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Research article

Random uniform attractors for fractional stochastic FitzHugh-Nagumo lattice systems

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Abstract: The present study focuses on the asymptotic behavior of fractional stochastic FitzHugh-Nagumo lattice systems with multiplicative noise. First, we investigate the well-posedness of solutions for these stochastic systems and subsequently establish the existence and uniqueness of tempered random uniform attractors.

Keywords: fractional stochastic FitzHugh-Nagumo lattice systems; multiplicative noise; random uniform attractor; almost periodic symbol
Mathematics Subject Classification: 35B40, 35B41, 37L30

1. Introduction

In this paper, we investigate the dynamics of a fractional stochastic FitzHugh-Nagumo lattice system defined on the integer set $\mathbb{Z}$:

$$\begin{cases} \dot{\bar{u}}_i + (-\Delta_d)^s \bar{u}_i + \alpha \bar{v}_i = f_i(t, \bar{u}_i) + g_i(t) + \varepsilon \bar{u}_i \circ \dot{\omega}, \ t > 0, \\ \dot{\bar{v}}_i + \sigma \bar{v}_i - \beta \bar{u}_i = h_i(t) + \varepsilon \bar{v}_i \circ \dot{\omega}, \ t > 0, \\ \bar{u}_i(0) = \bar{u}_{i,0}, \bar{v}_i(0) = \bar{v}_{i,0}, \end{cases} \tag{1.1}$$

where $\bar{u}_i, \bar{v}_i \in \mathbb{R}$, $(-\Delta_d)^s$ is the fractional discrete Laplacian, $s \in (0, 1)$, α, σ, β are positive real constants, $f(t, u) = (f_i(t, u_i))_{i\in\mathbb{Z}}$ is a nonlinear function that satisfies certain conditions, the terms $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $h(t) = (h_i(t))_{i\in\mathbb{Z}}$ are time-dependent, while ω is a two-sided real-valued Wiener process. The system should be understood in the Stratonovich-integral sense.

Lattice systems with standard discrete Laplacian has been extensively investigated in the literature. Previous studies [1, 2] have investigated the existence of traveling wave solutions in such systems, while other relevant research mentioned in [3, 4] has also analyzed the chaotic properties associated with these solutions. For a comprehensive understanding of the asymptotic behavior of lattice systems, interested readers are referred to references [5–17]. The framework of the pullback random attractor proposed by Wang [18] effectively captures the dynamics of non-autonomous stochastic systems in the pullback sense. However, it lacks information regarding the desired forward dynamics. Therefore, Cui and Langa [19] introduced the concept of a random uniform attractor as a random generalization of deterministic uniform attractors that exhibit uniform attraction in symbols from a symbol space. Furthermore, Cui et al. [20] investigated the conditions that ensure a random uniform attractor possesses a finite fractal dimension. A random uniform attractor is defined as being pathwise pullback-attracting, while also exhibiting weak forward attraction in terms of probability. Recently, Abdallah conducted a study on the existence of the random uniform attractors within the set of tempered closed bounded random sets for a family of first-order stochastic non-autonomous lattice systems with multiplicative white noise in [21].

The fractional discrete Laplacian, which extensively explores the fractional powers of the discrete Laplacian, has been thoroughly investigated in previous studies [22, 23]. In [23], the discrete diffusion systems with fractional discrete Laplacian were examined, and the pointwise nonlocal formula and various properties associated with this operator were derived. Additionally, Schauder estimates in discrete Hölder spaces and the existence and uniqueness of solutions for the considered system were established. By employing the theories of analytic semigroups and cosine operators, the existence and uniqueness of solutions to the Schrödinger, wave, and heat systems with the fractional discrete Laplacian were successfully established in [24]. Recent studies have focused on exploring the existence, uniqueness, and upper semi-continuity of random attractors in fractional stochastic lattice systems with linear or nonlinear multiplicative noise [25, 26].

The FitzHugh-Nagumo systems were employed to describe the transmission of signals across axons in neurobiology [27]. The long-term dynamics of FitzHugh-Nagumo systems have been investigated in both deterministic scenarios [28–30] and stochastic scenarios [31–35]. Among these studies, Wang et al. [34] derived the existence and upper semi-continuity of random attractors for FitzHugh-Nagumo lattice systems with multiplicative noise in $\ell^2 \times \ell^2$, while Chen et al. [35] obtained the existence and uniqueness of weak pullback mean random attractors for FitzHugh-Nagumo lattice systems driven by nonlinear noise in weighted space $\ell^2_\sigma \times \ell^2_\sigma$.

However, as far as we know, there is no result available regarding the stochastic dynamics of fractional FitzHugh-Nagumo lattice systems with multiplicative noise. The main challenge of this paper lies in establishing the asymptotic compactness of solutions to system (1.1), which bears resemblance to the scenario encountered in stochastic partial differential equations on unbounded domains where Sobolev embedding is no longer compact. More precisely, we will demonstrate, using a cut-off technique, that the tails of solutions for system (1.1) remain uniformly small as time approaches infinity. This aspect will play a crucial role in establishing the asymptotic compactness of solutions. By leveraging the asymptotic compactness of solutions and uniformly absorbing sets, we can derive the existence and uniqueness of random uniform attractors.

This paper is structured as follows: In Section 2, we establish the conditions for the Hilbert space and state fundamental assumptions regarding the nonlinearity and forcing terms of the system (1.1). We also present several significant lemmas and properties that greatly facilitate the analysis of solutions throughout this paper. Additionally, we investigate the well-posedness of solutions to the system (1.1). In Section 3, we derive all the necessary uniform estimates of the solutions. Section 4 is dedicated to proving the existence and uniqueness of random uniform attractors for system (1.1).

2. Spaces and assumptions

In this section, we will introduce the appropriate spaces and assumptions regarding the linear and nonlinear parts of a fractional stochastic non-autonomous lattice system (1.1).

Consider the Hilbert space

$$\ell^2 = \Big\{u = (u_i)_{i\in\mathbb{Z}} | u_i \in \mathbb{R}, \sum_{i\in\mathbb{Z}} |u_i|^2 < +\infty\Big\},$$

with the inner product and norm given by

$$(u, v) = \sum_{i\in\mathbb{Z}} u_i v_i,\ \|u\|^2 = (u, u),\ u, v \in \ell^2.$$

For $0 \leq s \leq 1$, define ℓ_s by

$$\ell_s = \{u : \mathbb{Z} \to \mathbb{R} | \|u\|_{\ell_s} := \sum_{i\in\mathbb{Z}} \frac{|u_i|}{(1 + |i|)^{1+2s}} < +\infty\}.$$

Obviously, $\ell^m \subset \ell^n \subset \ell_s$ if $1 \leq m \leq n \leq +\infty$ and $0 \leq s \leq 1$.

The fractional discrete Laplacian $(-\Delta_d)^s$ simplifies to the discrete Laplacian $-\Delta_d$ if $s = 1$. For $i \in \mathbb{Z}$, the discrete Laplacian $-\Delta_d$ is given by

$$-\Delta_d u_i = 2u_i - u_{i-1} - u_{i+1}.$$

For $1 < s < 1$ and $u_j \in \mathbb{R}$, the fractional discrete Laplacian $(-\Delta_d)^s$ is defined with the semigroup method in [36] as

$$(-\Delta_d)^s u_j = \frac{1}{\Gamma(-s)} \int_0^{+\infty} (e^{t\Delta_d} u_j - u_j) \frac{dt}{t^{1+s}}, \tag{2.1}$$

where Γ is the Gamma function with $\Gamma(-s) = -\frac{1}{s} \int_0^{+\infty} r^{-s} e^{-r} dr < 0$ and $v_j(t) = e^{t\Delta_d} u_j$ is the solution for the semidiscrete heat system

$$\begin{cases} \partial_t v_j = \Delta_d v_j, \text{ in } \mathbb{Z} \times (0, +\infty), \\ v_j(0) = u_j, \text{ on } \mathbb{Z}. \end{cases} \tag{2.2}$$

The solution to system (2.2) can be expressed by the semidiscrete Fourier transform

$$e^{t\Delta_d} u_j = \sum_{i\in\mathbb{Z}} G(j - i, t) u_i = \sum_{i\in\mathbb{Z}} G(i, t) u_{j-i},\ t \geq 0, \tag{2.3}$$

where the semidiscrete heat kernel $G(i, t)$ is defined as $e^{-2t} I_i(2t)$, and I_i represents the modified Bessel function of order i. Subsequently, the pointwise formula for $(-\Delta_d)^s$ has been presented as follows:

Lemma 2.1. *([23], Lemma 2.3) Let* $0 < s < 1$ *and* $u = (u_i)_{i\in\mathbb{Z}} \in \ell_s$. *Then, we have*

$$(-\Delta_d)^s u_i = \sum_{j\in\mathbb{Z}, j\neq i} (u_i - u_j) \tilde{K}_s(i - j),$$

where the discrete kernel $\tilde{K}_s$ is given by

$$\tilde{K}_s(j) = \begin{cases} \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}|\Gamma(-s)|} \cdot \frac{\Gamma(|j|-s)}{\Gamma(|j|+1+s)}, & j \in \mathbb{Z} \setminus \{0\}, \\ 0, & j = 0. \end{cases}$$

In addition, there exist positive constants $\check{c}_s \leq \hat{c}_s$ such that for any $j \in \mathbb{Z} \setminus \{0\}$,

$$\frac{\check{c}_s}{|j|^{1+2s}} \leq \tilde{K}_s(j) \leq \frac{\hat{c}_s}{|j|^{1+2s}}.$$

In this paper, we will consider the probability space $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Define the time shift by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \ t \in \mathbb{R}. \tag{2.4}$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system (refer to [37] for details).

