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Research article

Exponential stability of a type III thermo-porous-elastic system from a new approach in its coupling

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Abstract: In this article, we prove the well-posedness and stability of a one-dimensional thermo-porous-elastic system, considering a thermal coupling of the Green and Naghdi types III. In this scenario, we propose a new model where the temperature of the material directly affects the stress tensor of displacements and the equilibrated stress tensor on the volume fractions in the porous medium, thereby generalizing some existing results in the literature on the subject. Additionally, the exponential decay of energy is proven when the evolution equation corresponding to the dynamics of the elastic skeletal structure exhibits frictional-type damping.

Keywords: exponential decay; well-posedness; thermo-porous-elastic systems; hyperbolic heat conduction; Green-Naghdi heat transfer
Mathematics Subject Classification: 35L15, 35B40, 74F05, 93D20

1. Introduction

The study of porous-elastic systems focuses on analyzing materials that have an elastic matrix and regions devoid of material. In particular, this type of material allows the volumetric density to be described based on the density of the material within the elastic skeletal structure and the volumetric fraction over the void regions (see [2, 3, 9], among others). A more realistic approach to these types of materials considers changes in temperature and heat flow in the system's state variables. These state variables typically include the displacement field, temperature, and volume fractions corresponding to the porous part of the body. This is where thermo-porous-elastic systems become highly relevant. Such models have a wide range of applications in physical phenomena, ranging from the behavior of biological tissues to the dynamics of geological formations.

In engineering, for example, comprehending the thermal behavior of porous-elastic materials is crucial for the design and optimization of various structures and systems. In particular, in civil engineering, where porous concrete is widely used, the thermal behavior of the material affects its durability, energy efficiency, and resistance to thermal stress [15, 16]. On the other hand, in geophysics, studying porous-elastic materials with thermal couplings is essential for understanding various geophysical phenomena, such as thermal convection in porous rocks [14, Chapter 2] and the thermal behavior of geothermal reservoirs [8, 17].

In this context, an important class of systems that fits the description provided above occurs when the spatial domain is represented by the line segment [0, 1], following the constitutive equations given by:

$$\begin{aligned} T &= \mu\omega_x + b\varphi - \beta_1\theta + \boxed{\textbf{Structural/Memory damping on } \omega} \\ H &= \delta\varphi_x - \beta_2\theta + \boxed{\textbf{Structural/Memory damping on } \varphi} \\ G &= -\xi\varphi - b\omega_x + m\theta - \tau\varphi_t \\ \rho E &= \alpha\theta + \beta_1\omega_x + \beta_2\varphi_x + m\varphi \\ \tau_0 q_t + q &= -\kappa T_0\theta_x - \sigma T_0\Theta_x \\ \Theta_t &= \theta \end{aligned} \tag{1.1}$$

together with the evolution equations of porous materials with a one-dimensional temperature

$$\rho\omega_{tt} = T_x + \boxed{\textbf{Frictional damping on } \omega} \qquad J\varphi_{tt} = H_x + G \qquad \rho T_0 E_t + q_x = 0 \tag{1.2}$$

where ρ density, J is the product of the mass density by the equilibrated inertia, T is the stress, H is the equilibrated stress, G is the equilibrated body force, q is the heat flux, E is the entropy, and the variables ω, φ, Θ, and θ are the displacement, the volume fraction, the thermal displacement, and the temperature, respectively. All the constitutive constants $\mu, \xi, J, \delta, \rho, b, \kappa, \alpha$ are positive with the additional condition

$$\eta = \frac{b^2}{\mu\xi} < 1.$$

Note that if $\sigma = \tau_0 = 0$, thermal coupling follows Fourier's law. On the other hand, if only $\sigma = 0$, then it follows Cattaneo's law, and if $\tau_0 = 0$, then the thermal coupling is Green and Naghdi type III.

In the literature, extensive research has been conducted on the long-term behavior of the systems (1.1) and (1.2) by numerous authors over the past decades. For example, Maguiña and Quintanilla [5] analyzed the model without any additional damping on the displacements, specifically when $J = \beta_1 = \sigma = \tau_0 = 0$. In this case, under Dirichlet-Neumann-Dirichlet conditions, they showed that the solutions exhibit slow decay.

In the case of considering only an additional memory damping on φ, Messaoudi and Fareh [6] studied (1.1) and (1.2) with $\mu = b$, $m = \beta_1 = 1$, and $\sigma = \tau_0 = 0$ considering a condition of equal velocities, that is

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{1.3}$$

For this problem, the authors showed that the energy associated with the system decays exponentially as long as the Dirichlet-Dirichlet-Dirichlet type boundary condition is satisfied. In the same scenario, the authors later showed in [7] that if (1.3) is not considered, a general decay is achieved, which depends on the decay of the memory kernel. Similarly, with this same type of memory, Afilal and Soufyane [1] proved the exponential decay of the energy associated with the system when $\tau = \sigma = \beta_2 = 0$ and the boundary conditions are of the type

$$\omega(0) = \omega_x(1) = \varphi(0) = \varphi(1) = q(0) = \theta(1) = 0.$$

The property of equal velocities (1.3) has been extensively studied in contrast to the type of additional damping applied to the system. In addition to the reference [6], for example, Santos, Campero, and Almeida [12] demonstrated that if $q = T_0 = 0$ in (1.1) and (1.2) (i.e., the system lacks thermal coupling) and structural damping is considered on the displacements, then the property (1.3) implies the exponential decay of the energy provided the boundary condition is of the Dirichlet-Neumann type. However, if this equality of velocities is not satisfied, the decay is polynomial. In the case of considering Fourier-type thermal coupling instead of structural damping on the displacements, a similar result is achieved in [11] when $\beta_2 = \sigma = \tau = \tau_0 = 0$ and Dirichlet-Neumann-Neumann boundary conditions are fulfilled.

In the case of considering the systems (1.1) and (1.2) with Green and Naghdi type III thermal coupling, Said and Messaoudi [13] demonstrated that if $\tau_0 = \tau = \beta_2 = 0$ and $\beta_1 \neq 0$, then the solutions decay very slowly and lose regularity. The authors considered structural damping on the displacements and the entire $\mathbb{R}$ as the spatial domain. Later, Lacheheb, Messaoudi, and Zahri [4] provided a reinterpretation of the type III thermal coupling. In this case, they considered $\tau_0 = \tau = m = \beta_1 = 0$ and $\beta_2 \neq 0$. Note that in [13], temperature variations directly impact the tensor T, meaning they directly influence the displacements of the elastic matrix. In contrast, Lacheheb et al. coupled the thermal component to the tensor H, so temperature variations directly affect the volume fractions. For this latter study, the authors considered a bounded domain with Dirichlet-Neumann-Dirichlet boundary conditions and proved that if (1.3) is assumed, the energy decays exponentially. However, if this equality is not satisfied, the energy decays polynomially.

