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Research article

Estimates for functions of generalized Marcinkiewicz operators related to surfaces of revolution

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Abstract: In this paper, specific L^p estimates for generalized Marcinkiewicz operators correlated to surfaces of revolution are proved. These estimates and the extrapolation procedure of Yano are employed to confirm the L^p boundedness of the above-mentioned integrals under weaker assumptions on the singular kernels. Our findings generalize and improve several known results.

Keywords: Triebel-Lizorkin space; singular kernels; generalized Marcinkiewicz integrals; L^p estimates; extrapolation
Mathematics Subject Classification: 42B20, 42B25

1. Introduction

Throughout this article, assume that $\mathbb{S}^{\eta-1}$ ($\eta \geq 2$) is the unit sphere in the Euclidean space $\mathbb{R}^\eta$, that is equipped with the spherical measure $d\sigma_\eta(\cdot)$. Also, assume that $v' = v/|v|$ for $v \in \mathbb{R}^\eta \setminus \{0\}$.

For $n = \alpha + i\beta$ ($\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$), let $\mathsf{K}_{\Psi,h}(v) = \frac{\Psi(v)h(|v|)}{|v|^{\eta-n}}$, where h is a measurable mapping on $\mathbb{R}^+$ and $\Psi \in L^1(\mathbb{S}^{\eta-1})$ is a measurable mapping satisfying the following conditions:

$$\Psi(tv) = \Psi(v), \quad \forall t > 0, \tag{1.1}$$

$$\int_{\mathbb{S}^{\eta-1}} \Psi(v')d\sigma(v') = 0. \tag{1.2}$$

For an appropriate mapping $\phi : \mathbb{R}^+ \to \mathbb{R}$, we define the generalized Marcinkiewicz operator $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ by

$$\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)(\bar{w}) = \left(\int_{\mathbb{R}^+} \left| \frac{1}{t^n} \int_{|v|\leq t} F(w-v, w_{\eta+1} - \phi(|v|)) \mathrm{K}_{\Psi,h}(v) dv \right|^{\gamma} \frac{dt}{t} \right)^{1/\gamma},$$

where $F \in C_0^\infty(\mathbb{R}^{\eta+1})$, $\bar{w} = (w, w_{\eta+1}) \in \mathbb{R}^{\eta+1}$, and $\gamma > 1$.

When $\gamma = 2$, $\phi \equiv 0$, and $h \equiv 1$, we denote $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ by $\mathcal{G}_{\Psi,n}$, and when $n = 1$, we denote $\mathcal{G}_{\Psi,n}$ by $\mathcal{G}_\Psi$. The operator $\mathcal{G}_\Psi$ is basically the traditional Marcinkiewicz operator defined in [1] where the author studied the L^p $(1 < p \leq 2)$ boundedness of $\mathcal{G}_\Psi$ whenever the singular kernel Ψ belongs to the space $Lip_\tau(\mathbb{S}^{\eta-1})$ with $\tau \in (0, 1]$. This result was improved in [2], in which the author obtained the L^2 boundedness of $\mathcal{G}_\Psi$ under the condition $\Psi \in L(logL)^{1/2}(\mathbb{S}^{\eta-1})$. Also, he obtained that the assumption $\Psi \in L(\log L)^{1/2}(\mathbb{S}^{\eta-1})$ is optimal in the sense that when it is replaced by any weaker assumption $\Psi \in L(\log L)^{\eta}(\mathbb{S}^{\eta-1})$ with $\eta \in (0, 1/2)$, then $\mathcal{G}_\Psi$ will not be bounded on $L^2(\mathbb{R}^\eta)$. Later, the authors of [3] confirmed the results in [2] not only for $p = 2$, but for all $p \in (1,\infty)$. On the other side, the L^p boundedness of $\mathcal{G}_\Psi$ was proved by Al-Qassem and Al-Salman in [4] for all $p \in (1,\infty)$ provided that $\Psi \in B_q^{(0,-1/2)}(\mathbb{S}^{\eta-1})$ for some $q > 1$. Also, they proved the optimality of the assumption $\Psi \in B_q^{(0,-1/2)}(\mathbb{S}^{\eta-1})$. When $\gamma = 2$, $\Psi \in L(\log L)^{1/2}(\mathbb{S}^{\eta-1})$, $h \in \nabla_\kappa(\mathbb{R}^+)$ with $\Psi > 1$, and $\phi \in \mathcal{H}_d$, the L^p boundedness of $\mathcal{G}^{(2)}_{\Psi,\phi,h}$ was established in [5] for all $\left|\frac{2-p}{2p}\right| < \min\{1/\kappa', 1/2\}$. Here, $\nabla_\kappa(\mathbb{R}^+)$ indicates the set of measurable mappings h on $\mathbb{R}^+$,

$$\|h\|_{\nabla_\kappa(\mathbb{R}^+)} = \sup_{j\in\mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^\kappa \frac{dt}{t} \right)^{1/\kappa} < \infty.$$

The integral operator $\mathcal{G}^{(2)}_{\Psi,\phi,h}$ under several assumptions has been investigated by many researchers: For the case $h \in L^\infty(\mathbb{R})^+$ [6, 7], along surfaces [8–11], using extrapolation [12, 13].

The study of the generalized Marcinkiewicz operator $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ was started in [14], in which the authors proved that whenever $\Psi \in L^q(\mathbb{S}^{\eta-1})$ with $q > 1$, $\phi(t) = t$, $h \equiv 0$, and $1 < \gamma < \infty$, then the inequality

$$\left\| \mathcal{G}^{(\gamma)}_{\Psi,\phi,1}(F) \right\|_{L^p(\mathbb{R}^\eta)} \leq C \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^\eta)} \tag{1.3}$$

holds for all $p \in (1,\infty)$. This result was improved in [15] where the author satisfied inequality (1.3) under the weaker conditions that $h \in \nabla_{\max\{\kappa',2\}}(\mathbb{R}^+)$ and $\Psi \in L(\log L)(\mathbb{S}^{\eta-1})$.

Later, the authors of [16] extended and improved these results. Precisely, they used the extrapolation argument of Yano to show that if $\phi(t) = t$, $h \in \nabla_\kappa(\mathbb{R}^+)$ with $\kappa > 2$ and $\Psi \in L(\log L)^{1/\gamma}(\mathbb{S}^{\eta-1}) \cup B_q^{(0,\frac{1}{\gamma}-1)}(\mathbb{S}^{\eta-1})$, then $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ is bounded on $L^p(\mathbb{R}^\eta)$ for all $p \in (1,\gamma)$ with $\gamma' \geq \kappa$ and also for all $p \in (\kappa',\infty)$ with $\gamma > \kappa'$. For recent advances on the investigation of the operator $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ and their developments, the readers can refer to [17–22], among others.