Moreover, let us consider the stochastic equation as follows:

$$dz + zdt = d\omega. \tag{2.5}$$

In fact, the following lemma can be obtained:

Lemma 2.2. *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\Omega' \in \mathcal{F}$ of full measure such that*

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{t} \to 0 \quad \text{for all} \quad \omega \in \Omega', \tag{2.6}$$

and the random variable given by

$$z(\omega) = -\int_{-\infty}^{0} e^s\omega(s)ds$$

is well defined. Moreover, for $\omega \in \Omega'$, the mapping

$$(t, \omega) \to z(\theta_t\omega) = -\int_{-\infty}^{0} e^s\theta_t\omega(s)ds = -\int_{-\infty}^{0} e^s\omega(t+s)ds + \omega(t)$$

is a stationary solution of (2.5) with continuous trajectories. In addition, for $\omega \in \Omega'$,

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{t} \to 0, \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0. \tag{2.7}$$

For all $v = (v_i)_{i\in\mathbb{Z}}$, $v_i \in \mathbb{R}$, let p and q be two given functions satisfying the following conditions:

$$p(v) = (p_i(v_i))_{i\in\mathbb{Z}}, \ q(v) = (q_i(v_i))_{i\in\mathbb{Z}}, \ p_i, q_i \in C(\mathbb{R}, (0, +\infty)), \tag{2.8}$$

and there are continuous functions $L_1, L_2 : \mathbb{R}^+ \to (0, +\infty)$ such that for $s \geq 0$,

$$\|p(v)\| \leq L_1(s), \forall \|v\| \leq s, \quad q_i(r) \leq L_2(s), \forall i \in \mathbb{Z}, |r| \leq s. \tag{2.9}$$

For all $v = (v_i)_{i\in\mathbb{Z}}, v_i \in \mathbb{R}$, let W be the set of functions ϕ with

$$\phi(v) = (\phi_i(v_i))_{i\in\mathbb{Z}}, \ \phi_i \in C'(\mathbb{R}, \mathbb{R}), \ \phi_i(0) = 0, \forall i \in \mathbb{Z}, \tag{2.10}$$

and

$$\sup_{i\in\mathbb{Z}} \sup_{r\in\mathbb{R}} \Big(\frac{|\phi_i(r)|}{p_i(r)} + \frac{|\phi_i'(r)|}{q_i(r)} \Big) < +\infty. \tag{2.11}$$

By using the similar proof of Lemma 5.2 in [5], we obtain the following lemma:

Lemma 2.3. *W is a real Banach space with norm given by*

$$\|\phi\|_W = \sup_{i\in\mathbb{Z}} \sup_{r\in\mathbb{R}} \Big(\frac{|\phi_i(r)|}{p_i(r)} + \frac{|\phi_i'(r)|}{q_i(r)} \Big), \forall \phi \in W.$$

Moreover, we assume that

(H_1) $g_0, h_0 : \mathbb{R} \to \ell^2$ with $g_0(t) = (g_{0i}(t))_{i\in\mathbb{Z}}, h_0(t) = (h_{0i}(t))_{i\in\mathbb{Z}}$ are almost periodic functions of t.

(H_2) $f_0(t, u) = (f_{0i}(t, u_i))_{i\in\mathbb{Z}}$ is a nonlinear function of $t \in \mathbb{R}$ and $u = (u_i)_{i\in\mathbb{Z}}, u_i \in \mathbb{R}$ with $f_{0i} : \mathbb{R} \times \mathbb{R}$ such that $f_0(t, \cdot)$ is an almost periodic function of t with values in W and

$$f_{0i}(t, r)r \leq -\lambda r^2, \quad t, r \in \mathbb{R}, i \in \mathbb{Z} \tag{2.12}$$

for some $\lambda > 0$.

The space $C_b(\mathbb{R}, X)$ represents the Banach space of bounded continuous functions on $\mathbb{R}$, where the functions take values in a Banach space X and are equipped with a norm given by

$$\|\xi\|_{C_b(\mathbb{R},X)} = \sup_{t\in\mathbb{R}} \|\xi(t)\|_X, \ \xi \in C_b(\mathbb{R}, X). \tag{2.13}$$

Let $\xi_0 : \mathbb{R} \to X$ be an almost periodic function of t with values in X. By Bochner's criterion in [38], the set of translations $\{\xi_0(\cdot + s) : s \in \mathbb{R}\}$ is precompact in $C_b(\mathbb{R}, X)$. The closure of this set in $C_b(\mathbb{R}, X)$ is referred to as the hull $\mathcal{H}(\xi_0)$ of the function $\xi_0(t)$, i.e.,

$$\mathcal{H}(\xi_0) = \overline{\{\xi_0(\cdot + s) : s \in \mathbb{R}\}} \subset\subset C_b(\mathbb{R}, X). \tag{2.14}$$

Furthermore, for any $\xi(t) \in \mathcal{H}(\xi_0)$, ξ is almost periodic in X, and $\mathcal{H}(\xi_0) = \mathcal{H}(\xi)$.

In this study, we consider the time symbol $\xi_0(t) = (g_0(t), f_0(t, u), h_0(t))$, where g_0, f_0, and h_0 are determined by assumptions (H_1) and (H_2). It is observed that $\xi_0(t)$ exhibits almost periodic behavior with values in $\ell^2 \times W \times \ell^2$. Subsequently, we focus on the symbol space $\Sigma = \mathcal{H}(g_0) \times \mathcal{H}(f_0) \times \mathcal{H}(h_0)$, which is compact in $C_b(\mathbb{R}, \ell^2) \times C_b(\mathbb{R}, W) \times C_b(\mathbb{R}, \ell^2)$. Under these conditions, $\xi(t) = (\xi_1(t), \xi_2(t)) = ((g(t), f(t, u)), h(t)) \in \Sigma$ also demonstrates almost periodicity in $\ell^2 \times W \times \ell^2$ and $\Sigma = \mathcal{H}(g) \times \mathcal{H}(f) \times \mathcal{H}(h)$.

The objective of this study is to investigate the existence of random uniform attractors with respect to $\xi(t) = (g(t), f(t,u), h(t)) \in \Sigma$ for the fractional stochastic FitzHugh-Nagumo lattice system with multiplicative white noise as follows:

$$\begin{cases} \dot{\bar{u}} + (-\Delta_d)^s \bar{u} + \alpha \bar{v} = f(t,\bar{u}) + g(t) + \varepsilon \bar{u} \circ \dot{\omega},\ t > 0, \\ \dot{\bar{v}} + \sigma \bar{v} - \beta \bar{u} = h(t) + \varepsilon \bar{v} \circ \dot{\omega},\ t > 0, \\ \bar{u}(0) = \bar{u}_0, \bar{v}(0) = \bar{v}_0, \end{cases} \tag{2.15}$$

where $\bar{u} = (\bar{u}_i)_{i\in\mathbb{Z}}, \bar{v} = (\bar{v}_i)_{i\in\mathbb{Z}}, f(t,\bar{u}) = (f_i(t,\bar{u}_i))_{i\in\mathbb{Z}}, g(t) = (g_i(t))_{i\in\mathbb{Z}}$, and $h(t) = (h_i(t))_{i\in\mathbb{Z}}$.

Furthermore, it can be deduced from Lemma 2.1 that the fractional discrete Laplacian $(-\Delta_d)^s u$ is a nonlocal operator on $\mathbb{Z}$, and $(-\Delta_d)^s u$ is a well-defined bounded function wherever $u \in \ell^q (1 \le q \le +\infty)$. In particular, we obtain that, for $0 < s < 1$, if $u \in \ell^2$, then

$$(-\Delta_d)^s u \in \ell^2 \ \text{ satisfying } \ \|(-\Delta_d)^s u\| \le 4^s \|u\|. \tag{2.16}$$

The subsequent lemma will be repeatedly utilized in various estimations of solutions to system (1.1).

Lemma 2.4. *([26], Lemma 2.3) Let $u, v \in \ell^2$. Then, for every $s \in (0,1)$,*

$$\Big((-\Delta_d)^s u, v\Big) = \Big((-\Delta_d)^{\frac{s}{2}} u, (-\Delta_d)^{\frac{s}{2}} v\Big) = \frac{1}{2} \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}, j\neq i} (u_i - u_j)(v_i - v_j) \tilde{K}_s(i-j).$$

3. Continuous NRDS and uniform absorbing set

In this section, for $\omega \in \Omega$, $\xi = (\xi_1, \xi_2) = ((g,f),h) \in \Sigma$ and initial conditions $(u_0, v_0) \in \ell^2 \times \ell^2$, we aim to demonstrate the existence of global solutions $(u(t,\omega,\xi_1,u_0), v(t,\omega,\xi_2,v_0))$ to system (2.15) by transforming the original random system into a deterministic one. Specifically, we introduce the non-autonomous random dynamical system (NRDS) $\Psi : \mathbb{R}^+ \times \Omega \times \Sigma \times \ell^2 \times \ell^2 \to \ell^2 \times \ell^2$ associated with system (2.15), which is jointly continuous in both Σ and $\ell^2 \times \ell^2$ and possesses a closed random uniformly $\mathcal{D}$-(pullback) absorbing set.

