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***Research article***

# Some zero product preserving additive mappings of operator algebras

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**Abstract:** Let $\mathcal{M}$ be a von Neumann algebra without direct commutative summands, and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(\mathcal{M})$ containing $\mathcal{M}$, where $LS(\mathcal{M})$ is the $^*$-algebra of all locally measurable operators with respect to $\mathcal{M}$. Suppose $\delta$ is an additive mapping from $\mathcal{A}$ to $LS(\mathcal{M})$ that satisfies the condition $\delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) = 0$ whenever $AB = BA = 0$. In this paper, we prove that there exists an element $Y$ in $LS(\mathcal{M})$ such that $\delta(X) = XY - YX^*$, for every $X$ in $\mathcal{A}$.

**Keywords:** AF algebra; Jordan $^*$-derivation; locally measurable operator; properly infinite $C^*$-algebra; von Neumann algebra
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## 1. Introduction

Let $\mathcal{A}$ be a $^*$-ring, meaning a ring with involution $^*$, and let $\mathcal{B}$ be a subring of $\mathcal{A}$. An additive mapping $\delta : \mathcal{B} \to \mathcal{A}$ is called a Jordan $^*$-derivation (though in some literature, this term may carry a different meaning) if

$$\delta(T^2) = \delta(T)T^* + T\delta(T)$$

for all $T \in \mathcal{B}$. It can be easily verified that if $\mathcal{A}$ is 2-torsion-free, meaning $2A = 0$ implies $A = 0$ for every $A$ in $\mathcal{A}$, then a Jordan $^*$-derivation can be equivalently defined as

$$\delta(A \circ B) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A)$$

for all $A, B \in \mathcal{B}$, where $A \circ B = AB + BA$. For each $A \in \mathcal{A}$, one can define a Jordan $^*$-derivation $\delta_A$ by $\delta_A(T) = TA - AT^*$, for all $T \in \mathcal{B}$. Such Jordan $^*$-derivations are referred to as inner Jordan $^*$-derivations.

The significance of Jordan $^*$-derivations lies in their structural importance in problems concerning the representability of quadratic functionals by sesquilinear forms on modules (see [10–12]).

Brešar and Vukman [3] established that if a unital $^*$-ring $\mathcal{A}$ contains $\frac{1}{2}$ and a central invertible element $A$ such that $A^* = -A$, then every Jordan $^*$-derivation from $\mathcal{A}$ to itself is inner. Consequently, every Jordan $^*$-derivation on a unital complex $^*$-algebra is inner. To adapt the approach employed in the proof of [3, Theorem 1], the following lemma can be derived:

**Lemma 1.1.** *Let $\mathcal{A}$ be a complex $^*$-algebra with the unity* **1**, *and let $\mathcal{B}$ be an arbitrary subalgebra of $\mathcal{A}$. Then every Jordan $^*$-derivation from $\mathcal{B}$ into $\mathcal{A}$ is inner.*

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a real or complex Hilbert space $\mathcal{H}$ with dim $\mathcal{H} > 1$, and let $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$. Šemrl [11] proved that every Jordan $^*$-derivation from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is inner.

Let $\mathcal{R}$ be a 2-trision-free, noncommutative prime $^*$-ring with a nontrivial projection. Qi and Zhang [8] demonstrated that if $\delta : \mathcal{R} \to \mathcal{R}$ satisfies the condition

$$\delta(A \circ B) = \delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) \text{ whenever } AB = 0, \qquad (\mathbb{P}_1)$$

then $\delta$ is a Jordan $^*$-derivation.

Consider a real Hilbert space $\mathcal{H}$ with dim $\mathcal{H} = \infty$, and let $\delta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a real linear mapping. Qi and Wang in [9] established that if $\delta$ satisfies

$$\delta(A)A^* + A\delta(A) = 0 \text{ whenever } A^2 = 0, \qquad (\mathbb{P}_2)$$

then $\delta$ is inner. In the same paper, the authors constructed an example of an additive mapping that satisfies condition $(\mathbb{P}_2)$ but is not a Jordan $^*$-derivation on the algebra of all $2 \times 2$ real matrices. This implies that, on a large class of $^*$-rings, an additive mapping $\delta$ that only satisfies condition $(\mathbb{P}_2)$ is not sufficient to ensure it is a Jordan $^*$-derivation.

Motivated by these results, in this paper, we aim to characterize an additive mapping $\delta : \mathcal{B} \to \mathcal{A}$ satisfying the following condition:

$$\delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A) = 0 \text{ whenever } AB = BA = 0. \qquad (\mathbb{P})$$

Clearly, condition $(\mathbb{P})$ is weaker than condition $(\mathbb{P}_1)$.

In the paper, our main focus is on investigating the aforementioned preservation problem of operator algebras, specifically in von Neumann algebras and $C^*$-algebras. For a von Neumann algebra $\mathcal{M}$, we approach the study within a broader context by considering $\mathcal{M}$ as a subalgebra of the $^*$-algebra of all locally measurable operators with respect to $\mathcal{M}$. Regarding $C^*$-algebras, achieving results for the preservation problem discussed above is challenging in general $C^*$-algebras. Hence, we primarily focus on properly infinite, primitive, and AF (approximately finite) $C^*$-algebras. In the paper, we present the following main results:

(1) Let $\mathcal{M}$ be a von Neumann algebra without direct commutative summands, and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(\mathcal{M})$ containing $\mathcal{M}$. An additive mapping $\delta : \mathcal{A} \to LS(\mathcal{M})$ is an inner Jordan $^*$-derivation if and only if it satisfies condition $(\mathbb{P})$.
(2) Let $\mathcal{A}$ be a properly infinite $C^*$-algebra. An additive mapping $\delta : \mathcal{A} \to \mathcal{A}$ is inner if and only if it satisfies condition $(\mathbb{P})$.
(3) Let $\mathcal{A}$ be a unital noncommutative primitive $C^*$-algebra with a nonzero $soc(\mathcal{A})$, and let $\mathcal{B}$ be an arbitrary subalgebra of $\mathcal{A}$ containing $soc(\mathcal{A})$. An additive mapping $\delta : \mathcal{B} \to \mathcal{A}$ is inner if and only if it satisfies condition $(\mathbb{P})$.