For $r \in \mathbb{R}$ and $\gamma, p \in (1,\infty)$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{r,\gamma}(\mathbb{R}^\eta)$ is given by

$$\dot{F}_p^{r,\gamma}(\mathbb{R}^\eta) = \left\{ F \in \mathcal{S}'(\mathbb{R}^\eta) : \|F\|_{\dot{F}_p^{r,\gamma}(\mathbb{R}^\eta)} = \left\| \left(\sum_{j\in\mathbb{Z}} 2^{jr\gamma} \left| \vartheta_j * F \right|^\gamma \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^\eta)} < \infty \right\},$$

where $\mathcal{S}'$ is the tempered distribution class on $\mathbb{R}^\eta$, $\widehat{\vartheta}_j(\eta) = \mathcal{A}(2^{-j}\eta)$, and $\mathcal{A} \in C_0^\infty(\mathbb{R}^\eta)$ is a radial mapping with the following properties:

(a) $0 \leq \mathcal{A} \leq 1$,

(b) $\mathcal{A}(\eta) \geq K > 0$ if $|\eta| \in [\frac{3}{5}, \frac{5}{3}]$,
(c) $supp\,(\mathcal{A}) \subset \{\eta : |\eta| \in [1/2, 2]\}$,
(d) $\sum_{j\in\mathbb{Z}} \mathcal{A}(2^{-j}\eta) = 1$ if $\eta \neq 0$.

It was proved in [18] that the space $\dot{F}_p^{r,\gamma}(\mathbb{R}^\eta)$ satisfies the following:
(i) $\mathcal{S}(\mathbb{R}^\eta)$ is dense in $\dot{F}_p^{r,\gamma}(\mathbb{R}^\eta)$,
(ii) For $p \in (1, \infty)$, $L^p(\mathbb{R}^\eta) = \dot{F}_p^{0,2}(\mathbb{R}^\eta)$,
(iii) $\dot{F}_p^{s,\gamma_1}(\mathbb{R}^\eta) \subseteq \dot{F}_p^{s,\gamma_2}(\mathbb{R}^\eta)$ if $\gamma_1 \leq \gamma_2$.

For $d \neq 0$, let $\mathcal{H}_d$ be the set of all mappings $\phi : \mathbb{R}^+ \to \mathbb{R}$ that satisfies the following conditions:
(a) $|\phi(t)| \leq k_1 t^d$,
(b) $k_2 t^{d-1} \leq |\phi'(t)| \leq k_3 t^{d-1}$,
(c) $|\phi''(t)| \leq k_4 t^{d-2}$,
where k_1, k_2, k_3, and k_4 are positive numbers independent of t.

In the light of the findings in [16] about the estimates for the generalized Marcinkiewicz operator $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ whenever $\phi(t) = t$, and of the findings in [5] concerning the boundedness of Marcinkiewicz integral operator $\mathcal{G}^{(2)}_{\Psi,\phi,h}$, it is natural to ask whether the operator $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ is bounded under the same assumptions in [5] replacing $\gamma = 2$ by any $\gamma > 1$?

In this paper, the above question will be answered affirmatively. Our main results is described as follows.

Theorem 1.1. *Assume that* $\Psi \in L^q\left(\mathbb{S}^{\eta-1}\right)$, $q \in (1, 2]$ *satisfies* (1.1). *Let* $h \in \nabla_\kappa(\mathbb{R}^+)$ *with* $\kappa \in (1, 2]$ *and* $\phi \in \mathcal{H}_d$. *Then there is a constant* $C_p > 0$ *such that the inequalities*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\left(\frac{1}{(\kappa-1)(q-1)}\right)^{1/\gamma} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})} \;\; if\; \gamma \leq p \leq \frac{\kappa\gamma'}{\gamma'-\kappa},$$

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\left(\frac{1}{(\kappa-1)(q-1)}\right)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})} \;\; if\; \frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa} < p < \gamma,$$

and

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\left(\frac{1}{(\kappa-1)(q-1)}\right)^{\frac{\kappa\gamma-\gamma+1}{\gamma\kappa}} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})} \;\; if\; \frac{\gamma\kappa}{\kappa\gamma-\gamma+1} < p < \gamma$$

hold for all $F \in \dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})$, *where* $C_{p,\Psi,h} = C_p\,\|\Psi\|_{L^q(\mathbb{S}^{\eta-1})}\,\|h\|_{\nabla_\kappa(\mathbb{R}^+)}$.

Theorem 1.2. *Assume that* Ψ *and* ϕ *are given as in Theorem 1.1, and that* $h \in \nabla_\kappa(\mathbb{R}^+)$ *with* $2 < \kappa < \infty$. *Then, a bounded number* $C_p > 0$ *exists so that*
(a) *If* $\gamma > \kappa'$, *we have for* $\kappa' < p < \infty$,

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\left(\frac{1}{q-1}\right)^{1/\kappa'} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})}.$$

(b) *If* $\gamma \leq \kappa'$, *we have for* $1 < p < \gamma$,

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\left(\frac{1}{q-1}\right) \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})}.$$

The estimates come from Theorems 1.1 and 1.2 allow us to utilize the extrapolation argument of Yano (see also [23–25]) to obtain the following results.

Theorem 1.3. *Assume that $\phi \in \mathcal{H}_d$ and $h \in \nabla_\kappa(\mathbb{R}^+)$ with $\kappa \in (1,2]$.*
(a) *If $\Psi \in L(\log L)^{1/\gamma}(\mathbb{S}^{\eta-1})$, then for $p \in [\gamma, \frac{\kappa\gamma'}{\gamma'-\kappa}]$,*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\left(1+\|\Psi\|_{L(\log L)^{1/\gamma}(\mathbb{S}^{\eta-1})}\right)\|h\|_{\nabla_\kappa(\mathbb{R}^+)}\ C_p.$$

(b) *If $\Psi \in L(\log L)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}}(\mathbb{S}^{\eta-1})$, then for $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa}, \gamma)$,*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\left(1+\|\Psi\|_{L(\log L)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}}(\mathbb{S}^{\eta-1})}\right)\|h\|_{\nabla_\kappa(\mathbb{R}^+)}\ C_p.$$

(c) *If $\Psi \in L(\log L)^{\frac{\kappa\gamma-\gamma+1}{\gamma\kappa}}(\mathbb{S}^{\eta-1})$, then for $p \in (\frac{\gamma\kappa}{\gamma-\kappa\gamma+1}, \gamma)$,*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\left(1+\|\Psi\|_{L(\log L)^{\frac{\kappa\gamma-\gamma+1}{\kappa\gamma}}(\mathbb{S}^{\eta-1})}\right)\|h\|_{\nabla_\kappa(\mathbb{R}^+)}\ C_p.$$

(d) *If $\Psi \in B_q^{(0,-1/\gamma')}(\mathbb{S}^{\eta-1})$ with $q>1$, then for $p \in [\gamma, \frac{\kappa\gamma'}{\gamma'-\kappa}]$,*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\left(1+\|\Psi\|_{q^{(0,-1/\gamma')}(\mathbb{S}^{\eta-1})}\right)\|h\|_{\nabla_\kappa(\mathbb{R}^+)}\ C_p.$$