Considering the previously described literature review, it can be observed that the type III thermal coupling raises several questions regarding the thermal effects on the model. Additionally, the properties of exponential stability are not entirely clear when the *equal velocity* condition is not considered, which is often unrealistic in materials simulations. Therefore, this article makes the following contributions:

Concerning the study model, we establish the well-posedness of systems (1.1) and (1.2) with $\tau = \tau_0 = m = 0$ and Dirichlet-Neumann-Dirichlet boundary conditions. Additionally, we introduce frictional-type damping on the displacements as the sole additional damping mechanism. This new model extends those of [13] and [4] by incorporating temperature variations in both the tensor T and H.

In the stability analysis for this new model, we demonstrate exponential decay of energy by defining an equivalent Lyapunov functional without considering the equal velocities in the evolution equations.

This work is structured as follows: In Section 2, we state the new model. In Section 3, we establish and prove the well-posedness of our new model through semigroup theory. In Section 4, we prove the exponential decay of energy. Finally, in Section 5, we describe our conclusions.

2. Statement of the problem

Let $\rho, J, \alpha, \xi, \mu, \sigma, \delta, a, \beta_2$ be positive constants and β_1 be a non-negative constant. Let us consider the frictional-type damping $a\omega_t$, in (1.2), then after substituting (1.1) into (1.2), we obtain the system

$$\begin{cases} \rho\omega_{tt} - \mu\omega_{xx} - b\varphi_x + \beta_1\Theta_{tx} + a\omega_t & = & 0 \quad \text{in } (0,1)\times\mathbb{R}^+ \\ J\varphi_{tt} - \delta\varphi_{xx} + \beta_2\Theta_{tx} + \xi\varphi + b\omega_x & = & 0 \quad \text{in } (0,1)\times\mathbb{R}^+ \\ \alpha\Theta_{tt} - \sigma\Theta_{xx} + \beta_1\omega_{tx} + \beta_2\varphi_{tx} - \kappa\Theta_{txx} & = & 0 \quad \text{in } (0,1)\times\mathbb{R}^+. \end{cases} \tag{2.1}$$

In this work, we consider the system (2.1) together with the boundary conditions

$$\omega(0,t) = \omega(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = \Theta(0,t) = \Theta(1,t) = 0, \quad t \geq 0 \tag{2.2}$$

and the initial conditions

$$\begin{cases} \omega(x,0) = \omega_0(x), & \omega_t(x,0) = \omega_1(x) & x \in (0,1) \\ \varphi(x,0) = \varphi_0(x), & \varphi_t(x,0) = \varphi_1(x) & x \in (0,1) \\ \Theta(x,0) = \Theta_0(x), & \Theta_t(x,0) = \Theta_1(x) & x \in (0,1)\,. \end{cases} \tag{2.3}$$

Note that if $\beta_1 = a = 0$ in (2.1), the system studied in [5] is obtained with the difference that the authors considered the property of equality of velocities on the coefficients. On the other hand, if $\beta_2 = a = 0$, a variation of the model is addressed in [15] when the domain is all of $\mathbb{R}$.

Now, by integrating the second equation in (2.1) with respect to the spatial variable, we have

$$J\int_0^1 \varphi_{tt}(r,t)dr - \delta\int_0^1 \varphi_{xx}(r,t)dr + \beta_2\int_0^1 \Theta_{tx}(r,t)dr + \xi\int_0^1 \varphi(r,t)dr + b\int_0^1 \omega_x(r,t)dr = 0,$$

from (2.2), it follows that

$$\int_0^1 \varphi_{xx}(r,t)dr = \varphi_x(1,t) - \varphi_x(0,t) = 0$$
$$\int_0^1 \Theta_{tx}(r,t)dr = \Theta_t(1,t) - \Theta_t(0,t) = 0$$
$$\int_0^1 \omega_x(r,t)dr = \omega(1,t) - \omega(0,t).$$

Therefore, we obtain the following ODE:

$$J\frac{d}{dt^2}\int_0^1 \varphi(r,t)dr + \xi\int_0^1 \varphi(r,t)dr = 0$$

with initial conditions

$$\int_0^1 \varphi(r,0)dr = \int_0^1 \varphi_0(r)dr \qquad \int_0^1 \varphi_t(r,0)dr = \int_0^1 \varphi_1(r)dr.$$

This ODE has the solution

$$G(t) = a_0\cos\left(\sqrt{\frac{\xi}{J}}t\right) + \sqrt{\frac{J}{\xi}}b_0\sin\left(\sqrt{\frac{\xi}{J}}t\right)$$

where

$$a_0 = \int_0^1 \varphi_0(x)dx, \qquad b_0 = \int_0^1 \varphi_1(x)dx.$$

So, by taking $\overline{\varphi}(x,t) = \varphi(x,t) - G(t)$, we have $\int_0^1 \overline{\varphi}(r,t)dr = 0, t \geq 0$ and $\omega, \overline{\varphi}, \Theta$ satisfy the system

$$\begin{cases} \rho\omega_{tt} - \mu\omega_{xx} - b\overline{\varphi}_x + \beta_1\Theta_{tx} + a\omega_t & = 0 \quad \text{in } (0,1)\times\mathbb{R}^+ \\ J\overline{\varphi}_{tt} - \delta\overline{\varphi}_{xx} + \beta_2\Theta_{tx} + \xi\overline{\varphi} + b\omega_x & = 0 \quad \text{in } (0,1)\times\mathbb{R}^+ \\ \alpha\Theta_{tt} - \sigma\Theta_{xx} + \beta_1\omega_{tx} + \beta_2\overline{\varphi}_{tx} - \kappa\Theta_{txx} & = 0 \quad \text{in } (0,1)\times\mathbb{R}^+ \end{cases} \tag{2.4}$$

$$\omega(0,t) = \omega(1,t) = \overline{\varphi}_x(0,t) = \overline{\varphi}_x(1,t) = \Theta(0,t) = \Theta(1,t) = 0, \quad t \geq 0 \tag{2.5}$$

$$\begin{cases} \omega(x,0) = \omega_0(x), & \omega_t(x,0) = \omega_1(x) & x \in (0,1) \\ \overline{\varphi}(x,0) = \varphi_0(x) - a_0, & \overline{\varphi}_t(x,0) = \varphi_1(x) - b_0 & x \in (0,1) \\ \Theta(x,0) = \Theta_0(x), & \Theta_t(x,0) = \Theta_1(x) & x \in (0,1). \end{cases} \tag{2.6}$$

This allows us to use Poincare's inequality on $\overline{\varphi}$. Based on what has been observed, we work with the systems (2.4)–(2.6) and solutions $\omega, \overline{\varphi}, \Theta$ but we still refer to systems (2.1)–(2.3) and write ω, φ, Θ for simplicity.

