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Research article

Analytical formulae for variance and volatility swaps with stochastic volatility, stochastic equilibrium level and regime switching

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Abstract: The CIR stochastic volatility model is modified to introduce nonlinear mean reversion, with the long-run volatility average as a random variable controlled by two parts being modeled through a Brownian motion and a Markov chain, respectively. This model still possesses an analytical formulation of the forward characteristic function, from which we establish variance swap prices as well as volatility swap ones with a nonlinear payoff in closed form. The numerical implementation of the two formulae demonstrates the significant impact of regime switching.

Keywords: nonlinear mean reversion; regime switching; stochastic volatility; analytical; variance and volatility swaps
Mathematics Subject Classification: 91G20

1. Introduction

Trading derivatives is of great popularity in today's financial markets, and thus, how to manage the risk caused by trading attracts much attention from researchers and market practitioners. Volatility is often used to measure asset price risk, and it is common knowledge that volatility values change randomly over time [1, 2], which makes it even more difficult to figure out the nature of the volatility variations. As a result, volatility derivatives are developed to manage the relative risk associated with the asset.

In practice, variance and volatility swaps enjoy high trading volumes, which has resulted in the desire to evaluate the two contracts correctly. To be more specific, while the results in [3] are more general and model-independent, most other research results depend on different adopted models. For example, Grunbichler and Longstaff [4] valued the two contracts with the CIR stochastic volatility process, which is the same process used in the Heston model [5] to describe volatility evolution, while how to price the two contracts under stochastic volatility without giving a specific stochastic volatility dynamic is illustrated by Javaheri et al. [6]. However, these results rely on the realized variance as well as volatility used in final pay-off functions being continuous, which is certainly inappropriate since the realized variance and volatility in real markets are discretely sampled, and such a mis-specification could result in a pricing bias, making these results less valuable. Therefore, pricing discretely-sampled swaps is more meaningful since that is the definition used in reality.

The literature has already witnessed the explosive expansion of evaluating discrete swap contracts. For example, analytical solutions for these two contracts under the Heston model have already been presented in [7, 8], respectively. Of course, the Heston model is unable to capture the significantly nonlinear mean revision property exhibited by asset volatility [9], which prompts research into developing more sophisticated models. In particular, He and Chen [10] introduced the concept of the stochastic mean-reversion level in the Heston (CIR) volatility process so that the nonlinear mean revision property can be partially addressed. Recently, the economic status has been shown to be changing [11], and such effects on derivative prices have been shown to be significant [12]. This has led to the emergence of regime switching models [13–16].

Motivated by the fact that the implied volatility as well as the variance swap curve cannot be appropriately fitted using the classical Heston model [17] and the significance of combining stochastic volatility models with regime switching [12], the He–Chen model is further modified by adding regime switching [10], so that the long-run mean is divided into two parts. One part is characterized by a stochastic process following a normal distribution, and another is a regime switching part. Based on the successful presentation of the forward characteristic function, analytical solutions used to price the two swap contracts are shown under the newly established model.

We organize the remainder of the article as follows: Section 2 briefly introduces the adopted model, and then the forward characteristic function is derived in closed form, leading to analytical solutions of the target contracts. In Section 3, we discuss the numerical results after implementing the solutions we obtain. The last section concludes.

2. Analytical solution

The regime switching dynamics are embraced into the He–Chen model [10] so that the effects of economic cycles are taken into consideration. After that, we derive the two swap pricing formulae under this new model analytically.

2.1. Our model

As our model modifies the Heston model and the He–Chen model, we first introduce the Heston model. With the assumption that S is the stock price and v denotes asset volatility, the Heston model in a risk-neutral world is characterized through

$$\begin{aligned} \frac{dS}{S} &= rdt + \sqrt{v}dW_t^1, \\ dv &= k(\bar{v} - v)dt + \sigma_1 \sqrt{v}dW_t^2. \end{aligned} \tag{2.1}$$

Here, W_t^1 and W_t^2 are two Wiener processes, with $dW_t^1 dW_t^2 = \rho dt$. The He–Chen model modifies it by proposing the concept of stochastic long-run mean, which formulates

$$\begin{aligned} \frac{dS}{S} &= rdt + \sqrt{v}dW_t^1, \\ dv &= k(\bar{v} + \theta - v)dt + \sigma_1 \sqrt{v}dW_t^2, \\ d\theta &= \lambda dt + \sigma_2 dB_t, \end{aligned} \tag{2.2}$$

where θ is the newly introduced stochastic source in the long-term mean. B_t is another Wiener process that is unrelated to the other two. Our model further revises the He–Chen model through the consideration of economic cycles on volatility

$$\begin{aligned} \frac{dS}{S} &= rdt + \sqrt{v}dW_t^1, \\ dv &= k(\bar{v}_{X_t} + \theta - v)dt + \sigma_1 \sqrt{v}dW_t^2, \\ d\theta &= \lambda dt + \sigma_2 dB_t. \end{aligned} \tag{2.3}$$

