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Research article

On pointwise convergence of sequential Boussinesq operator

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Abstract: We study the almost everywhere pointwise convergence of the Boussinesq operator along sequences $\{t_n\}_{n=1}^{\infty}$ with $\lim\limits_{n\to\infty} t_n = 0$ in one dimension. We obtain a characterization of convergence almost everywhere when $\{t_n\} \in l^{r,\infty}(\mathbb{N})$ for all $f \in H^s(\mathbb{R})$ provided $0 < s < \frac{1}{2}$.

Keywords: Boussinesq operator; pointwise convergence; Sobolev space; Lorentz space
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1. Introduction

1.1. The pointwise convergence of the Schrödinger operator

The formal solution to the free Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta_x u = 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}; \\ u(x,0) = f(x), x \in \mathbb{R}^n \end{cases}$$

is defined by

$$e^{it\triangle} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|^2)} \hat{f}(\xi) d\xi,$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

Carleson [6] considered the following problem: Determine the optimal s for which

$$\lim_{t\to 0} e^{it\triangle} f(x) = f(x), \quad \text{a. e. } \ x \in \mathbb{R}^n \tag{1.1}$$

whenever $f \in H^s(\mathbb{R}^n)$, where $H^s(\mathbb{R}^n)$ is the L^2-Sobolev space of order s, which is given by

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\}.$$

In 1979, Carleson [6] first showed that the almost everywhere convergence (1.1) holds for any $f \in H^{\frac{1}{4}}(\mathbb{R})$. Dahlberg-Kenig [10] proved (1.1) fails for $s < \frac{1}{4}$ when $n \geq 1$. For the situation in higher dimensions, many researchers such as Carbery [5] and Cowling [9] considered this problem, and Sjölin [31] and Vega [38] proved independently that (1.1) holds when $s > \frac{1}{2}$ in any dimensions. The sufficient condition of (1.1) has been obtained in [1,2,7,11,13,15,20,21,23,28–30,37]. Bourgain [3] gave counterexamples demonstrating that (1.1) fails provided $s < \frac{n}{2(n+1)}$. The best sufficient condition was improved by Du-Guth-Li [14] and Du-Zhang [16] in general dimension $n \geq 2$. Hence, the Carleson problem was essentially solved except for the endpoint.

1.2. The pointwise convergence of Boussinesq operator

As a nonlinear variant of (1.1), the Boussinesq operator acting on $f \in S(\mathbb{R}^n)$ is given by

$$\mathcal{B}f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + t|\xi|\sqrt{1+|\xi|^2})} \hat{f}(\xi) d\xi,$$

which occurs in many physical situations. The name of this operator comes from the Boussinesq equation (cf. [4])

$$u_{tt} - u_{xx} \pm u_{xxxx} = (u^2)_{xx}, \quad \forall (t,x) \in \mathbb{R}^2$$

modelling the propagation of long waves on the surface of water with small amplitude.

We are motivated by Section 1.1 and the similarity between the Schrödinger operator and the Boussinesq operator to study the pointwise convergence of $\mathcal{B}f(x,t)$: Evaluate the optimal s_c such that

$$\lim_{t\to 0} \mathcal{B}f(x,t) = f(x), \quad \text{a. e. } x \in \mathbb{R}^n \tag{1.2}$$

holds for any $f \in H^s(\mathbb{R}^n)$ with $s > s_c$.

Cho-Ko [7] improved the convergence result on the Schrödinger operator to some generalized dispersive operators excluding the Boussinesq operator. Li-Li [22] proved that almost everywhere convergence (1.2) holds for any $f \in H^{\frac{1}{4}}(\mathbb{R})$ and Li-Li [22] also proved the condition $s \geq \frac{1}{4}$ is sharp. Li-Wang [25] obtained almost everywhere convergence (1.2) holds for the optimal $s_c = \frac{1}{3}$ when $n = 2$.

In this paper, we are interested in a related problem: To study the pointwise convergence of $\mathcal{B}f(x,t_n)$, where $\{t_n\}_{n=1}^{\infty}$ is a decreasing sequence with $\lim_{n\to\infty} t_n = 0$. One may expect less regularity on f is enough to obtain convergence in the discrete case. Let's review the convergence of the Schrödinger operator. When $t_n = \frac{1}{n}, n = 1,2,\cdots$, Carleson [6] proved that the convergence result holds provided that $s > \frac{1}{4}$ but fails for $s < \frac{1}{8}$ in one dimension. Indeed, it actually fails for $s < \frac{1}{4}$ by the counterexample in Dahlberg-Kenig [10]; see Lee-Rogers [21] for more details. Recently, this problem was further studied by [8,12,24,26,32,33]. In particular, under the assumption that $\{t_n\}_{n=1}^{\infty}$ belongs to Lorentz space $l^{r,\infty}(\mathbb{N})$, $0 < r < \infty$, i.e.,

$$\sup_{b>0} b^r \#\{n \in \mathbb{N} : t_n > b\} < \infty,$$

Dimou-Seeger [12] considered the fractional Schrödinger operator, which is defined by

$$e^{it(-\Delta)^{\frac{a}{2}}}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi+t|\xi|^a)}\hat{f}(\xi)d\xi,$$

and obtained a characterization of convergence for all functions in $H^s(\mathbb{R})$ when $0 < s < \min\left\{\frac{a}{4}, \frac{1}{4}\right\}$ and $a \neq 1$. Li-Wang-Yan [26] and Cho-Ko-Koh-Lee [8] extended the result of Dimou-Seeger [12] to higher dimensions by different methods.

In this paper, we study the almost everywhere pointwise convergence problem of the sequential Boussinesq operator. The fractional Schrödinger operator and Boussinesq operator are different operators, and the result from Dimou-Seeger [12] cannot cover our work. We obtain a characterization of convergence almost everywhere for any $f \in H^s(\mathbb{R})$ when $0 < s < \frac{1}{2}$. Our main results are as follows:

Theorem 1.1. *Suppose* $0 < s < \frac{1}{2}$. *Let* $\{t_n\}_{n=1}^{\infty}$ *be a decreasing sequence with* $\lim_{n\to\infty} t_n = 0$, *and suppose that* $\{t_n - t_{n+1}\}_{n=1}^{\infty}$ *is also decreasing. Then the following four statements are equivalent.*

(i) Let $r(s) = \frac{s}{1-2s}$, *the sequence* $\{t_n\} \in l^{r(s),\infty}(\mathbb{N})$.

(ii) There exists a constant C_1 *such that for any* $f \in H^s(\mathbb{R})$ *and for all sets* B *with* $diam(B) \leq 1$, *we obtain*

$$\left\|\sup_{n\in\mathbb{N}} |\mathcal{B}f(x, t_n)|\right\|_{L^2(B)} \leq C_1\|f\|_{H^s(\mathbb{R})}. \tag{1.3}$$

(iii) There exists a constant C_2 *such that for any* $f \in H^s(\mathbb{R})$, *for all sets* B *with* $diam(B) \leq 1$, *and for any* $\alpha > 0$, *we obtain*

$$\left|\left\{x \in B : \sup_{n\in\mathbb{N}} |\mathcal{B}f(x, t_n)| > \alpha\right\}\right| \leq C_2\alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}. \tag{1.4}$$

(iv) For all $f \in H^s(\mathbb{R})$, *we have*

$$\lim_{n\to\infty} \mathcal{B}f(x, t_n) = f(x), \quad \text{a. e. } x \in \mathbb{R}.$$

It is easy to see that (ii)⇒(iii). However, the opposite result (iii)⇒(ii) seems nontrivial, so we do not have a direct proof for it. Next, we introduce the outline of proving Theorem 1.1 briefly, as follows: We prove the following five statements: (i)⇒(ii), (i)⇒(iv), (ii)⇒(iii), (iii)⇒(i), and (iv)⇒(iii).

