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Research article

Notes on the generalized Perron complements involving inverse N_0-matrices

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Abstract: In the context of inverse N_0-matrices, this study focuses on the closure of generalized Perron complements by utilizing the characteristics of M-matrices, nonnegative matrices, and inverse N_0-matrices. In particular, we illustrate that the inverse N_0-matrix and its Perron complement matrix possess the same spectral radius. Furthermore, we present certain general inequalities concerning generalized Perron complements, Perron complements, and submatrices of inverse N_0-matrices. Finally, we provide specific examples to verify our findings.

Keywords: N_0-matrix; inverse N_0-matrix; generalized Perron complement; spectral radius; Schur complement
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1. Introduction

In this study, we work with real matrices of order n. As the results for $n = 1$ are trivial, we assume $n \geq 2$ throughout. If the matrix $A = \left(a_{ij}\right)$ is nonpositive (nonnegative), that is, $a_{ij} \leq 0$ $\left(a_{ij} \geq 0\right)$, we write $A \leq 0$ $(A \geq 0)$. Similarly, we write $A \geq B$ if and only if $A - B \geq 0$. According to the Perron-Frobenius theorem, if A is a nonnegative matrix, then the spectral radius, written as $\rho(A)$, is a characteristic root of the matrix A.

The collection of Z-matrices comprises real matrices that possess nonpositive elements in their off-diagonal positions. This collection of matrices is commonly encountered in various mathematical and physical science contexts and is generally referred to as

$$Z = \lambda I - A, A \geq 0, \tag{1}$$

where λ is a real number and I is the n-square identity matrix. According to (1), there is a strong connection between the Z-matrices and the nonnegative matrices. Fielder and Markham [1] conducted the first comprehensive examination of the entire collection of Z-matrices.

If we set $\lambda > \rho(A)$ in (1), we obtain the most famous subclass of Z-matrices, referred to as the M-matrices. The collection of M-matrices has many elegant characteristics and properties [2]. Many scholars have conducted extensive research on inverse M-matrices, that is, nonsingular matrices whose inverses are M-matrices [3]. One of the most important conclusions is that inverse M-matrices are nonnegative.

For any positive integer n, let $\{1, 2, \cdots, n\} = \langle n \rangle$. Throughout this study, we assume that β is an increasing sequence of integers chosen from $\{1, 2, \cdots, n\}$ and $\alpha = \langle n \rangle \setminus \beta$. By $A[\alpha, \beta]$, we denote the submatrix of the matrix A with rows α and columns β. In particular, if $\alpha = \beta$, $A[\alpha, \alpha]$ is abbreviated as $A[\alpha]$.

The following is the notion of the Schur complement. If $A[\beta]$ is nonsingular, then the Schur complement with respect to $A[\beta]$ in A, which is expressed as $A/A[\beta]$, is defined as

$$A/A[\beta] = A[\alpha] - A[\alpha, \beta](A[\beta])^{-1} A[\beta, \alpha].$$

The Schur complement has been extensively utilized in various fields, including applied mathematics and statistics [4, 5], particularly as an effective technique for deriving matrix inequalities, determining determinants, traces, norms, and handling large-scale matrix calculations. A significant amount of research on the Schur complements of specific matrices has been conducted since the late 1960s.

The Perron complement refers to a smaller square matrix obtained naturally from a given square matrix. This concept was initially introduced in [6] for computing the Perron vectors of finite-state Markov processes. The Perron complement is named after the Schur complement and has similar properties.

For an irreducible nonnegative matrix A of order n, Meyer [6] introduced the concept of Perron complement with respect to $A[\beta]$ as follows:

$$P(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](\rho(A) I - A[\beta])^{-1} A[\beta, \alpha]. \tag{2}$$

Recall that A is irreducible, it holds that $\rho(A) > \rho(A[\beta])$. This means that $\rho(A) I - A[\beta]$ is a nonsingular M-matrix.

Lu [7] substituted λ for $\rho(A)$ in (2) and defined the generalized Perron complement of $A[\beta]$ as the matrix:

$$P_\lambda(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](\lambda I - A[\beta])^{-1} A[\beta, \alpha],$$

where $\lambda > \rho(A[\beta])$. Clearly, $P_\lambda(A/A[\beta])$ is well-defined for $\lambda > \rho(A[\beta])$.