The two-state Markov chain X_t can take the value within the set $\{(1,0)^T, (0,1)^T\}$ depending on the different states that the economy belongs to, so that $\bar{v}_{X_t} = <\bar{v}, X_t>$, with $\bar{v} = (v_1, v_2)^T$. $\lambda_{ij}, i, j = 1, 2,\ i \neq j$ denotes the transition rate of the Markov chain transferring from State i to j. Clearly, the presented model is actually a combination of the Heston stochastic volatility model with regime switching dynamics and a stochastic mean-reversion level. It will become the He–Chen model [10] if we remove regime switching mechanics by setting $\lambda_{12} = \lambda_{21} = 0$. If we further make $\lambda = \sigma_2 = 0$, it will go back to the Heston model.

2.2. *Pricing variance and volatility swaps*

To price the two swap contracts, one should be very clear about the cash flow between the long and short positions. If one longs a variance and volatility swap, he or she needs to pay a strike price specified in the contract at maturity while receiving a floating leg of realized variance and volatility written in an annualized form, whose computation depends on the actual asset price fluctuation during the lifetime of the swaps.

In fact, one can compute the values of the two contracts with a notional amount L at maturity through

$$V_{var} = (RV_{var} - K_{var})L, \quad V_{vol} = (RV_{vol} - K_{vol})L,$$

where RV_{var} and K_{var} are the annualized realized variance and variance swap strike price, respectively, with RV_{vol} as well as K_{vol} representing the same meaning for a volatility swap. As swap contracts are worth zero values when they are initiated, we have the following to be fair to both long and short positions:

$$K_{var} = E(RV_{var}), \quad K_{vol} = E(RV_{vol}).$$

Clearly, the determination of K_{var} (K_{vol}) requires that we compute the two expectations, implying that it is necessary for us to know how RV_{var} (RV_{vol}) is defined before we proceed to figure out the expectation. Here, we adopt the expression widely used in the literature [13, 18]

$$RV_{var} = \frac{100^2}{T} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2,$$

$$RV_{vol} \quad = \quad 100\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right|.$$

t_i is ith observation time equally spaced within [0,T] with $t_0 = 0$ and $t_N = T$. Therefore, once we are able to calculate the expectation $E\left[\left(\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right]$ and $E\left[\left|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right|\right]$ for every $i = 0, .., N$, then we could obtain the delivery price straightforwardly by adding these expectations together.

To analytically work out these expectations, we shall first derive the forward characteristic function of the underlying price. In particular, if we write $y_T = \ln(\frac{S_T}{S_t})$, the forward characteristic function is formulated using

$$\mu(\phi; v_0, \theta_0, X_0, t, T) = E\left(e^{j\phi y_T}|y_0, v_0, \theta_0, X_0\right). \tag{2.4}$$

The direct evaluation of this particular function is very difficult, and such difficulty arises from the stochastic nature of both the Winier processes and the Markov chain. To simplify the problem for the time being, we assume that the values of $X_t, t \in [0, T]$ are given at the current time so that the target characteristic function becomes a conditional one

$$\mu(\phi; v_0, \theta_0, t, T|X_t, t \in [0, T]) = E\left(e^{j\phi y_T}|y_0, v_0, \theta_0, X_T\right). \tag{2.5}$$

The solution to this particular function is presented below.

Proposition 2.1. *If Eq (2.3) provides information on the model dynamics, the solution of $\mu(\phi; v_0, \theta_0, t, T|X_t, t \in [0, T])$ satisfies*

$$\mu(\phi; v_0, \theta_0, t, T|X_t, t \in [0, T]) = e^{\widetilde{C}(\phi;\tau,t)+\bar{D}(\phi;\tau,t)v_0+\bar{E}(\phi;\tau,t)\theta_0}e^{\int_0^T <J(s),X_s>ds}, \tag{2.6}$$

where

$$\begin{aligned}
&\bar{D}(\phi;\tau,t) = \frac{2k}{\sigma_1^2}\frac{1}{1-\left[1-\frac{2k}{\sigma_1^2 D(\phi;\tau)}\right]e^{kt}},\\
&\bar{E}(\phi;\tau,t) = E(\phi;\tau) + \frac{2k}{\sigma_1^2}\left\{kt - \ln\left(1-\left[1-\frac{2k}{\sigma_1^2 D(\phi;\tau)}\right]e^{kt}\right) + \ln\left[\frac{2k}{\sigma_1^2 D(\phi;\tau)}\right]\right\},\\
&\widetilde{C}(\phi;\tau,t) = \int_0^\tau \frac{1}{2}\sigma_2^2 E^2(\phi;s) + \lambda E(\phi;s)ds + \int_0^t \frac{1}{2}\sigma_2^2\bar{E}(\phi;\tau,s) + \lambda\bar{E}(\phi;\tau,s)ds + jr\phi\tau,\\
&J(s) = k\bar{v}_s D(\phi;T-s)H(s) + k\bar{v}_s\bar{D}(\phi;\tau,s)[1-H(s)],\\
&D(\phi;\tau) = \frac{d-(\rho\sigma_1 j\phi - k)}{\sigma_1^2}\left(\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right),\\
&E(\phi;\tau) = \frac{k}{\sigma_1^2}\left\{[d-(\rho\sigma_1 j\phi - k)]\tau - 2\ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right\},\\
&d = \sqrt{(\rho\sigma_1 j\phi - k)^2 + \sigma_1^2(j\phi+\phi^2)}, \quad g = \frac{(\rho\sigma_1 j\phi - k) - d}{(\rho\sigma_1 j\phi - k) + d},\\
&H(s) = \begin{cases} 1, & s > t,\\ 0, & s \le t,\end{cases}
\end{aligned}$$