Remark 1.1. *We can drop the convexity assumption in Theorem 1.1. In fact, statements* (ii), (iii), *and* (iv) *hold whenever* t_n *is decreasing and* $\{t_n\} \in l^{\frac{s}{1-2s},\infty}(\mathbb{N})$, *see Proposition 2.1 for more details.*

We also have a global version of the maximal function inequalities, as follows:

Theorem 1.2. *Suppose* $0 < s < \frac{1}{2}$. *Let* $\{t_n\}_{n=1}^{\infty}$ *be a decreasing sequence with* $\lim_{n\to\infty} t_n = 0$, *and suppose that* $\{t_n - t_{n+1}\}_{n=1}^{\infty}$ *is also decreasing. Then the following three statements are equivalent.*

(i) The sequence $\{t_n\} \in l^{\frac{s}{1-2s},\infty}(\mathbb{N})$.

(ii) There exists a constant C_1 *such that for any* $f \in H^s(\mathbb{R})$, *we have*

$$\left\|\sup_{n\in\mathbb{N}} |\mathcal{B}f(x, t_n)|\right\|_{L^2(\mathbb{R})} \leq C_1\|f\|_{H^s(\mathbb{R})}. \tag{1.5}$$

(iii) There exists a constant C_2 such that for any $f \in H^s(\mathbb{R})$ and for any $\alpha > 0$,

$$\left|\left\{x \in \mathbb{R} : \sup_{n\in\mathbb{N}} |\mathcal{B}f(x,t_n)| > \alpha\right\}\right| \leq C_2\alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}. \tag{1.6}$$

The proof of Theorem 1.2 is similar to that of Theorem 1.1. It suffices to prove the following three statements: (i)⇒(ii), (ii)⇒(iii), (iii)⇒(i).

Throughout this paper, we always use C to denote a positive constant, independent of the main parameters involved, whose value may change at each occurrence. The positive constants with subscripts, such as C_1 and C_2, do not change in different occurrences. For two real functions f and g, we always use $f \lesssim g$ or $g \gtrsim f$ to denote that f is smaller than a positive constant C times g, and we always use $f \sim g$ as shorthand for $f \lesssim g \lesssim f$. We shall use the notation $f \gg g$, which means that there is a sufficiently large constant C, which does not depend on the relevant parameters arising in the context in which the quantities f and g appear, such that $f \geq Cg$. If the function f has compact support, we use supp$f$ to denote the support of f. We write $|A|$ for the Lebesgue measure of $A \subset \mathbb{R}$. We use $S(\mathbb{R}^n)$ to denote the Schwartz functions on $\mathbb{R}^n$. The notation diam(B) denotes the diameter of set B.

2. The estimates of maximal functions

In this part, we have proved the boundedness of the maximal function provided $\{t_n\} \in l^{\frac{s}{1-2s},\infty}(\mathbb{N})$, which implies that (i)⇒(ii) and (i)⇒(iv) in Theorem 1.1. At the same time, we also obtain (i)⇒(ii) in Theorem 1.2.

2.1. The main lemma

Without loss of generality, we can suppose that $t_n \in (0,1)$ for all $n \in \mathbb{N}$. Next, we study the frequency truncated operator

$$\mathcal{B}_\lambda f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x\cdot\xi+t|\xi|\sqrt{1+|\xi|^2})}\hat{f}(\xi)\chi\left(\frac{\xi}{\lambda}\right)d\xi,$$

where $\chi \in C^\infty$ is a real-value, smooth function, supp $\chi \subset \left\{\frac{1}{2} \leq |\xi| \leq 1\right\}$. We can obtain the following result by [19], and the following conclusion will play a crucial role in our proof.

Lemma 2.1. *Let $J \subset [0,1]$ be an interval, then*

$$\left\|\sup_{t\in J}|\mathcal{B}_\lambda f(x,t)|\right\|_{L^2(\mathbb{R})} \leq C(1+|J|^{\frac{1}{4}}\lambda^{\frac{1}{2}})\|f\|_{L^2(\mathbb{R})}.$$

In order to prove Lemma 2.1, we review the following three lemmas first: Oscillatory integrals have played a key role in harmonic analysis. So we introduce the following well-known variant of Van der Corput's lemma:

Lemma 2.2. *(Van der Corput's lemma [36]) For $a < b$, let $F \in C^\infty([a,b])$ be real valued and $\psi \in C^\infty([a,b])$.*

(i) *If* $|F'(x)| \geq \lambda > 0, \quad \forall\, x \in [a,b]$ *and* $F'(x)$ *is monotonic on* $[a,b]$*, then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right| \leq \frac{C}{\lambda}\left(|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right),$$

where C *does not depend on* F*,* ψ*, or* $[a,b]$*.*

(ii) *If* $|F''(x)| \geq \lambda > 0, \quad \forall\, x \in [a,b]$*, then*

$$\left|\int_a^b e^{iF(x)}\psi(x)\,dx\right| \leq \frac{C}{\lambda^{\frac{1}{2}}}\left(|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right),$$

where C *does not depend on* F*,* ψ*, or* $[a,b]$*.*

Schur's lemma, which is described as follows, provides sufficient conditions for linear operators to be bounded on $L^p(\mathbb{R}^n)$.

Lemma 2.3. *(Schur's lemma [18]) Assume that* $K(x,y)$ *is a locally integral function on a product of two* σ*-finite measure spaces* $(X,\mu)\times(Y,\nu)$*, and let* T *be a linear operator defined by*

$$Tf(x) = \int_Y K(x,y)f(y)d\nu(y)$$

when f *is bounded and compactly supported. Suppose*

$$\sup_{x\in X}\int_Y |K(x,y)|\,d\nu(y) = A < \infty,$$

$$\sup_{y\in Y}\int_X |K(x,y)|\,d\mu(x) = B < \infty.$$

Then the operator T *extends to a bounded operator from* $L^p(Y)$ *to* $L^p(X)$ *with norm* $A^{1-\frac{1}{p}}B^{\frac{1}{p}}$ *for* $1 \leq p \leq \infty$*.*

The following lemma is well known.