(4) Suppose $\mathcal{A} = \overline{\bigcup \mathcal{A}_n}$ is an AF algebra such that $\mathcal{A}_1$ has no direct commutative summands. Let $\mathcal{B}$ be chosen from $\mathcal{A}$ or $\bigcup_{n=1}^{N} \mathcal{A}_n$, where $N$ is a finite integer or infinite. Then an additive mapping $\delta : \mathcal{B} \to \mathcal{A}$ is inner if and only if it satisfies condition $(\mathbb{P})$.

## 2. Main results

An element $P$ in a $^*$-ring is called a projection if $P^* = P = P^2$. Let $\mathcal{G}$ be a $^*$-ring with unity $\mathbf{1}$ and a nontrivial projection $P_1$. By $P_2$, we shall always mean $\mathbf{1} - P_1$ unless otherwise specified. To obtain the main results of the paper, we first present the following theorem, which generalizes [8, Theorem 2.2]. Additionally, we place the proof of the theorem at the end of the paper to keep the focus on its main points.

**Theorem 2.1.** *Let $\mathcal{G}$ be a 2-torsion-free $^*$-ring with unity $\mathbf{1}$ and a nontrivial projection $P_1$. Suppose $\mathcal{U}$ is a subring of $\mathcal{G}$ that satisfies the following conditions:*

(1) $P_1, P_2 \in \mathcal{U}$;
(2) $\mathcal{U}P_2$ *left separates* $P_1\mathcal{G}P_1$, *i.e. for each $A$ in $P_1\mathcal{G}P_1$, $A\mathcal{U}P_2 = \{0\}$ implies that $A = 0$;*
(3) $P_1\mathcal{U}$ *right separates* $P_2\mathcal{G}P_2$, *i.e. for each $A$ in $P_2\mathcal{G}P_2$, $P_1\mathcal{U}A = \{0\}$ implies that $A = 0$.*

*If $\delta : \mathcal{U} \to \mathcal{G}$ is an additive mapping satisfying condition $(\mathbb{P})$, then it is a Jordan $^*$-derivation.*

Recall that a ring $\mathcal{R}$ is prime if $A\mathcal{R}B = 0$ $(A, B \in \mathcal{R})$, which implies that $A = 0$ or $B = 0$.

Applying Theorem 2.1, we can get the following corollary immediately.

**Corollary 2.1.** *Let $\mathcal{A}$ be a unital, 2-torsion-free, noncommutative prime $^*$-ring with a nontrivial projection. If $\delta : \mathcal{A} \to \mathcal{A}$ is an additive mapping satisfying condition $(\mathbb{P})$, then $\delta$ is a Jordan $^*$-derivation.*

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{M}$ be a von Neumann algebra in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{P}(\mathcal{M})$ be the set of all projections in $\mathcal{M}$, and $\mathcal{P}_{\text{fin}}(\mathcal{M})$ be the subset of all finite projections of $\mathcal{P}(\mathcal{M})$.

A linear subspace $\mathcal{D}$ in $\mathcal{H}$ is affiliated with $\mathcal{M}$ (denoted as $\mathcal{D}\ \eta\ \mathcal{M}$) if $u(\mathcal{D}) \subseteq \mathcal{D}$ for every unitary operator $u$ in $\mathcal{M}'$, the commutant of $\mathcal{M}$. $\mathcal{D}$ is strongly dense in $\mathcal{H}$ with respect to $\mathcal{M}$, if $\mathcal{D}\ \eta\ \mathcal{M}$ and there is a sequence of projections $\{P_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathcal{M})$, such that $P_n \uparrow \mathbf{1}$, $p_n(\mathcal{H}) \subseteq \mathcal{D}$, and $\mathbf{1} - P_n \in \mathcal{P}_{\text{fin}}(\mathcal{M})$, for every $n \in \mathbb{N}$. A linear operator $x$ on $\mathcal{H}$ with a dense domain $\mathcal{D}(x)$ is said to be affiliated with $\mathcal{M}$ (denoted as $x\ \eta\ \mathcal{M}$) if $\mathcal{D}(x)\ \eta\ \mathcal{M}$ and $ux(\xi) = xu(\xi)$ for all $\xi \in \mathcal{D}(x)$ and for every unitary operator in $\mathcal{M}'$. A closed linear operator $x$ acting in $\mathcal{H}$ is measurable with respect to $\mathcal{M}$ if $x\ \eta\ \mathcal{M}$ and $\mathcal{D}(x)$ strongly dense in $\mathcal{H}$. Let $S(\mathcal{M})$ denote the set of all measurable operators.

A closed linear operator $x$ acting in $\mathcal{H}$ is called locally measurable with respect to $\mathcal{M}$ if $x\ \eta\ \mathcal{M}$ and there is a sequence $\{P_n\}_{n=1}^{\infty}$ of central projections in $\mathcal{M}$ such that $P_n \uparrow \mathbf{1}$ and $xP_n \in S(\mathcal{M})$ for every $n \in \mathbb{N}$.

The set $LS(\mathcal{M})$ of all locally measurable operators with respect to $\mathcal{M}$ forms a unital $*$-algebra with respect to algebraic operators of strong addition and multiplication and taking the adjoint of an operator. Both $\mathcal{M}$ and $S(\mathcal{M})$ are subalgebras of $LS(\mathcal{M})$. Refer to [1,5] and related literature for further details.