with $\tau = T - t$.

We prove the proposition in Appendix A.

Of course, the characteristic function presented in (2.6) is a conditional one, and one still has to work out its expectation since the forward characteristic function satisfies

$$\begin{aligned} \mu(\phi; v_0, \theta_0, X_0, t, T) &= E\left(e^{j\phi y_T}|y_0, v_0, \theta_0, X_0\right), \\ &= E\left[E\left(e^{j\phi y_T}|y_0, v_0, \theta_0, X_T\right)|y_0, v_0, \theta_0, X_0\right], \\ &= E\left[\mu(\phi; v_0, \theta_0, t, T|X_t, t \in [0, T])|y_0, v_0, \theta_0, X_0\right]. \end{aligned} \tag{2.7}$$

By noticing the fact that $\bar{v}$ depends on the Markov chain X_t with $t \in [0, T]$, substituting Eq (2.6) into (2.7) could yield

$$\mu(\phi; v_0, \theta_0, X_0, t, T) = e^{\widetilde{C}(\phi;\tau,t)+\bar{D}(\phi;\tau,t)v_0+\bar{E}(\phi;\tau,t)\theta_0} E\left[e^{\int_0^T <J(s),X_s>ds}|X_0\right], \tag{2.8}$$

which implies that $E\left[e^{\int_0^T <J(s),X_s>ds}|X_0\right]$ has to be derived. Actually, this expectation can be figured out by simply following Elliott and Lian [13]. In this case, we could obtain

$$E\left[e^{\int_0^T <J(s),X_s>ds}|X_0\right] =< e^M X_0, I >, \tag{2.9}$$

where $X_0 \in \{(1,0)', (0,1)'\}$, and $I = (1,1)'$. With A being the matrix formulated by the transition rates associated with X_t, Matrix M is defined as

$$M = \int_0^T A' + diag[J(s)]ds, \tag{2.10}$$

yielding

$$M = \begin{pmatrix} \bar{v}_1\bar{E}(\phi;\tau,t) - \lambda_{12}T & \lambda_{21}T \\ \lambda_{12}T & \bar{v}_2\bar{E}(\phi;\tau,t) - \lambda_{21}T \end{pmatrix}.$$

Therefore, we have finally obtained

$$\mu(\phi; v_0, \theta_0, X_0, t, T) = e^{\widetilde{C}(\phi;\tau,t)+\bar{D}(\phi;\tau,t)v_0+\bar{E}(\phi;\tau,t)\theta_0} < e^M X_0, I > . \tag{2.11}$$

With Eq (2.11), we are now able to present the formulae for pricing both swaps in the following proposition.

Proposition 2.2. *If Eq (2.3) provides information on the model dynamics, the prices of variance and volatility swaps can be formulated as*

$$\begin{aligned} K_{var} &= \frac{100^2}{T}\sum_{i=1}^{N}\left[\mu(-2j; t_{i-1}, t_i, v_0, \theta_0, X_0) - 2\mu(-j; t_{i-1}, t_i, v_0, \theta_0, X_0) + 1\right], \\ K_{vol} &= 100\sqrt{\frac{2}{\pi NT}}\int_0^{+\infty}\sum_{i=1}^{N} RE\left[\frac{\mu(\phi - j; v_0, \theta_0, X_0, t_{i-1}, t_i) - \mu(\phi; v_0, \theta_0, X_0, t_{i-1}, t_i)}{j\phi}\right]d\phi. \end{aligned} \tag{2.12}$$