Lemma 2.4. *(Lemma 2.4.2 [34] or [27]) Suppose that* F *is* $C^1(\mathbb{R})$*. Then, if* $q > 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$*,*

$$\sup_{u\in[1,2]} |F(u)|^q \leq |F(1)|^q + q\left(\int_1^2 |F(u)|^q\,du\right)^{\frac{1}{q'}}\left(\int_1^2 |F'(u)|^q\,du\right)^{\frac{1}{q}}.$$

Proof of Lemma 2.1. Let us take the first $\lambda > 1$. The proof follows from the idea of Kolmogorov-Seliverstov-Plessner method. By linearizing the maximal operator, that is, let $x \to t(x)$ be a measurable function and $t(x) \in J$. It suffices to prove

$$\|\mathcal{B}_\lambda f(x,t(x))\|_{L^2(\mathbb{R})} \leq C(1 + |J|^{\frac{1}{4}}\lambda^{\frac{1}{2}})\|f\|_{L^2(\mathbb{R})},$$

where the constant C does not depend on $t(\cdot)$ and f. Denote

$$\mathcal{B}_\lambda f(x,t(x)) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x\cdot\xi + t(x)|\xi|\sqrt{1+|\xi|^2})}\hat{f}(\xi)\chi\left(\frac{\xi}{\lambda}\right)d\xi := \lambda T_\lambda[\hat{f}(\lambda\cdot)](x),$$

where

$$T_\lambda g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\lambda x\cdot\xi + t(x)|\lambda\xi|\sqrt{1+|\lambda\xi|^2})} g(\xi)\chi(\xi)d\xi.$$

Since $\|\hat{f}(\lambda\cdot)\|_{L^2(\mathbb{R})} = \lambda^{-\frac{1}{2}}\|f\|_{L^2(\mathbb{R})}$, our goal translates into proving the following inequality:

$$\|T_\lambda\|_{L^2\to L^2} \lesssim |J|^{\frac{1}{4}} + \lambda^{-\frac{1}{2}},$$

which can further transform into demonstrating

$$\|T_\lambda(T_\lambda)^*\|_{L^2\to L^2} \lesssim |J|^{\frac{1}{2}} + \lambda^{-1}. \tag{2.1}$$

We use the idea of TT^* to complete the proof. After some computation, the kernel of $T_\lambda(T_\lambda)^*$ is

$$K_\lambda(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\lambda(x-y)\cdot\xi + (t(x)-t(y))|\lambda\xi|\sqrt{1+|\lambda\xi|^2})}\chi^2(\xi)d\xi.$$

Denote

$$\begin{aligned}\Phi_\lambda(\xi) &:= \lambda(x-y)\cdot\xi + (t(x)-t(y))|\lambda\xi|\sqrt{1+|\lambda\xi|^2}\\ &= \lambda(x-y)\cdot\xi + (t(x)-t(y))\lambda\xi\sqrt{1+\lambda^2\xi^2},\end{aligned}$$

where $\xi > 0$. Thus,

$$\Phi'_\lambda(\xi) = \lambda(x-y) + \lambda(t(x)-t(y))\frac{1+2\lambda^2\xi^2}{\sqrt{1+\lambda^2\xi^2}}, \quad \frac{1}{2}\le\xi\le 1.$$

On the one hand, if $|x-y| \ge 100\lambda|t(x)-t(y)|$, we have

$$\begin{aligned}|\Phi'_\lambda(\xi)| &\ge \lambda|x-y| - \lambda|t(x)-t(y)|\frac{1+2\lambda^2\xi^2}{\sqrt{1+\lambda^2\xi^2}}\\ &\ge \lambda|x-y| - \lambda|t(x)-t(y)|\frac{1+2\lambda^2}{\sqrt{1+\lambda^2}}\\ &\ge \lambda|x-y| - 4\lambda^2|t(x)-t(y)|\\ &\ge \frac{24}{25}\lambda|x-y|.\end{aligned}$$

The first inequality follows from $\frac{1+2\lambda^2\xi^2}{\sqrt{1+\lambda^2\xi^2}}$ is increasing on $[\frac{1}{2},1]$, and the second inequality follows from $\lambda > 1$.

Therefore, we have

$$|K_\lambda(x,y)| \lesssim (\lambda|x-y|)^{-N}.$$

By the definition of $K_\lambda(x,y)$, we have

$$|K_\lambda(x,y)| \lesssim 1.$$

Since $|K_\lambda(x,y)| \lesssim (\lambda|x-y|)^{-N}$ when $|x-y| \ge 100\lambda|t(x)-t(y)|$, we have

$$|K_\lambda(x,y)| \lesssim (1+\lambda|x-y|)^{-N} \tag{2.2}$$

when $|x-y| \geq 100\lambda|t(x)-t(y)|$.

On the other hand, if $|x-y| \leq 100\lambda|t(x)-t(y)|$, we have

$$\begin{aligned}
|\Phi_\lambda''(\xi)| &= \left|\lambda(t(x)-t(y))\frac{3\lambda^2\xi+2\lambda^4\xi^3}{(1+\lambda^2\xi^2)^{\frac{3}{2}}}\right| \\
&\geq \lambda|t(x)-t(y)|\frac{\frac{3}{2}\lambda^2+\frac{1}{4}\lambda^4}{(1+\frac{1}{4}\lambda^2)^{\frac{3}{2}}} \\
&\geq \lambda|t(x)-t(y)|\frac{\frac{1}{4}\lambda^4}{(4\lambda^2)^{\frac{3}{2}}} \\
&= \frac{1}{32}\lambda^2|t(x)-t(y)| \\
&\geq \frac{\lambda}{3200}|x-y|.
\end{aligned}$$

The first inequality follows from the fact that $\frac{3\lambda^2\xi+2\lambda^4\xi^3}{(1+\lambda^2\xi^2)^{\frac{3}{2}}}$ is increasing on $[\frac{1}{2},1]$. And the second inequality follows from $\lambda > 1$.

Using Lemma 2.2, we obtain

$$|K_\lambda(x,y)| \lesssim \lambda^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}. \tag{2.3}$$

The case $\xi \in [-1,-\frac{1}{2}]$ is similar to case $\xi \in [\frac{1}{2},1]$, and we neglect the details here. (2.2) and (2.3) imply that

$$\begin{aligned}
\int_{\mathbb{R}} |K_\lambda(x,y)|dy &\lesssim \int_{|x-y|\lesssim\lambda|t(x)-t(y)|} \lambda^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}dy + \int_{|x-y|\gg\lambda|t(x)-t(y)|}(1+\lambda|x-y|)^{-N}dy \\
&\lesssim \int_{|x-y|\lesssim\lambda|J|} \lambda^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}dy + \int_{\mathbb{R}}(1+\lambda|x-y|)^{-N}dy \\
&\lesssim |J|^{\frac{1}{2}} + \lambda^{-1},
\end{aligned}$$

which implies that

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |K_\lambda(x,y)|dy \lesssim |J|^{\frac{1}{2}} + \lambda^{-1}.$$

By symmetry, we have the same upbound for $\sup_{y\in\mathbb{R}} \int_{\mathbb{R}} |K_\lambda(x,y)|dx$. By Lemma 2.3, we obtain the desired conclusion (2.1).

The case $\lambda \leq 1$ follows from Lemma 2.4. By Lemma 2.4, we obtain

$$\sup_{t\in J} |B_\lambda f(x,t)|^2 \leq |B_\lambda f(x,t_0)|^2 + 2\left(\int_J |B_\lambda f(x,t)|^2\,dt\right)^{\frac{1}{2}}\left(\int_J |\partial_t B_\lambda f(x,t)|^2\,dt\right)^{\frac{1}{2}}.$$

By Hölder's inequality, we have

$$\left\|\sup_{t\in J}|B_\lambda f(\cdot,t)|\right\|_{L^2(\mathbb{R})} \lesssim \|B_\lambda f(x,t_0)\|_{L^2(\mathbb{R})} + \left\|\|B_\lambda f(x,t)\|_{L^2_x(\mathbb{R})}\right\|^{\frac{1}{2}}_{L^2_t(J)}\left\|\|\partial_t B_\lambda f(x,t)\|_{L^2_x(\mathbb{R})}\right\|^{\frac{1}{2}}_{L^2_t(J)}.$$

Plancherel's theorem implies that

$$\begin{aligned}\left\|\sup_{t\in J}|B_\lambda f(\cdot,t)|\right\|_{L^2(\mathbb{R})} &\lesssim \|f\|_{L^2(\mathbb{R})} + |J|^{\frac14}\|f\|_{L^2(\mathbb{R})}^{\frac12}|J|^{\frac14}\|f\|_{L^2(\mathbb{R})}^{\frac12}\\ &= \|f\|_{L^2(\mathbb{R})} + |J|^{\frac12}\|f\|_{L^2(\mathbb{R})}\\ &\lesssim \|f\|_{L^2(\mathbb{R})}\\ &\lesssim (1+|J|^{\frac14}\lambda^{\frac12})\|f\|_{L^2(\mathbb{R})},\end{aligned}$$

where we use $J \subset [0,1]$ in the second inequality.