(e) *If $\Psi \in B_q^{(0,\frac{\kappa-\gamma}{\kappa\gamma})}(\mathbb{S}^{\eta-1})$ with $q>1$, then for $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa}, \gamma)$,*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\left(1+\|\Psi\|_{B_q^{(0,\frac{\kappa-\gamma}{\kappa\gamma})}(\mathbb{S}^{\eta-1})}\right)\|h\|_{\nabla_\kappa(\mathbb{R}^+)}\ C_p.$$

(f) *If $\Psi \in B_q^{(0,\frac{1-\gamma}{\gamma\kappa})}(\mathbb{S}^{\eta-1})$ with $q>1$, then for $p \in (\frac{\gamma\kappa}{\gamma-\gamma\kappa+1}, \gamma)$,*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\left(1+\|\Psi\|_{B_q^{(0,\frac{1-\gamma}{\gamma\kappa})}(\mathbb{S}^{\eta-1})}\right)\|h\|_{\nabla_\kappa(\mathbb{R}^+)}\ C_p.$$

Theorem 1.4. *Assume that $\phi \in \mathcal{H}_d$ and $h \in \nabla_\kappa(\mathbb{R}^+)$ for some $2<\kappa<\infty$.*
(a) *If $\Psi \in L(\log L)^{1/\kappa'}(\mathbb{S}^{\eta-1})$, then*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|h\|_{\nabla_\kappa(\mathbb{R}^+)}\left(1+\|\Psi\|_{L(\log L)^{1/\kappa'}(\mathbb{S}^{\eta-1})}\right)\|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\ C_p,$$

for $\kappa'<p<\infty$ and $\gamma>\kappa'$.
(b) *If $\Psi \in L(\log L)(\mathbb{S}^{\eta-1})$, then we have*

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \|h\|_{\nabla_\kappa(\mathbb{R}^+)}\left(1+\|\Psi\|_{L(\log L)(\mathbb{S}^{\eta-1})}\right)\|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}\ C_p,$$

for $1 < p < \gamma$ *and* $\gamma \le \kappa'$.

(c) *If* $\Psi \in B_q^{(0,-1/\kappa)}(\mathbb{S}^{\eta-1})$ *with* $q > 1$, *then*

$$\left\|\mathcal{G}_{\Psi,\phi,h}^{(\gamma)}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \le \|h\|_{\nabla_\kappa(\mathbb{R}^+)} \left(1 + \|\Psi\|_{q^{(0,-1/\kappa)}(\mathbb{S}^{\eta-1})}\right) \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})} \ C_p,$$

for $\kappa' < p < \infty$ *and* $\kappa' < \gamma$.

(d) *If* $\Psi \in B_q^{(0,0)}(\mathbb{S}^{\eta-1})$ *with* $q > 1$, *then*

$$\left\|\mathcal{G}_{\Psi,\phi,h}^{(\gamma)}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \le \|h\|_{\nabla_\kappa(\mathbb{R}^+)} \left(1 + \|\Psi\|_{q^{(0,0)}(\mathbb{S}^{\eta-1})}\right) \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})} \ C_p,$$

for all $1 < p < \gamma$ *and* $\gamma \le \kappa'$.

Remark 1.5. **(i)** For the special cases $\phi \equiv 0$, $h \equiv 1$, $\gamma = 2$, and $n = 1$, the L^p $(1 < p \le 2)$ boundedness of $\mathcal{G}_{\Psi,\phi,h}^{(\gamma)}$ was established in [1] only whenever $\Psi \in Lip_\tau(\mathbb{S}^{\eta-1})$ for some $\tau \in (0,1]$. As $Lip_\tau(\mathbb{S}^{\eta-1}) \subset L(\log L)^r(\mathbb{S}^{\eta-1}) \cup B_q^{(0,r)}(\mathbb{S}^{\eta-1})$, then our results generalize, extend, and also improve what was proved in [1].

(ii) For the cases $h \in \nabla_\kappa(\mathbb{R}^+)$, $\phi \equiv 0$, and $\gamma = 2$, $n = 1$, the authors of [8] only obtained the L^2 boundedness of $\mathcal{G}_{\Psi,\phi,h}^{(\gamma)}$ under the condition $\Psi \in L(\log L)(\mathbb{S}^{\eta-1})$. Hence, our results are essential generalization and improvement to the results in [8].

(iii) For the cases $\phi \equiv 0$, $h \equiv 0$, and $\gamma = 2$, the conditions on Ψ in our results are the best possible among their respective classes, (see [2,4]).

(iv) In Theorem 1.3, if we take $\gamma = 2$ and $\kappa \in (1,2]$, then the range of p is better than the range of p in the results found in [5]: $(\frac{2\kappa'}{\kappa'-2}, \frac{2\kappa}{2-\kappa})$.

(v) In Theorem 1.3, the conditions on Ψ in **(c)** and **(e)** are stronger than the conditions on Ψ in **(b)** and **(d)**. However, the range of p in **(c)** and **(e)** are better than the range of p in **(b)** and **(d)**.

(vi) In Theorem 1.4, the spaces that the singular kerenels belong to in **(a)** and **(c)** are better than the spaces in **(b)** and **(d)**.

2. Some lemmas

In this section, we prove some auxiliary results which will be the key role in the proof of the main results. For $\mu \ge 2$ and appropriate mappings $h : \mathbb{R}^+ \to \mathbb{C}$, $\Psi : \mathbb{S}^{\eta-1} \to \mathbb{R}$, and $\phi : \mathbb{R}^+ \to \mathbb{R}$, we consider the family of measures $\{\mho_{\Psi,\phi,h,t} : \mho_{h,t} : t \in \mathbb{R}^+\}$ and their related maximal operators $\mho_{\Psi,h}^*$ and $M_{\Psi,h,\mu}$ on $\mathbb{R}^{\eta+1}$ by

$$\int_{\mathbb{R}^{\eta+1}} F d\mho_{h,t} = \frac{1}{t^n} \int_{t/2 \le |v| \le t} F(v, \phi(|v|)) \mathrm{K}_{\Psi,h}(v) dv,$$

$$\mho_{\Psi,h}^* F(\bar{w}) = \sup_{t \in \mathbb{R}^+} \||\mho_{h,t}| * F(\bar{w})|$$

and

$$M_{\Psi,h,\mu} F(\bar{w}) = \sup_{j \in \mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \||\mho_{h,l}| * F(\bar{w})| \frac{dt}{t},$$

where $|\mho_{h,t}|$ is defined similar to $\mho_{h,t}$ with replacing $h\Psi$ by $|h\Psi|$.

Utilizing similar arguments (with minor modifications) employed in the proof of Theorem 1.3 in [5] gives the following.