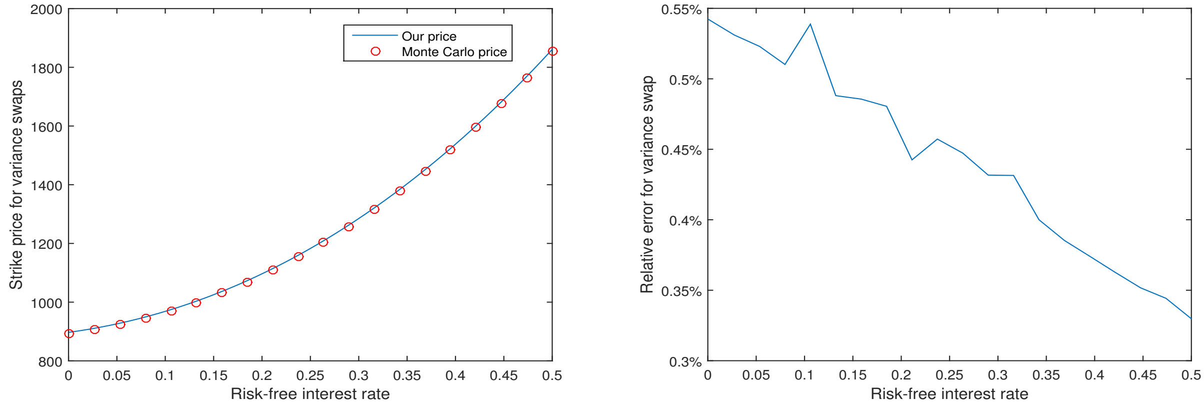
We prove the proposition in Appendix B.

By now, we have successfully worked out the two swap prices analytically. Once a formula for derivative prices has been derived under a new model, one may be very interested in the influence that adopting the new model would bring about, which makes it rather important to make a comparison between our model and the existing related models to find the different numerical behavior. The related discussions are provided below.

3. Numerical examples

This section will mainly analyze how the results provided by the He–Chen model [10] can be affected by adding regime switching. In particular, we will first check whether our model can represent the He–Chen model if transition rates are zero. Then, we will compare our results with those under the He–Chen model without regime switching to identify the difference caused by introducing regime switching mechanics. The parameter values for the analysis are defaulted as follows, whose magnitude is consistent with a number of different pieces of literature [13, 19, 20]. Both parameters in the process describing the stochastic part of the long-term mean, i.e., λ and σ_2, are set to be 0.01, and the initial value of this particular stochastic process, θ_0, is 0.03. The expiration date T is set to be 1 year. We also assume that $\bar{v}$ in the He–Chen model takes the value of $\bar{v}_1$ for comparison. Other model parameters include $r = 0.05,\ k = 10,\ \sigma_1 = 0.1,\ v_0 = 0.03$ and $\rho = -0.5$.

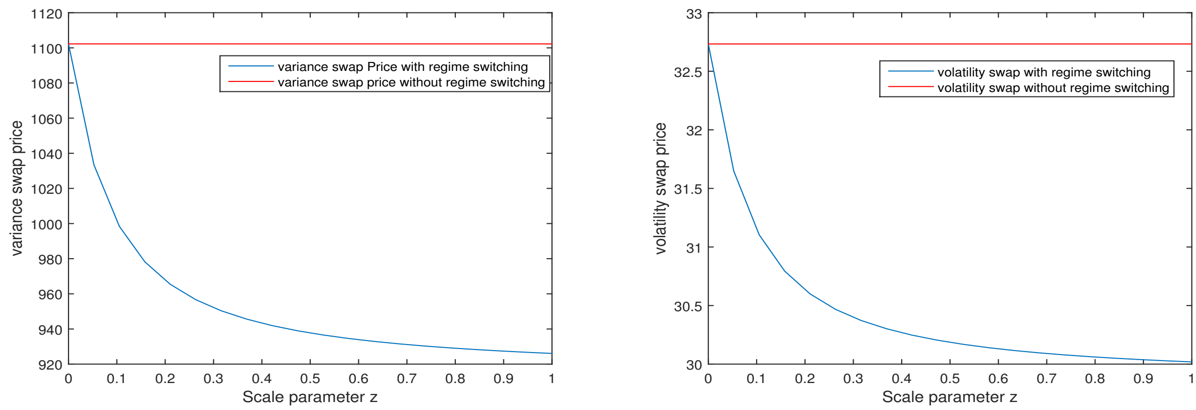
Before we numerically study the swap pricing formulae, their accuracy should first be checked using the semi-Monte Carlo benchmark, which is much more efficient than the standard Monte Carlo approach. The detailed procedure in a single simulation will be illustrated in the following: First, by simulating the jump moments using the exponential distribution and the transition rates, a Markov chain within the period $[0, T]$ is generated so that λ_{X_t} and η_{X_t} become time-dependent parameters. Then, we are able to evaluate the two swaps through the conditional characteristic function (2.6). Finally, by repeating 500,000 times the above steps, we could obtain one variance and one volatility swap price. In the following, we will only present the verification results of the variance swap pricing formula, which is shown in Figure 1, since the two formulae are quite similar to each other and there is no need to check the volatility swap pricing formula once variance swap pricing formula is proved. Specifically, what could be detected first in Figure 1(a) is that our variance swap prices are point-wise close to those from the Monte Carlo approach, and the relative difference displayed in Figure 1(b) is below 0.6%, demonstrating that our formulae are clearly accurate.



(a) Two prices. (b) Relative difference.