By using the similar proof of Lemma 3.1 in [30], we obtain the following lemma:

Lemma 3.1. *Suppose assumptions* (H_1) *and* (H_2) *hold. For $(\xi_1, \xi_2) \in \Sigma$, there exist non-negative constants $\delta_1(g_0)$, $\delta_2(f_0)$, and $\delta_3(h_0)$ such that*

$$\|g\|_{C_b(\mathbb{R},\ell^2)} = \sup_{t\in\mathbb{R}} \|g(t)\| = \delta_1(g_0), \ \ \|h\|_{C_b(\mathbb{R},\ell^2)} = \sup_{t\in\mathbb{R}} \|h(t)\| = \delta_2(h_0), \tag{3.1}$$

and for $t, r \in \mathbb{R}$, $i \in \mathbb{Z}$,

$$f_i(t,r)r \le -\lambda r^2, \tag{3.2}$$

$$|f_i(t,r)| \le \delta_3(f_0) p_i(r), \tag{3.3}$$

$$\Big|\frac{\partial f_i(t,r)}{\partial r}\Big| \le \delta_3(f_0) q_i(r). \tag{3.4}$$

Next, we will define NRDS Ψ for system (2.15). To this end, we need to transform the stochastic system (2.15) into a deterministic one through the utilization of $z(\theta_t\omega)$. Let

$$\begin{aligned} u(t,\omega,\xi_1,u_0(\omega)) &= e^{-\varepsilon z(\theta_t\omega)}\bar{u}(t,\omega,\xi_1,\bar{u}_0(\omega)),\\ v(t,\omega,\xi_2,v_0(\omega)) &= e^{-\varepsilon z(\theta_t\omega)}\bar{v}(t,\omega,\xi_2,\bar{v}_0(\omega)), \end{aligned} \tag{3.5}$$

where $(\bar{u},\bar{v})$ is a solution of system (2.15), $u_0(\omega) = e^{-\varepsilon z(\omega)}\bar{u}_0(\omega)$ and $v_0(\omega) = e^{-\varepsilon z(\omega)}\bar{v}_0(\omega)$. Then (u,v) satisfies

$$\begin{cases} \dot{u} + (-\Delta_d)^s u - \varepsilon z(\theta_t\omega)u + \alpha v = e^{-\varepsilon z(\theta_t\omega)} f(t, e^{\varepsilon z(\theta_t\omega)}u) + e^{-\varepsilon z(\theta_t\omega)} g(t),\ t>0,\\ \dot{v} + (\sigma - \varepsilon z(\theta_t\omega))v - \beta u = e^{-\varepsilon z(\theta_t\omega)} h(t),\ t>0,\\ u(0)=u_0, v(0)=v_0. \end{cases} \tag{3.6}$$

According to Lemma 4.4 of [21], for $f \in \mathcal{H}(f_0) \subset\subset C_b(\mathbb{R}, W)$, it can be inferred that the function $f : \mathbb{R}\times\ell^2 \to \ell^2$ with $(t,u) \to f(t, e^{\varepsilon z(\theta_t\omega)}u)$ is a continuous function of t. Moreover, for $R>0$ and $T>0$, $u^1 = (u_i^1)_{i\in\mathbb{Z}}, u^2 = (u_i^2)_{i\in\mathbb{Z}} \in \ell^2$ satisfying $\|u^1\| \le R$, $\|u^2\| \le R$, and $t\in[0,T]$, we obtain that

$$\|f(t, e^{\varepsilon z(\theta_t\omega)}u^1) - f(t, e^{\varepsilon z(\theta_t\omega)}u^2)\| \le \delta_3(f_0)L_2(e^{a_1(T)}R)e^{a_1(T)}\|u^1 - u^2\|, \tag{3.7}$$

where $a_1(T) = \max_{t\in[0,T]} |\varepsilon z(\theta_t\omega)|$. Then, it follows from (2.16), (3.7), and the standard theory of ordinary differential equations that system (3.6) has a unique local solution $(u(t), v(t)) \in C([0,T], \ell^2\times\ell^2)$ for some $T>0$. The following estimates show that this local solution is actually defined globally.

Lemma 3.2. *For every $\omega\in\Omega$, $(\xi_1,\xi_2)\in\Sigma$, and $(u_0(\omega), v_0(\omega)) \in \ell^2\times\ell^2$, then the solution $(u(t,\omega,\xi_1,u_0(\omega)), v(t,\omega,\xi_2,v_0(\omega)))$ of system (3.6) satisfies*

$$\begin{aligned} \|u(t,\omega,\xi_1,u_0(\omega))\|^2 + \|v(t,\omega,\xi_2,v_0(\omega))\|^2 &\le e^{-\kappa t + 2\varepsilon\int_0^t z(\theta_l\omega)dl}(\|u_0(\omega)\|^2 + \|v_0(\omega)\|^2)\\ &\quad + \frac{2}{\gamma\kappa}(\beta(\delta_1(g_0))^2 + \alpha(\delta_2(h_0))^2)\int_0^t e^{\kappa(r-t) - 2\varepsilon\int_t^r z(\theta_l\omega)dl - 2\varepsilon z(\theta_r\omega)}dr. \end{aligned}$$

Proof. By (3.6), we have

$$\begin{aligned} &\frac{\beta}{2}\frac{d}{dt}\|u\|^2 + \frac{\alpha}{2}\frac{d}{dt}\|v\|^2 + \beta\Big((-\triangle_d)^s u, u\Big) - \beta\varepsilon z(\theta_t\omega)\|u\|^2 + \alpha(\sigma - \varepsilon z(\theta_t\omega))\|v\|^2\\ &= \beta e^{-\varepsilon z(\theta_t\omega)}\Big(f(t, e^{\varepsilon z(\theta_t\omega)}u), u\Big) + \beta e^{-\varepsilon z(\theta_t\omega)}\Big(g(t), u\Big) + \alpha e^{-\varepsilon z(\theta_t\omega)}\Big(h(t), v\Big). \end{aligned} \tag{3.8}$$

By Lemma 2.4, we obtain

$$\beta\Big((-\triangle_d)^s u, u\Big) = \beta\Big((-\triangle_d)^{\frac{s}{2}}u, (-\triangle_d)^{\frac{s}{2}}u\Big) = \beta\|(-\triangle_d)^{\frac{s}{2}}u\|^2. \tag{3.9}$$

By (3.2), we find

$$\beta e^{-\varepsilon z(\theta_t\omega)}\Big(f(t, e^{\varepsilon z(\theta_t\omega)}u), u\Big) = \beta e^{-2\varepsilon z(\theta_t\omega)}\Big(f(t, e^{\varepsilon z(\theta_t\omega)}u), e^{\varepsilon z(\theta_t\omega)}u\Big) \le -\lambda\beta\|u\|^2. \tag{3.10}$$

By (3.1) and Young's inequality, we obtain

$$\begin{aligned}
&\beta e^{-\varepsilon z(\theta_t\omega)}\Big(g(t),u\Big)+\alpha e^{-\varepsilon z(\theta_t\omega)}\Big(h(t),v\Big)\\
&\leq \frac{\lambda\beta}{4}\|u\|^2+\frac{\beta}{\lambda}e^{-2\varepsilon z(\theta_t\omega)}\|g(t)\|^2+\frac{\sigma\alpha}{4}\|v\|^2+\frac{\alpha}{\sigma}e^{-2\varepsilon z(\theta_t\omega)}\|h(t)\|^2\\
&\leq \frac{\lambda\beta}{4}\|u\|^2+\frac{\beta}{\lambda}e^{-2\varepsilon z(\theta_t\omega)}(\delta_1(g_0))^2+\frac{\sigma\alpha}{4}\|v\|^2+\frac{\alpha}{\sigma}e^{-2\varepsilon z(\theta_t\omega)}(\delta_2(h_0))^2.
\end{aligned}\tag{3.11}$$