3. Well-posedness

Let us consider the Hilbert spaces $L^2(0,1)$, $H^1(0,1)$, and $H^2(0,1)$ together with their usual inner product and usual norm, so we define the vector spaces

$$\begin{aligned} L^2_*(0,1) &= \left\{ u \in L^2(0,1) \,\Big|\, \int_0^1 u dx = 0 \right\} \\ H^1_*(0,1) &= \left\{ u \in H^1(0,1) \,\Big|\, \int_0^1 u dx = 0 \right\} = H^1(0,1) \cap L^2_*(0,1) \\ H^2_*(0,1) &= \left\{ u \in H^2(0,1) \,\Big|\, u_x(0) = u_x(0) = 0 \right\} \end{aligned}$$

which are still Hilbert spaces with the inner product induced from $L^2(0,1)$, $H^1(0,1)$, and $H^2(0,1)$, respectively. For simplicity, we denote by $\langle\cdot,\cdot\rangle$, $|\cdot|_2$ the usual inner product and usual norm on $L^2(0,1)$.

The weak space is defined as the Hilbert space

$$\mathcal{H} = H^1_0(0,1) \times L^2(0,1) \times H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1)$$

with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by:

$$\begin{aligned} \langle U, W\rangle_{\mathcal{H}} = {}& \rho\langle v_1, v_2\rangle + J\langle \psi_1, \psi_2\rangle + \alpha\langle \vartheta_1, \vartheta_2\rangle + \xi\langle \varphi_1, \varphi_2\rangle + \mu\langle \omega_{1x}, \omega_{2x}\rangle + \sigma\langle \Theta_{1x}, \Theta_{2x}\rangle \\ & + \delta\langle \varphi_{1x}, \varphi_{2x}\rangle + b\langle \omega_{1x}, \varphi_2\rangle + b\langle \omega_{2x}, \varphi_1\rangle \end{aligned}$$

for all $U = (\omega_1, v_1, \varphi_1, \psi_1, \Theta_1, \vartheta_1), W = (\omega_2, v_2, \varphi_2, \psi_2, \Theta_2, \vartheta_2) \in \mathcal{H}$.

This inner product induces the norm $\|\cdot\|_{\mathcal{H}}$, which is equivalent to the usual norm on $\mathcal{H}$, defined by

$$\begin{aligned}\|U\|_{\mathcal{H}}^2 &= \rho|v|_2^2 + J|\psi|_2^2 + \alpha|\vartheta|_2^2 + \xi|\varphi|_2^2 + \mu|\omega_x|^2 + \sigma|\Theta_x|_2^2 + \delta|\varphi_x|_2^2 + 2b\langle\omega_x,\varphi\rangle \\ &= \rho|v|_2^2 + \left(\xi - \frac{b^2}{\mu}\right)|\varphi|_2^2 + J|\psi|_2^2 + \alpha|\vartheta|_2^2 + \mu\left|\omega_x + \frac{b}{\mu}\varphi\right|_2^2 + \sigma|\Theta_x|_2^2 + \delta|\varphi_x|_2^2\end{aligned}$$

for all $U = (\omega, v, \varphi, \psi, \Theta, \vartheta) \in \mathcal{H}$.

In order to establish the well-posedness, let us denote $v = \omega_t, \psi = \varphi_t, \vartheta = \Theta_t$, $U = (\omega, v, \varphi, \psi, \Theta, \vartheta)^T$ and $U_0 = (\omega_0, \omega_1, \varphi_0, \varphi_1, \Theta_0, \Theta_1)^T$, then systems (2.1)–(2.3) are equivalent to the abstract Cauchy problem

$$\begin{cases} U_t = AU & t > 0 \\ U(0) = U_0, \end{cases} \tag{3.1}$$

where the operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$AU = \begin{pmatrix} v \\ \frac{1}{\rho}(\mu\omega_{xx} + b\varphi_x - \beta_1\vartheta_x - av) \\ \psi \\ \frac{1}{J}(\delta\varphi_{xx} - b\omega_x - \xi\varphi - \beta_2\vartheta_x) \\ \vartheta \\ \frac{1}{\alpha}(\sigma\Theta_{xx} + \kappa\vartheta_{xx} - \beta_1 v_x - \beta_2\psi_x) \end{pmatrix} \tag{3.2}$$

and its domain is

$$D(A) = \left\{ U \in \mathcal{H} \;\middle|\; \begin{array}{l} \omega, \sigma\Theta + k\vartheta \in H^2(0,1) \cap H_0^1(0,1), \\ v, \vartheta \in H_0^1(0,1), \\ \varphi \in H_*^2(0,1) \cap H_*^1(0,1), \psi \in H_*^1(0,1) \end{array} \right\}.$$

Consequently, we can state the result of well-posedness as follows:

Theorem 3.1 (Well-posedness)**.** *Under the hypothesis described in Section 2 on the systems* (2.1)–(2.3)*, it resulted that:*

i) For $U_0 \in \mathcal{H}$, the systems (2.1)–(2.3) *possess a unique solution*

$$U \in C(\mathbb{R}^+; \mathcal{H}).$$

ii) For $U_0 \in D(A)$, the systems (2.1)–(2.3) *possess a unique solution*

$$U \in C(\mathbb{R}^+; D(A)) \cap C^1(\mathbb{R}^+; \mathcal{H}).$$

The above theorem is a consequence of the following lemmas:

Lemma 3.2. *The operator A is an infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof. From (2.1) and (3.2), through a simple calculation, we have

$$\langle AU, U\rangle_{\mathcal{H}} = -a|v|_2^2 - \kappa|\vartheta_x|_2^2,$$

therefore, A is dissipative.