Meyer [6] explored the properties of $P(A/A[\beta])$ in detail and obtained elegant results. For example, $P(A/A[\beta])$ inherits the nonnegativity and irreducibility of the matrix A. Moreover, $P(A/A[\beta])$ and the irreducible nonnegative matrix A share a common spectral radius. The Perron complement possesses various intriguing properties and applications. Notably, it can facilitate the analysis of the eigenvalues and eigenvectors of a matrix. Researchers can leverage the properties and relationships involving the Perron complement to gain insights into the structure and properties of the original matrix. In fact, Neumann [8] utilized the Perron complement to examine the characteristics of inverse M-matrices. For an irreducible nonnegative matrix A, Lu [7], Yang [9], and Huang [10] utilized the generalized Perron complement to determine the Perron root of A. Adm [11] investigated the extended Perron complement of a principal submatrix in a totally nonnegative matrix. Additionally, inequalities between minors of the extended Perron complement and the Schur complement are presented. Furthermore, Wang [12] and Zeng [13] analyzed the closure property for the Perron complement of several diagonally dominant matrices by using the entries and spectral radius of the original matrix.

Johnson [14] introduced the collection of N_0-matrices as follows:

$$N = \lambda I - A, \beta \leq \lambda < \rho(A), A \geq 0,$$

where $\beta = \max\left\{\rho\left(\tilde{A}\right) : \tilde{A}\right.$ denotes the principal submatrix of A with order $n-1\}$.

Research on N_0-matrices has led to the study of inverse N_0-matrices, that is, nonsingular matrices whose inverses are N_0-matrices. A systematic effort to consider N_0-matrices and inverse N_0-matrices was made in [14, 15].

This study discusses the collection of inverse N_0-matrices. Before beginning our study, we provide the following notions:

Again, let $\varnothing \neq \beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \setminus \beta$. For an irreducible nonpositive matrix N of order n, the generalized Perron complement with respect to $N[\beta]$ is given by:

$$P_\lambda(N/N[\beta]) = N[\alpha] - N[\alpha,\beta](\lambda I + N[\beta])^{-1} N[\beta,\alpha], \tag{3}$$

where $\lambda > \rho(N[\beta])$. According to (3), the conditions $\lambda > \rho(N[\beta])$ and $N[\beta] \leq 0$ ensure that the matrix $\lambda I + N[\beta]$ is a nonsingular M-matrix. This means that $P_\lambda(N/N[\beta])$ is well-defined.

If we set $\lambda = \rho(N)$ in (3), we will acquire

$$P(N/N[\beta]) = N[\alpha] - N[\alpha,\beta](\rho(N) I + N[\beta])^{-1} N[\beta,\alpha], \tag{4}$$

and $P(N/N[\beta])$ is referred to as the Perron complement of $N[\beta]$. Because N is irreducible, $\rho(N) > \rho(N[\beta])$. Therefore, $\rho(N) I + N[\beta]$ is a nonsingular M-matrix, and the expression $(\rho(N) I + N[\beta])^{-1}$ continues to be well-defined.

The Perron complements and submatrices of special matrices are vital topics that have attracted the attention of many experts and scholars. Notably, the Perron complements of inverse M-matrices and Z-matrices have been the subject of extensive research [8, 16]. Additionally, some interesting findings on Perron complements of special matrices [11–13, 16] informed our investigation of inverse N_0-matrices. In particular, Zhou [17] proposed the notion of an extended Perron complement of an irreducible nonpositive matrix by restricting $\lambda \geq \rho(N)$ in (3) and demonstrated that the Perron complements of irreducible N_0-matrices and irreducible inverse N_0-matrices are closed under the Perron complement. Based on the study of Zhou [17], this study aims to investigate the related properties of the generalized Perron complement involving inverse N_0-matrices.

The remainder of this paper is organized as follows: Initially, we prove the closure of the generalized Perron complement for the collection of inverse N_0-matrices in Section 2. We demonstrate that for an inverse N_0-matrix N, the Perron complement of N and the matrix N share the same spectral radius.