**Theorem 2.2.** *Let $\mathcal{M}$ be a von Neumann algebra without direct commutative summands, and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(\mathcal{M})$ containing $\mathcal{M}$. If $\delta : \mathcal{A} \to LS(\mathcal{M})$ is an additive mapping satisfying condition $(\mathbb{P})$, then it is an inner Jordan $^*$-derivation.*

*Proof.* By assumption, $\mathcal{M}$ has no direct commutative summands; there exists a projection $P_1$ in $\mathcal{M}$ such that $C_{P_1} = C_{P_2} = \mathbf{1}$, where $C_{P_i}$ denotes the central carrier of $P_i$ for $i = 1, 2$.

To prove that $\delta$ is an inner Jordan $^*$-derivation, according to Lemma 1.1 and Theorem 2.1, it is sufficient to show that $\mathcal{M}P_2$ left separates $P_1LS(\mathcal{M})P_1$ and $P_1\mathcal{M}$ right separates $P_2LS(\mathcal{M})P_2$.

Assume that $A \in P_1LS(\mathcal{M})P_1$ and $AX = 0$ for each $X \in \mathcal{M}P_2$. It follows from [4, Proposition 6.1.8] that there are projections $Q_1$ and $T_1$ such that $Q_1 \leq P_1$, $T_1 \leq P_2$, and $Q_1 \sim T_1$. Then there exists a partial isometry $V \in \mathcal{M}$ such that $V^*V = Q_1$ and $VV^* = T_1$. Thus,

$$P_1AQ_1 = P_1AP_1Q_1 = AP_1V^*VQ_1 = AP_1V^*VV^*V = A(P_1V^*T_1P_2)V = 0.$$

If $Q_1 = P_1$, then the proof is complete. If $P_1 - Q_1 \neq 0$, it implies that $C_{P_1-Q_1}C_{P_2} \neq 0$. By [4, Proposition 6.1.8], there exist $Q_2 \leq P_1 - Q_1$ and $T_2 \leq P_2$ with $Q_2 \sim T_2$. Let $Q_\alpha$ be an orthogonal family of projections in $\mathcal{M}$ maximal with respect to the property that $Q_\alpha \leq P_1$, and $P_1AQ_\alpha = 0$ for each $\alpha$. By maximality of $Q_\alpha$, we have $P_1 = \sum Q_\alpha$. Therefore,

$$A = P_1AP_1 = \sum P_1AQ_\alpha = 0.$$

Using a similar technique, we can show that $P_1\mathcal{M}$ right separates $P_2LS(\mathcal{M})P_2$, and we omit it here. The proof is complete. □

In a $C^*$-algebra $\mathcal{A}$, projections $P$ and $Q$ are considered (Murray-von Neumann ) equivalent, denoted by $P \sim Q$, if there exists a partial isometry $V \in \mathcal{A}$ such that $V^*V = P$, $VV^* = Q$, and $P \precsim Q$ if $P$ is equivalent to a subprojection of $Q$. Note that $P \precsim Q$ and $Q \precsim P$ do not necessarily imply $P \sim Q$ in general $C^*$-algebras. In other words, there is no Schröder–Bernstein theorem for the equivalence of projections in general $C^*$-algebras.

A nonzero projection $P$ in $\mathcal{A}$ is termed properly infinite if there exist mutually orthogonal subprojections $Q_1$ and $Q_2$ of $P$ such that $Q_1 \sim P \sim Q_2$. A unital $C^*$-algebra is properly infinite if its unity $\mathbf{1}$ is properly infinite. For example, the Calkin algebra and Cuntz algebras are properly infinite.

**Theorem 2.3.** *Let $\mathcal{A}$ be a properly infinite $C^*$-algebra. If $\delta : \mathcal{A} \to \mathcal{A}$ is an additive mapping satisfying condition* $(\mathbb{P})$*, then $\delta$ is an inner Jordan $^*$-derivation.*

*Proof.* The first step is to establish the following claim:
**Claim 1.** If $P$ and $Q$ are two projections in $\mathcal{A}$ such that $P \precsim Q$, then $\mathcal{A}Q$ ($Q\mathcal{A}$) left (right) separates $P\mathcal{A}P$.

Since $P \precsim Q$, there exists a partial isometry $V \in \mathcal{A}$ with $V^*V = P$ and $VV^* = Q_1 \leqslant Q$. Now, suppose $A \in P\mathcal{A}P$ such that $A\mathcal{A}Q = 0$. This implies

$$0 = AV^*Q = AV^*VV^*Q = AV^*Q_1 = AV^*,$$

which leads to $A = 0$. Similarly, we can demonstrate that $Q\mathcal{A}$ right separates $P\mathcal{A}P$. Thus, Claim 1 is validated.

Next, consider mutually orthogonal projections $P_1$, $Q_1$ in $\mathcal{A}$ such that $P_1 \sim \mathbf{1} \sim Q_1$.
**Claim 2.** $P_2 \precsim P_1$.

Let $U \in \mathcal{A}$ be a partial isometry such that $U^*U = \mathbf{1}$ and $UU^* = P_1$. Define $T = UP_2U^*$, which is a projection satisfying $T \leq P_1$. Let $S = UP_2$. Then $S$ is a partial isometry operator. Clearly, $SS^* = T$ and $S^*S = P_2$. Therefore,

$$P_2 \sim T \leq P_1.$$

Moreover, it is evident that $P_1 \precsim P_2$. From Claims 1 and 2, we conclude that $\mathcal{A}P_2$ left separates $P_1\mathcal{A}P_1$, and $P_1\mathcal{A}$ right separates $P_2\mathcal{A}P_2$. By Lemma 1.1 and Theorem 2.1, we conclude that the statement holds. The proof is complete. □

A complex unital Banach $^*$-algebra $\mathcal{A}$ is called proper if $A^*A = 0$ implies $A = 0$ for each $A \in \mathcal{A}$. Suppose a proper $^*$-algebra $\mathcal{A}$ has a minimal left ideal $\mathcal{J}$, or equivalently, there exists a minimal projection $P \in \mathcal{A}$ such that $\mathcal{J} = \mathcal{A}P$. The sum of all minimal left ideals is referred to as the socle of $\mathcal{A}$, denoted by soc($\mathcal{A}$). If $\mathcal{A}$ does not have minimal left ideal, we define soc($\mathcal{A}$) = 0. It is well known that the socle of $\mathcal{B}(\mathcal{H})$ is identical to $\mathcal{F}(\mathcal{H})$, the ideal of all finite rank operators in $\mathcal{B}(\mathcal{H})$ (cf. [6, p.1142 and 1143]).