Now, we prove that $I - A$ is surjective; that is, for any $Z = (g_1, g_2, g_3, g_4, g_5, g_6) \in \mathcal{H}$, there exists a $U = (\omega, v, \varphi, \psi, \Theta, \vartheta) \in D(A)$ satisfying

$$(I - A)U = Z$$

or equivalently

$$\begin{cases} \omega - v & = & g_1 \\ v - \dfrac{1}{\rho}(\mu\omega_{xx} + b\varphi_x - \beta_1\vartheta_x - av) & = & g_2 \\ \varphi - \psi & = & g_3 \\ \psi - \dfrac{1}{J}(\delta\varphi_{xx} - b\omega_x - \xi\varphi - \beta_2\vartheta_x) & = & g_4 \\ \Theta - \vartheta & = & g_5 \\ \vartheta - \dfrac{1}{\alpha}(\sigma\Theta_{xx} + \kappa\vartheta_{xx} - \beta_1 v_x - \beta_2\psi_x) & = & g_6 \end{cases}$$

hence, we obtain the equations

$$\begin{aligned} (\rho + a)\omega - \mu\omega_{xx} - b\varphi_x + \beta_1\Theta_x &= (\rho + a)g_1 + \rho g_2 + \beta_1 g_{5x} \\ (J + \xi)\varphi - \delta\varphi_{xx} + b\omega_x + \beta_2\Theta_x &= J(g_3 + g_4) + \beta_2 g_{5x} \\ \alpha\Theta - (\sigma + \kappa)\Theta_{xx} + \beta_1\omega_x + \beta_2\varphi_x &= \alpha(g_6 + g_5) + \beta_2 g_{3x} + \beta_1 g_{1x} - \kappa g_{5xx}. \end{aligned}$$

We prove the existence of U by using Lax-Milgram. Denote by

$$X = H_0^1(0, 1) \times H_*^1(0, 1) \times H_0^1(0, 1)$$

and consider the variational problem

$$B((\omega, \varphi, \Theta), (\overline{\omega}, \overline{\varphi}, \overline{\Theta})) = L(\overline{\omega}, \overline{\varphi}, \overline{\Theta}), \quad \forall(\overline{\omega}, \overline{\varphi}, \overline{\Theta}) \in X$$

where $B : X \times X \to \mathbb{R}$ is defined by

$$\begin{aligned} B((\omega, \varphi, \Theta), (\overline{\omega}, \overline{\varphi}, \overline{\Theta})) =& (\rho + a)\langle \omega, \overline{\omega}\rangle + \mu\langle \omega_x, \overline{\omega}_x\rangle - b\langle \varphi_x, \overline{\omega}\rangle + \beta_1\langle \Theta_x, \overline{\omega}\rangle \\ &+ (J + \xi)\langle \varphi, \overline{\varphi}\rangle + \delta\langle \varphi_x, \overline{\varphi}_x\rangle + b\langle \omega_x, \overline{\varphi}\rangle + \beta_2\langle \Theta_x, \overline{\varphi}\rangle \\ &+ \alpha\left\langle \Theta, \overline{\Theta}\right\rangle + (\sigma + \kappa)\left\langle \Theta_x, \overline{\Theta}_x\right\rangle + \beta_1\left\langle \omega_x, \overline{\Theta}\right\rangle + \beta_2\left\langle \varphi_x, \overline{\Theta}\right\rangle \end{aligned}$$

and $L : X \to \mathbb{R}$ is defined by

$$\begin{aligned} L(\omega, \varphi, \Theta) =& \langle (\rho + a)g_1 + \rho g_2 + \beta_1 g_{5x}, \overline{\omega}\rangle + \langle J(g_3 + g_4) + \beta_2 g_{5x}, \overline{\varphi}\rangle \\ &+ \kappa\left\langle g_{5x}, \overline{\Theta}_x\right\rangle + \left\langle \alpha(g_6 + g_5) + \beta_2 g_{3x} + \beta_1 g_{1x}, \overline{\Theta}\right\rangle. \end{aligned}$$

Clearly, B is a bilinear form and L is linear; moreover from Cauchy-Schwartz and Poincare's inequality, we have that B and L are continuous. Now

$$
\begin{aligned}
B(((\omega,\varphi,\Theta),(\omega,\varphi,\Theta)) &= (\rho+a)|\omega|_2^2 + \left(\mu - \frac{b^2}{\xi}\right)|\omega_x|_2^2 + \xi\left|\varphi + \frac{b}{\xi}\omega_x\right|_2^2 + J|\varphi|_2^2 + \\
&\quad + \delta|\varphi_x|_2^2 + \alpha|\theta|_2^2 + (\sigma+\kappa)|\theta_x|_2^2 \\
&\geq \min\left\{\mu - \frac{b^2}{\xi}, \delta, J, \sigma+\kappa\right\}\left(|u_x|_2^2 + |\phi_x|_2^2 + |\phi|_2^2 + |\theta_x|_2^2\right).
\end{aligned}
$$

Hence B is coercive. Thus, from the Lax-Milgram theorem and regularity properties, we have that $I-A$ is surjective. Therefore, the lemma follows from the Lummer-Philips theorem. □

Now, let (ω,φ,Θ) be the solution of (2.1)–(2.3), and the energy associated is defined by

$$E(t) = \frac{1}{2}\|(\omega,\omega_t,\varphi,\varphi_t,\Theta,\Theta_t)\|_{\mathcal{H}}^2.$$

Lemma 3.3. *The function $E(t)$ is non-creasing; moreover,*

$$E'(t) = -\kappa|\Theta_{tx}|_2^2 - a|\omega_t|_2^2 \leq 0. \tag{3.3}$$

Proof. By differentiating E, we have

$$
\begin{aligned}
E'(t) &= \langle(\omega,\omega_t,\varphi,\varphi_t,\Theta,\Theta_t),(\omega_t,\omega_{tt},\varphi_t,\varphi_{tt},\Theta_t,\Theta_{tt})\rangle_{\mathcal{H}} \\
&= \rho\langle\omega_t,\omega_{tt}\rangle + \xi\langle\varphi,\varphi_t\rangle + J\langle\varphi_t,\varphi_{tt}\rangle + \alpha\langle\Theta_t,\Theta_{tt}\rangle + \mu\langle\omega_x,\omega_{tx}\rangle + \sigma\langle\Theta_x,\Theta_{tx}\rangle + \\
&\quad + \delta\langle\varphi_x,\varphi_{tx}\rangle + b\langle\omega_x,\varphi_t\rangle + b\langle\omega_{tx},\varphi\rangle \\
&= \langle\omega_t,\mu\omega_{xx} + b\varphi_x - \beta_1\Theta_{tx} - a\omega_t\rangle + \langle\varphi_t,\delta\varphi_{xx} - \beta_2\Theta_{tx} - \xi\varphi - b\omega_x\rangle + \\
&\quad + \langle\Theta_t,\sigma\Theta_{xx} - \beta_1\omega_{tx} - \beta_2\varphi_{tx} + \kappa\Theta_{txx}\rangle + \xi\langle\varphi,\varphi_t\rangle + \mu\langle\omega_x,\omega_{tx}\rangle + \sigma\langle\Theta_x,\Theta_{tx}\rangle + \\
&\quad + \delta\langle\varphi_x,\varphi_{tx}\rangle + b\langle\omega_x,\varphi_t\rangle + b\langle\omega_{tx},\varphi\rangle \\
&= -a|\omega_t|_2^2 - \kappa|\Theta_{tx}|_2^2.
\end{aligned}
$$

□

Proof of Theorem 3.1. From Lemma 3.2, the system (3.1) admits a unique local solution $U : [0,T_{\max}) \to \mathcal{H}$ given by a semigroup $(\mathcal{H},S(t))$, and from Lemma 3.3, those solutions can be extended to $[0,+\infty)$. □

4. Exponential decay

In this section, we are devoted to providing the exponential decay of the energy $E(t)$. To achieve this goal, we must first prove several estimates.

