2014), 1087–1101. https://doi.org/10.2298/FIL1406087A

18. L. B. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267–273. https://doi.org/10.2307/2040075

19. N. Lu, F. He, W. S. Du, Fundamental questions and new counterexamples for b-Metric spaces and fatou property, *Mathematics*, **7** (2019), 1107. https://doi.org/10.3390/math7111107

20. P. Kumam, N. V. Dung, K. Sitthithakerngkiet, A generalization of Ćirić fixed point theorems, *Filomat*, **29** (2015), 1549–1556. https://doi.org/10.2298/FIL1507549K

21. F. He, X. Zhao, Y. Sun, Cyclic quasi-contractions of Ćirić type in b-metric spaces, *J. Nonlinear Sci. Appl.*, **10** (2017), 1075–1088. https://doi.org/10.22436/jnsa.010.03.18

22. A. Amini-Harandi, D. Mihet, Quasi-contractive mappings in fuzzy metric spaces, *Iran. J. Fuzzy Syst.*, **12** (2015), 147–153. https://doi.org/10.22111/IJFS.2015.2090

23. F. Vetro, S. Radenovic, Nonlinear ψ-quasi-contractions of Ćirić-type in partial metric spaces, *Appl. Math. Comput.*, **219** (2012), 1594–1600. https://doi.org/10.1016/j.amc.2012.07.061

24. A. H. Ansari, M. A. Barakat, H. Aydi, New approach for common fixed point theorems via C-class functions in G_p-metric spaces, *J. Funct. Space.*, **2017** (2017), 2624569. https://doi.org/10.1155/2017/2624569

25. M. Liang, C. Zhu, C. Chen, Z. Wu, Some new theorems for cyclic contractions in G_b-metric spaces and some applications, *Appl. Math. Comput.*, **346** (2019), 545–558. https://doi.org/10.1016/j.amc.2018.10.028

26. H. Aydi, D. Rakic, A. Aghajani, T. Dosenovic, M. S. M. Noorani, H. Qawaqneh, On fixed point results in G_b-metric spaces, *Mathematics*, **7** (2019), 617. https://doi.org/10.3390/math7070617

27. V. Gupat, O. Ege, R. Saini, M. D. L. Sen, Various fixed point results in complete G_b-metric spaces, *Dynam. Syst. Appl.*, **30** (2021), 277–293. https://doi.org/10.46719/dsa20213028