Lemma 2.1. *Let $\mu \geq 2$, $h \in \nabla_\kappa(\mathbb{R}^+)$, and $\Psi \in L^q\left(\mathbb{S}^{\eta-1}\right)$ for some $\kappa, q > 1$. Let ϕ be an arbitrary function on $\mathbb{R}^+$. Then, there are positive constants C and $\delta < 1/(2q')$ such that*

$$\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_{h,t}(\zeta, \zeta_{\eta+1})\right|^2 \frac{dt}{t} \leq C(\ln \mu),$$

$$\int_{\mu^j}^{\mu^{j+1}} \left|\hat{\mho}_{h,t}(\zeta, \zeta_{\eta+1})\right|^2 \frac{dt}{t} \leq C(\ln \mu) \|\Psi\|^2_{L^q(\mathbb{S}^{\eta-1})} \|h\|^2_{\nabla_\kappa(\mathbb{R}^+)} \min\left\{\left|\mu^j \zeta\right|^{-\frac{\delta}{\ln \mu}}, \left|\mu^j \zeta\right|^{\frac{\delta}{\ln \mu}}\right\}.$$

Lemma 2.2. *Let Ψ, h and ϕ be given as in Theorem 1.1. Then there exists a constant $C_{p,\Psi,h} > 0$ such that for all $p > \kappa'$,*

$$\|M_{\Psi,h,\mu}(F)\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}(\ln \mu)\|F\|_{L^p(\mathbb{R}^{\eta+1})} \tag{2.1}$$

and

$$\|\mho^*_{\Psi,h}(F)\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}(\ln \mu)^{1/\kappa'}\|F\|_{L^p(\mathbb{R}^{\eta+1})}. \tag{2.2}$$

By employing similar arguments as employed in [16], we get the following.

Lemma 2.3. *Let Ψ, ϕ, and γ be given as in Theorem 1.2. Suppose that $h \in \nabla_\kappa(\mathbb{R}^+)$ with $2 < \kappa < \infty$. Then, for $\mu \geq 2$, a constant $C_{p,\Psi,h}$ exists such that:*
(a) *If $\gamma > \kappa'$, we have for $\kappa' < p < \infty$,*

$$\left\|\left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left|\mho_{h,t} * \mathcal{U}_j\right|^\gamma \frac{dt}{t}\right)^{\frac{1}{\gamma}}\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}(\ln \mu)^{1/\kappa'} \left\|\left(\sum_{j\in\mathbb{Z}} \left|\mathcal{U}_j\right|^\gamma\right)^{1/\gamma}\right\|_{L^p(\mathbb{R}^{\eta+1})}.$$

(b) *If $\gamma \leq \kappa'$, we have for $1 < p < \gamma$,*

$$\left\|\left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left|\mho_{h,t} * \mathcal{U}_j\right|^\gamma \frac{dt}{t}\right)^{\frac{1}{\gamma}}\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}(\ln \mu) \left\|\left(\sum_{j\in\mathbb{Z}} \left|\mathcal{U}_j\right|^\gamma\right)^{1/\gamma}\right\|_{L^p(\mathbb{R}^{\eta+1})},$$

where $\{\mathcal{U}_j(\cdot), j \in \mathbb{Z}\}$ is any sequence of functions on $\mathbb{R}^{\eta+1}$.

Proof. One can easily check that

$$\begin{aligned} \left\|\sup_{j\in\mathbb{Z}} \sup_{t\in[1,\mu]} \left|\mho_{h,t\mu^j} * \mathcal{U}_j\right|\right\|_{L^p(\mathbb{R}^{\eta+1})} &\leq \left\|\mho^*_{\Psi,h}\left(\sup_{j\in\mathbb{Z}} \left|\mathcal{U}_j\right|\right)\right\|_{L^p(\mathbb{R}^{\eta+1})} \\ &\leq C_{p,\Psi,h} \ln(\mu)^{1/\kappa'} \left\|\sup_{j\in\mathbb{Z}} \left|\mathcal{U}_j\right|\right\|_{L^p(\mathbb{R}^{\eta+1})}, \end{aligned}$$

which means that

$$\left\|\left\|\|\mho_{h,t\mu^j} * \mathcal{U}_j\|_{L^\infty([1,\mu],\frac{dt}{t})}\right\|_{l^\infty(\mathbb{Z})}\right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h} \ln(\mu)^{1/\kappa'} \left\|\left\|\mathcal{U}_j\right\|_{l^\infty(\mathbb{Z})}\right\|_{L^p(\mathbb{R}^{\eta+1})}. \tag{2.3}$$

If $p > \kappa' < \gamma$, then the duality gives that a function $\mathcal{J} \in L^{(p/\kappa')'}(\mathbb{R}^{\eta+1})$ with $\|\mathcal{J}\|_{L^{(p/\kappa')'}(\mathbb{R}^{\eta+1})} \leq 1$ and

$$\left\| \left(\sum_{j\in\mathbb{Z}} \int_1^{\mu} \left| \mho_{h,t\mu^j} * \mathcal{U}_j \right|^{\kappa'} \frac{dt}{t} \right)^{\frac{1}{\kappa'}} \right\|_{L^p(\mathbb{R}^{\eta+1})}^{\kappa'} = \int_{\mathbb{R}^{\eta+1}} \sum_{j\in\mathbb{Z}} \int_1^{\mu} \left| \mho_{h,t\mu^j} * \mathcal{U}_j(\bar{w}) \right|^{\kappa'} \frac{dt}{t} \mathcal{J}(w, w_{\eta+1}) dw dw_{\eta+1}$$

$$\leq C \left\|\Psi\right\|_{L^1(\mathbb{S}^{\eta-1})}^{(\kappa'/\kappa)} \left\|h\right\|_{\nabla_\kappa(\mathbb{R}_+)}^{\kappa'} \int_{\mathbb{R}^{\eta+1}} \sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j(w, w_{\eta+1}) \right|^{\kappa'} \mho^*_{\Psi,1} \mathcal{J}^{\bullet}(w, w_{\eta+1}) dw dw_{\eta+1}$$

$$\leq C \left\|\Psi\right\|_{L^1(\mathbb{S}^{\eta-1})}^{(\kappa'/\kappa)} \left\|h\right\|_{\nabla_\kappa(\mathbb{R}^+)}^{\kappa'} \left\| \sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\kappa'} \right\|_{L^{(p/\kappa')}(\mathbb{R}^{\eta+1})} \left\| \mho^*_{\Psi,1}(\mathcal{J}^{\bullet}) \right\|_{L^{(p/\kappa')'}(\mathbb{R}^{\eta})}$$

$$\leq C(\ln\mu) \left\|\Psi\right\|_{L^q(\mathbb{S}^{\eta-1})}^{(\kappa'/\kappa)+1} \left\|h\right\|_{\nabla_\kappa(\mathbb{R}^+)}^{\kappa'} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\kappa'} \right)^{\frac{1}{\kappa'}} \right\|_{L^p(\mathbb{R}^{\eta+1})}^{\kappa'}, \tag{2.4}$$

where $\mathcal{J}^{\bullet}(w, w_{\eta+1}) = \mathcal{J}(-w, -w_{\eta+1})$. This leads to

$$\left\| \left(\sum_{j\in\mathbb{Z}} \int_1^{\mu} \left| \mho_{h,t\mu^j} * \mathcal{U}_j \right|^{\kappa'} \frac{dt}{t} \right)^{1/\kappa'} \right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h} \ln(\mu)^{1/\kappa'} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\kappa'} \right)^{1/\kappa'} \right\|_{L^p(\mathbb{R}^{\eta+1})}. \tag{2.5}$$