In Section 3, some general inequalities concerning the generalized Perron complement $P_\lambda(N/N[\beta])$, Perron complement $P(N/N[\beta])$, and submatrix $N[\alpha]$ of an inverse N_0-matrix N are presented. In addition, we discuss the monotonicity of $P_\lambda(N/N[\beta])$ on $(\rho(N[\beta]), +\infty)$ and present the following result:

$$\lim_{\lambda\to\infty} P_\lambda(N/N[\beta]) = N[\alpha].$$

These findings are contained in Theorems 3.1 and 3.2.

Finally, we compare $\rho\left[P_\lambda\left(N/N\left[\beta\right]\right)\right]$ and $\rho\left(N\right)$ when N is an inverse N_0-matrix.

2. Generalized Perron complements and Perron complements of inverse N_0-matrices

For the convenience of writing in the following work, without confusion, $P_\lambda\left(N/N\left[\beta\right]\right)$ $\left(P\left(N/N\left[\beta\right]\right)\right)$ is abbreviated $P_\lambda\left(N/\beta\right)$ $\left(P\left(N/\beta\right)\right)$, and $N/N\left[\beta\right]$ is denoted N/β.

We begin this section by giving the following concept:

Definition 2.1. *[18] A matrix A is defined as a reducible matrix if there exists a permutation matrix P in the form:*

$$PAP^T = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are square matrices. Otherwise, A is irreducible.

According to Definition 2.1, we obtain the important conclusion given below:

Lemma 2.1. *Let A be a nonsingular matrix. Then, A is reducible if and only if A^{-1} is reducible.*

Proof. A is reducible and nonsingular.

$\Leftrightarrow$ There must exist a permutation matrix P with the following form:

$$PAP^{\mathrm{T}} = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix},$$

where A_{11} and A_{22} are nonsingular.

$\Leftrightarrow$ There must exist a permutation matrix P with the following form:

$$\left(PAP^{\mathrm{T}}\right)^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}^{-1},$$

where A_{11} and A_{22} are nonsingular. Note that

$$P^{\mathrm{T}} = P^{-1}, \quad \begin{pmatrix} A_{11} & A_{12} \\ O & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{pmatrix}.$$

$\Leftrightarrow$ There must exist a permutation matrix P with the following form:

$$PA^{-1}P^{\mathrm{T}} = \begin{pmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{pmatrix}.$$

$\Leftrightarrow$ A^{-1} is reducible. □

For the collection of N_0-matrices, Johnson [14] presented the following results:

Lemma 2.2. *Suppose that A is an Z-matrix. Then, the following are equivalent:*

(1) $A^{-1} \leq 0$ and A is irreducible;

(2) A is an N_0-matrix.

We can reach a similar result by substituting A for A^{-1} in Lemma 2.2.

Lemma 2.3. *Suppose that A^{-1} is an Z-matrix. Then, the following are equivalent:*
(1) $A \leq 0$ and A^{-1} is irreducible;
(2) A^{-1} is an N_0-matrix; that is, A is an inverse N_0-matrix.

To derive our conclusions, we must recall some essential lemmas. These lemmas are mainly concerned with the properties of inverse N_0-matrices, M-matrices, and nonnegative matrices. These play an important role in later proofs.

Lemma 2.4. *[19] All the (inverse) N_0-matrices are irreducible.*

According to Lemmas 2.3 and 2.4, we discover that all inverse N_0-matrices are nonpositive and irreducible.

Lemma 2.5. *[14] If A is an inverse N_0-matrix of order n and $A[\beta]$ is nonsingular, then A/β is an inverse M-matrix for any $\varnothing \neq \beta \subset \langle n\rangle$.*

Lemma 2.6. *[20] The principal submatrix of order k $(k \geq 2)$ of an inverse N_0-matrix is also an inverse N_0-matrix.*

Lemma 2.7. *[3] If D is a nonnegative diagonal matrix and A is an inverse M-matrix, then $A + D$ is an inverse M-matrix.*

Lemma 2.8. *[21] If the nonsingular M-matrices A and B satisfy $A \geq B$, then $0 \leq A^{-1} \leq B^{-1}$.*

Lemma 2.9. *[18] If $A \geq B \geq 0$, then $\rho(A) \geq \rho(B)$.*

We present the first conclusion of this study.