2.2. The boundedness of a maximal function

Proposition 2.1. *Let* $\{t_n\} \in l^{r,\infty}(\mathbb{N})$ *be decreasing. Then*

$$\left\|\sup_{n\in\mathbb{N}}|\mathcal{B}f(\cdot,t_n)|\right\|_{L^2(\mathbb{R})} \le C\|f\|_{H^s(\mathbb{R})}, \quad s = \frac{r}{1+2r}. \tag{2.4}$$

Moreover, $\mathcal{B}f(x,t_n) \to f(x)$, a. e. $x \in \mathbb{R}$ *whenever* $f \in H^\kappa(\mathbb{R})$ *for* $\kappa \ge \min\left\{\frac14, \frac{r}{1+2r}\right\}$.

Proof of Proposition 2.1. We use the idea of [12] to complete the proof of Proposition 2.1. We use a standard inhomogeneous frequency decomposition, that is, $\sum_{k\ge0} P_k f = f$, where

$$\begin{aligned}\widehat{P_0 f}(\xi) &= 1_{[-\frac12,\frac12]}(\xi)\hat{f}(\xi),\\ \widehat{P_k f}(\xi) &= \left(1_{[2^{k-1},2^k]}(\xi) + 1_{[-2^k,-2^{k-1}]}(\xi)\right)\hat{f}(\xi), \quad k \ge 1.\end{aligned}$$

Obviously, $P_kP_k = P_k$.

For each integer $l \ge 0$, we define

$$\mathfrak{n}_l := \left\{n \in \mathbb{N} : 2^{-(l+1)\frac{2}{1+2r}} < t_n \le 2^{-l\frac{2}{1+2r}}\right\}.$$

Since $\{t_n\} \in l^{r,\infty}(\mathbb{N})$, there exists $C > 0$ such that

$$\#\mathfrak{n}_l \le C2^{l\frac{2r}{1+2r}} = C2^{2ls}, \tag{2.5}$$

where $s = \frac{r}{1+2r}$.

According to the frequency, we divide our proof into three parts, that is,

$$\sup_n |\mathcal{B}f(x,t_n)| \le A_1(x) + A_2(x) + A_3(x),$$

where

$$A_1(x) := \sup_l \sup_{n\in\mathfrak{n}_l}\left|\sum_{k<\frac{l}{1+2r}} \mathcal{B}P_kf(x,t_n)\right|,$$

$$A_2(x) := \sup_l \sup_{n\in\mathfrak{n}_l} \left| \sum_{\frac{l}{1+2r}\le k<l} \mathcal{B}P_k f(x,t_n) \right|,$$

$$A_3(x) := \sup_l \sup_{n\in\mathfrak{n}_l} \left| \sum_{k\ge l} \mathcal{B}P_k f(x,t_n) \right|.$$

By the definition of $\#\mathfrak{n}_l$, we have

$$t_n \in J_l := \left[0, 2^{-l\frac{2}{1+2r}}\right],$$

where $n \in \mathfrak{n}_l$.

Firstly, we consider the term $\|A_1\|_{L^2(\mathbb{R})}$. By Minkowski's inequality, we obtain

$$\begin{aligned}\|A_1(x)\|_{L^2(\mathbb{R})} &\le \left\| \sup_l \sup_{n\in\mathfrak{n}_l} \sum_{k<\frac{l}{1+2r}} |\mathcal{B}P_k f(x,t_n)| \right\|_{L^2(\mathbb{R})} \\ &\le \sum_{k\ge 0} \left\| \sup_{l>k(1+2r)} \sup_{n\in\mathfrak{n}_l} |\mathcal{B}P_k f(\cdot,t_n)| \right\|_{L^2(\mathbb{R})}.\end{aligned}$$

For $n \in \cup_{l>k(1+2r)}\mathfrak{n}_l$, t_n lies in an interval of length $O(2^{-2k})$, that is, $J_{l(k)}$ with $l(k) = \lfloor k(1+2r)\rfloor$. We use Lemma 2.1 and take $J = J_{l(k)}$ and $\lambda = 2^k$. Thus

$$\left\| \sup_{l>k(1+2r)} \sup_{n\in\mathfrak{n}_l} |\mathcal{B}P_k f(\cdot,t_n)| \right\|_{L^2(\mathbb{R})} \lesssim (1 + 2^{\frac{k}{2}}|J_{l(k)}|^{\frac{1}{4}})\, \|P_k f\|_{L^2(\mathbb{R})} \lesssim \|P_k f\|_{L^2(\mathbb{R})}.$$

The last inequality follows from the fact that $2^{\frac{k}{2}}|J_{l(k)}|^{\frac{1}{4}} \lesssim 1$. Therefore, we obtain

$$\begin{aligned}\|A_1(x)\|_{L^2(\mathbb{R})} &\lesssim \sum_{k\ge 0} \|P_k f\|_{L^2(\mathbb{R})} \\ &\lesssim \sum_{k\ge 0} \left\|\widehat{P_k f}\right\|_{L^2(\mathbb{R})} \\ &= \sum_{k\ge 0} \left\|\left(\chi_{[2^{k-1},2^k]} + \chi_{[-2^k,-2^{k-1}]}\right)\hat{f}\right\|_{L^2(\mathbb{R})} \\ &= \sum_{k\ge 0} \left\|\chi_{[2^{k-1},2^k]}(|\xi|)\hat{f}(\xi)\right\|_{L^2(\mathbb{R})} \\ &= \sum_{k\ge 0} \left\|\chi_{[2^{k-1},2^k]}(|\xi|)\left(1+|\xi|^2\right)^{-\frac{s}{2}}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{f}(\xi)\right\|_{L^2(\mathbb{R})} \\ &\lesssim \sum_{k\ge 0} 2^{-ks}\left\|\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{f}(\xi)\right\|_{L^2(\mathbb{R})} \\ &\le C(s)\|f\|_{H^s(\mathbb{R})}, \quad s>0.\end{aligned}$$

Secondly, we study the term $\|A_2\|_{L^2(\mathbb{R})}$. For simplicity of notation, we take the change of variables $k = l - j$. By Minkowski's inequality, we have

$$A_2(x) = \sup_l \sup_{n\in\mathfrak{n}_l} \left| \sum_{0<j\le\frac{2r}{1+2r}l} \mathcal{B}P_{l-j} f(x,t_n) \right|$$

$$\leq \left(\sum_{l\geq 0} \left(\sum_{0<j\leq \frac{2r}{1+2r}l} \sup_{n\in \mathfrak{n}_l} |\mathcal{B}P_{l-j}f(x,t_n)| \right)^2 \right)^{\frac{1}{2}}$$

$$\leq \sum_{j>0} \left(\sum_{l\geq j\frac{1+2r}{2r}} \sup_{n\in \mathfrak{n}_l} |\mathcal{B}P_{l-j}f(x,t_n)|^2 \right)^{\frac{1}{2}}.$$