Define a linear operator $\mathcal{T}$ on any function $\mathcal{U} = \mathcal{U}_j(w, w_{\eta+1})$ by $\mathcal{T}(\mathcal{U}) = \mho_{h,t\mu^j} * \mathcal{U}_j(w, w_{\eta+1})$, then interpolate the estimate in (2.3) with the estimate in (2.5) to get

$$\left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left| \mho_{h,t} * \mathcal{U}_j \right|^{\gamma} \frac{dt}{t} \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^{\eta+1})} \leq \left\| \left(\sum_{j\in\mathbb{Z}} \int_1^{\mu} \left| \mho_{h,t\mu^j} * \mathcal{U}_j \right|^{\gamma} \frac{dt}{t} \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^{\eta+1})}$$

$$\leq C_{p,\Psi,h} (\ln\mu)^{1/\kappa'} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\mathbb{R}^{\eta+1})}, \tag{2.6}$$

for all $\kappa' < p < \infty$ with $\gamma > \kappa'$ and $\kappa > 2$. Hence, the proof of first estimate of this lemma is complete.

Now, if $1 < p < \gamma \leq \kappa'$, then $p'/\gamma' > 1$. Thanks to the duality, there are functions $g_j(\bar{w}, t)$ on $\mathbb{R}^{\eta+1} \times \mathbb{R}^+$ with $\left\| \left\| \|g_j\|_{L^{\gamma'}([\mu^j, \mu^{j+1}], \frac{dt}{t})} \right\|_{l^{\gamma'}} \right\|_{L^{p'}(\mathbb{R}^{\eta+1})} \leq 1$ and

$$\left\| \sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^j} \left| \mho_{h,t} * \mathcal{U}_j \right|^{\gamma} \frac{dt}{t} \right\|_{L^{p/\gamma}(\mathbb{R}^{\eta+1})}^{1/\gamma} = \int_{\mathbb{R}^{\eta+1}} \sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left(\mho_{h,t} * \mathcal{U}_j(\bar{w}) \right) g_j(\bar{w}, t) \frac{dt}{t} d\bar{w}$$

$$\leq C_p (\ln\mu)^{1/\gamma} \left\| (\Gamma(g_j))^{1/\gamma'} \right\|_{L^{p'}(\mathbb{R}^{\eta+1})} \left\| \sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\gamma} \right\|_{L^{p/\gamma}(\mathbb{R}^{\eta+1})}^{1/\gamma}, \tag{2.7}$$

where

$$\Gamma(g_j)(\bar{w}) = \sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left| \mho_{h,t} * g_j(\bar{w}, t) \right|^{\gamma'} \frac{dt}{t}.$$

Notice that $\gamma \le \kappa' \le 2 \le \kappa$. So, Hölder's inequality leads to

$$\begin{aligned} \left|\mho_{h,t} * g_j(\bar{w},t)\right|^{\gamma'} \le \quad & C\,\|\Psi\|_{L^1(\mathbb{S}^{\eta-1})}^{(\gamma'/\gamma)}\,\|h\|_{\nabla_\kappa(\mathbb{R}_+)}^{\gamma'} \int_{\mu^j}^{\mu^{j+1}}\int_{\mathbb{S}^{\eta-1}}|\Psi(v)| \\ & \times\left|g_j(w-rv, w_{\eta+1}-\phi(r),t)\right|^{\gamma'} d\sigma_\eta(v)\tfrac{dr}{r}. \end{aligned} \tag{2.8}$$

Again, we employ the duality, so we obtain a function $\varphi \in L^{(p'/\gamma')'}(\mathbb{R}^{\eta+1})$,

$$\left\|\left(\Gamma(g_j)\right)^{1/\gamma'}\right\|_{L^{p'}(\mathbb{R}^{\eta+1})}^{\gamma'} = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{\eta+1}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * g_j(\bar{w},t)\right|^{\gamma'}\frac{dt}{t}\varphi(\bar{w})d\bar{w}.$$

Thus, by Hölder's inequality and the inequalities (2.2) and (2.8), we conclude

$$\begin{aligned} \left\|\left(\Gamma(g_j)\right)^{1/\gamma'}\right\|_{L^{p'}(\mathbb{R}^{\eta+1})}^{\gamma'} \le\quad & C\|\Psi\|_{L^1(\mathbb{S}^{\eta-1})}^{(\gamma'/\gamma)}\left\|\mho^*{}_{|\Psi|,1}(\varphi)\right\|_{L^{(p'/\gamma')'}(\mathbb{R}^{\eta+1})}\|h\|_{\nabla_\kappa(\mathbb{R}_+)}^{\gamma'} \\ & \times\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|g_j(\bar{w},t)\right|^{\gamma'}\frac{dt}{t}\right)\right\|_{L^{(p'/\gamma')}(\mathbb{R}^{\eta+1})} \\ \le\quad & C_p(\ln\mu)\,\|\Psi\|_{L^q(\mathbb{S}^{\gamma-1})}^{(\gamma'/\gamma)+1}\|g\|_{\nabla_\kappa(\mathbb{R}_+)}^{\gamma'}\|\varphi\|_{L^{(p'/\gamma')'}(\mathbb{R}^{\eta+1})}. \end{aligned} \tag{2.9}$$

Therefore, by the last inequality and (2.7), we complete the proof of Lemma 2.3 for the case $1 < p < \gamma$ with $\gamma \le \kappa' < 2$. □

Lemma 2.4. *Let Ψ, ϕ, and γ be given as in Theorem 1.1. Suppose that $\mu \ge 2$ and $h \in \nabla_\kappa(\mathbb{R}^+)$ for some $\kappa \in (1,2]$. Then, a positive number $C_{p,\Psi,h}$ exists such that, for any sequence of functions $\{\mathcal{U}_j\}$ on $\mathbb{R}^{\eta+1}$, we have*

$$\textbf{(a)}\quad \left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j\right|^{\gamma}\frac{dt}{t}\right)^{\frac{1}{\gamma}}\right\|_{L^p(\mathbb{R}^{\eta+1})} \le C_{p,\Psi,h}(\ln\mu)^{1/\kappa'}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j\right|^{\gamma}\right)^{1/\gamma}\right\|_{L^p(\mathbb{R}^{\eta+1})}, \tag{2.10}$$

for all $p \in [\gamma, \frac{\kappa\gamma'}{\gamma'-\kappa}]$,

$$\textbf{(b)}\quad \left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j\right|^{\gamma}\frac{dt}{t}\right)^{\frac{1}{\gamma}}\right\|_{L^p(\mathbb{R}^{\eta+1})} \le C_{p,\Psi,h}(\ln\mu)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j\right|^{\gamma}\right)^{1/\gamma}\right\|_{L^p(\mathbb{R}^{\eta+1})}, \tag{2.11}$$

for all $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa}, \gamma)$, and

$$\textbf{(c)}\quad \left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j\right|^{\gamma}\frac{dt}{t}\right)^{\frac{1}{\gamma}}\right\|_{L^p(\mathbb{R}^{\eta+1})} \le C_{p,\Psi,h}(\ln\mu)^{\frac{\kappa\gamma-\gamma+1}{\gamma\kappa}}\left\|\left(\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j\right|^{\gamma}\right)^{1/\gamma}\right\|_{L^p(\mathbb{R}^{\eta+1})}, \tag{2.12}$$

for all $p \in (\frac{\gamma\kappa}{\kappa\gamma-\gamma+1}, \gamma)$.