Theorem 2.1. *Let N be an inverse N_0-matrix of order n, and let $\varnothing \neq \beta \subset \langle n\rangle, \alpha = \langle n\rangle \backslash \beta$. We have that:*
(1) For any $\lambda > \rho(N[\beta])$, the generalized Perron complement

$$P_\lambda(N/\beta) = N[\alpha] - N[\alpha,\beta](\lambda I + N[\beta])^{-1} N[\beta,\alpha]$$

is an inverse N_0-matrix. In particular, the Perron complement

$$P(N/\beta) = N[\alpha] - N[\alpha,\beta](\rho(N)I + N[\beta])^{-1} N[\beta,\alpha]$$

is an inverse N_0-matrix;
(2) The matrix N and the Perron complement $P(N/\beta)$ share the same spectral radius.

Proof. Note that the inverse N_0-matrices are invariant under the simultaneous permutations of rows and columns. Hence, if N is an inverse N_0-matrix, for any $\varnothing \neq \beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \backslash \beta$, we may assume that N can be partitioned as:

$$N = \begin{pmatrix} N[\alpha] & N[\alpha,\beta] \\ N[\beta,\alpha] & N[\beta] \end{pmatrix} = \begin{pmatrix} B & C \\ D & E \end{pmatrix}. \tag{5}$$

We have

$$P_\lambda(N/\beta) = B - C(\lambda I + E)^{-1} D, \lambda > \rho(E).$$

We begin by showing that the generalized Perron complement $P_\lambda(N/\beta)$ is nonpositive and $[P_\lambda(N/\beta)]^{-1}$ is irreducible. Obviously, the inverse N_0-matrix N is nonpositive and irreducible. This means that B, C, D, and E are all nonpositive. Note that if $\lambda > \rho(E)$, then $\lambda I + E$ is an M-matrix, and hence, $(\lambda I + E)^{-1} \geq 0$. Therefore, we obtain $P_\lambda(N/\beta) \leq 0$.

In the following, we consider the irreducibility of $[P_\lambda(N/\beta)]^{-1}$. Let

$$-N = \begin{pmatrix} -B & -C \\ -D & -E \end{pmatrix}.$$

It is obvious that $-N$ is nonnegative and irreducible. For any $\lambda > \rho(E)$, the generalized Perron complement of $-N$ at λ (see [7]) is

$$\begin{aligned} P_\lambda(-N/\beta) &= -B + (-C)(\lambda I + E)^{-1}(-D) \\ &= -B + C(\lambda I + E)^{-1}D \\ &= -P_\lambda(N/\beta). \end{aligned}$$

Because the generalized Perron complement $P_\lambda(-N/\beta)$ of an irreducible nonnegative matrix $-N$ is also nonnegative and irreducible (see Lemma 2 in [7]), we conclude that $P_\lambda(N/\beta)$ is irreducible.

Now, assume that $[P_\lambda(N/\beta)]^{-1}$ is reducible, according to Lemma 2.1, so is $P_\lambda(N/\beta)$. This finding contradicts the fact that $P_\lambda(N/\beta)$ is irreducible. Therefore, $[P_\lambda(N/\beta)]^{-1}$ is irreducible.

Finally, we demonstrate that the matrix $[P_\lambda(N/\beta)]^{-1}$ is an Z-matrix. We need the following result:

Let the m-square matrix K and the k-square matrix L be nonsingular. Let M be an $m \times k$ matrix, and let H be an $k \times m$ matrix. If $L^{-1} + HK^{-1}M$ is nonsingular, then $K + MLH$ is nonsingular, and the following equation holds:

$$(K + MLH)^{-1} = K^{-1} - K^{-1}M\left(L^{-1} + HK^{-1}M\right)^{-1}HK^{-1}.$$

This result is called the Sherman-Morrison formula [18].