Since $l \geq j\frac{1+2r}{2r}$, we have

$$|J_l|^{\frac{1}{4}} 2^{\frac{1}{2}(l-j)} = 2^{-l\frac{1}{2(1+2r)}} 2^{\frac{1}{2}(l-j)} \geq 1.$$

By Minkowski's inequality and using Lemma 2.1 again with $J = J_l$ and $\lambda = 2^{l-j}$, we then obtain

$$\begin{aligned}
\|A_2(x)\|_{L^2(\mathbb{R})} &\leq \sum_{j\geq 0} \left\| \left(\sum_{l\geq j\frac{1+2r}{2r}} \sup_{n\in \mathfrak{n}_l} \left|\mathcal{B}P_{l-j}f(\cdot,t_n)\right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R})} \\
&\leq \sum_{j\geq 0} \left(\sum_{l\geq j\frac{1+2r}{2r}} \| \sup_{n\in \mathfrak{n}_l} |\mathcal{B}P_{l-j}f(\cdot,t_n)| \|^2_{L^2(\mathbb{R})} \right)^{\frac{1}{2}} \\
&\lesssim \sum_{j\geq 0} \left(\sum_{l\geq j\frac{1+2r}{2r}} \left[(1 + |J_l|^{\frac{1}{4}} 2^{\frac{1}{2}(l-j)} \|P_{l-j}f\|_{L^2(\mathbb{R})} \right]^2 \right)^{\frac{1}{2}} \\
&= \sum_{j\geq 0} \left(\sum_{l\geq j\frac{1+2r}{2r}} \left[(1 + 2^{\frac{1}{2}(l-j)} 2^{-l\frac{1}{2(1+2r)}} \|P_{l-j}f\|_{L^2(\mathbb{R})} \right]^2 \right)^{\frac{1}{2}} \\
&\lesssim \sum_{j\geq 0} \left(\sum_{l\geq j\frac{1+2r}{2r}} \left[2^{\frac{1}{2}(l-j)(1-\frac{1}{1+2r})} 2^{-j\frac{1}{2(1+2r)}} \|P_{l-j}f\|_{L^2(\mathbb{R})} \right]^2 \right)^{\frac{1}{2}} \\
&= \sum_{j\geq 0} 2^{-j\frac{1}{2(1+2r)}} \left(\sum_{l\geq j\frac{1+2r}{2r}} \left[2^{s(l-j)} \|P_{l-j}f\|_{L^2(\mathbb{R})} \right]^2 \right)^{\frac{1}{2}} \\
&\lesssim \|f\|_{H^s(\mathbb{R})},
\end{aligned}$$

where $s = \frac{1}{2}\left(1 - \frac{1}{1+2r}\right) = \frac{r}{1+2r}$.

Finally, we consider the estimate $\|A_3\|_{L^2(\mathbb{R})}$. We also make the change of variable $k = l + m$. By Minkowski's inequality, we have

$$A_3(x) \leq \sum_{m\geq 0} \left(\sum_{l\geq 0} \sup_{n\in \mathfrak{n}_l} |\mathcal{B}P_{l+m}f(x,t_n)|^2 \right)^{\frac{1}{2}}.$$

By Minkowski's inequality and Plancherel's theorem, we obtain

$$\|A_3\|_{L^2(\mathbb{R})} \leq \sum_{m\geq 0} \left(\sum_{l\geq 0} \left\| \sup_{n\in \mathfrak{n}_l} |\mathcal{B}P_{l+m}f(\cdot,t_n)| \right\|^2_{L^2(\mathbb{R})} \right)^{\frac{1}{2}}$$

$$\leq \sum_{m\geq 0}\left(\sum_{l\geq 0}\sum_{n\in \mathfrak{n}_l}\|\mathcal{B}P_{l+m}f(\cdot,t_n)\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}$$
$$\leq \sum_{m\geq 0}\left(\sum_{l\geq 0}\#(\mathfrak{n}_l)\,\|P_{l+m}f\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}.$$

From (2.5), it follows that

$$\|A_3\|_{L^2(\mathbb{R})} \lesssim \sum_{m\geq 0}\left(\sum_{l\geq 0}2^{2sl}\,\|P_{l+m}f\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}$$
$$= \sum_{m\geq 0}2^{-ms}\left(\sum_{l\geq 0}2^{2s(l+m)}\,\|P_{l+m}f\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}$$
$$\lesssim \|f\|_{H^s(\mathbb{R})}.$$

We are ready to combine all our ingredients and finish the proof.

$$\left\|\sup_{n\in\mathbb{N}}|\mathcal{B}f(\cdot,t_n)|\right\|_{L^2(\mathbb{R})} \leq \|A_1\|_{L^2(\mathbb{R})}+\|A_2\|_{L^2(\mathbb{R})}+\|A_3\|_{L^2(\mathbb{R})}\lesssim \|f\|_{H^s(\mathbb{R})},$$

which means that the maximal inequality (2.4) is established. For any $f\in S(\mathbb{R})$, we have $\lim_{t\to 0}\mathcal{B}f(x,t) = f(x)$ for all $x\in\mathbb{R}$. By [22], it holds

$$\left\|\sup_{t\in[0,1]}|\mathcal{B}f(x,t)|\right\|_{L^2(B)} \leq C\|f\|_{H^{\frac{1}{4}}(\mathbb{R})}. \tag{2.6}$$

Since Schwartz functions are dense in $H^\kappa(\mathbb{R})$, by (2.4) and (2.6), we obtain

$$\lim_{t\to 0}\mathcal{B}f(x,t) = f(x), \quad \text{a. e. } x\in\mathbb{R},$$

whenever $f\in H^\kappa(\mathbb{R}), \kappa\geq \min\left\{\frac{1}{4},\frac{r}{1+2r}\right\}$.

3. Necessary conditions

In this part, we use ideas from Nikishin-Stein theory to prove necessity in Theorems 1.1 and 1.2; that is, we obtain the following statements: (iii)⇒(i), (iv)⇒(iii) in Theorem 1.1, and (iii)⇒(i) in Theorem 1.2.

3.1. The proofs of Theorems 1.1 and 1.2

Firstly, we introduce the following proposition:

Proposition 3.1. *Suppose that for all $f\in H^s(\mathbb{R})$, the limit $\lim_{n\to\infty}\mathcal{B}f(x,t_n)$ exists for almost every $x\in\mathbb{R}$. Then for all compact sets $K\subset\mathbb{R}$, there exists a constant C_k, such that for any $\alpha>0$,*

$$\left|\left\{x\in K:\sup_{n\in\mathbb{N}}|\mathcal{B}f(x,t_n)|>\alpha\right\}\right| \leq C_K\alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}.$$

We postpone the proof of Proposition 3.1 here, and the details will be shown in Section 3.3.

Secondly, we also need the following key lemma, which is proved in [12].

Lemma 3.1. *[12] Let $\{t_n\}$ be a sequence of positive numbers in $[0,1]$, let $0 < r < \infty$, and suppose that*

$$\sup_{b>0} b^r \#(\{n : b < t_n \le 2b\}) \le A.$$

Then $\{t_n\} \in l^{r,\infty}(\mathbb{N})$.

Thirdly, we are now ready to prove the necessity of the $l^{r,\infty}(\mathbb{N})$ condition in Theorems 1.1 and 1.2. In fact, we summarize the necessity of the $l^{r,\infty}(\mathbb{N})$ condition into the following Proposition 3.2 which plays a key role in this paper.