Let (ω,φ,Θ) be the solution of (2.1)–(2.3), and we consider the functionals (multipliers)

$$
\begin{aligned}
\chi_1(t) &= \rho\langle\omega,\omega_t\rangle \\
\chi_2(t) &= J\langle\varphi,\varphi_t\rangle
\end{aligned}
$$

$$\chi_3(t) = \alpha \langle \Theta, \Theta_t \rangle + \frac{\kappa}{2}|\Theta_x|_2^2 + \beta_2 \langle \Theta, \varphi_x \rangle$$
$$\chi_4(t) = -\alpha J \left\langle \int_0^x \varphi_t(y)dy, \Theta_t \right\rangle$$
$$\Upsilon(t) = \epsilon E(t) + \frac{\beta_2^2}{12J\sigma}\chi_1(t) + \frac{\beta_2^2}{12J\sigma}\chi_2(t) + \chi_3(t) + \frac{\beta_2}{4J\sigma}\chi_4(t),$$

where the constant $\epsilon > 0$ will be chosen later.

Lemma 4.1. *There exist constants K_1 and $K_2 > 0$ such that*

$$K_2 E(t) \leq \Upsilon(t) \leq K_1 E(t). \tag{4.1}$$

Proof. It is clear that

$$E(t) = \frac{\rho}{2}|\omega_t|_2^2 + \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)|\varphi|_2^2 + \frac{J}{2}|\varphi_t|_2^2 + \frac{\alpha}{2}|\Theta_t|_2^2 + \frac{\mu}{2}\left|\omega_x + \frac{b}{\mu}\varphi\right|_2^2 + \frac{\sigma}{2}|\Theta_x|_2^2 + \frac{\delta}{2}|\varphi_x|_2^2.$$

Thus, from Cauchy-Schwartz, Young and Poincare's inequalities, we have the estimates

$$|\chi_1(t)| \leq \left(1 + \frac{2\rho}{\lambda_1\mu} + \frac{2\rho b^2}{\lambda_1\mu(\xi\mu - b^2)}\right)E(t),$$

$$|\chi_2(t)| \leq \left(1 + \frac{J\mu}{\xi\mu - b^2}\right)E(t),$$

$$|\chi_3(t)| \leq \left(1 + \frac{\alpha + \beta_2}{\lambda_1\sigma} + \frac{\kappa}{\sigma} + \frac{\beta_2}{\delta}\right)E(t)$$

and

$$\begin{aligned}
|\chi_4(t)| &\leq \frac{\alpha J}{2}\int_0^1 \left|\int_0^x \varphi_t(y)dy\right|_2^2 dx + \frac{\alpha J}{2}|\Theta_t|_2^2 \\
&\leq \frac{\alpha J}{2}|\varphi_t|_2^2 + \frac{\alpha J}{2}|\Theta_t|_2^2 \\
&\leq (J + \alpha)\, E(t)
\end{aligned}$$

where λ_1 denotes the first eigenvalue of $-\Delta$. Then we take

$$\begin{aligned}
K_0 = {} & \frac{\beta_2^2}{12J\sigma}\left(1 + \frac{2\rho}{\lambda_1\mu} + \frac{2\rho b^2}{\lambda_1\mu(\xi\mu - b^2)}\right) + \frac{\beta_2^2}{12J\sigma}\left(1 + \frac{J\mu}{\xi\mu - b^2}\right) + \frac{\beta_2}{4J\sigma}(J + \alpha) \\
& + \left(1 + \frac{\alpha + \beta_2}{\lambda_1\sigma} + \frac{\kappa}{\sigma} + \frac{\beta_2}{\delta}\right).
\end{aligned} \tag{4.2}$$

The lemma follows, taking $K_1 = \epsilon + K_0$ and $K_2 = \epsilon - K_0 > 0$.

□

Hereinafter, c always denotes a constant to simplify computations.

Lemma 4.2 (estimating χ_1'). *Let (ω,φ,Θ) be the solution of (2.1)–(2.3), then the function χ_1 satisfies the estimate*

$$\chi_1'(t) \le -\left(\frac{\mu}{2} - \mu p_0\right)|\omega_x|_2^2 + \frac{b^2}{2\mu}|\varphi|_2^2 + c|\omega_t|_2^2 + c|\Theta_{tx}|_2^2 \tag{4.3}$$

where $0 < p_0 := \dfrac{1-\eta}{4} < \dfrac{1}{2}$.

Proof. Multiplying, first equation on (2.1) by ω and integrating on $[0,1]$

$$\begin{aligned}\chi_1'(t) &= \rho|\omega_t|_2^2 + \langle \omega, \rho\omega_{tt}\rangle \\ &= \rho|\omega_t|_2^2 + \langle \omega, \mu\omega_{xx} + b\varphi_x - \beta_1\Theta_{tx} - a\omega_t\rangle \\ &= \rho|\omega_t|_2^2 - \mu|\omega_x|_2^2 + b\langle \omega,\varphi_x\rangle - \beta_1\langle \omega,\Theta_{tx}\rangle - a\langle\omega,\omega_t\rangle\end{aligned}$$

now, from the boundary conditions, Cauchy-Schwartz, Young, and Poincare's inequalities, we have

$$\begin{aligned} b\langle \omega,\varphi_x\rangle = -b\langle\omega_x,\varphi\rangle &\le \frac{\mu}{2}|\omega_x|_2^2 + \frac{b^2}{2\mu}|\varphi|_2^2, \\ -a\langle\omega,\omega_t\rangle &\le \frac{\mu p_0\lambda_1}{2}|\omega|_2^2 + \frac{a^2}{2\mu p_0\lambda_1}|\omega_t|_2^2 \le \frac{\mu p_0}{2}|\omega_x|_2^2 + c|\omega_t|_2^2, \\ -\beta_1\langle\omega,\Theta_{tx}\rangle &\le \frac{\mu p_0}{2}|\omega_x|_2^2 + c|\Theta_{tx}|_2^2. \end{aligned}$$

Thus,

$$\chi_1'(t) \le -\left(\frac{\mu}{2} - \mu p_0\right)|\omega_x|_2^2 + \frac{b^2}{2\mu}|\varphi|_2^2 + c|\omega_t|_2^2 + c|\Theta_{tx}|_2^2.$$