It follows from (3.8)–(3.11) that

$$\begin{aligned}
&\frac{d}{dt}(\beta\|u\|^2+\alpha\|v\|^2)+(\kappa-2\varepsilon z(\theta_t\omega))(\beta\|u\|^2+\alpha\|v\|^2)+2\beta\|(-\triangle_d)^{\frac{s}{2}}u\|^2+\frac{\lambda\beta}{2}\|u\|^2\\
&\leq \frac{2e^{-2\varepsilon z(\theta_t\omega)}}{\kappa}(\beta(\delta_1(g_0))^2+\alpha(\delta_2(h_0))^2),
\end{aligned}\tag{3.12}$$

where $\kappa=\min\{\lambda,\sigma\}$. Let $\gamma=\min\{\alpha,\beta\}$, multiplying (3.12) by $e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl}$, we have

$$\begin{aligned}
&\frac{d}{dt}\Big(e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl}(\|u\|^2+\|v\|^2)\Big)+e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl}\Big(2\|(-\triangle_d)^{\frac{s}{2}}u\|+\frac{\lambda}{2}\|u\|^2\Big)\\
&\leq e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl-2\varepsilon z(\theta_t\omega)}\frac{2}{\gamma\kappa}(\beta(\delta_1(g_0))^2+\alpha(\delta_2(h_0))^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|u(t,\omega,\xi_1,u_0(\omega))\|^2+\|v(t,\omega,\xi_2,v_0(\omega))\|^2\\
&\quad+2\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_l\omega)dl}\|(-\triangle_d)^{\frac{s}{2}}u(r,\omega,\xi_1,u_0(\omega))\|^2dr\\
&\quad+\frac{\lambda}{2}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_l\omega)dl}\|u(r,\omega,\xi_1,u_0(\omega))\|^2dr\\
&\leq e^{-\kappa t+2\varepsilon\int_0^t z(\theta_l\omega)dl}(\|u_0(\omega)\|^2+\|v_0(\omega)\|^2)\\
&\quad+\frac{2}{\gamma\kappa}(\beta(\delta_1(g_0))^2+\alpha(\delta_2(h_0))^2)\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_l\omega)dl-2\varepsilon z(\theta_r\omega)}dr.
\end{aligned}\tag{3.13}$$

This completes the proof. □

For $\omega\in\Omega$, $(\xi_1,\xi_2)\in\Sigma$, and $(u_0,v_0)\in\ell^2\times\ell^2$, by Lemma 3.2, we get that the solution $(u(t,\omega,\xi_1,u_0(\omega)),v(t,\omega,\xi_2,v_0(\omega)))$ to the system (3.6) is defined globally in $\ell^2\times\ell^2$ which is measurable. Then, system (3.6) generates the NRDS $\Phi:\mathbb{R}^+\times\Omega\times\Sigma\times\ell^2\times\ell^2\to\ell^2\times\ell^2$, where

$$\Phi(t,\omega,\xi,u_0(\omega),v_0(\omega))=(u(t,\omega,\xi_1,u_0(\omega)),v(t,\omega,\xi_2,v_0(\omega))).\tag{3.14}$$

By the fact of (3.5), the system (2.15) generates the NRDS $\Psi:\mathbb{R}^+\times\Omega\times\Sigma\times\ell^2\times\ell^2\to\ell^2\times\ell^2$, where

$$\begin{aligned}
\Psi(t,\omega,\xi,\bar{u}_0(\omega),\bar{v}_0(\omega))&=(\bar{u}(t,\omega,\xi_1,\bar{u}_0(\omega)),\bar{v}(t,\omega,\xi_2,\bar{v}_0(\omega)))\\
&=e^{\varepsilon z(\theta_t\omega)}(u(t,\omega,\xi_1,u_0(\omega)),v(t,\omega,\xi_2,v_0(\omega))),
\end{aligned}\tag{3.15}$$

where $(u_0(\omega),v_0(\omega))=e^{-\varepsilon z(\omega)}(\bar{u}_0(\omega),\bar{v}_0(\omega))$.

Lemma 3.3. *The NRDS Ψ associated with system (2.15) is jointly continuous in Σ and $\ell^2 \times \ell^2$, i.e., for $\omega \in \Omega$ and $t \geq 0$, the mapping $(\xi, \bar{u}_0(\omega), \bar{v}_0(\omega)) \to \Psi(t, \omega, \xi, \bar{u}_0(\omega), \bar{v}_0(\omega))$ is continuous from $\Sigma \times \ell^2 \times \ell^2$ into $\ell^2 \times \ell^2$.*

Proof. For $\omega \in \Omega$, $t \geq 0$, $r \in [0, t]$, and $n = 1, 2$, let $(u^n(r, \omega, \xi_1^n, u_0^n(\omega)), v^n(r, \omega, \xi_2^n, v_0^n(\omega)))$ be the solution of system (3.6) with symbol $\xi^n = (\xi_1^n, \xi_2^n) = ((g^n, f^n), h^n)$ and initial data $(u^n(0), v^n(0)) = (u_0^n(\omega), v_0^n(\omega))$. Set

$$(\tilde{u}(r), \tilde{v}(r)) = (u^1(r, \omega, \xi_1^1, u_0^1(\omega)) - u^2(r, \omega, \xi_1^2, u_0^2(\omega)), v^1(r, \omega, \xi_2^1, v_0^1(\omega)) - v^2(r, \omega, \xi_2^2, v_0^2(\omega))).$$

By system (3.6), we obtain

$$\begin{cases} \dot{\tilde{u}} + (-\Delta_d)^s \tilde{u} - \varepsilon z(\theta_r \omega)\tilde{u} + \alpha \tilde{v} = e^{-\varepsilon z(\theta_r \omega)}(f^1(r, e^{\varepsilon z(\theta_r \omega)} u^1) - f^2(r, e^{\varepsilon z(\theta_r \omega)} u^2) + g^1(r) - g^2(r)), \\ \dot{\tilde{v}} + (\sigma - \varepsilon z(\theta_r \omega))\tilde{v} - \beta \tilde{u} = e^{-\varepsilon z(\theta_r \omega)}(h^1(r) - h^2(r)), \end{cases}$$

which implies that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}(\beta\|\tilde{u}\|^2 + \alpha\|\tilde{v}\|^2) + \beta\Big((-\Delta_d)^s \tilde{u}, \tilde{u}\Big) - \beta\varepsilon z(\theta_r\omega)\|\tilde{u}\|^2 + \alpha(\sigma - \varepsilon z(\theta_r\omega))\|\tilde{v}\|^2 \\ &= \beta e^{-\varepsilon z(\theta_r\omega)}\Big(f^1(r, e^{\varepsilon z(\theta_r\omega)}u^1) - f^2(r, e^{\varepsilon z(\theta_r\omega)}u^2), \tilde{u}\Big) + \beta e^{-\varepsilon z(\theta_r\omega)}\Big(g^1(r) - g^2(r), \tilde{u}\Big) \\ &\quad + \alpha e^{-\varepsilon z(\theta_r\omega)}\Big(h^1(r) - h^2(r), \tilde{v}\Big). \end{aligned} \tag{3.16}$$

Let $a_2 = a_2(\omega) = \max\{\|u_0^1(\omega)\|^2 + \|v_0^1(\omega)\|^2, \|u_0^2(\omega)\|^2 + \|v_0^2(\omega)\|^2\}$. By (3.13), we obtain

$$\|u^n\|^2 + \|v^n\|^2 \leq a_3, n = 1, 2, \ \|\tilde{u}\|^2 + \|\tilde{v}\|^2 \leq 2a_3, \tag{3.17}$$

where

$$\begin{aligned} a_3 = a_3(t, \omega) = &a_2 \max_{r\in[0,t]} e^{-\kappa r + 2\varepsilon \int_0^r z(\theta_l \omega)dl} \\ &+ \frac{2}{\gamma\kappa}(\beta(\delta_1(g_0))^2 + \alpha(\delta_2(h_0))^2) \max_{r\in[0,t]} \int_0^r e^{\kappa(s-r) - 2\varepsilon\int_r^s z(\theta_l\omega)dl - 2\varepsilon z(\theta_s\omega)} ds. \end{aligned} \tag{3.18}$$

By (2.9) and (3.4), we have

$$\begin{aligned} &\beta e^{-\varepsilon z(\theta_r\omega)}\Big|\Big(f^1(r, e^{\varepsilon z(\theta_r\omega)}u^1) - f^2(r, e^{\varepsilon z(\theta_r\omega)}u^2), \tilde{u}\Big)\Big| \\ &\leq \beta e^{-\varepsilon z(\theta_r\omega)}\|\tilde{u}\|\|f^2(r, e^{\varepsilon z(\theta_r\omega)}u^2) - f^2(r, e^{\varepsilon z(\theta_r\omega)}u^1)\| \\ &\quad + \beta e^{-\varepsilon z(\theta_r\omega)}\|\tilde{u}\|\|f^2(r, e^{\varepsilon z(\theta_r\omega)}u^1) - f^1(r, e^{\varepsilon z(\theta_r\omega)}u^1)\| \\ &\leq \beta\delta_3(f_0)L_2(\sqrt{2a_3}e^{\varepsilon z(\theta_r\omega)})\|\tilde{u}\|^2 + \sqrt{2a_3}\beta e^{-\varepsilon z(\theta_r\omega)}\|f^2(r, e^{\varepsilon z(\theta_r\omega)}u^1) - f^1(r, e^{\varepsilon z(\theta_r\omega)}u^1)\|. \end{aligned} \tag{3.19}$$

Note that

$$\begin{aligned} &\beta e^{-\varepsilon z(\theta_r\omega)}\Big(g^1(r) - g^2(r), \tilde{u}\Big) + \alpha e^{-\varepsilon z(\theta_r\omega)}\Big(h^1(r) - h^2(r), \tilde{v}\Big) \\ &\leq \sqrt{2a_3}e^{-\varepsilon z(\theta_r\omega)}(\beta\|g^1(r) - g^2(r)\| + \alpha\|h^1(r) - h^2(r)\|). \end{aligned} \tag{3.20}$$