**Theorem 2.4.** *Let $\mathcal{A}$ be a complex proper Banach $^*$-algebra with unity* **1***, and $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ containing $soc(\mathcal{A})$. Suppose there is a minimal projection $P_1$ in $\mathcal{A}$ such that $P_1 soc(\mathcal{A})$ right separates $P_2\mathcal{A}P_2$. If $\delta : \mathcal{B} \to \mathcal{A}$ is an additive mapping satisfying condition* ($\mathbb{P}$)*, then $\delta$ is an inner Jordan $^*$-derivation.*

*Proof.* Without loss of generality, we assume that $\mathbf{1}$ in $\mathcal{B}$. If $\mathbf{1} \notin \mathcal{B}$, let $\mathcal{B}_1 = \mathcal{B} + \mathbb{C}\mathbf{1}$. In this case, consider the mapping $\widetilde{\delta} : \mathcal{B}_1 \to \mathcal{A}$ by $\widetilde{\delta}(B + \lambda\mathbf{1}) = \delta(B)$ for each $B \in \mathcal{B}$. Clearly, $\widetilde{\delta}|_{\mathcal{B}} = \delta$.

To prove that $\delta$ is an inner Jordan $^*$-derivation, according to Lemma 1.1 and Theorem 2.1, it is sufficient to show that $soc(\mathcal{A})P_2$ left separates $P_1\mathcal{A}P_1$.

For any $A \in \mathcal{A}$, it follows from [6, Theorem 10.6.2, p.1143] that there exists a continuous linear functional $f$ on $\mathcal{A}$ such that $P_1AP_1 = f(A)P_1$. Given the assumption that $P_1 soc(\mathcal{A})$ right separates $P_2\mathcal{A}P_2$, it implies $P_1 soc(\mathcal{A})P_2 \neq 0$. If $P_1AP_1 soc(\mathcal{A})P_2 = 0$, it follows that $f(A) = 0$. Consequently, $P_1AP_1 = 0$. □

Let $\mathcal{A}$ be a $C^*$-algebra. A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be irreducible if $\pi(\mathcal{A})$ has no nontrivial invariant subspace. $\mathcal{A}$ is called primitive if it has a faithful irreducible representation. It is easy to verify that every primitive $C^*$-algebra is prime, and for separable algebras, the converse is also true (cf. [2, p. 112]).

**Corollary 2.2.** *Let $\mathcal{A}$ be a unital noncommutative primitive $C^*$-algebra with a nonzero $soc(\mathcal{A})$, and let $\mathcal{B}$ be an arbitrary subalgebra of $\mathcal{A}$ containing $soc(\mathcal{A})$. If $\delta : \mathcal{B} \to \mathcal{A}$ is an additive mapping satisfying condition* ($\mathbb{P}$)*, then it is an inner Jordan $^*$-derivation.*

*Proof.* Consider $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, a faithful irreducible representation of $\mathcal{A}$. If $soc(\mathcal{A}) \neq 0$, it implies $soc(\pi(\mathcal{A})) \neq 0$. According to [7, Theorem 6.1.5], we have $soc(\pi(\mathcal{A})) \supseteq \mathcal{F}(\mathcal{H})$. This implies that $P_1 soc(\mathcal{A})$ right separates $P_2\mathcal{A}P_2$ for every minimal projection $P_1$ in $\mathcal{A}$. The conclusion then follows from Theorem 2.4. □

The following theorem improves the main result of [11].

**Theorem 2.5.** *Let $\mathcal{H}$ be a real or complex Hilbert space, dim $\mathcal{H} > 1$, and let $\mathcal{A}$ be a standard operator algebra on $\mathcal{H}$. Suppose that $\delta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is an additive mapping satisfying condition* ($\mathbb{P}$)*. Then there exists a unique linear operator $A \in \mathcal{B}(\mathcal{H})$ such that $\delta(X) = XA - AX^*$ for all $X \in \mathcal{A}$.*

*Proof.* In the real space setting of $\mathcal{H}$, Theorem 2.1 establishes $\delta$ as a Jordan $^*$-derivation. When $\mathcal{H}$ is a complex space, an immediate application of Corollary 2.2 confirms $\delta$ as an inner Jordan $^*$-derivation. Therefore, the conclusion holds true in both cases, as supported by [11, Theorem]. □

Let $\mathcal{B}$ be a $C^*$-subalgebra of a $C^*$-algebra $\mathcal{A}$. A *conditional expectation* from $\mathcal{A}$ to $\mathcal{B}$ is a completely positive contraction $\phi : \mathcal{A} \to \mathcal{B}$ such that $\phi(B) = B$, $\phi(BA) = B\phi(A)$, and $\phi(AB) = \phi(A)B$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. If $\mathcal{B}$ is injective, then there exists a conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ (cf. [2, IV.2.1]).