□

Lemma 4.3 (estimating χ_2'). *Let (ω,φ,Θ) be the solution of (2.1)–(2.3), then χ_2 satisfies*

$$\chi_2'(t) \le -\frac{\delta}{2}|\varphi_x|_2^2 - \frac{\xi}{2}|\varphi|_2^2 + J|\varphi_t|_2^2 + \frac{b^2}{2\xi}|\omega_x|_2^2 + c|\Theta_{tx}|_2^2. \tag{4.4}$$

Proof. From the second equation on (2.1), we have

$$\begin{aligned}\chi_2'(t) &= J|\varphi_t|_2^2 + \langle\varphi, J\varphi_{tt}\rangle \\ &= J|\varphi_t|_2^2 + \langle\varphi, \delta\varphi_{xx} - b\omega_x - \xi\varphi - \beta_2\Theta_{tx}\rangle \\ &= J|\varphi_t|_2^2 - \delta|\varphi_x|_2^2 - b\langle\varphi,\omega_x\rangle - \xi|\varphi|_2^2 - \beta_2\langle\varphi,\Theta_{tx}\rangle,\end{aligned}$$

now, from the boundary conditions, Cauchy-Schwartz, Young, and Poincare's inequalities, we have

$$\begin{aligned} -b\langle\varphi,\omega_x\rangle &\le \frac{\xi}{2}|\varphi|_2^2 + \frac{b^2}{2\xi}|\omega_x|_2^2 \\ -\beta_2\langle\varphi,\Theta_{tx}\rangle = \beta_2\langle\varphi_x,\Theta_t\rangle &\le \frac{\delta}{2}|\varphi_x|_2^2 + \frac{\beta_2^2}{2\delta}|\Theta_t|_2^2 \le \frac{\delta}{2}|\varphi_x|_2^2 + c|\Theta_{tx}|_2^2. \end{aligned}$$

Hence

$$\chi_2'(t) \le -\frac{\delta}{2}|\varphi_x|_2^2 - \frac{\xi}{2}|\varphi|_2^2 + J|\varphi_t|_2^2 + \frac{b^2}{2\xi}|\omega_x|_2^2 + c|\Theta_{tx}|_2^2.$$

□

Lemma 4.4 (estimating χ_3'). *Let* $(\omega, \varphi, \Theta)$ *be the solution of* (2.1)–(2.3)*, and the function* χ_3 *satisfies*

$$\chi_3'(t) \leq -\frac{\sigma}{2}|\Theta_x|_2^2 + \frac{\delta\beta_2^2}{48J\sigma}|\varphi_x|_2^2 + c|\omega_t|_2^2 + c|\Theta_{tx}|_2^2. \tag{4.5}$$

Proof. From the third equation of (2.1), we deduce

$$\begin{aligned}\chi_3'(t) &= \alpha|\Theta_t|_2^2 + \langle\Theta, \alpha\Theta_{tt}\rangle + \kappa\langle\Theta_x, \Theta_{xt}\rangle + \beta_2\langle\Theta_t, \varphi_x\rangle + \beta_2\langle\Theta, \varphi_{tx}\rangle \\ &= \alpha|\Theta_t|_2^2 + \langle\Theta, \sigma\Theta_{xx} + \kappa\Theta_{txx} - \beta_1\omega_{tx} - \beta_2\varphi_{tx}\rangle - \kappa\langle\Theta, \Theta_{txx}\rangle + \beta_2\langle\varphi_{xt}, \Theta\rangle + \beta_2\langle\varphi_x, \Theta_t\rangle \\ &= \alpha|\Theta_t|_2^2 - \sigma|\Theta_x|_2^2 - \beta_1\langle\Theta, \omega_{tx}\rangle + \beta_2\langle\varphi_x, \Theta_t\rangle.\end{aligned}$$

Furthermore,

$$\begin{aligned}-\beta_1\langle\Theta,\ \omega_{tx}\rangle = \beta_1\langle\Theta_x,\ \omega_t\rangle &\leq \frac{\sigma}{2}|\Theta_x|_2^2 + c|\omega_t|_2^2 \\ \beta_2\langle\varphi_x, \Theta_t\rangle &\leq \frac{\delta\beta_2^2}{48J\sigma}|\varphi_x|_2^2 + c|\Theta_{tx}|_2^2.\end{aligned}$$

Hence

$$\chi_3'(t) \leq -\frac{\sigma}{2}|\Theta_x|_2^2 + \frac{\delta\beta_2^2}{48J\sigma}|\varphi_x|_2^2 + c|\omega_t|_2^2 + c|\Theta_{tx}|_2^2.$$

□

Lemma 4.5 (estimating χ_4'). *Let* $(\omega, \varphi, \Theta)$ *be the solution of* (2.1)–(2.3)*, and the function* χ_4 *satisfies the estimate*

$$\chi_4'(t) \leq -\frac{J\beta_2}{2}|\varphi_t|_2^2 + \frac{3J\sigma^2}{2\beta_2}|\Theta_x|_2^2 + \epsilon_1|\varphi_x|_2^2 + \epsilon_2|\omega_x|_2^2 + \epsilon_3|\varphi|_2^2 + c|\omega_t|_2^2 + c_{(\epsilon_1,\epsilon_2,\epsilon_3)}|\Theta_{tx}|_2^2 \tag{4.6}$$

for some $\epsilon_1, \epsilon_2, \epsilon_3 > 0$*, which will be chosen later. Here* $c_{(\epsilon_1,\epsilon_2,\epsilon_3)}$ *denotes a constant that depends on* $\epsilon_1, \epsilon_2, \epsilon_3$.

Proof. From (2.1) multiplying appropriately and integrating on $[0, 1]$

$$\begin{aligned}\chi_4'(t) &= -\alpha\left\langle\int_0^x J\varphi_{tt}(y)dy, \Theta_t\right\rangle - J\left\langle\int_0^x \varphi_t(y)dy, \alpha\Theta_{tt}\right\rangle \\ &= -\alpha\left\langle\int_0^x \left[\delta\varphi_{xx}(y) - b\omega_x(y) - \xi\varphi(y) - \beta_2\Theta_{tx}(y)\right]dy, \Theta_t\right\rangle - \\ &\quad - J\left\langle\int_0^x \varphi_t(y)dy, \sigma\Theta_{xx} + \kappa\Theta_{txx} - \beta_1\omega_{tx} - \beta_2\varphi_{tx}\right\rangle \\ &= -\alpha\delta\left\langle\int_0^x \varphi_{xx}(y)dy, \Theta_t\right\rangle + \alpha b\left\langle\int_0^x \omega_x(y)dy, \Theta_t\right\rangle + \alpha\xi\left\langle\int_0^x \varphi(y)dy, \Theta_t\right\rangle + \\ &\quad + \alpha\beta_2\left\langle\int_0^x \Theta_{tx}(y)dy, \Theta_t\right\rangle - J\sigma\left\langle\int_0^x \varphi_t(y)dy, \Theta_{xx}\right\rangle - J\kappa\left\langle\int_0^x \varphi_t(y)dy, \Theta_{txx}\right\rangle + \\ &\quad + J\beta_1\left\langle\int_0^x \varphi_t(y)dy, \omega_{tx}\right\rangle + J\beta_2\left\langle\int_0^x \varphi_t(y)dy, \varphi_{tx}\right\rangle + J\left\langle\int_0^x \varphi_t(y)dy, \Delta_2\right\rangle.\end{aligned}$$