It follows from (3.16) and (3.19)–(3.20) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}(\beta\|\tilde{u}\|^2+\alpha\|\tilde{v}\|^2)-\beta\varepsilon z(\theta_r\omega)\|\tilde{u}\|^2+\alpha(\sigma-\varepsilon z(\theta_r\omega))\|\tilde{v}\|^2\\ &\le \beta\delta_3(f_0)L_2(\sqrt{2a_3}e^{\varepsilon z(\theta_r\omega)})\|\tilde{u}\|^2+\sqrt{2a_3}\beta e^{-\varepsilon z(\theta_r\omega)}\|f^2(r,e^{\varepsilon z(\theta_r\omega)}u^1)-f^1(r,e^{\varepsilon z(\theta_r\omega)}u^1)\|\\ &\quad+\sqrt{2a_3}e^{-\varepsilon z(\theta_r\omega)}(\beta\|g^1(r)-g^2(r)\|+\alpha\|h^1(r)-h^2(r)\|). \end{aligned} \tag{3.21}$$

Let $\gamma=\min\{\alpha,\beta\}$, $a_4=a_4(t,\omega)=\frac{2\beta}{\gamma}\max_{r\in[0,t]}\Big(\varepsilon|z(\theta_r\omega)|+\delta_3(f_0)L_2(\sqrt{2a_3}e^{\varepsilon z(\theta_r\omega)})\Big)+\frac{2\alpha}{\gamma}\max_{r\in[0,t]}\Big(\sigma+\varepsilon|z(\theta_r\omega)|\Big)$, $a_5=\frac{2\sqrt{2a_3}}{\gamma}\max_{r\in[0,t]}e^{\varepsilon|z(\theta_r\omega)|}$. Then, by (3.21), we obtain

$$\begin{aligned} \frac{d}{dr}(\|\tilde{u}\|^2+\|\tilde{v}\|^2)-a_4(\|\tilde{u}\|^2+\|\tilde{v}\|^2)\le a_5\beta\|f^2(r,e^{\varepsilon z(\theta_s\omega)}u^1)-f^1(r,e^{\varepsilon z(\theta_r\omega)}u^1)\|\\ +a_5(\beta\|g^1(r)-g^2(r)\|+\alpha\|h^1(r)-h^2(r)\|), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{d}{dr}[e^{-a_4r}(\|\tilde{u}\|^2+\|\tilde{v}\|^2)]\le a_5\beta e^{-a_4r}\|f^2(r,e^{\varepsilon z(\theta_r\omega)}u^1)-f^1(r,e^{\varepsilon z(\theta_r\omega)}u^1)\|\\ +a_5e^{-a_4r}(\beta\|g^1(r)-g^2(r)\|+\alpha\|h^1(r)-h^2(r)\|). \end{aligned}$$

Integrating both sides of the above inequality from 0 into t, we obtain

$$\begin{aligned} \|\tilde{u}(t)\|^2+\|\tilde{v}(t)\|^2\le& e^{a_4t}(\|\tilde{u}(0)\|^2+\|\tilde{v}(0)\|^2)+\frac{a_5\beta}{a_4}e^{a_4t}\max_{r\in[0,t]}\|f^2(r,e^{\varepsilon z(\theta_r\omega)}u^1(r))-f^1(r,e^{\varepsilon z(\theta_r\omega)}u^1(r))\|\\ &+\frac{a_5}{a_4}e^{a_4t}\max_{r\in[0,t]}(\beta\|g^1(r)-g^2(r)\|+\alpha\|h^1(r)-h^2(r)\|). \end{aligned} \tag{3.22}$$

By (2.9) and (3.17), we have

$$\begin{aligned} &\sup_{r\in[0,t]}\|f^2(r,e^{\varepsilon z(\theta_r\omega)}u^1(r))-f^1(r,e^{\varepsilon z(\theta_r\omega)}u^1(r))\|^2\\ &=\sup_{r\in[0,t]}\sum_{i\in\mathbb{Z}}\left|p_i(e^{\varepsilon z(\theta_r\omega)}u_i^1(r))\right|^2\frac{|f_i^2(r,e^{\varepsilon z(\theta_r\omega)}u_i^1(r))-f_i^1(r,e^{\varepsilon z(\theta_r\omega)}u_i^1(r))|^2}{\left|p_i(e^{\varepsilon z(\theta_r\omega)}u_i^1(r))\right|^2}\\ &\le\sup_{r\in\mathbb{R}}\sup_{i\in\mathbb{Z}}\sup_{\upsilon\in\mathbb{R}}\frac{|f_i^2(r,\upsilon)-f_i^1(r,\upsilon)|^2}{|p_i(\upsilon)|^2}\sup_{r\in[0,t]}\sum_{i\in\mathbb{Z}}\left|p_i(e^{\varepsilon z(\theta_r\omega)}u_i^1(r))\right|^2\\ &\le\|f^2-f^1\|^2_{C_b(\mathbb{R},W)}\sup_{r\in[0,t]}\|p(e^{\varepsilon z(\theta_r\omega)}u^1(r))\|^2\le\Big(L_1\Big(\frac{a_5\gamma}{2\sqrt{2}}\Big)\Big)^2\|f^2-f^1\|^2_{C_b(\mathbb{R},W)}. \end{aligned} \tag{3.23}$$

Observe that

$$\begin{aligned} \sup_{r\in[0,t]}(\beta\|g^1(r)-g^2(r)\|+\alpha\|h^1(r)-h^2(r)\|)&\le\sup_{r\in\mathbb{R}}(\beta\|g^1(r)-g^2(r)\|+\alpha\|h^1(r)-h^2(r)\|)\\ &\le\beta\|g^1-g^2\|_{C_b(\mathbb{R},\ell^2)}+\alpha\|h^1-h^2\|_{C_b(\mathbb{R},\ell^2)}. \end{aligned} \tag{3.24}$$

It follows from (3.22)–(3.24) that

$$\begin{aligned} \|\tilde{u}(t)\|^2+\|\tilde{v}(t)\|^2\le& e^{a_4t}(\|\tilde{u}(0)\|^2+\|\tilde{v}(0)\|^2)+\frac{a_5\beta}{a_4}e^{a_4t}\Big(L_1\Big(\frac{a_5\gamma}{2\sqrt{2}}\Big)\Big)^2\|f^2-f^1\|^2_{C_b(\mathbb{R},W)}\\ &+\frac{a_5}{a_4}e^{a_4t}\Big(\beta\|g^1-g^2\|_{C_b(\mathbb{R},\ell^2)}+\alpha\|h^1-h^2\|_{C_b(\mathbb{R},\ell^2)}\Big), \end{aligned}$$

which shows the desired result. □

For $\omega \in \Omega$, let $D = D(\omega)$ be a family of nonempty subsets of $\ell^2 \times \ell^2$. D is called tempered if for every $\varsigma > 0$, the following holds:

$$\lim_{t \to +\infty} e^{-\varsigma t} \|D(\theta_{-t}\omega)\| = 0, \tag{3.25}$$

where $\|D\| = \sup_{x \in D} \|x\|$. In the sequel, we denote by $\mathcal{D}$ the collection of all families of tempered nonempty subsets of $\ell^2 \times \ell^2$.

Lemma 3.4. *For $\omega \in \Omega$, the NRDS Ψ associated with system (2.15) has a closed random uniformly $\mathcal{D}$-(pullback) absorbing set $Q \subset \mathcal{D}$ such that*

$$Q(\omega) = \{(\bar{u}, \bar{v}) \in \ell^2 \times \ell^2 : \|\bar{u}\|^2 + \|\bar{v}\|^2 \le R(\omega)\}, \tag{3.26}$$

where $R(\omega)$ is given by (3.29).

Proof. Replacing ω by $\theta_{-t}\omega$ in (3.13), we have

$$\begin{aligned}
&\|u(t, \theta_{-t}\omega, \xi_1, u_0(\theta_{-t}\omega))\|^2 + \|v(t, \theta_{-t}\omega, \xi_2, v_0(\theta_{-t}\omega))\|^2 \\
&\le e^{-\kappa t + 2\varepsilon \int_0^t z(\theta_{l-t}\omega) dl} (\|u_0(\theta_{-t}\omega)\|^2 + \|v_0(\theta_{-t}\omega)\|^2) \\
&\quad + \frac{2}{\gamma\kappa} (\beta(\delta_1(g_0))^2 + \alpha(\delta_2(h_0))^2) \int_0^t e^{\kappa(r-t) - 2\varepsilon \int_t^r z(\theta_{l-t}\omega) dl - 2\varepsilon z(\theta_{r-t}\omega)} dr \\
&\le e^{-\kappa t + 2\varepsilon \int_{-t}^0 z(\theta_l \omega) dl} (\|u_0(\theta_{-t}\omega)\|^2 + \|v_0(\theta_{-t}\omega)\|^2) \\
&\quad + \frac{2}{\gamma\kappa} (\beta(\delta_1(g_0))^2 + \alpha(\delta_2(h_0))^2) \int_{-\infty}^0 e^{\kappa r - 2\varepsilon \int_0^r z(\theta_l \omega) dl - 2\varepsilon z(\theta_r \omega)} dr.
\end{aligned} \tag{3.27}$$