Recall that an approximately finite (AF) algebra is a unital $C^*$-algebra $\mathcal{A}$, which is an inductive limit of an increasing sequence of finite-dimensional $C^*$-algebras $\mathcal{A}_n$, $1 \leq n < \infty$, with unital embeddings $\jmath_n : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$. Equivalently, $\mathcal{A}$ is an AF algebra if it can be represented as the closed union of an ascending sequence of finite-dimensional $C^*$-algebras. Clearly, every finite-dimensional $C^*$-algebra is injective; thus, there exists a sequence $\phi_n : \mathcal{A} \to \mathcal{A}_n$ of conditional expectations such that

$$\lim_{n\to\infty} \phi_n(A) = A, \ A \in \mathcal{A}. \tag{2.1}$$

**Theorem 2.6.** *Suppose $\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n}$ is an AF algebra such that $\mathcal{A}_1$ has no direct commutative summands. Let $\mathcal{B}$ be either $\mathcal{A}$ or $\bigcup_{n=1}^{N} \mathcal{A}_n$, where N is a finite integer or infinite. If $\delta : \mathcal{B} \to \mathcal{A}$ is an additive mapping satisfying condition* ($\mathbb{P}$)*, then $\delta$ is an inner Jordan $^*$-derivation.*

*Proof.* We divide the proof into two cases.

**Case 1.** Let $\mathcal{B} = \bigcup_{n=1}^{N} \mathcal{A}_n$. For any positive integer $k$ ($k < N + 1$), we consider the mapping $\phi_k \circ \delta : \mathcal{B} \to \mathcal{A}_k$. Let $A, B \in \mathcal{A}_k$ such that $AB = BA = 0$, then

$$\begin{aligned}\phi_k \circ \delta(A)B^* + A\phi_k \circ \delta(B) + \phi_k \circ \delta(B)A^* + B\phi_k \circ \delta(A) \\ = \phi_k \left(\delta(A)B^* + A\delta(B) + \delta(B)A^* + B\delta(A)\right) \\ = \phi_k(0) = 0.\end{aligned}$$

Thus, $\phi_k \circ \delta \mid_{\mathcal{A}_k}: \mathcal{A}_k \to \mathcal{A}_k$ satisfies condition ($\mathbb{P}$). Since $\mathcal{A}_1$ has no direct commutative summands, it implies that $\mathcal{A}_k$ has no direct commutative summands. Thus, $\mathcal{A}_k = \mathbf{M}_{k_1} \oplus \cdots \oplus \mathbf{M}_{k_l}$, where $k_i \geq 2$ for each $i$. Through routine calculation, we can show that $\phi_k \circ \delta \mid_{\mathcal{A}_k} (\mathbf{M}_{k_i}) \subseteq \mathbf{M}_{k_i}$. By Theorem 2.1, we have $\phi_k \circ \delta \mid_{\mathcal{A}_k}$ is a Jordan $^*$-derivation.

Fix $n$ and let $n \leq k < N + 1$. For each $A_n \in \mathcal{A}_n \subseteq \mathcal{A}_k$, we have

$$\phi_k \circ \delta(A_n^2) = \phi_k \circ \delta(A_n)A_n^* + A_n\phi_k \circ \delta(A_n).$$

It follows from Eq (2.1) that

$$\delta(A_n^2) = \delta(A_n)A_n^* + A_n\delta(A_n)$$

on $\bigcup_{n=1}^{N} \mathcal{A}_n$. Therefore, $\delta$ is a Jordan $^*$-derivation. By Lemma 1.1, $\delta$ is inner. Hence, we finish the proof of the first statement.

**Case 2.** Assume $\delta$ is defined from $\mathcal{A}$ to itself. Fix $n$ and choose a nontrivial projection $P_1$ in $\mathcal{A}_n$. Now, consider an element $A \in \mathcal{A}$ such that $P_1AP_1 \neq 0$. This implies the existence of a subsequence $A_{k_m}$ converging to $A$, where $P_1A_{k_m}P_1 \neq 0$ and $A_{k_m} \in \mathcal{A}_{k_m}$ for each $k_m$. Since $\mathcal{A}_{k_m}$ is finite dimensional, it is also prime. Therefore,

$$P_1A_{k_m}P_1\mathcal{A}_{k_m}P_2 = P_1A_{k_n}P_1\mathcal{A}_{k_m}(\mathbf{1}_{\mathcal{A}_{k_m}} - P_1) \neq 0.$$

Hence, $P_1A_{k_m}P_1\mathcal{A}P_2 \neq 0$, implying $P_1AP_1\mathcal{A}P_2 \neq 0$.

Similarly, we can show that $P_1\mathcal{A}$ right separates $P_2\mathcal{A}P_2$. By applying Lemma 1.1 and Theorem 2.1, we conclude that $\delta$ is an inner Jordan $^*$-derivation. □

Next, we prove Theorem 2.1. Before providing its proof, we introduce the following lemmas, established under the assumptions of Theorem 2.1. For convenience, we denote $P_i\mathcal{G}P_j$ and $P_i\mathcal{U}P_j$ as $\mathcal{G}_{ij}$ and $\mathcal{U}_{ij}$, respectively. Then, the Peirce decomposition of $\mathcal{G}$ and $\mathcal{U}$ is as follows:

$$\mathcal{G} = \mathcal{G}_{11} + \mathcal{G}_{12} + \mathcal{G}_{21} + \mathcal{G}_{22}, \ \ \mathcal{U} = \mathcal{U}_{11} + \mathcal{U}_{12} + \mathcal{U}_{21} + \mathcal{U}_{22}.$$

**Lemma 2.1.** *The following statements hold:*

(1) $P_1\delta(P_2)P_1 = P_2\delta(P_1)P_2 = 0$;
(2) $P_1\delta(\mathbf{1})P_1 = P_1\delta(P_1)P_1$;
(3) $P_2\delta(\mathbf{1})P_2 = P_2\delta(P_2)P_2$.