Proposition 3.2. *Suppose that $\{t_n\}_{n=1}^{\infty}$ is a decreasing sequence such that $\{t_n - t_{n+1}\}_{n=1}^{\infty}$ is decreasing and $\lim_{n\to\infty} t_n = 0$. For $0 < s < \frac{1}{2}$, let $r(s) = \frac{s}{1-2s}$.*

(i) If $s < \frac{1}{4}$ and

$$\left|\left\{x \in [0,1] : \sup_{n\in\mathbb{N}} |\mathcal{B}f(x,t_n)| > \frac{1}{2}\right\}\right| \le C\|f\|_{H^s(\mathbb{R})}^2 \tag{3.1}$$

holds for any $f \in H^s(\mathbb{R})$, then $\{t_n\} \in l^{r(s),\infty}(\mathbb{N})$.

(ii) If $s < \frac{1}{2}$ and the global weak type inequality

$$\left|\left\{x \in \mathbb{R} : \sup_{n\in\mathbb{N}} |\mathcal{B}f(x,t_n)| > \frac{1}{2}\right\}\right| \le C\|f\|_{H^s(\mathbb{R})}^2 \tag{3.2}$$

holds for any $f \in H^s(\mathbb{R})$, then $\{t_n\} \in l^{r(s),\infty}(\mathbb{N})$.

We will prove Proposition 3.2 in Section 3.2. We are now ready to combine all our ingredients and finish the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By Proposition 2.1, we can prove the implications of (i)⇒(ii) and (i)⇒(iv). By Tshebyshev's inequality, we have the result (ii)⇒(iii). By the first part of Proposition 3.2, we obtain the implication (iii)⇒(i). Finally, using Proposition 3.1, we obtain the conclusion (iv)⇒(iii). Thus, the four statements (i), (ii), (iii), and (iv) are equivalent.

Proof of Theorem 1.2. From Proposition 2.1, we have the implication (i)⇒(ii). It is easy to obtain the implication (ii)⇒(iii) by Tshebyshev's inequality. From the second part of Proposition 3.2, we see the implication (iii)⇒(i). Thus, the three statements (i), (ii), and (iii) are equivalent.

3.2. The proof of Proposition 3.2

We can divide the proof of part (ii) of Proposition 3.2 into two cases: $s < \frac{1}{4}$ and $\frac{1}{4} \le s < \frac{1}{2}$. Furthermore, since (3.2) yields (3.1), we have (i) implies (ii) when $s < \frac{1}{4}$. So we only need to consider $\frac{1}{4} \le s < \frac{1}{2}$ when we prove part (ii).

We use a contradiction argument. Assume that $\{t_n\} \notin l^{r(s),\infty}(\mathbb{N})$, while (3.1) holds for $s < \frac{1}{4}$ or (3.2) holds in the case $\frac{1}{4} \le s < \frac{1}{2}$. By Lemma 3.1, we obtain

$$\sup_{0<b<\frac{1}{2}} b^{r(s)} \#(\{n : b < t_n \le 2b\}) = \infty.$$

Thus, there exists an increasing sequence $\{K_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} K_j = \infty$ and a sequence of positive numbers with $\lim_{j\to\infty} b_j = 0$, such that

$$\#(\{n : b_j < t_n \leq 2b_j\}) \geq K_j b_j^{-r(s)}. \tag{3.3}$$

We choose another sequence $L_j \leq K_j$ with $\lim_{j\to\infty} L_j = \infty$ so that in the case where $s < \frac{1}{4}$

$$2L_j b_j^{\frac{1-4s}{2-4s}} \leq \frac{1}{2}. \tag{3.4}$$

In the case $\frac{1}{4} \leq s < \frac{1}{2}$, we let $L_j = K_j$.

Using the idea originally proposed by Dahlberg-Kenig [10], we complete the construction of a counterexample. We introduce a family of Schwartz functions that are used to test (3.1). Take a C^∞ function g with supp $g \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$ so that $\int_{\mathbb{R}} g(\xi)d\xi = 1$ and $g(\xi) \geq 0$ and study a family of functions $f_{\lambda,\rho}$, where λ is a large number $\rho \ll \lambda$, and λ, ρ will be given later. $f_{\lambda,\rho}$ is defined via the Fourier transform by

$$\widehat{f_{\lambda,\rho}}(\eta) = \rho^{-1} g\left(\frac{\eta + \lambda}{\rho}\right).$$

Thus, supp $\widehat{f_{\lambda,\rho}}$ belongs to an interval of length $\rho \ll \lambda$ contained in $\left[-2\lambda, -\frac{\lambda}{2}\right]$. By the definition of $f_{\lambda,\rho}$, we obtain

$$\|f_{\lambda,\rho}\|_{H^s(\mathbb{R})} \lesssim \lambda^s \rho^{-\frac{1}{2}}. \tag{3.5}$$

We now study the property of $\mathcal{B}$ on $f_{\lambda,\rho}$. We have

$$\left|\mathcal{B} f_{\lambda,\rho}(x, t_n)\right| = \left|\frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\cdot\eta + t_n|\eta|\sqrt{1+|\eta|^2})} \rho^{-1} g\left(\frac{\eta+\lambda}{\rho}\right) d\eta\right| = \left|\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\Phi_{\lambda,\rho}(\xi; x, t_n)} g(\xi) d\xi\right|,$$

where

$$\Phi_{\lambda,\rho}(\xi; x, t_n) := x(\rho\xi - \lambda) + t_n(\lambda - \rho\xi)\sqrt{1 + (\lambda - \rho\xi)^2}.$$

For x in a suitable interval $I_j \subset I$, and for suitable choices of λ_j, ρ_j and $n(x, j)$, we obtain

$$\begin{aligned}\left|\mathcal{B} f_{\lambda_j,\rho_j}(x, t_{n(x,j)})\right| &\geq \int_{\mathbb{R}} g(\xi) d\xi - \frac{1}{2\pi} \int_{\mathbb{R}} \left|e^{i\Phi_{\lambda_j,\rho_j}(\xi; x, t_{n(x,j)})} - 1\right| g(\xi) d\xi \\ &\geq 1 - \max_{|\xi| \leq \frac{1}{2}} \left|e^{i\Phi_{\lambda_j,\rho_j}(\xi; x, t_{n(x,j)})} - 1\right|.\end{aligned} \tag{3.6}$$

In order to prove $\left|\mathcal{B} f_{\lambda_j,\rho_j}(x, t_{n(x,j)})\right| \geq \frac{1}{2}$, we only need to demonstrate that

$$\max_{|\xi| \leq \frac{1}{2}} \left|e^{i\Phi_{\lambda_j,\rho_j}(\xi; x, t_{n(x,j)})} - 1\right| \leq \frac{1}{2}$$

for our choices of $x, n(x, j)$ and (λ_j, ρ_j).