Figure 1. Variance swap prices with two approaches. Parameters are $N = 4$; $\bar{v}_1 = 0.08$; $\bar{v}_2 = 0.02$.

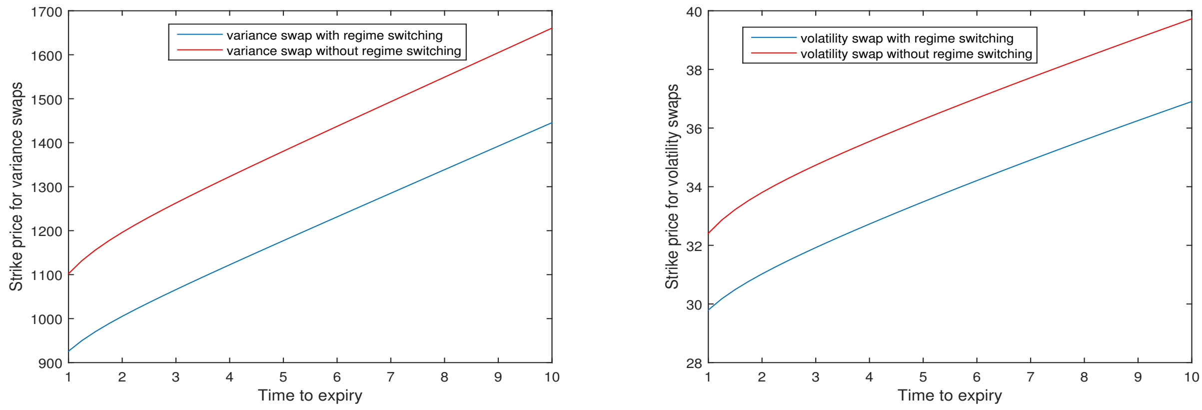
The He–Chen model [10] being a special case of ours, can be easily observed from the model dynamics if the regime switching mechanics are removed. In this case, whether our formulae can degenerate to those under the He–Chen model is of interest, and this is also a sign of the correctness of the formulae. A scaling parameter $z \in [0, 1]$ is employed with $\lambda_{12} = 10 * z, \lambda_{21} = 20 * z$, and the corresponding swap prices are shown in Figure 2. Specifically, Figure 2 shows the variance and volatility swap price with and without regime switching, and as expected, the two prices are exactly the same when the two transition rates become zero. It can also be easily observed that with the selected set of parameters, our price is no larger than that without regime switching, and it decreases with the scaling parameter z.



(a) Variance swaps. (b) Volatility swaps.

Figure 2. Strike prices of the He–Chen model and ours with respect to the scaling parameter. Parameters are $N = 4$; $\bar{v}_1 = 0.08$; $\bar{v}_2 = 0.02$.

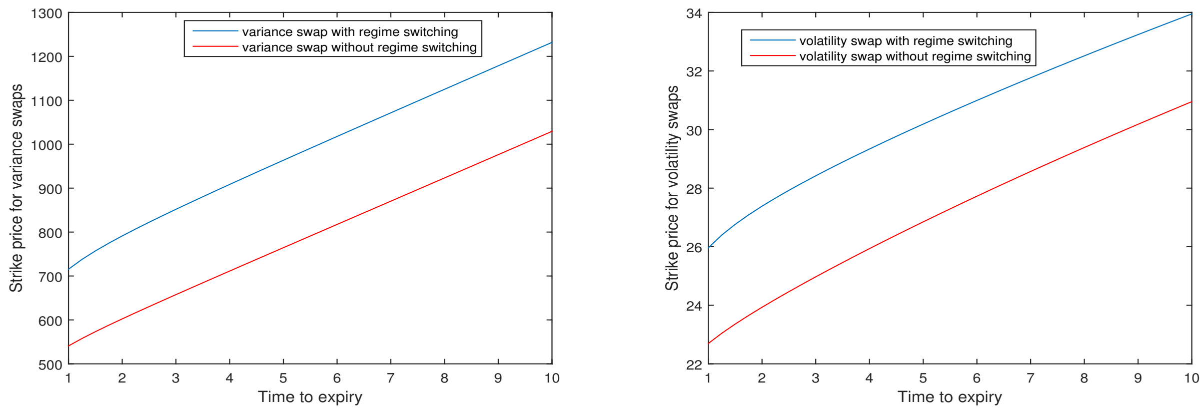
What can be seen in Figure 3 are different swap prices when the constant long-term mean for State 1 is greater than State 2. As shown in Figure 3(a), variance swap prices keep linearly increasing as the time to expiry increases. Moreover, the introduction of regime switching under the current settings will lower swap prices. A very similar decreasing pattern is also shown in Figure 3(b) when we consider volatility swap prices, except that the curve is much more flattening, which is mainly due to the much lower magnitude of volatility swap prices.



(a) Variance swaps.

(b) Volatility swaps.