*Proof.* Since $P_1P_2 = P_2P_1 = 0$, it follows from the assumption that

$$\delta(P_1)P_2 + P_2\delta(P_1) + \delta(P_2)P_1 + P_1\delta(P_2) = 0.$$

Multiplying the above equation from both sides by $P_1$, we have $2P_1\delta(P_2)P_1 = 0$. Given that $\mathcal{G}_{11}$ is 2-torsion-free by assumption, we have $P_1\delta(P_2)P_1 = 0$. Similarly, we have $P_2\delta(P_1)P_2 = 0$. Thus, (1) holds. Statements (2) and (3) are easily verified from (1), and we omit the details. Hence, the proof is complete. □

**Lemma 2.2.** *If $E$ is an idempotent in $\mathcal{U}$, then $E\delta(\mathbf{1}) = \delta(\mathbf{1})E^*$.*

*Proof.* Since $E(\mathbf{1} - E) = (\mathbf{1} - E)E = 0$, it follows from the assumption that

$$\delta(E)(\mathbf{1} - E)^* + E\delta(\mathbf{1} - E) + \delta(\mathbf{1} - E)E^* + (\mathbf{1} - E)\delta(E) = 0.$$

Hence,

$$2\delta(E) + \delta(\mathbf{1})E^* + E\delta(\mathbf{1}) = 2\delta(E)E^* + 2E\delta(E). \tag{2.2}$$

Multiplying by $E^*$ from the right side in Eq (2.2), we have

$$\delta(\mathbf{1})E^* + E\delta(\mathbf{1})E^* = 2E\delta(E)E^*.$$

Multiplying by $E$ from the left side in Eq (2.2), we have

$$E\delta(\mathbf{1})E^* + E\delta(\mathbf{1}) = 2E\delta(E)E^*.$$

Combining the above two equations, we obtain $E\delta(\mathbf{1}) = \delta(\mathbf{1})E^*$. □

Applying the above result, we can get the following lemma immediately.

**Lemma 2.3.** $\delta(\mathbf{1}) = P_1\delta(\mathbf{1})P_1 + P_2\delta(\mathbf{1})P_2$.

**Lemma 2.4.** $\delta(\mathbf{1}) = 0$.

*Proof.* For any $G_{12} \in \mathcal{U}_{12}$, then $P_1 + G_{12}$ is an idempotent in $\mathcal{U}$. By Lemma 2.2, we have

$$(P_1 + G_{12})^* \delta(\mathbf{1}) = \delta(\mathbf{1})(P_1 + G_{12}).$$

It follows from Lemma 2.2 that

$$G_{12}^* \delta(\mathbf{1}) = \delta(\mathbf{1}) G_{12}.$$

By Lemma 2.3, we have $\delta(\mathbf{1})G_{12} \in \mathcal{G}_{12}$ and $G_{12}^*\delta(\mathbf{1}) \in \mathcal{G}_{21}$. This means that $\delta(\mathbf{1})G_{12} = 0$ for any $G_{12} \in \mathcal{U}_{12}$. By assumption, $\mathcal{U}P_2$ left separates $\mathcal{G}_{11}$, it follows that $P_1\delta(\mathbf{1})P_1 = 0$. Similarly, $P_2\delta(\mathbf{1})P_2 = 0$. Using Lemma 2.3, then $\delta(\mathbf{1}) = 0$. The proof is complete. □

For every $A, B \in \mathcal{G}$, let $[A, B]_* = AB - BA^*$. Define an additive mapping $\sigma : \mathcal{U} \to \mathcal{G}$ by the formula:

$$\sigma(G) = [G, P_1\delta(P_1)P_2 + P_2\delta(P_2)P_1]_* - \delta(G), \ G \in \mathcal{U}.$$

It is evident that $\sigma$ satisfies condition ($\mathbb{P}$). Additionally, it is straightforward to verify that $\sigma(P_1) = \sigma(P_2) = 0$.

**Lemma 2.5.** *For each $G \in \mathcal{U}$, the following statements hold:*

(1) $\sigma(G_{11}) \in \mathcal{G}_{11}$;
(2) $\sigma(G_{22}) \in \mathcal{G}_{22}$;
(3) $P_1\sigma(G_{12})P_1 = P_2\sigma(G_{12})P_2 = 0$;
(4) $P_1\sigma(G_{21})P_1 = P_2\sigma(G_{21})P_2 = 0$.

*Proof.* For any $G_{11} \in \mathcal{U}_{11}$, we have $G_{11}P_2 = P_2G_{11} = 0$. Therefore,

$$\sigma(G_{11})P_2 + G_{11}\sigma(P_2) + \sigma(P_2)G_{11}^* + P_2\sigma(G_{11}) = 0.$$

Simplifying further, we obtain

$$\sigma(G_{11})P_2 + P_2\sigma(G_{11}) = 0. \tag{2.3}$$

Multiplying both sides of Eq (2.3) by $P_1$ from the left side, we have

$$P_1\sigma(G_{11})P_2 = 0.$$

Similarly, multiplying both sides of Eq (2.3) by $P_1$ from the right side, we obtain

$$P_2\sigma(G_{11})P_1 = 0.$$

Furthermore, multiplying both sides of Eq (2.3) by $P_2$ and using the assumption that $\mathcal{G}$ is 2-torsion-free, we have $P_2\sigma(G_{11})P_2 = 0$. This implies that

$$\sigma(G_{11}) \in \mathcal{G}_{11},$$

which proves statement (1). The proof for statement (2) follows a similar pattern as statement (1). Therefore, we omit it here.

$$(G_{12} + P_1)(P_2 - G_{12}) = (P_2 - G_{12})(G_{12} + P_1) = 0$$

implies that

$$\sigma(G_{12} + P_1)(P_2 - G_{12})^* + (G_{12} + P_1)\sigma(P_2 - G_{12}) \\ + \sigma(P_2 - G_{12})(G_{12} + P_1)^* + (P_2 - G_{12})\sigma(G_{12} + P_1) = 0.$$

Simplifying further, we obtain

$$\sigma(G_{12})P_1 + P_1\sigma(G_{12}) = \sigma(G_{12})P_2 + P_2\sigma(G_{12}).$$

Consequently, we have

$$P_1\sigma(G_{12})P_1 = P_2\sigma(G_{12})P_2 = 0.$$

Thus, (3) holds. The proof of statement (4) follows a similar approach to that of (3), so we omit it. The proof is complete. □