We work out the estimates of these integrals as follows:

By solving the integrals using the boundary conditions, Cauchy-Schwartz, Young, and Poincare's inequalities, we have

$$\begin{aligned} -\alpha\delta\left\langle\int_0^x \varphi_{xx}(y)dy, \Theta_t\right\rangle &= -\alpha\delta\left\langle\varphi_x, \Theta_t\right\rangle &\leq \epsilon_1|\varphi_x|_2^2 + \frac{\alpha^2\delta^2}{4\epsilon_1\lambda_1}|\Theta_{tx}|_2^2, \\ \alpha b\left\langle\int_0^x \omega_x(y)dy, \Theta_t\right\rangle &= \alpha b\left\langle\omega, \Theta_t\right\rangle &\leq \epsilon_2|\omega_x|_2^2 + \frac{\alpha^2 b^2}{4\epsilon_2\lambda_1^2}|\Theta_{tx}|_2^2, \\ \alpha\beta_2\left\langle\int_0^x \Theta_{tx}(y)dy, \Theta_t\right\rangle &= \alpha\beta_2|\Theta_t|_2^2 &\leq c|\Theta_{tx}|_2^2. \end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned} \alpha\xi\left\langle\int_0^x \varphi(y)dy, \Theta_t\right\rangle &\leq \epsilon_3\left|\int_0^x \varphi(y)dy\right|_2^2 + \frac{\alpha^2\xi^2}{4\epsilon_3}|\Theta_t|_2^2 \\ &\leq \epsilon_3\int_0^1 \varphi(y)^2dy + \frac{\alpha^2\xi^2}{4\epsilon_3\lambda_1}|\Theta_{tx}|_2^2 \\ &= \epsilon_3|\varphi|_2^2 + \frac{\alpha^2\xi^2}{4\epsilon_3\lambda_1}|\Theta_{tx}|_2^2. \end{aligned} \tag{4.8}$$

Since $\int_0^1 \varphi(y)dy = 0$ for all $t > 0$, then

$$\left\langle\int_0^x \varphi_t(y)dy, f_x\right\rangle = -\left\langle\varphi_t, f\right\rangle, \text{ for all } f \in H^1(0,1). \tag{4.9}$$

Thus, from (4.9),

$$J\beta_2\left\langle\int_0^x \varphi_t(y)dy, \varphi_{tx}\right\rangle = -J\beta_2|\varphi_t|_2^2.$$

Again, from (4.9), we have

$$\begin{aligned} J\beta_1\left\langle\int_0^x \varphi_t(y)dy, \omega_{tx}\right\rangle &= -J\beta_1\left\langle\varphi_t, \omega_t\right\rangle &\leq \frac{J\beta_2}{6}|\varphi_t|_2^2 + c|\omega_t|_2^2, \\ -J\sigma\left\langle\int_0^x \varphi_t(y)dy, \Theta_{xx}\right\rangle &= J\sigma\left\langle\varphi_t, \Theta_x\right\rangle &\leq \frac{J\beta_2}{6}|\varphi_t|_2^2 + \frac{3J\sigma^2}{2\beta_2}|\Theta_x|_2^2, \\ -J\kappa\left\langle\int_0^x \varphi_t(y)dy, \Theta_{txx}\right\rangle &= J\kappa\left\langle\varphi_t, \Theta_{tx}\right\rangle &\leq \frac{J\beta_2}{6}|\varphi_t|_2^2 + c|\Theta_{tx}|_2^2. \end{aligned} \tag{4.10}$$

In conclusion, from (4.7), (4.8), and (4.10) we obtain

$$\chi_4'(t) \leq -\frac{J\beta_2}{2}|\varphi_t|_2^2 + \frac{3J\sigma^2}{2\beta_2}|\Theta_x|_2^2 + \epsilon_1|\varphi_x|_2^2 + \epsilon_2|\omega_x|_2^2 + \epsilon_3|\varphi|_2^2 + c|\omega_t|_2^2 + c_{(\epsilon_1,\epsilon_2,\epsilon_3)}|\Theta_{tx}|_2^2.$$

□

Lemma 4.6 (estimating Υ')**.** *Let* $(\omega, \varphi, \Theta)$ *be the solution of* (2.1)–(2.3)*, then there exist constants* $\epsilon, \epsilon_1, \epsilon_2, \epsilon_3 > 0$ *and* $\lambda > 0$ *such that* (4.1) *is valid and*