Figure 3. Strike prices of the He–Chen model and ours when $\bar{v}_1 > \bar{v}_2$. Parameters are $\lambda_{12} = 10$; $\lambda_{21} = 20$; $\bar{v}_1 = 0.08$; $\bar{v}_2 = 0.02$.



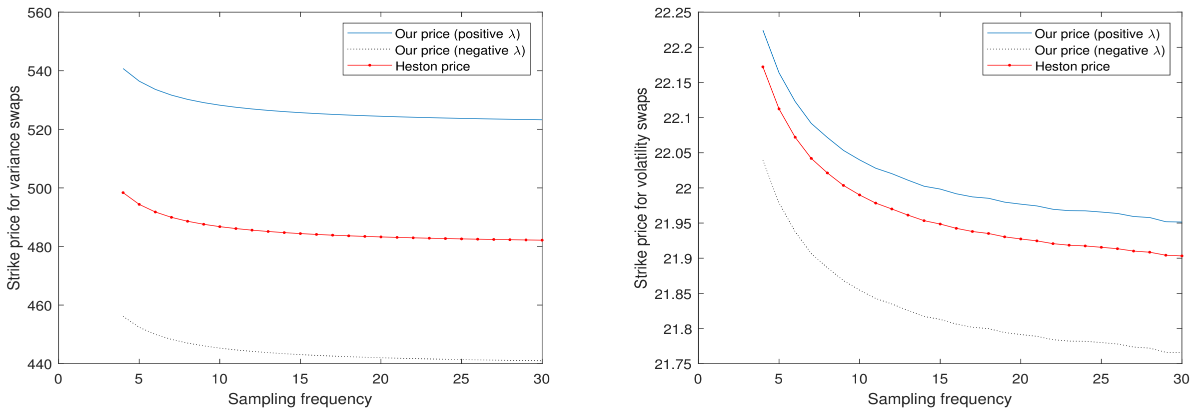
(a) Variance swaps.

(b) Volatility swaps.

Figure 4. Strike prices of the He–Chen model and ours when $\bar{v}_1 < \bar{v}_2$. Parameters are $\lambda_{12} = 10$; $\lambda_{21} = 20$; $\bar{v}_1 = 0.02$; $\bar{v}_2 = 0.08$.

Naturally, one may draw the conclusion that the variance and volatility swap prices with regime switching will always be lower than those without regime switching. In order to prove the incorrectness of this viewpoint, we reverse the values of $\bar{v}_1$ and $\bar{v}_2$, and the results in this situation are presented in Figure 4. Clearly, if the constant mean-reversion level for current state is lower compared with the other one, our variance and volatility swap price are larger than those obtained under the He–Chen model without regime switching, which is shown in Figure 4. Also, the variance and volatility swap prices in both models decrease when we lower the constant long-term mean for the current state.

One may also be interested in the difference between our model and the well-known Heston model. We show the corresponding comparison results in Figure 5. One can clearly see that variance (volatility) swap prices under our model exceed those under the Heston model when λ is positive, while our prices would be lower than the Heston prices with a negative λ. This can be understood from the fact that λ controls the trend of long-run volatility, with positive (or negative) λ indicating increasing (or decreasing) long-run volatility, leading to higher (or smaller) risks. This also reflects the greater flexibility of our model compared to the Heston model.



(a) Variance swaps. (b) Volatility swaps.

Figure 5. Strike prices of the Heston model and ours. Parameters are $\lambda_{12} = 10$; $\lambda_{21} = 20$; $\bar{v}_1 = \bar{v}_2 = 0.02$.

4. Conclusions

This article proposes a particular regime switching stochastic volatility model, dividing the long-run mean into two parts, with one being characterized by a normal distribution and the other incorporating regime switching mechanics. After successfully working out the analytical pricing formulae, we have also shown through the numerical experiments that adding regime switching in the modeling framework has a large impact, which thus implies that this model can be used as an alternative to the existing models in practice.

Author contributions

Xin-Jiang He: investigation, methodology, software, writing-original draft; Sha Lin: conceptualization, software, validation, writing-review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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Appendix A

We now prove Proposition 2.1. We can express

$$\begin{aligned}\mu(\phi; v_0, \theta_0, t, T | X_t, t \in [0, T]) &= E\left(e^{j\phi y_T} | y_0, v_0, \theta_0, X_T\right) \\ &= E\left[E\left(e^{j\phi y_T} | y_t, v_t, \theta_t, X_T\right) | y_0, v_0, \theta_0, X_T\right]. \quad \text{(A-1)}\end{aligned}$$