By (3.15), we note that for $D \in \mathcal{D}(\ell^2 \times \ell^2)$ and $(u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \in D(\theta_{-t}\omega)$,

$$\begin{aligned}
&(\bar{u}(t, \theta_{-t}\omega, \xi_1, \bar{u}_0(\theta_{-t}\omega)), \bar{v}(t, \theta_{-t}\omega, \xi_2, \bar{v}_0(\theta_{-t}\omega))) \\
&= e^{\varepsilon z(\omega)} (u(t, \theta_{-t}\omega, \xi_1, u_0(\theta_{-t}\omega)), v(t, \theta_{-t}\omega, \xi_2, v_0(\theta_{-t}\omega))),
\end{aligned}$$

which, along with (3.27) implies that

$$\begin{aligned}
&\|\bar{u}(t, \theta_{-t}\omega, \xi_1, \bar{u}_0(\theta_{-t}\omega))\|^2 + \|\bar{v}(t, \theta_{-t}\omega, \xi_2, \bar{v}_0(\theta_{-t}\omega))\|^2 \\
&\le e^{2\varepsilon z(\omega) - \kappa t + 2\varepsilon \int_{-t}^0 z(\theta_l \omega) dl - 2\varepsilon z(\theta_{-t}\omega)} (\|\bar{u}_0(\theta_{-t}\omega)\|^2 + \|\bar{v}_0(\theta_{-t}\omega)\|^2) \\
&\quad + \frac{2e^{2\varepsilon z(\omega)}}{\gamma\kappa} (\beta(\delta_1(g_0))^2 + \alpha(\delta_2(h_0))^2) \int_{-\infty}^0 e^{\kappa r - 2\varepsilon \int_0^r z(\theta_l \omega) dl - 2\varepsilon z(\theta_r \omega)} dr.
\end{aligned} \tag{3.28}$$

By (2.7), we obtain

$$R(\omega) = \frac{4e^{2\varepsilon z(\omega)}}{\gamma\kappa} (\beta(\delta_1(g_0))^2 + \alpha(\delta_2(h_0))^2) \int_{-\infty}^0 e^{\kappa r - 2\varepsilon \int_0^r z(\theta_l \omega) dl - 2\varepsilon z(\theta_r \omega)} dr < +\infty, \tag{3.29}$$

and there exists $T_1 = T_1(\omega) > 0$ such that

$$-\kappa t + 2\varepsilon \int_{-t}^0 z(\theta_l \omega) dl - 2\varepsilon z(\theta_{-t}\omega) < -\frac{\kappa}{2} t, \quad t \ge T_1,$$

which, together with (3.25) implies that

$$\begin{aligned}&\lim_{t\to+\infty} e^{-\kappa t+2\varepsilon\int_{-t}^{0} z(\theta_l\omega)dl-2\varepsilon z(\theta_{-t}\omega)}(\|\bar{u}_0(\theta_{-t}\omega)\|^2+\|\bar{v}_0(\theta_{-t}\omega)\|^2)\\ &\le \lim_{t\to+\infty} e^{-\frac{\kappa}{2}t}(\|\bar{u}_0(\theta_{-t}\omega)\|^2+\|\bar{v}_0(\theta_{-t}\omega)\|^2)=0.\end{aligned}\tag{3.30}$$

Then, there exists $T_2 = T_2(\omega, D) > 0$ such that for $t \ge T_2$,

$$e^{2\varepsilon z(\omega)-\kappa t+2\varepsilon\int_{-t}^{0} z(\theta_l\omega)dl-2\varepsilon z(\theta_{-t}\omega)}(\|\bar{u}_0(\theta_{-t}\omega)\|^2+\|\bar{v}_0(\theta_{-t}\omega)\|^2)\le \frac{1}{2}R(\omega).\tag{3.31}$$

By (3.28), (3.29), and (3.31), we have

$$\|\bar{u}(t,\theta_{-t}\omega,\xi_1,\bar{u}_0(\theta_{-t}\omega))\|^2+\|\bar{v}(t,\theta_{-t}\omega,\xi_2,\bar{v}_0(\theta_{-t}\omega))\|^2\le R(\omega),\quad t\ge T_2.\tag{3.32}$$

Then, the set Q given by (3.26) is a closed random uniformly $\mathcal{D}$-(pullback) absorbing set for Ψ. Next, we need to obtain $Q \in \mathcal{D}(\ell^2 \times \ell^2)$. Indeed, for any $\zeta > 0$, we obtain

$$\begin{aligned}e^{-\zeta t}R(\theta_{-t}\omega) &= 4(\beta(\delta_1(g_0))^2+\alpha(\delta_2(h_0))^2)\frac{e^{-\zeta t+2\varepsilon z(\theta_{-t}\omega)}}{\gamma\kappa}\int_{-\infty}^{0} e^{\kappa r-2\varepsilon\int_0^r z(\theta_{l-t}\omega)dl-2\varepsilon z(\theta_{r-t}\omega)}dr\\ &= 4(\beta(\delta_1(g_0))^2+\alpha(\delta_2(h_0))^2)\frac{e^{-\zeta t+2\varepsilon z(\theta_{-t}\omega)}}{\gamma\kappa}\int_{-\infty}^{-t} e^{\kappa(r+t)-2\varepsilon\int_{-t}^r z(\theta_l\omega)dl-2\varepsilon z(\theta_r\omega)}dr.\end{aligned}$$

By (2.7), we find that there exists $T_3 = T_3(\omega)$ such that

$$2\varepsilon z(\theta_{-t}\omega)\le \frac{1}{2}\zeta t,\quad t\ge T_3.$$

Then,

$$\lim_{t\to+\infty} e^{-\zeta t+2\varepsilon z(\theta_{-t}\omega)}\le \lim_{t\to+\infty} e^{-\frac{1}{2}\zeta t}=0,$$

which implies that

$$\lim_{t\to+\infty} e^{-\zeta t}R(\theta_{-t}\omega)=0.$$

This completes the proof. □

4. Random uniform attractors

In this section, we will derive uniform estimates for the tails of solutions to system (2.15), which is crucial in establishing the asymptotic compactness of solutions. To this end, we select a smooth function $\vartheta(r)$ that satisfies $0 \le \vartheta(r) \le 1$ for all $s \in \mathbb{R}^+$, and

$$\vartheta(r)=\begin{cases}0, & 0\le r\le 1,\\ 1, & r\ge 2.\end{cases}$$

Moreover, given $s \in (0, 1)$, by Lemma 3.3 of [6], we note that for all $i \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$\sum_{j\in\mathbb{Z},j\ne i}\left|\vartheta\Big(\frac{|i|}{k}\Big)-\vartheta\Big(\frac{|j|}{k}\Big)\right|^2\tilde{K}_s(i-j)\le \frac{L_s^2}{k^{2s}}.\tag{4.1}$$

Lemma 4.1. *For $\epsilon > 0$, $\omega \in \Omega$, and $D \in \mathcal{D}(\ell^2 \times \ell^2)$, there are $K = K(\omega, \epsilon) > 0$ and $T = T(\omega, \epsilon, D) > 0$, such that for all $(\xi_1, \xi_2) \in \Sigma$, $(\bar{u}_0(\theta_{-t}\omega), \bar{v}_0(\theta_{-t}\omega)) \in D(\theta_{-t}\omega)$, $t \geq T$ and $k \geq K$, the solution $(\bar{u}, \bar{v})$ to system (2.15) satisfies*

$$\sum_{|i|\geq k} \left(|\bar{u}_i(t, \theta_{-t}\omega, \xi_1, \bar{u}_0(\theta_{-t}\omega))|^2 + |\bar{v}_i(t, \theta_{-t}\omega, \xi_2, \bar{v}_0(\theta_{-t}\omega))|^2\right) \leq \epsilon.$$

Proof. By (3.6), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)(\beta|u_i|^2 + \alpha|v_i|^2) + \beta\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)(-\triangle_d)^s u_i \cdot u_i \\
&\quad - \beta\varepsilon z(\theta_t\omega)\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)|u_i|^2 + \alpha(\sigma - \varepsilon z(\theta_t\omega))\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)|v_i|^2 \\
&= \beta e^{-\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) f_i(t, e^{\varepsilon z(\theta_t\omega)} u_i)u_i + \beta e^{-\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) g_i(t)u_i \\
&\quad + \alpha e^{-\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) h_i(t)v_i.
\end{aligned} \tag{4.2}$$

By Lemma 2.4 and (4.1), we obtain

$$\begin{aligned}
-\beta\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)(-\triangle_d)^s u_i \cdot u_i &= -\frac{\beta}{2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}, j\neq i}(u_i - u_j)\Big(\vartheta\Big(\frac{|i|}{k}\Big)u_i - \vartheta\Big(\frac{|j|}{k}\Big)u_j\Big)\tilde{K}_s(i-j) \\
&= -\frac{\beta}{2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}, j\neq i}\Big(\vartheta\Big(\frac{|i|}{k}\Big) - \vartheta\Big(\frac{|j|}{k}\Big)\Big)(u_i - u_j)u_i\tilde{K}_s(i-j) \\
&\quad - \frac{\beta}{2}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}, j\neq i}\vartheta\Big(\frac{|j|}{k}\Big)|u_i - u_j|^2\tilde{K}_s(i-j) \\
&\leq \frac{\beta}{2}\|u\|\Big[\sum_{i\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}, j\neq i}\Big|\vartheta\Big(\frac{|i|}{k}\Big) - \vartheta\Big(\frac{|j|}{k}\Big)\Big|^2\tilde{K}_s(i-j)\Big) \\
&\quad \times\Big(\sum_{i\in\mathbb{Z}, j\neq i}|u_i - u_j|^2\tilde{K}_s(i-j)\Big)\Big]^{\frac{1}{2}} \\
&\leq \frac{\sqrt{2}\beta}{4}\frac{L_s}{k^s}(\|u\|^2 + \|(-\triangle_d)^{\frac{s}{2}}u\|^2).
\end{aligned} \tag{4.3}$$