Proof. Let us first prove inequality (2.10). Notice that

$$\begin{aligned}\left|\mho_{h,t} * \mathcal{U}_j(\bar{w})\right|^\gamma &\leq C\,\|h\|^{(\gamma/\gamma')}_{\nabla_\kappa(\mathbb{R}_+)}\,\|\Psi\|^{(\gamma/\gamma')}_{L^1(\mathbb{S}^{\eta-1})} \int_{\frac{1}{2}t}^{t}\int_{\mathbb{S}^{\eta-1}} \left|\mathcal{U}_j(w-rv, w_{\eta+1}-\phi(r))\right|^\gamma \\ &\quad \times |\Psi(v)|\, d\sigma_\eta(v)\ |h(r)|^{\gamma-\frac{\gamma\kappa}{\gamma'}}\,\frac{dr}{r}. \qquad (2.13)\end{aligned}$$

If $p=\gamma$, then by using Hölder's inequality, (2.1), and (2.13), we get

$$\begin{aligned}&\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j\right|^\gamma \frac{dt}{t}\right)^{\frac{1}{\gamma}}\right\|^\gamma_{L^p(\mathbb{R}^{\eta+1})} \\ \leq\;& C\,\|h\|^{(\gamma/\gamma')}_{\nabla_\kappa(\mathbb{R}_+)}\,\|\Psi\|^{(\gamma/\gamma')}_{L^1(\mathbb{S}^{\eta-1})} \\ &\times \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^{\eta+1}}\int_{\mu^j}^{\mu^{j+1}}\int_{\frac{1}{2}t}^{t}\int_{\mathbb{S}^{\eta-1}}\left|\mathcal{U}_j(w-rv, w_{\eta+1}-\phi(r))\right|^\gamma |\Psi(v)|\,|h(r)|^{\gamma-\frac{\gamma\kappa}{\gamma'}}\,d\sigma_\eta(v)\frac{dr}{r}\frac{dt}{t}d\bar{w} \\ \leq\;& C(\ln\mu)\,\|h\|^{(\gamma/\gamma')+1}_{\nabla_\kappa(\mathbb{R}_+)}\,\|\Psi\|^{(\gamma/\gamma')+1}_{L^1(\mathbb{S}^{\eta-1})}\int_{\mathbb{R}^{\eta+1}}\left(\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j(\bar{w})\right|^\gamma\right)^p d\bar{w} \\ \leq\;& (C_{p,\Psi,h})^\gamma(\ln\mu)\left\|\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j\right|^\gamma\right\|^\gamma_{L^p(\mathbb{R}^{\eta+1})}. \qquad (2.14)\end{aligned}$$

If $p>\gamma$, then by duality, there exists a function $\mathcal{Z}$ lies in the space $L^{(p/\gamma)'}(\mathbb{R}^{\eta+1})$ with $\|\mathcal{Z}\|_{L^{(p/\gamma)'}(\mathbb{R}^{\eta+1})}\leq 1$ and

$$\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j\right|^\gamma \frac{dt}{t}\right)^{1/\gamma}\right\|^\gamma_{L^p(\mathbb{R}^{\eta+1})} = \int_{\mathbb{R}^{\eta+1}}\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j(\bar{w})\right|^\gamma \frac{dt}{t}\mathcal{Z}(\bar{w})d\bar{w}. \qquad (2.15)$$

Thus, the estimates in (2.13) and (2.15) along with Lemma 2.2 lead to

$$\begin{aligned}&\left\|\left(\sum_{j\in\mathbb{Z}}\int_{\mu^j}^{\mu^{j+1}}\left|\mho_{h,t} * \mathcal{U}_j\right|^\gamma \frac{dt}{t}\right)^{1/\gamma}\right\|^\gamma_{L^p(\mathbb{R}^{\eta+1})} \\ \leq\;& C\,\|h\|^{(\gamma/\gamma')}_{\nabla_\kappa(\mathbb{R}_+)}\,\|\Psi\|^{(\gamma/\gamma')}_{L^1(\mathbb{S}^{\eta-1})}\int_{\mathbb{R}^{\eta+1}}\left(\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j(\bar{w})\right|^\gamma\right)M_{|\Psi|,|h|^{\gamma-\frac{\gamma\kappa}{\gamma'}},\mu}\mathcal{Z}^\bullet(\bar{w})d\bar{w} \\ \leq\;& C\,\|h\|^{(\gamma/\gamma')}_{\nabla_\kappa(\mathbb{R}_+)}\,\|\Psi\|^{(\gamma/\gamma')}_{L^1(\mathbb{S}^{\eta-1})}\left\|\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j\right|^\gamma\right\|_{L^{(p/\gamma)}(\mathbb{R}^{\eta+1})}\left\|M_{|\Psi|,|h|^{\frac{\gamma(\gamma'-\kappa)}{\gamma'}},\mu}(\mathcal{Z}^\bullet)\right\|_{L^{(p/\gamma)'}(\mathbb{R}^{\eta+1})} \\ \leq\;& C(\ln\mu)\,\|h\|^{(1+\gamma/\gamma')}_{\nabla_\kappa(\mathbb{R}_+)}\,\|\Psi\|^{(1+\gamma/\gamma')}_{L^q(\mathbb{S}^{\eta-1})}\left\|\sum_{j\in\mathbb{Z}}\left|\mathcal{U}_j\right|^\gamma\right\|_{L^{(p/\gamma)}(\mathbb{R}^{\eta+1})}\|\mathcal{Z}^\bullet\|_{L^{(p/\gamma)'}(\mathbb{R}^{\eta+1})},\end{aligned}$$

where $\mathcal{Z}^{\bullet}(\bar{w}) = \mathcal{Z}(-\bar{w})$. Therefore, by the last inequality and (2.14), we obtain that (2.10) holds for all $p \in [\gamma, \frac{\kappa\gamma'}{\gamma'-\kappa}]$.