As a result, we will firstly deal with the inner expectation

$$h(\phi; v_s, \theta_s, X_s, s, T) = E\left(e^{j\phi y_T}|y_s, v_s, \theta_s, X_T\right), s \in [t, T],$$

which should satisfy the following PDE

$$\begin{cases} \dfrac{\partial h}{\partial s} + \dfrac{1}{2}v\dfrac{\partial^2 h}{\partial y^2} + \dfrac{1}{2}\sigma_1^2 v\dfrac{\partial^2 h}{\partial v^2} + \dfrac{1}{2}\sigma_2^2\dfrac{\partial^2 h}{\partial \theta^2} + \rho\sigma_1 v\dfrac{\partial^2 h}{\partial v \partial y} \\ \quad + (r - \frac{1}{2}v)\frac{\partial h}{\partial y} + k(\bar{v} + \theta - v)\frac{\partial h}{\partial v} + \lambda\frac{\partial h}{\partial \theta} = 0, \\ h(\phi; v_s, \theta_s, X_s, s, T)|_{s=T} = e^{j\phi y_T}. \end{cases} \tag{A-2}$$

Following a number of different literature [21–23], if we write

$$h(\phi; v_s, \theta_s, X_s, s, T) = e^{C(\phi,\tau)+D(\phi,\tau)v_s+E(\phi,\tau)\theta_s+j\phi y_s}, \tag{A-3}$$

with τ defined as $T - s$, then substituting it into PDE (A-2) could yield the three ODEs (ordinary differential equations)

$$\begin{aligned} \frac{\partial D}{\partial \tau_i} &= \frac{1}{2}\sigma_1^2 D^2 + (\rho\sigma\phi j - k)D - \frac{1}{2}(j\phi + \phi^2), \\ \frac{\partial E}{\partial \tau_i} &= kD, \\ \frac{\partial C}{\partial \tau_i} &= \frac{1}{2}\sigma_2^2 E^2 + \lambda E + k\bar{v}D + r(j\phi - 1). \end{aligned}$$

From this, we could obtain the solutions to the first two ODEs presented in Proposition 2.1 with $s = t$. Then, directly integrating the ODE for C derives

$$C(\phi, \tau) = \int_0^\tau \frac{1}{2}\sigma_2^2 E^2(\phi; s) + \lambda E(\phi; s)ds + jr\phi\tau + \int_t^T k\bar{v}_t D(T - s)ds. \tag{A-4}$$

Since we have $y_s = 0$ when $s = t$, we can write

$$h(\phi; v_t, \theta_t, X_t, t, T) = e^{C(\phi,\tau)+D(\phi,\tau)v_t+E(\phi,\tau)\theta_t}. \tag{A-5}$$

We now reformulate Eq (A-1) so that

$$\mu(\phi; v_0, \theta_0, t, T|X_t, t \in [0, T]) = e^{C(\phi,\tau)}E[e^{D(\phi,\tau)v_t+E(\phi,\tau)\theta_t}|y_0, v_0, \theta_0, X_T]. \tag{A-6}$$

If we define

$$f(\phi; v_0, \theta_0, t, T|X_t, t \in [0, T]) = E[e^{D(\phi,\tau)v_t+E(\phi,\tau)\theta_t}|y_0, v_0, \theta_0, X_T],$$

and considering the fact that f is independent of y, $f(\phi; v_s, \theta_s, s, t, T|X_a, a \in [0, T])$ is a solution to

$$\begin{cases} \dfrac{\partial f}{\partial s} + \dfrac{1}{2}\sigma_1^2 v\dfrac{\partial^2 f}{\partial v^2} + \dfrac{1}{2}\sigma_2^2\dfrac{\partial^2 f}{\partial \theta^2} + k(\bar{v} + \theta - v)\dfrac{\partial f}{\partial v} + \lambda\dfrac{\partial f}{\partial \theta} = 0, \\ f(\phi; v_s, \theta_s, s, t, T|X_a, a \in [0, T])|_{s=t} = e^{D(\phi,\tau)v_t+E(\phi,\tau)\theta_t}. \end{cases} \tag{A-7}$$

Similarly, by guessing that the solution to PDE (A-7) be

$$f(\phi; v_s, \theta_s, s, t, T|X_a, a \in [0, T]) = e^{\bar{C}(\phi;\tau,t-s)+\bar{D}(\phi;\tau,t-s)v_s+\bar{E}(\phi;\tau,t-s)\theta_s}, \tag{A-8}$$

and substituting it into PDE (A-7), we could obtain

$$\begin{aligned} \frac{\partial \bar{D}}{\partial \tau_s} &+ \frac{1}{2}\sigma_1^2 \bar{D}^2 - k\bar{D} = 0, \\ \frac{\partial \bar{E}}{\partial \tau_s} &+ k\bar{D} = 0, \\ \frac{\partial \bar{C}}{\partial \tau_s} &+ \frac{1}{2}\sigma_2^2 \bar{E}^2 + k\bar{v}\bar{D} + \lambda \bar{E} = 0, \end{aligned}$$

with $\tau_s = t - s$, and the initial condition

$$\bar{D}(\phi; \tau, 0) = D(\phi, \tau),\ \bar{E}(\phi; \tau, 0) = E(\phi, \tau),\ \bar{C}(\phi; \tau, 0) = 0.$$

Obviously, the first two ODEs can be directly solved, which could contribute to solving the ODE for $\bar{C}(\phi; \tau, \tau_s)$, so that

$$\bar{C}(\phi; \tau, \tau_s) = \int_0^{\tau_s} \frac{1}{2}\sigma_2^2 \bar{E}^2(\phi; \tau, p) + \lambda \bar{E}(\phi; \tau, p) dp + \int_0^{\tau_s} k\bar{v}_t \bar{D}(\phi; \tau, p) dp. \tag{A-9}$$

As a result, by letting $s = 0$, we could finally reach our target. This has completed the proof.