By (3.2), we obtain

$$\begin{aligned}
\beta e^{-\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) f_i(t, e^{\varepsilon z(\theta_t\omega)}u_i)u_i &= \beta e^{-2\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) f_i(t, e^{\varepsilon z(\theta_t\omega)}u_i)e^{\varepsilon z(\theta_t\omega)}u_i \\
&\leq -\lambda\beta\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)|u_i|^2.
\end{aligned} \tag{4.4}$$

By Young's inequality and $\kappa = \min\{\lambda, \sigma\}$, we have

$$\begin{aligned}
\beta e^{-\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) g_i(t)u_i + \alpha e^{-\varepsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big) h_i(t)v_i &\leq \frac{\lambda\beta}{2}\sum_{n\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)|u_i|^2 + \frac{\sigma\alpha}{2}\sum_{n\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)|v_i|^2 \\
&\quad + \frac{e^{-2\varepsilon z(\theta_t\omega)}}{2\kappa}\sum_{i\in\mathbb{Z}} \vartheta\Big(\frac{|i|}{k}\Big)(\beta|g_i(t)|^2 + \alpha|h_i(t)|^2).
\end{aligned} \tag{4.5}$$

It follows from (4.2)–(4.5) that

$$\frac{d}{dt}\Big[\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(\beta|u_i|^2+\alpha|v_i|^2)\Big]+(\kappa-2\varepsilon z(\theta_t\omega))\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(\beta|u_i|^2+\alpha|v_i|^2)$$
$$\le\frac{\sqrt{2}\beta}{2}\frac{L_s}{k^s}(\|u\|^2+\|(-\triangle_d)^{\frac{s}{2}}u\|^2)+\frac{e^{-2\varepsilon z(\theta_t\omega)}}{\kappa}\sum_{i\in\mathbb{Z}}(\beta|g_i(t)|^2+\alpha|h_i(t)|^2),$$

which implies that for $t\ge 0$,

$$\begin{aligned}\frac{d}{dt}\Big[e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl}\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(\beta|u_i|^2+\alpha|v_i|^2)\Big]&\le\frac{\sqrt{2}\beta}{2}\frac{L_s}{k^s}e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl}(\|u\|^2+\|(-\triangle_d)^{\frac{s}{2}}u\|^2)\\&\quad+\frac{e^{-2\varepsilon z(\theta_t\omega)}}{\kappa}e^{\kappa t-2\varepsilon\int_0^t z(\theta_l\omega)dl}\sum_{i\in\mathbb{Z}}(\beta|g_i(t)|^2+\alpha|h_i(t)|^2).\end{aligned}\tag{4.6}$$

Integrating both sides of (4.6) from 0 into t, we have

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_i(t,\omega,\xi_1,u_0(\omega))|^2+|v_i(t,\omega,\xi_2,v_0(\omega))|^2)\\&\le e^{-\kappa t+2\varepsilon\int_0^t z(\theta_l\omega)dl}\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_{0i}(\omega)|^2+|v_{0i}(\omega)|^2)\\&\quad+\frac{\sqrt{2}\beta}{2\gamma}\frac{L_s}{k^s}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_l\omega)dl}(\|u(r,\omega,\xi_1,u_0(\omega))\|^2+\|(-\triangle_d)^{\frac{s}{2}}u(r,\omega,\xi_1,u_0(\omega))\|^2)dr\\&\quad+\frac{1}{\gamma\kappa}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_l\omega)dl-2\varepsilon z(\theta_r\omega)}\sum_{i\in\mathbb{Z}}(\beta|g_i(r)|^2+\alpha|h_i(r)|^2)dr,\end{aligned}$$

where $\gamma=\min\{\alpha,\beta\}$. Replacing ω by $\theta_{-t}\omega$, we find

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_i(t,\theta_{-t}\omega,\xi_1,u_0(\theta_{-t}\omega))|^2+|v_i(t,\theta_{-t}\omega,\xi_2,v_0(\theta_{-t}\omega))|^2)\\&\le e^{-\kappa t+2\varepsilon\int_0^t z(\theta_{l-t}\omega)dl}\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_{0i}(\theta_{-t}\omega)|^2+|v_{0i}(\theta_{-t}\omega)|^2)\\&\quad+\frac{\sqrt{2}\beta}{2\gamma}\frac{L_s}{k^s}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_{l-t}\omega)dl}\|u(r,\theta_{-t}\omega,\xi_1,u_0(\theta_{-t}\omega))\|^2dr\\&\quad+\frac{\sqrt{2}\beta}{2\gamma}\frac{L_s}{k^s}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_{l-t}\omega)dl}\|(-\triangle_d)^{\frac{s}{2}}u(r,\theta_{-t}\omega,\xi_1,u_0(\theta_{-t}\omega))\|^2dr\\&\quad+\frac{1}{\gamma\kappa}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_{l-t}\omega)dl-2\varepsilon z(\theta_{r-t}\omega)}\sum_{i\in\mathbb{Z}}(\beta|g_i(r)|^2+\alpha|h_i(r)|^2)dr.\end{aligned}\tag{4.7}$$

Since $s\in(0,1)$ and L_s is independent of s, given $\epsilon_0>0$, there exists $K_1=K_1(\epsilon_0)\ge 1$ such that for all $k\ge K_1$,

$$\frac{\sqrt{2}\beta}{2\gamma}\frac{L_s}{k^s}(\|u\|^2+\|(-\triangle_d)^{\frac{s}{2}}u\|^2)\le\epsilon_0(\|u\|^2+\|(-\triangle_d)^{\frac{s}{2}}u\|^2),$$

which, along with (3.13) and (4.7), implies that

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_i(t,\theta_{-t}\omega,\xi_1,u_0(\theta_{-t}\omega))|^2+|v_i(t,\theta_{-t}\omega,\xi_2,v_0(\theta_{-t}\omega))|^2)\\
&\le(1+\eta)e^{-\kappa t-2\varepsilon\int_0^t z(\theta_{l-t}\omega)dl}\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_{0i}(\theta_{-t}\omega)|^2+|v_{0i}(\theta_{-t}\omega)|^2)\\
&\quad+\frac{1+\eta}{\gamma\kappa}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_{l-t}\omega)dl-2\varepsilon z(\theta_{r-t}\omega)}\sum_{i\in\mathbb{Z}}(\beta|g_i(r)|^2+\alpha|h_i(r)|^2)dr,
\end{aligned}
\tag{4.8}
$$

where $\eta=\max\{\frac{\epsilon_0}{2},\frac{2\epsilon_0}{\lambda}\}$. Since g and h are almost periodic functions, the sets $\{(g_i(t))_{i\in\mathbb{Z}}:t\in\mathbb{R}\}$ and $\{(h_i(t))_{i\in\mathbb{Z}}:t\in\mathbb{R}\}$ are precompact in ℓ^2. Then, for $\epsilon>0$, there exists $K_2=K_2(g,h,\omega,\epsilon)>0$ such that for all $k\ge K_2$,

$$
\frac{1+\eta}{\gamma\kappa}\sum_{|i|\ge k}(\beta|g_i(r)|^2+\alpha|h_i(r)|^2)\le\frac{\epsilon}{2a_6},
\tag{4.9}
$$

where

$$
a_6=a_6(\omega)=\int_{-\infty}^0 e^{\kappa r-2\varepsilon\int_0^r z(\theta_l\omega)dl-2\varepsilon z(\theta_r\omega)}dr.
\tag{4.10}
$$

By (4.9), $g\in\mathcal{H}(g_0)$, and $h\in\mathcal{H}(h_0)$, we get that there exists a constant $K_3=K_3(\omega,\epsilon)>0$ such that for all $k\ge K_3$,

$$
\frac{1+\eta}{\gamma\kappa}\sum_{|i|\ge k}(\beta|g_i(r)|^2+\alpha|h_i(r)|^2)\le\frac{\epsilon}{2a_6},
$$

which implies that for all $k\ge K_3$,

$$
\frac{1+\eta}{\gamma\kappa}\int_0^t e^{\kappa(r-t)-2\varepsilon\int_t^r z(\theta_{l-t}\omega)dl-2\varepsilon z(\theta_{r-t}\omega)}\sum_{|i|\ge k}(\beta|g_i(r)|^2+\alpha|h_i(r)|^2)dr\le\frac{\epsilon}{2}.
\tag{4.11}
$$