**Lemma 2.6.** *For each $G_{ij}$ in $\mathcal{U}_{ij}$ $(i, j = 1, 2)$, the following statements hold:*

(1) $\sigma(G_{11} \circ G_{12}) = \sigma(G_{11})G_{12}^* + G_{11}\sigma(G_{12}) + \sigma(G_{12})G_{11}^* + G_{12}\sigma(G_{11})$;
(2) $\sigma(G_{22} \circ G_{21}) = \sigma(G_{22})G_{21}^* + G_{22}\sigma(G_{21}) + \sigma(G_{21})G_{22}^* + G_{21}\sigma(G_{22})$;
(3) $\sigma(G_{11} \circ G_{21}) = \sigma(G_{11})G_{21}^* + G_{11}\sigma(G_{21}) + \sigma(G_{21})G_{11}^* + G_{21}\sigma(G_{11})$;
(4) $\sigma(G_{22} \circ G_{12}) = \sigma(G_{22})G_{12}^* + G_{22}\sigma(G_{12}) + \sigma(G_{12})G_{22}^* + G_{12}\sigma(G_{22})$;
(5) $\sigma(G_{11}^2) = \sigma(G_{11})G_{11}^* + G_{11}\sigma(G_{11})$;
(6) $\sigma(G_{22}^2) = \sigma(G_{22})G_{22}^* + G_{22}\sigma(G_{22})$;
(7) $\sigma(G_{12} \circ G_{21}) = \sigma(G_{12})G_{21}^* + G_{12}\sigma(G_{21}) + \sigma(G_{21})G_{12}^* + G_{21}\sigma(G_{12})$.

*Proof.*

$$(G_{11} - G_{11}G_{12})(G_{12} + P_2) = (G_{12} + P_2)(G_{11} - G_{11}G_{12}) = 0$$

implies

$$\sigma(G_{11} - G_{11}G_{12})(G_{12} + P_2)^* + (G_{11} - G_{11}G_{12})\sigma(G_{12} + P_2) \\ + \sigma(G_{12} + P_2)(G_{11} - G_{11}G_{12})^* + (G_{12} + P_2)\sigma(G_{11} - G_{11}G_{12}) = 0.$$

Simplifying further, we have

$$\sigma(G_{12})G_{11}^* + G_{11}\sigma(G_{12}) - \sigma(G_{11}G_{12})P_2 - P_2\sigma(G_{11}G_{12}) = 0.$$

Lemma 2.5 (3) implies that

$$\sigma(G_{11}G_{12})P_2 + P_2\sigma(G_{11}G_{12}) = \sigma(G_{11}G_{12}).$$

Combining the above two equations, we obtain

$$\sigma(G_{11}G_{12}) = \sigma(G_{12})G_{11}^* + G_{11}\sigma(G_{12}).$$

It follows from Lemma 2.5 (1) that

$$\sigma(G_{11} \circ G_{12}) = \sigma(G_{11})G_{12}^* + G_{11}\sigma(G_{12}) + \sigma(G_{12})G_{11}^* + G_{12}\sigma(G_{11}).$$

Thus, statement (1) holds. The proof of statement (2) follows a similar approach to that of (1), so we omit it.

$$(G_{11} + G_{21}G_{11})(G_{21} - P_2) = (G_{21} - P_2)(G_{11} + G_{21}G_{11}) = 0$$

implies

$$\begin{aligned}&\sigma(G_{11} + G_{21}G_{11})(G_{21} - P_2)^* + (G_{11} + G_{21}G_{11})\sigma(G_{21} - P_2)\\&\quad + \sigma(G_{21} - P_2)(G_{11} + G_{21}G_{11})^* + (G_{21} - P_2)\sigma(G_{11} + G_{21}G_{11}) = 0.\end{aligned}$$

In Lemma 2.5, we have

$$\begin{aligned}&\sigma(G_{11})G_{21}^* + G_{11}\sigma(G_{21}) + G_{21}\sigma(G_{11}) + \sigma(G_{21})G_{11}^*\\&\qquad = \sigma(G_{21}G_{11})P_2 + P_2\sigma(G_{21}G_{11}) = \sigma(G_{21}G_{11}),\end{aligned}$$

which means that

$$\sigma(G_{11} \circ G_{21}) = \sigma(G_{11})G_{21}^* + G_{11}\sigma(G_{21}) + G_{21}\sigma(G_{11}) + \sigma(G_{21})G_{11}^*. \tag{2.4}$$

Thus, (3) holds. Similarly, statement (4) is true.

For each $G_{11} \in \mathcal{U}_{11}$, by Eq (2.4), we have

$$\sigma(G_{12}^*G_{11}^2) = \sigma(G_{11}^2)G_{12} + G_{11}^2\sigma(G_{12}^*) + \sigma(G_{12}^*)(G_{11}^2)^* + G_{12}^*\sigma(G_{11}^2),$$

where $\sigma(G_{11}^2)G_{12} + G_{11}^2\sigma(G_{12}^*) \in \mathcal{G}_{12}$. On the other hand,

$$\begin{aligned}&\sigma(G_{12}^*G_{11}G_{11})\\&\quad = \sigma(G_{12}^*G_{11})G_{11}^* + G_{12}^*G_{11}\sigma(G_{11}) + \sigma(G_{11})(G_{12}^*G_{11})^* + G_{11}\sigma(G_{12}^*G_{11})\\&\quad = \sigma(G_{11})(G_{12}^*G_{11})^* + G_{11}\sigma(G_{11})G_{12} + G_{11}^2\sigma(G_{12}^*) + \sigma(G_{12}^*)(G_{11}^2)^*\\&\quad + G_{12}^*\sigma(G_{11})G_{11}^* + G_{12}^*G_{11}\sigma(G_{11}),\end{aligned}$$

where $\sigma(G_{11})(G_{12}^*G_{11})^* + G_{11}\sigma(G_{11})G_{12} + G_{11}^2\sigma(G_{12}^*) \in \mathcal{G}_{12}$. Thus

$$\sigma(G_{11}^2)G_{12} + G_{11}^2\sigma(G_{12}^*) = \sigma(G_{11})(G_{12}^*G_{11})^* + G_{11}\sigma(G_{11})G_{12} + G_{11}^2\sigma(G_{12}^*).$$