Appendix B

We now prove Proposition 2.2. The variance swap price is equal to

$$\begin{aligned} K_{var} &= \frac{100^2}{T} \sum_{i=1}^{N} E\left[\left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2\right] \\ &= \frac{100^2}{T} \sum_{i=1}^{N} E\left[(e^{y_{t_i}} - 1)^2\right] \\ &= \frac{100^2}{T} \sum_{i=1}^{N} E\left(e^{2y_{t_i}} - 2e^{y_{t_i}} + 1\right) \\ &= \frac{100^2}{T} \sum_{i=1}^{N} [\mu(-2j; t_{i-1}, t_i, v_0, \theta_0, X_0) - 2\mu(-j; t_{i-1}, t_i, v_0, \theta_0, X_0) + 1], \end{aligned} \tag{B-1}$$

according to the specific formulation of the forward characteristic function.

We now start to deal with volatility swaps. We can compute

$$\begin{aligned} E[|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}|] &= \int_0^{+\infty} (e^{y_{t_i}} - 1) p(y_{t_i}) dy_{t_i} + \int_{-\infty}^{0} (1 - e^{y_{t_i}}) p(y_{t_i}) dy_{t_i}, \\ &= -\int_0^{+\infty} p(y_{t_i}) dy_{t_i} + \int_{-\infty}^{0} p(y_{t_i}) dy_{t_i} \\ &+ \int_0^{+\infty} e^{y_{t_i}} p(y_{t_i}) dy_{t_i} - \int_{-\infty}^{0} e^{y_{t_i}} p(y_{t_i}) dy_{t_i}, \end{aligned} \tag{B-2}$$

with $p(y_{t_i})$ defined as the forward density function of y_{t_i}. Clearly, all we need to do is to work out the four integrations. In fact, considering the fact that $p(y_{t_i})$ is a density function, we could easily obtain

$$\int_0^{+\infty} p(y_{t_i})dy_{t_i} = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} RE\left[\frac{\mu(\phi; v_0, \theta_0, X_0, t_{i-1}, t_i)}{j\phi}\right]d\phi, \tag{B-3}$$

which is a famous theory relating the characteristic function and the density. Moreover, it should also be pointed out that

$$\int_{-\infty}^{+\infty} e^{y_{t_i}} p(y_{t_i})dy_{t_i} = \mu(-j; v_0, \theta_0, X_0, t_{i-1}, t_i), \tag{B-4}$$

which would lead to the conclusion that

$$\bar{\mu}(\phi; v_0, \theta_0, X_0, t_{i-1}, t_i) = \frac{\mu(\phi - j; v_0, \theta_0, X_0, t_{i-1}, t_i)}{\mu(-j; v_0, \theta_0, X_0, t_{i-1}, t_i)} \tag{B-5}$$

denotes a characteristic function corresponding to a density $\frac{e^{y_{t_i}} p(y_{t_i})}{\mu(-j; v_0, \theta_0, X_0, t_{i-1}, t_i)}$. Therefore, one could obtain

$$\int_0^{+\infty} \frac{e^{y_{t_i}} p(y_{t_i})}{\mu(-j; v_0, \theta_0, X_0, t_{i-1}, t_i)}dy_{t_i} = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} RE\left[\frac{\mu(\phi - j; v_0, \theta_0, X_0, t_{i-1}, t_i)}{j\phi\mu(-j; v_0, \theta_0, X_0, t_{i-1}, t_i)}\right]d\phi. \tag{B-6}$$

Combining Eqs (B-3) and (B-6) will lead us to the following equality

$$E\left[\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|\right] = \frac{2}{\pi}\int_0^{+\infty} RE\left[\frac{\mu(\phi - j; v_0, \theta_0, X_0, t_{i-1}, t_i) - \mu(\phi; v_0, \theta_0, X_0, t_{i-1}, t_i)}{j\phi}\right]d\phi, \tag{B-7}$$

from which we could reach

$$\begin{aligned} K_{vol} &= 100\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N} E\left[\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|\right] \\ &= 100\sqrt{\frac{2}{\pi NT}}\int_0^{+\infty}\sum_{i=1}^{N} RE\left[\frac{\mu(\phi - j; v_0, \theta_0, X_0, t_{i-1}, t_i) - \mu(\phi; v_0, \theta_0, X_0, t_{i-1}, t_i)}{j\phi}\right]d\phi. \end{aligned} \tag{B-8}$$