$$\Upsilon'(t) \leq -\lambda E(t). \tag{4.11}$$

Proof. By using (3.3) and (4.3)–(4.6), we have

$$
\begin{aligned}
\Upsilon'(t) &= \epsilon E'(t) + \frac{\beta_2^2}{12J\sigma}\chi_1'(t) + \frac{\beta_2^2}{12J\sigma}\chi_2'(t) + \chi_3'(t) + \frac{\beta_2}{4J\sigma}\chi_4'(t) \\
&\le \epsilon\left[-\kappa|\Theta_{tx}|_2^2 - a|\omega_t|_2^2\right] + \frac{\beta_2^2}{12J\sigma}\left[-\left(\frac{\mu}{2} - \mu p_0\right)|\omega_x|_2^2 + \frac{b^2}{2\mu}|\varphi|_2^2 + c|\omega_t|_2^2 + c|\Theta_{tx}|_2^2\right] \\
&\quad + \frac{\beta_2^2}{12J\sigma}\left[-\frac{\delta}{2}|\varphi_x|_2^2 - \frac{\xi}{2}|\varphi|_2^2 + J|\varphi_t|_2^2 + \frac{b^2}{2\xi}|\omega_x|_2^2 + c|\Theta_{tx}|_2^2\right] - \frac{\sigma}{2}|\Theta_x|_2^2 + \frac{\delta\beta_2^2}{48J\sigma}|\varphi_x|_2^2 \\
&\quad + c|\omega_t|_2^2 + c|\Theta_{tx}|_2^2 + \frac{\beta_2}{4J\sigma}\left[-\frac{J\beta_2}{2}|\varphi_t|_2^2 + \frac{3J\sigma^2}{2\beta_2}|\Theta_x|_2^2 + \epsilon_1|\varphi_x|_2^2 + \epsilon_2|\omega_x|_2^2 + \epsilon_3|\varphi|_2^2\right] \\
&\quad + c|\omega_t|_2^2 + c_{(\epsilon_1,\epsilon_2,\epsilon_3)}|\Theta_{tx}|_2^2 \\
&\le \left[-\frac{\beta_2^2}{12J\sigma}\left(\frac{\mu}{2} - \mu p_0\right) + \frac{b^2\beta_2^2}{24J\sigma\xi} + \frac{\beta_2}{4J\sigma}\epsilon_2\right]|\omega_x|_2^2 + \left(-\epsilon a + c\right)|\omega_t|_2^2 \\
&\quad \left[-\frac{\delta\beta_2^2}{48J\sigma} + \frac{\beta_2}{4J\sigma}\epsilon_1\right]|\varphi_x|_2^2 - \frac{\beta_2^2}{14\sigma}|\varphi_t|_2^2 + \left[-\frac{\beta_2^2}{24J\sigma}\left(\xi - \frac{b^2}{\mu}\right) + \frac{\beta_2}{4J\sigma}\epsilon_3\right]|\varphi|_2^2 \\
&\quad - \frac{\sigma}{8}|\Theta_x|_2^2 + \left(-\epsilon\kappa + c + c_{(\epsilon_1,\epsilon_2,\epsilon_3)}\right)|\Theta_{tx}|_2^2.
\end{aligned}
$$

Clearly, there exist $\epsilon_1, \epsilon_3 > 0$ such that

$$
-\frac{\delta\beta_2^2}{48J\sigma} + \frac{\beta_2}{4J\sigma}\epsilon_1 < 0 \qquad \text{and} \qquad -\frac{\beta_2^2}{24J\sigma}\left(\xi - \frac{b^2}{\mu}\right) + \frac{\beta_2}{4J\sigma}\epsilon_3 < 0. \tag{4.12}
$$

Now,

$$
-\frac{\beta_2^2}{12J\sigma}\left(\frac{\mu}{2} - \mu p_0\right) + \frac{b^2\beta_2^2}{24J\sigma\xi} = \frac{\beta_2^2\mu}{12J\sigma}\left[-\frac{1}{2} + \frac{1-\eta}{4} + \frac{\eta}{2}\right] = -\frac{\beta_2^2\mu}{12J\sigma}p_0 < 0,
$$

which guarantee the existences of $\epsilon_2 > 0$ such that

$$
-\frac{\beta_2^2}{12J\sigma}\left(\frac{\mu}{2} - \mu p_0\right) + \frac{b^2\beta_2^2}{24J\sigma\xi} + \frac{\beta_2}{4J\sigma}\epsilon_2 < 0, \tag{4.13}
$$

by fixing $\epsilon_1, \epsilon_2, \epsilon_3$ such that (4.12) and 4.13 are valid, then $c_{(\epsilon_1,\epsilon_2,\epsilon_3)}$ is a constant. Thus, we just take $\epsilon > 0$ bigger enough that the next three inequalities are true

$$
K_0 < \epsilon, \qquad c < a\epsilon, \qquad c + c_{(\epsilon_1,\epsilon_2,\epsilon_3)} < \epsilon\kappa,
$$

where K_0 has been defined in (4.2). From Poincare's inequality, there exists $\overline{\lambda} > 0$ such that

$$
\Upsilon'(t) \le -\overline{\lambda}\left(|\omega_x|_2^2 + |\omega_t|_2^2 + |\varphi_x|_2^2 + |\varphi_t|_2^2 + |\varphi|_2^2 + |\Theta_x|_2^2 + |\Theta_t|_2^2\right),
$$

and from the equivalence between the usual norm of $\mathcal{H}$ and $\|\cdot\|_{\mathcal{H}}$, we obtain the existence of $\lambda > 0$ such that

$$
\Upsilon'(t) \le -\lambda E(t).
$$

□

Consequently, our main result can be established.

Theorem 4.7 (Exponential decay)**.** *Let* $\rho, J, \alpha, \xi, \mu, \sigma, \delta, a, \beta_2$ *be positive constants and* β_1 *be non-negative. Let* $(\omega, \varphi, \Theta)$ *be the solution of* (2.1)–(2.3)*, then there exist constants* $M, r > 0$ *such that*

$$E(t) \leq Me^{-rt}, \quad \textit{for all } t \geq 0.$$

Proof. From (4.11) and (4.1), we obtain

$$\Upsilon'(t) \leq -\lambda E(t) \leq -\frac{\lambda}{K_1}\Upsilon(t).$$

Using Gronwall's inequality, one has

$$\Upsilon(t) \leq \Upsilon(0)e^{-rt},$$

where $r = \dfrac{\lambda}{K_1}$. Again, from (4.1), we have

$$E(t) \leq \frac{K_1 E(0)}{K_2} e^{-rt}.$$

Therefore, the proof of the theorem follows by taking $M = \dfrac{K_1 E(0)}{K_2}$. □

5. Conclusions

In this work, we studied the well-posedness and stability of the systems (2.1)–(2.3), which were derived from (1.1) and (1.2) by assuming the thermal effect on the stress tensor corresponding to the elastic matrix and the equilibrated stress, allowing the generalization of some current thermoelastic-porous models of type III.

In particular, it was proved that the system decays exponentially when the frictional damping coefficient a is positive, that is, when extra damping is considered in the first equation of the system. It is important to note that if the temperature effect does not directly affect the displacements in the body, i.e., if $\beta_1 = 0$, the result still holds. This confirms the results obtained in [4], with the main difference being that in this case, the authors consider $a = 0$ but assume the equality of velocities (1.3).

With respect to the techniques used to demonstrate the well-posedness of the system, a change of variable on φ was employed, which generated the equivalent system (2.4). This technique is an adaptation of what was used in [18], which was applied to the thermal part. It is worth mentioning that this technique has also been used previously in [4, 13], among others. Subsequently, the existence of solutions directly follows from the classical theory on semigroups and autonomous Cauchy problems. Regarding the exponential stability of the system, the multipliers technique was used. This allowed defining a Lyapunov functional, which was mainly controlled by the multipliers defined in Section 4.

As for future work, it is believed that by reconfiguring the thermal effect in the constitutive equations, exponential energy decay can be achieved directly without considering frictional damping on the displacements or the equality of velocities. Additionally, the model (2.1)–(2.3) may admit the presence of non-conservative structural forces, which would propose a new study focusing on the search for compact attracting sets in the system's dynamics.

Author contributions

All authors contributed to the manuscript equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that they have no conflicts of interest.

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