Therefore,

$$(\sigma(G_{11}^2) - \sigma(G_{11})G_{11}^* - G_{11}\sigma(G_{11}))\mathcal{U}_{12} = 0.$$

By assumption, $\mathcal{U}_{12}$ left separates $\mathcal{G}_{11}$; it follows that

$$\sigma(G_{11}^2) - \sigma(G_{11})G_{11}^* - G_{11}\sigma(G_{11}) = 0.$$

Thus, (5) holds. Similarly, statement (6) is also true.

Since

$$\begin{aligned}0 &= (G_{12}G_{21} + G_{12} + G_{21} + P_2)(G_{21}G_{12} - G_{12} - G_{21} + P_1)\\&= (G_{21}G_{12} - G_{12} - G_{21} + P_1)(G_{12}G_{21} + G_{12} + G_{21} + P_2).\end{aligned}$$

It follows that

$$\begin{aligned}0 = &\sigma(G_{12}G_{21} + G_{12} + G_{21} + P_2)(G_{21}G_{12} - G_{12} - G_{21} + P_1)^* \\ &+ (G_{12}G_{21} + G_{12} + G_{21} + P_2)\sigma(G_{21}G_{12} - G_{12} - G_{21} + P_1) \\ &+ \sigma(G_{21}G_{12} - G_{12} - G_{21} + P_1)(G_{12}G_{21} + G_{12} + G_{21} + P_2)^* \\ &+ (G_{21}G_{12} - G_{12} - G_{21} + P_1)\sigma(G_{12}G_{21} + G_{12} + G_{21} + P_2).\end{aligned}$$

The above equation implies that

$$2(\sigma(G_{12} \circ G_{21}) - \sigma(G_{12})G_{21}^* - G_{12}\sigma(G_{21}) - \sigma(G_{21})G_{12}^* - G_{21}\sigma(G_{12})) = 0.$$

By assumption, $\mathcal{G}$ is 2-torsion-free; it follows that

$$\sigma(G_{12} \circ G_{21}) - \sigma(G_{12})G_{21}^* - G_{12}\sigma(G_{21}) - \sigma(G_{21})G_{12}^* - G_{21}\sigma(G_{12}) = 0.$$

Thus, (7) holds. The proof is complete. □

Using Lemma 2.6, we obtain

**Lemma 2.7.** *$\sigma$ is a Jordan $^*$-derivation.*

**Proof of Theorem 2.1**. Based on the definition of $\sigma$, we have

$$\delta(G) = [G, P_1\delta(P_1)P_2 + P_2\delta(P_2)P_1]_* - \sigma(G),\ G \in \mathcal{U}.$$

Since the mapping $G \longmapsto [G, P_1\delta(P_1)P_2 + P_2\delta(P_2)P_1]_*$ is a Jordan $^*$-derivation, by Lemma 2.7, it follows that $\delta$ is a Jordan $^*$-derivation. □

## Author contributions

Wenbo Huang: validation, resources, writing original draft preparation, writing review and editing, funding acquisition; Jiankui Li: methodology, validation, resources, writing original draft preparation, writing review and editing, funding acquisition; Shaoze Pan: resources, writing original draft preparation, writing review and editing.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflicts of interest.

## References


1. S. Albeverio, S. Ayupov, K. Kudaydergenov, Structure of derivations on various algebras of measurable operators for type I von Neumann algebras, *J. Funct. Anal.*, **256** (2009), 2917–2943. https://doi.org/10.1016/j.jfa.2008.11.003
2. B. Blackadar, *Operator algebras: Theory of $C^*$-algebras and von Neumann algebras*, Berlin: Springer-Verlag, 2006. https://doi.org/10.1007/3-540-28517-2
3. M. Brešar, J. Vukman, On some additive mappings in rings with involution, *Aeq. Math.*, **38** (1989), 178–185. https://doi.org/10.1007/BF01840003
4. R. Kadison, J. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II*, American Mathematical Society, 1997.
5. M. Muratov, V. Chilin, Topological algebras of measurable and locally measurable operators, *J. Math. Sci.*, **239** (2019), 654–705. https://doi.org/10.1007/s10958-019-04320-y
6. T. Palmer, *Banach algebras and the general theory of $*$-algebras. Vol. II*, Cambridge: Cambridge University Press, 2001.
7. G. K. Pedersen, *$C^*$-algebras and their automorphism groups*, London: Academic Press, 2018. https://doi.org/10.1016/C2016-0-03431-9
8. X. Qi, F. Zhang, Multiplicative Jordan $^*$-derivations on rings with involution, *Linear Multilinear A.*, **64** (2016), 1145–1162. https://doi.org/10.1080/03081087.2015.1073217
9. X. Qi, M. Wang, Local characterization of Jordan $^*$-derivations on $\mathcal{B}(\mathcal{H})$, *Publ. Math. Debrecen*, **94** (2019), 421–434.
10. P. Šemrl, Quadratic functionals and Jordan $^*$-derivations, *Studia Math.*, **97** (1991), 157–165.
11. P. Šemrl, Jordan $^*$-derivations of standard operator algebras, *Proc. Amer. Math. Soc.*, **120** (1994), 515–518. https://doi.org/10.2307/2159889
12. J. Vukman, Some functional equations in Banach algebras and an application, *Proc. Amer. Math. Soc.*, **100** (1987), 133–136. https://doi.org/10.1090/S0002-9939-1987-0883415-0


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