By (2.7), we note that there exists $T_4=T_4(\omega,\epsilon,D)$ such that for $(u_0(\theta_{-t}\omega),v_0(\theta_{-t}\omega))\in D(\theta_{-t}\omega)$ and $t\ge T_4$,

$$
\begin{aligned}
&(1+\eta)e^{-\kappa t+2\varepsilon\int_0^t z(\theta_{l-t}\omega)dl-2\varepsilon z(\theta_{-t}\omega)}\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_{0i}(\theta_{-t}\omega)|^2+|v_{0i}(\theta_{-t}\omega)|^2)\\
&\le(1+\eta)e^{-\frac{\kappa}{2}t}\|D(\theta_{-t}\omega)\|^2\le\frac{\epsilon}{2}.
\end{aligned}
\tag{4.12}
$$

Note that

$$
\begin{aligned}
&\sum_{|i|\ge 2k}(|\bar{u}_i(t,\theta_{-t}\omega,\xi_1,\bar{u}_0(\theta_{-t}\omega))|^2+|\bar{v}_i(t,\theta_{-t}\omega,\xi_2,\bar{v}_0(\theta_{-t}\omega))|^2)\\
&\le\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|\bar{u}_i(t,\theta_{-t}\omega,\xi_1,\bar{u}_0(\theta_{-t}\omega))|^2+|\bar{v}_i(t,\theta_{-t}\omega,\xi_2,\bar{v}_0(\theta_{-t}\omega))|^2)\\
&=e^{2\varepsilon z(\omega)}\sum_{i\in\mathbb{Z}}\vartheta\Big(\frac{|i|}{k}\Big)(|u_i(t,\theta_{-t}\omega,\xi_1,u_0(\theta_{-t}\omega))|^2+|v_i(t,\theta_{-t}\omega,\xi_2,v_0(\theta_{-t}\omega))|^2),
\end{aligned}
$$

which, along with (4.8), (4.11), and (4.12) conclude the proof. □

Lemma 4.2. *The NRDS Ψ associated with system (2.15) is uniformly $\mathcal{D}$-(pullback) asymptotically compact, i.e., for $D \in \mathcal{D}(\ell^2 \times \ell^2)$, $\omega \in \Omega$, any sequence $\{(t_n, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega))\}_{n=1}^{+\infty}$ with $(t_n, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega)) \in \mathbb{R}^+ \times \Sigma \times D(\theta_{-t_n}\omega)$ and $\lim_{n\to+\infty} t_n = +\infty$, the sequence $\{\Psi(t_n, \theta_{-t_n}\omega, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega))\}_{n=1}^{+\infty}$ has a convergent subsequence.*

Proof. By the boundedness of $D(\omega)$, for sufficiently large n, we obtain

$$\Psi(t_n, \theta_{-t_n}\omega, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega)) \in Q(\omega).$$

Then, there is $(\bar{u}^*, \bar{v}^*) \in \ell^2 \times \ell^2$ and a subsequence of $\{\Psi(t_n, \theta_{-t_n}\omega, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega))\}_{n=1}^{+\infty}$ (still denoted by $\{\Psi(t_n, \theta_{-t_n}\omega, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega))\}_{n=1}^{+\infty}$) such that

$$\{\Psi(t_n, \theta_{-t_n}\omega, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega))\} \to (\bar{u}^*, \bar{v}^*) \text{ weakly in } \ell^2 \times \ell^2. \tag{4.13}$$

The present study aims to demonstrate the equivalence between weak convergence and strong convergence, i.e., for $\epsilon > 0$ there is $N = N(\epsilon, \omega, D) > 0$ such that

$$\|\bar{u}(t_n, \theta_{-t_n}\omega, \xi_1^n, \bar{u}_0^n(\theta_{-t_n}\omega)) - \bar{u}^*\|^2 + \|\bar{v}(t_n, \theta_{-t_n}\omega, \xi_2^n, \bar{v}_0^n(\theta_{-t_n}\omega) - \bar{v}^*\|^2 \le \epsilon^2, \ n \ge N. \tag{4.14}$$

From Lemma 4.1, there are $N_1 = N_1(\epsilon, \omega, D) > 0$ and $K_4 = K_4(\epsilon, \omega) > 0$ such that

$$\sum_{|i|\ge K_4} (|\bar{u}_i(t_n, \theta_{-t_n}\omega, \xi_1^n, \bar{u}_0^n(\theta_{-t_n}\omega))|^2 + |\bar{v}_i(t_n, \theta_{-t_n}\omega, \xi_2^n, \bar{v}_0^n(\theta_{-t_n}\omega))|^2) \le \frac{\epsilon^2}{8}, \ n \ge N_1. \tag{4.15}$$

Since $(\bar{u}^*, \bar{v}^*) \in \ell^2 \times \ell^2$, then there is $K_5 = K_5(\epsilon) > 0$ such that

$$\sum_{|i|\ge K_5} (|\bar{u}_i^*|^2 + |\bar{v}_i^*|^2) \le \frac{\epsilon^2}{8}. \tag{4.16}$$

Choosing $K = K(\epsilon, \omega) = \max\{K_4(\epsilon, \omega), K_5(\epsilon)\}$ and by (4.13), we have for $|i| \le K$ as $n \to +\infty$,

$$\Psi_i(t_n, \theta_{-t_n}\omega, \xi^n, \bar{u}_0^n(\theta_{-t_n}\omega), \bar{v}_0^n(\theta_{-t_n}\omega)) \to (\bar{u}_i^*, \bar{v}_i^*) \text{ strongly in } \mathbb{R}, \tag{4.17}$$

and so there is $N_2 = N_2(\epsilon, \omega, D) > 0$ such that for all $n \ge N_2$,

$$\sum_{|i|\le K} (|\bar{u}_i(t_n, \theta_{-t_n}\omega, \xi_1^n, \bar{u}_0^n(\theta_{-t_n}\omega)) - \bar{u}_i^*|^2 + |\bar{v}_i(t_n, \theta_{-t_n}\omega, \xi_2^n, \bar{v}_0^n(\theta_{-t_n}\omega)) - \bar{v}_i^*|^2) \le \frac{\epsilon^2}{2}. \tag{4.18}$$

Let $N = N(\epsilon, \omega, D) = \max\{N_1(\epsilon, \omega, D), N_2(\epsilon, \omega, D)\}$. Then, by (4.15)–(4.17), we find that for $n \ge N$,

$$\begin{aligned}
&\|\bar{u}(t_n, \theta_{-t_n}\omega, \xi_1^n, \bar{u}_0^n(\theta_{-t_n}\omega)) - \bar{u}^*\|^2 + \|\bar{v}(t_n, \theta_{-t_n}\omega, \xi_2^n, \bar{v}_0^n(\theta_{-t_n}\omega)) - \bar{v}^*\|^2 \\
&= \sum_{|i|\le K} (|\bar{u}_i(t_n, \theta_{-t_n}\omega, \xi_1^n, \bar{u}_0^n(\theta_{-t_n}\omega)) - \bar{u}_i^*|^2 + |\bar{v}_i(t_n, \theta_{-t_n}\omega, \xi_2^n, \bar{v}_0^n(\theta_{-t_n}\omega)) - \bar{v}_i^*|^2) \\
&\quad + \sum_{|i|> K} (|\bar{u}_i(t_n, \theta_{-t_n}\omega, \xi_1^n, \bar{u}_0^n(\theta_{-t_n}\omega)) - \bar{u}_i^*|^2 + |\bar{v}_i(t_n, \theta_{-t_n}\omega, \xi_2^n, \bar{v}_0^n(\theta_{-t_n}\omega)) - \bar{v}_i^*|^2) \\
&\le \epsilon^2.
\end{aligned}$$

This completes the proof. □

The main result can be readily derived by applying Theorem 2.7 in [21], along with Lemmas 3.3, 3.4 and 4.2 from this paper.

Theorem 4.1. *The NRDS* Ψ *associated with system (2.15) has a unique random* $\mathcal{D}$*-uniform attractor* $\mathcal{A} \in \mathcal{D}(\ell^2 \times \ell^2)$ *given by*

$$\mathcal{A}(\omega) = \mathcal{W}(\omega, \Sigma, Q), \forall \omega \in \Omega.$$

Furthermore, for all $t \geq 0$ *and* $\omega \in \Omega$*, the attractor* $\mathcal{A}$ *is negatively semi-invariant, i.e.,*

$$\mathcal{A}(\theta_t \omega) \subset \Psi(t, \omega, \Sigma, \mathcal{A}(\omega)),$$

and is uniformly $\mathcal{D}$*-forward-attracting in probability, i.e.,*

$$\lim_{t \to +\infty} P\Big(\omega \in \Omega, \sup_{\xi \in \Sigma} d(\Psi(t, \omega, \xi, D(\omega)), \mathcal{A}(\theta_t \omega)) > \epsilon\Big) = 0, \forall \epsilon > 0, D \in \mathcal{D}(\ell^2 \times \ell^2).$$

5. Conclusions

The current focus lies in the theoretical proof of the well-posedness of solutions, as well as the existence and uniqueness of random $\mathcal{D}$-uniform attractors for a fractional stochastic FitzHugh-Nagumo lattice systems. In future research, we will explore the convergence and approximation of these systems' random $\mathcal{D}$-uniform attractors under noise perturbation. Furthermore, we can investigate these asymptotic behavior for retarded lattice systems on $\mathbb{Z}^k$ in weighted spaces.

Author contributions

Xintao Li: Conceptualization, Writing original draft and writing-review and editing; Yunlong Gao: Writing original draft and writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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