Now, let us prove (2.11). As $p < \gamma$, we have $\gamma' < p'$, which by the duality gives that a set of functions $\{\varphi_j(\bar{w}, t)\}$ defined on $\mathbb{R}^{\eta+1} \times \mathbb{R}^+$ exists and satisfies $\left\| \left\| \|\varphi_j\|_{L^{\gamma'}([\mu^j,\mu^{j+1}],\frac{dt}{t})} \right\|_{l^{\gamma'}} \right\|_{L^{p'}(\mathbb{R}^{\eta+1})} \leq 1$ and

$$\left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^j} \left| \mho_{h,t} * \mathcal{U}_j \right|^{\gamma} \frac{dt}{t} \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^{\eta+1})} = \int_{\mathbb{R}^{\eta+1}} \sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left(\mho_{h,t} * \mathcal{U}_j(\bar{w}) \right) \varphi_j(\bar{w}, t) \frac{dt}{t} d\bar{w}. \tag{2.16}$$

Define the operator $\Upsilon : \mathbb{R}^{\eta+1} \times \mathbb{R}^+ \to \mathbb{R}$ by

$$\Upsilon(\varphi_j)(\bar{w}, t) = \sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^j} \left| \mho_{h,t} * \varphi_j(\bar{w}, t) \right|^{\gamma'} \frac{dt}{t}.$$

Thus, thanks to the duality, a function $\Omega \in L^{(p'/\gamma')'}(\mathbb{R}^{\eta+1})$ with norm 1 exists such that

$$\begin{aligned} \left\| \left(\Upsilon(\varphi_j) \right)^{1/\gamma'} \right\|_{L^{p'}(\mathbb{R}^{\eta+1})}^{\gamma'} &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^{\eta+1}} \int_{\mu^j}^{\mu^{j+1}} \left| \mho_{h,t} * \varphi_j(\bar{w}, t) \right|^{\gamma'} \frac{dt}{t} \Omega(\bar{w}) d\bar{w} \\ &\leq C \|h\|_{\nabla_\kappa(\mathbb{R}_+)}^{(\gamma'/\gamma)} \|\Psi\|_{L^1(\mathbb{S}^{\eta-1})}^{(\gamma'/\gamma)} \left\| \mho^*_{|\Psi|,|h|^{\frac{\gamma'(\gamma-\gamma)}{\gamma}}} (\Omega^{\bullet}) \right\|_{L^{(p'/\gamma')'}(\mathbb{R}^{\eta+1})} \left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left| \varphi_j(\bar{w}, t) \right|^{\gamma'} \frac{dt}{t} \right) \right\|_{L^{(p'/\gamma')}(\mathbb{R}^{\eta+1})} \\ &\leq (C_{p,\Psi,h})^{\gamma'} (\ln \mu)^{1/\left(\frac{\gamma\kappa}{\gamma-\kappa}\right)'} \|\Omega\|_{L^{(p'/\gamma')'}(\mathbb{R}^{\eta+1})}, \end{aligned} \tag{2.17}$$

for all $\left(\frac{\gamma\kappa}{\gamma-\kappa}\right)' < p < \gamma$, where $\Omega^{\bullet}(\bar{w}) = \Omega(-\bar{w})$. Therefore, by inequalities (2.16)–(2.17) and Hölder's inequality, we conclude

$$\begin{aligned} &\left\| \left(\sum_{j\in\mathbb{Z}} \int_{\mu^j}^{\mu^{j+1}} \left| \mho_{h,t} * \mathcal{U}_j \right|^{\gamma} \frac{dt}{t} \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^{\eta+1})} \\ \leq\ & C_{p,\Psi,h} (\ln \mu)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}} \left\| (\Upsilon(\varphi_j))^{1/\gamma'} \right\|_{L^{p'}(\mathbb{R}^{\eta+1})} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\gamma} \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^{\eta+1})} \\ \leq\ & C_{p,\Psi,h} (\ln \mu)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}} \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right|^{\gamma} \right)^{1/\gamma} \right\|_{L^p(\mathbb{R}^{\eta+1})}, \end{aligned} \tag{2.18}$$

holds for all $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa}, \gamma)$. This finishes the proof of (2.11).

To prove (2.12), we use the linear operator $\mathcal{T}$ that was defined in the proof of Lemma 2.3. Hence, we have

$$\left\| \left\| \|\mathcal{U}(\mathcal{A})\|_{L^1(1,\mu),\frac{dt}{t}} \right\|_{l^1(\mathbb{Z})} \right\|_{L^1(\mathbb{R}^{\eta+1})} \leq C(\ln \mu) \left\| \left(\sum_{j\in\mathbb{Z}} \left| \mathcal{U}_j \right| \right) \right\|_{L^1(\mathbb{R}^{\eta+1})}, \tag{2.19}$$

which, when interpolated with (2.3), directly gives (2.11). □

3. Proof of Theorems 1.1 and 1.2

Let us first prove Theorem 1.1. Similar technique found in [16] will be employed here. Assume that $\phi \in \mathcal{H}_d$ and $h \in \nabla_\kappa(\mathbb{R}^+)$, $\Psi \in L^q\left(\mathbb{S}^{\eta-1}\right)$ for some $1 < \kappa, q \leq 2$. It is easy to verify that Minkowski's inequality gives

$$\begin{aligned} \mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F)(\bar{w}) &\leq \left(\sum_{j=0}^{\infty} \int_{\mathbb{R}^+} \left| \frac{1}{t^n} \int_{2^{-j-1}t<|v|\leq 2^{-j}t} F(w-v, w_{\eta+1} - \phi(|v|)) \mathrm{K}_{\Psi,h}(v) dv \right|^\gamma \frac{dt}{t} \right)^{1/\gamma} \\ &= \frac{2^\alpha}{2^\alpha - 1} \left(\int_{\mathbb{R}^+} \left| \mho_{h,t} * F(\bar{w}) \right|^\gamma \frac{dt}{t} \right)^{1/\gamma}. \end{aligned} \tag{3.1}$$

Set $\mu = 2^{\kappa' q'}$. So, $\ln(\mu) \leq \frac{1}{(\kappa-1)(q-1)}$. For $j \in \mathbb{Z}$, let $\left\{\Theta_j\right\}_{-\infty}^{\infty}$ be the set of a partition of unity in the space $C^\infty(0,\infty)$ such that

$$0 \leq \Theta_j \leq 1, \ \sum_{j\in\mathbb{Z}} \Theta_j(t) = 1,$$

$$\operatorname{supp} \Theta_j \subseteq [\mu^{-j-1}, \mu^{-j+1}] \equiv \mathbf{I}_{j,\mu}, \ and \ \left| \frac{d^l \Theta_j(t)}{dt^l} \right| \leq \frac{C_l}{t^l}.$$

Define the multiplier operator $\widehat{\mathbf{J}_j F}(\bar{\zeta}) = \Theta_j(|\zeta|)\widehat{F}(\bar{\zeta})$. So, we deduce that for any $F \in \mathcal{S}(\mathbb{R}^{\eta+1})$,

$$\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}(F) \leq C \sum_{j\in\mathbb{Z}} \mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F), \tag{3.2}$$

where

$$\mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F)(\bar{w}) = \left(\int_{\mathbb{R}^+} \left| \mathcal{V}_{\Psi,\phi,h,j,\mu}(\bar{w},t) \right|^\gamma \frac{dt}{t} \right)^{1/\gamma},$$

$$\mathcal{V}_{\Psi,\phi,h,j,\mu}(\bar{w},t) = \sum_{s\in\mathbb{Z}} (\Theta_{s+j} * \mho_{h,t} * F)(\bar{w}) \chi_{[\mu^s,\mu^{s+1})}(t).$$