From the definition of $\Phi_{\lambda,\rho}(\xi; x, t_n)$, it follows that

$$\Phi'_{\lambda,\rho}(\xi; x, t_n) = \rho\left[x - t_n \frac{1 + 2(\lambda - \rho\xi)^2}{\sqrt{1 + (\lambda - \rho\xi)^2}}\right],$$

$$\Phi''_{\lambda,\rho}(\xi;x,t_n) = t_n\rho^2(\lambda-\rho\xi)\frac{3+2(\lambda-\rho\xi)^2}{(1+(\lambda-\rho\xi)^2)^{\frac{3}{2}}},$$
$$\Phi'''_{\lambda,\rho}(\xi;x,t_n) = \frac{-3t_n\rho^3}{(1+(\lambda-\rho\xi)^2)^{\frac{5}{2}}}.$$

By Taylor expansion, we obtain

$$\begin{aligned}\Phi_{\lambda,\rho}(\xi;x,t_n) &= \Phi_{\lambda,\rho}(0;x,t_n) + \Phi'_{\lambda,\rho}(0;x,t_n)\xi + \frac{\Phi''_{\lambda,\rho}(0;x,t_n)}{2!}\xi^2 + \frac{1}{2!}\int_0^{\xi}\Phi'''_{\lambda,\rho}(t;x,t_n)(\xi-t)^2dt \\ &= -\lambda x + t_n\lambda\sqrt{1+\lambda^2} + \rho\left[x - t_n\frac{1+2\lambda^2}{\sqrt{1+\lambda^2}}\right]\xi + \frac{1}{2}t_n\rho^2\lambda\frac{3+2\lambda^2}{(1+\lambda^2)^{\frac{3}{2}}}\xi^2 \\ &\quad + \frac{1}{2!}\int_0^{\xi}\frac{-3t_n\rho^3}{(1+(\lambda-\rho t)^2)^{\frac{5}{2}}}(\xi-t)^2dt \\ &:= -\lambda x + t_n\lambda\sqrt{1+\lambda^2} + I + II + III. \end{aligned} \tag{3.7}$$

Noting that terms $-\lambda x+t_n\lambda\sqrt{1+\lambda^2}$ do not depend on ξ, we have terms $-\lambda x+t_n\lambda\sqrt{1+\lambda^2}$ do not affect our integral. We may neglect the terms $-\lambda x + t_n\lambda\sqrt{1+\lambda^2}$ and only need to consider the last three terms *I–III*. We consider t_n with $t_n \le \frac{b_j}{2}$ and let ϵ be such that $\epsilon < \frac{1}{100}$. We choose

$$\lambda_j = L_j b_j^{-\frac{1}{2-4s}}, \quad \rho_j = \epsilon b_j^{-\frac{1-2s}{2-4s}} = \epsilon b_j^{-\frac{1}{2}}.$$

Firstly, we study the upper bound of $|I|$. We consider x in the interval

$$I_j := \left[0, \frac{b_j}{2}\frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}}\right].$$

Observe that in the case $s < \frac{1}{4}$, by (3.4), we obtain

$$\frac{b_j}{2}\frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}} \le 2b_j\lambda_j = 2L_j b_j^{\frac{1-4s}{2(1-2s)}} \le \frac{1}{2},$$

which implies that $I_j \subset \left[0, \frac{1}{2}\right]$ in this case. Each $x \in I_j$ implies $x \in \left(\frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}}t_{n+1}, \frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}}t_n\right]$ for a unique n, which we label $n(x,j)$.

We now claim that

$$t_n - t_{n+1} \le 2L_j^{-1}b_j^{\frac{1-s}{1-2s}}, \tag{3.8}$$

where $t_n \le b_j$.

We can see this as follows: Since $\{t_n - t_{n+1}\}_{n=1}^{\infty}$ is decreasing, for $t_n \le b_j$, by (3.3), we obtain

$$t_n - t_{n+1} \le \min\{t_m - t_{m+1} : t_m > b_j\} \le \frac{2b_j}{\#(\{n : b_j < t_n \le 2b_j\})} \le \frac{2b_j}{K_j b_j^{-r(s)}} \le \frac{2b_j}{L_j b_j^{-r(s)}} = 2L_j^{-1}b_j^{\frac{1-s}{1-2s}}.$$

By (3.8), we have that

$$0 \le t_{n(x,j)} - t_{n(x,j)+1} \le 2L_j^{-1} b_j^{\frac{1-s}{1-2s}}. \tag{3.9}$$

By (3.9) and the definitions of λ_j and ρ_j, we obtain

$$\begin{aligned} |I| &= \left| \rho_j \left[x - t_n \frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}} \right] \xi \right| \le \frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}} (t_{n(x,j)} - t_{n(x,j)+1}) \rho_j \\ &\le 4\lambda_j L_j^{-1} b_j^{\frac{1-s}{1-2s}} \rho_j = 4\epsilon. \end{aligned} \tag{3.10}$$

Secondly, we consider the upper bound of $|II|$. From the definitions of λ_j and ρ_j, it follows that

$$\begin{aligned} |II| &= \left| \frac{1}{2} t_n \rho_j^2 \lambda_j \frac{3+2\lambda_j^2}{(1+\lambda_j^2)^{\frac{3}{2}}} \xi^2 \right| \le \frac{1}{4} b_j \rho_j^2 \lambda_j \frac{3+2\lambda_j^2}{(1+\lambda_j^2)^{\frac{3}{2}}} \frac{1}{4} \\ &\le \frac{1}{16} 4 b_j \rho_j^2 = 4\epsilon^2. \end{aligned} \tag{3.11}$$

Finally, we provide the estimate of the last term III. We obtain

$$\frac{\rho_j}{\lambda_j} = \epsilon L_j^{-1} b_j^{\frac{s}{1-2s}} \le \epsilon.$$

Using the change of variables $t = \xi s$ and the choices of λ_j, ρ_j, we obtain

$$\begin{aligned} |III| &= \left| \frac{1}{2!} \int_0^{\xi} \frac{-3t_n \rho_j^3}{(1+(\lambda_j - \rho_j t)^2)^{\frac{5}{2}}} (\xi - t)^2 dt \right| = \frac{3}{2} t_n \rho_j^3 \xi^3 \left| \int_0^1 \frac{(1-s)^2}{(1+(\lambda_j - \rho_j \xi s)^2)^{\frac{5}{2}}} ds \right| \\ &\le \frac{3}{32} b_j \rho_j^3 \left| \int_0^1 \frac{1}{(\lambda_j - \rho_j \xi s)^2} ds \right| = \frac{3}{32} b_j \rho_j^3 \frac{1}{\lambda_j(\lambda_j - \rho_j \xi)} \\ &\le \frac{3}{32} b_j \rho_j^3 \frac{1}{\lambda_j} \le \frac{3}{32} b_j \rho_j^2 \epsilon = \frac{3}{32} \epsilon^3. \end{aligned} \tag{3.12}$$

Since $\epsilon < \frac{1}{100}$, from (3.7) and (3.10)–(3.12), we obtain

$$\max_{|\xi| \le \frac{1}{2}} \left| e^{i\Phi_{\lambda_j,\rho_j}(\xi; x, t_{n(x,j)})} - 1 \right| \le \frac{1}{2},$$

which implies that

$$\sup_{n \in \mathbb{N}} \left| \mathcal{B} f_{\lambda_j,\rho_j}(x, t_n) \right| \ge 1 - \max_{|\xi| \le \frac{1}{2}} \left| e^{i\Phi_{\lambda_j,\rho_j}(\xi; x, t_{n(x,j)})} - 1 \right| \ge \frac{1}{2},$$

for $x \in I_j = \left[0, \frac{b_j}{2} \frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}} \right] \subset [0,1]$. By (3.1) or (3.2), we have

$$\text{meas}(I_j) \lesssim \|f_{\lambda_j,\rho_j}\|_{H^s(\mathbb{R})}^2 \approx \lambda_j^{2s} \rho_j^{-1},$$

which implies that

$$\frac{b_j}{2}\frac{1+2\lambda_j^2}{\sqrt{1+\lambda_j^2}} \lesssim \lambda_j^{2s}\rho_j^{-1}.$$

Since $\lambda_j = L_j b_j^{-\frac{1}{2-4s}}$, $\rho_j = \epsilon b_j^{-\frac{1}{2}}$, we have $b_j \lesssim \left(L_j b_j^{-\frac{1}{2-4s}}\right)^{2s-1}\left(\epsilon b_j^{-\frac{1}{2}}\right)^{-1}$, which implies that

$$\epsilon \lesssim L_j^{2s-1}.$$

Since $\lim_{j\to\infty} L_j = \infty$, we have $\lim_{j\to\infty} L_j^{2s-1} = 0$ with $0 < s < \frac{1}{2}$, which leads to a contradiction. This means that if $\{t_n\} \notin l^{r(s),\infty}(\mathbb{N})$, then (3.1) (and therefore (3.2)) fails if $s < \frac{1}{4}$ and (3.2) fails if $\frac{1}{4} \leq s < \frac{1}{2}$. Thus we complete the proof of the proposition.