So, to prove Theorem 1.1, it suffices to show that a positive constant τ exists such that the following inequalities hold:

$$\left\| \mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F) \right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\, 2^{-\tau|j|} \left(\frac{1}{(q-1)(\kappa-1)} \right)^{1/\gamma} \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}, \tag{3.3}$$

for all $p \in [\gamma, \frac{\kappa\gamma'}{\gamma'-\kappa}]$,

$$\left\| \mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F) \right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\, 2^{-\tau|j|} \left(\frac{1}{(q-1)(\kappa-1)} \right)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}} \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}, \tag{3.4}$$

for all $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa}, \gamma)$, and

$$\left\| \mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F) \right\|_{L^p(\mathbb{R}^{\eta+1})} \leq C_{p,\Psi,h}\, 2^{-\tau|j|} \left(\frac{1}{(q-1)(\kappa-1)} \right)^{\frac{\kappa\gamma-\gamma+1}{\kappa\gamma}} \|F\|_{\dot{F}^{0,\gamma}_p(\mathbb{R}^{\eta+1})}, \tag{3.5}$$

for all $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+1}, \gamma)$.

On one side, we prove the estimate (3.3) when $p = \gamma = 2$. In this case, we have $\|F\|_{\dot{F}_2^{0,2}(\mathbb{R}^{\eta+1})} = \|F\|_{L^2(\mathbb{R}^{\eta+1})}$. So, Plancherel's theorem along with Lemma 2.1 produce

$$\begin{aligned}
\left\|\mathcal{G}^{(2)}_{\Psi,\phi,h,j}(F)\right\|^2_{L^2(\mathbb{R}^{\eta+1})} &\le \sum_{s\in\mathbb{Z}} \int_{\mathcal{D}_{s+j,\mu}} \left(\int_{\mu^s}^{\mu^{s+1}} \left|\hat{\mho}_{h,t}(\zeta,\zeta_{\eta+1})\right|^2 \frac{dt}{t}\right) \left|\widehat{F}(\zeta,\zeta_{\eta+1})\right|^2 d\zeta d\zeta_{\eta+1} \\
&\le C^2_{2,\Psi,h}(\ln\mu) \sum_{s\in\mathbb{Z}} \int_{\mathcal{D}_{s+j,\mu}} \left(\min\left\{\left|\mu^{j-1}\zeta\right|^{-\frac{\delta}{\ln\mu}}, \left|\mu^{j+1}\zeta\right|^{\frac{\delta}{\ln\mu}}\right\}\right) \left|\widehat{F}(\zeta,\zeta_{\eta+1})\right|^2 d\zeta d\zeta_{\eta+1} \\
&\le C^2_{2,\Psi,h}(\ln\mu)\ 2^{-2\delta|j|} \sum_{s\in\mathbb{Z}} \int_{\mathcal{D}_{s+j,\mu}} \left|\widehat{F}(\zeta,\zeta_{\eta+1})\right|^2 d\zeta d\zeta_{\eta+1} \\
&\le C^2_{2,\Psi,h}(\ln\mu)\ 2^{-2\delta|j|}\, \|F\|^2_{L^2(\mathbb{R}^{\eta+1})},
\end{aligned}$$

where $\mathcal{D}_{s,\mu} = \left\{(\zeta,\zeta_{\eta+1}) \in \mathbb{R}^\eta \times \mathbb{R} : \left|(\zeta,\zeta_{\eta+1})\right| \in \mathbf{I}_{s,\mu}\right\}$. Therefore, we have

$$\left\|\mathcal{G}^{(2)}_{\Psi,\phi,h,j}(F)\right\|^2_{L^2(\mathbb{R}^{\eta+1})} \le C_{2,\Psi,h} 2^{-\delta|j|} \left[(q-1)(\kappa-1)\right]^{-1/2} \|F\|_{\dot{F}_0^{2,2}(\mathbb{R}^{\eta+1})}. \tag{3.6}$$

On the other side, by invoking Lemma 2.1 in [16] and Lemma 2.4, we have

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \le C_{p,\Psi,h} \left(\frac{1}{(q-1)(\kappa-1)}\right)^{1/\gamma} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})}, \tag{3.7}$$

for all $p \in [\gamma, \frac{\kappa\gamma'}{\gamma'-\kappa}]$,

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \le C_{p,\Psi,h} \left(\frac{1}{(q-1)(\kappa-1)}\right)^{\frac{\kappa\gamma-\gamma+\kappa}{\kappa\gamma}} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})}, \tag{3.8}$$

for all $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+\kappa}, \gamma)$, and

$$\left\|\mathcal{G}^{(\gamma)}_{\Psi,\phi,h,j}(F)\right\|_{L^p(\mathbb{R}^{\eta+1})} \le C_{p,\Psi,h} \left(\frac{1}{(q-1)(\kappa-1)}\right)^{\frac{\kappa\gamma-\gamma+1}{\kappa\gamma}} \|F\|_{\dot{F}_p^{0,\gamma}(\mathbb{R}^{\eta+1})}, \tag{3.9}$$

for all $p \in (\frac{\kappa\gamma}{\kappa\gamma-\gamma+1}, \gamma)$. Therefore, when we interpolate (3.6) with (3.7)–(3.9), we directly obtain (3.3)–(3.5), which in turn with (3.2) finishes the proof of Theorem 1.1.

In the same manner employed in the proof of Theorem 1.1, except employing Lemma 2.3 instead of Lemma 2.4 and taking $\mu = 2^{q'}$ instead of $\mu = 2^{\kappa' q'}$, we immediately prove Theorem 1.2.

4. Conclusions

In this work, we obtained specific L^p bounds for the generalized Marcinkiewicz operator $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ whenever the rough kernel Ψ lies in the space $L^q(\mathbb{S}^{\eta-1})$. These bounds allow us to utilize Yano's extrapolation technique to confirm the boundedness of $\mathcal{G}^{(\gamma)}_{\Psi,\phi,h}$ under weaker conditions on Ψ; that is, Ψ belongs to either the space $L(\log L)^s(\mathbb{S}^{\eta-1})$ or to the space $B_q^{(0,s-1)}(\mathbb{S}^{\eta-1})$. The results of this article generalize and improve many previously know results, as the results in [1–5, 14–16, 22].

Author contributions

Mohammed Ali: Writing–original draft, commenting; Qutaibeh Katatbeh: Formal analysis, commenting; Oqlah Al-Refai: Writing–original draft, Funding acquisition, commenting; Basma Al-Shatnawi: Writing–original draft, commenting. All authors have read and approved the final version of the manuscript for publication

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest in this paper.

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