Remark 3.1. *We explain the choices of the parameters λ_j and ρ_j. We divide the estimate of $\Phi_{\lambda,\rho}(\xi; x, t_n)$ into three terms I, II and III. From the estimate of $|I|$, we have*

$$\lambda_j L_j^{-1} b_j^{\frac{1-s}{1-2s}} \rho_j = \epsilon. \tag{3.13}$$

From the estimate of $|II|$, we have

$$b_j \rho_j^2 = \epsilon^2. \tag{3.14}$$

From the estimate of $|III|$, we have

$$b_j \rho_j^2 = \epsilon^2. \tag{3.15}$$

We obtain (3.14)*, which is the same as* (3.15)*. By* (3.13) *and* (3.14)*, we obtain*

$$\lambda_j = L_j b_j^{-\frac{1}{2-4s}}, \quad \rho_j = \epsilon b_j^{-\frac{1}{2}}.$$

3.3. The proof of Proposition 3.1

We use Nikishin's theorem here, whose proof can be found in [17, 35]. Nikishin's theorem asserts that if $M : L^2(Y, \mu) \to L^0(\mathbb{R}^n, |\cdot|)$ is a continuous sublinear operator (with (Y, μ) an arbitrary measure space), then there exists a measurable function $\omega(x)$ with $\omega(x) > 0$ such that

$$\int_{\{x:|Mf(x)|>\alpha\}} \omega(x)dx \leq \alpha^{-2}\|f\|_{L^2(\mu)}^2.$$

Let $M^{\mathcal{B}} f(x) = \sup_{n\in\mathbb{N}} |\mathcal{B}f(x, t_n)|$ and $T_n^{\mathcal{B}} g(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{i(x\cdot\xi + t_n|\xi|\sqrt{1+|\xi|^2})} g(\xi)d\xi$. We obtain

$$T_n^{\mathcal{B}} \hat{f}(x) = (2\pi)^{-1} \int_{\mathbb{R}} e^{i(x\cdot\xi + t_n|\xi|\sqrt{1+|\xi|^2})} \hat{f}(\xi)d\xi = \mathcal{B}f(x, t_n).$$

Then $T_n^{\mathcal{B}}$ acts on functions in the weighted L^2 space $L^2(\mu_s)$, where $d\mu_s = (1 + |\xi|^2)^s d\xi$. Define the maximal operator $\widetilde{M}^{\mathcal{B}} g = \sup_{n\in\mathbb{N}} |T_n^{\mathcal{B}} g|$. Since $\lim_{n\to\infty} \mathcal{B}f(x, t_n)$ exists almost everywhere for every $f \in H^s(\mathbb{R})$, we have $\widetilde{M}^{\mathcal{B}}g < \infty$ almost everywhere for every $g \in L^2(\mu_s)$. Then [17] implies that the sublinear operator $\widetilde{M}^{\mathcal{B}} : L^2(\mu_s) \to L^0(|\cdot|)$ is continuous. By Nikishin's theorem, we obtain

$$\int_{\{x:|\widetilde{M}^{\mathcal{B}}g(x)|>\alpha\}} \omega(x)dx \le \alpha^{-2}\|g\|^2_{L^2(\mu_s)} \tag{3.16}$$

for some weight $\omega(x)$, with $\omega(x) > 0$ almost everywhere. Without loss of generality, we may further assume that ω is bounded.

Next, for $f \in H^s(\mathbb{R})$, we obtain

$$\widetilde{M}^{\mathcal{B}}\hat{f}(x) = \sup_{n\in\mathbb{N}} \left|T_n^{\mathcal{B}}\hat{f}(x)\right| = \sup_{n\in\mathbb{N}} |\mathcal{B}f(x,t_n)| = M^{\mathcal{B}}f(x), \tag{3.17}$$

and

$$\|\hat{f}\|_{L^2(\mu_s)} = \|f\|_{H^s(\mathbb{R})}. \tag{3.18}$$

(3.16)–(3.18) imply that

$$\int_{\{x:|M^{\mathcal{B}}f(x)|>\alpha\}} \omega(x)dx \le \alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}.$$

Using the change of variables $x \to x - y$, we have

$$\int_{\{x:|M^{\mathcal{B}}f(x)|>\alpha\}} \omega(x-y)dx \le \alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}. \tag{3.19}$$

We multiply both sides of (3.19) by $h(y)$, where h is a strictly positive continuous function with $\int_{\mathbb{R}} h(y)dy = 1$, we obtain

$$\int_{\{x:|M^{\mathcal{B}}f(x)|>\alpha\}} \omega(x-y)h(y)dx \le \alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}h(y).$$

Then we integrate in y to obtain that

$$\int_{\mathbb{R}}\int_{\{x:|M^{\mathcal{B}}f(x)|>\alpha\}} \omega(x-y)h(y)dxdy \le \alpha^{-2}\|f\|^2_{H^s(\mathbb{R})},$$

which yields that

$$\int_{\{x:|M^{\mathcal{B}}f(x)|>\alpha\}} h*\omega(x)dx \le \alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}. \tag{3.20}$$

Since $h*\omega$ is continuous, it attains a minimum over any compact set. For every compact set K, by (3.20), we obtain

$$\begin{aligned}\left|\left\{x \in K : \sup_{n\in\mathbb{N}} |\mathcal{B}f(x,t_n)| > \alpha\right\}\right| &= \left|\left\{x \in K : \left|M^{\mathcal{B}}f(x)\right| > \alpha\right\}\right| \\ &= \int_{\{x\in K:|M^{\mathcal{B}}f(x)|>\alpha\}} dx \\ &\le C_K \int_{\{x\in K:|M^{\mathcal{B}}f(x)|>\alpha\}} h*\omega(x)dx \\ &\le C_K\alpha^{-2}\|f\|^2_{H^s(\mathbb{R})}.\end{aligned}$$

Therefore, Proposition 3.1 is established.

4. Conclusions

In this paper, we study the almost everywhere pointwise convergence problem of the sequential Boussinesq operator. The fractional Schrödinger operator and Boussinesq operator are different operators, and the result from Dimou-Seeger [12] cannot cover our work. The Boussinesq operator along sequences $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} t_n = 0$ in one dimension is studied. We obtain a characterization of convergence almost everywhere when $\{t_n\} \in l^{r,\infty}(\mathbb{N})$ for all $f \in H^s(\mathbb{R})$ provided $0 < s < \frac{1}{2}$.

Author contributions

Dan Li: Conceptualization, Methodology, Investigation, Writing – Original Draft; Fangyuan Chen: Writing – Review and Editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare they have no conflict of interest.

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