It is clear that N is nonsingular because N is an inverse N_0-matrix, that is, N^{-1} exists and N^{-1} is an N_0-matrix. According to (5), the inverse of the matrix N has the following structure:

$$N^{-1} = \begin{bmatrix} B^{-1} + B^{-1}C(N/\alpha)^{-1}DB^{-1} & -B^{-1}C(N/\alpha)^{-1} \\ -(N/\alpha)^{-1}DB^{-1} & (N/\alpha)^{-1} \end{bmatrix}.$$

On the other hand, according to Lemma 2.2, the N_0-matrix N^{-1} is an Z-matrix, and the Z-matrices have nonpositive off-diagonal elements. This implies that

$$B^{-1}C(N/\alpha)^{-1} \geq 0. \tag{6}$$

Since N/α is the Schur complement of an inverse N_0-matrix, from Lemma 2.5, we find that N/α is an inverse M-matrix. Therefore, one can obtain $N/\alpha \geq 0$. By multiplying the two sides of inequality (6) by the nonnegative matrix N/α, we obtain $B^{-1}C \geq 0$. By utilizing a similar approach, we obtain $DB^{-1} \geq 0$. Additionally, according to Lemma 2.7, $\lambda I + N/\alpha$ is an inverse M-matrix when $\lambda > \rho(E)$. Therefore, $(\lambda I + N/\alpha)^{-1}$ and $(N/\alpha)^{-1}$ are M-matrices. Assuming that $(\lambda I + N/\alpha)^{-1} \geq (N/\alpha)^{-1}$, by Lemma 2.8, one can obtain $\lambda I + N/\alpha \leq N/\alpha$. This contradicts $\lambda I + N/\alpha > N/\alpha$ when $\lambda > \rho(E)$. Therefore, we have $(\lambda I + N/\alpha)^{-1} < (N/\alpha)^{-1}$. Further, we obtain

$$B^{-1}C(\lambda I + N/\alpha)^{-1}DB^{-1} \leq B^{-1}C(N/\alpha)^{-1}DB^{-1} \tag{7}$$

using the fact that $B^{-1}C \geq 0$ and $DB^{-1} \geq 0$. By the Sherman-Morrison formula, we obtain

$$\begin{aligned}
[P_\lambda(N/\beta)]^{-1} &= \left[B - C(\lambda I + E)^{-1}D\right]^{-1} \\
&= \left[B + (-C)(\lambda I + E)^{-1}D\right]^{-1} \\
&= B^{-1} - B^{-1}(-C)\left[(\lambda I + E) + DB^{-1}(-C)\right]^{-1}DB^{-1} \\
&= B^{-1} + B^{-1}C\left(\lambda I + E - DB^{-1}C\right)^{-1}DB^{-1} \\
&= B^{-1} + B^{-1}C(\lambda I + N/\alpha)^{-1}DB^{-1}. \qquad (8)
\end{aligned}$$

The last equality holds since

$$N/\alpha = E - DB^{-1}C.$$

Combined with (7) and (8), we get

$$[P_\lambda(N/\beta)]^{-1} \leq B^{-1} + B^{-1}C(N/\alpha)^{-1}DB^{-1}. \qquad (9)$$

In addition, we have

$$N/\beta = B - CE^{-1}D.$$

Using the Sherman-Morrison formula again, one can get

$$\begin{aligned}
(N/\beta)^{-1} &= \left(B - CE^{-1}D\right)^{-1} \\
&= \left[B + (-C)E^{-1}D\right]^{-1} \\
&= B^{-1} - B^{-1}(-C)\left[E + DB^{-1}(-C)\right]^{-1}DB^{-1} \\
&= B^{-1} + B^{-1}C\left(E - DB^{-1}C\right)^{-1}DB^{-1} \\
&= B^{-1} + B^{-1}C(N/\alpha)^{-1}DB^{-1}. \qquad (10)
\end{aligned}$$

It follows from (9) and (10) that

$$[P_\lambda(N/\beta)]^{-1} \leq (N/\beta)^{-1}. \qquad (11)$$

By Lemma 2.5, we see N/β is an inverse M-matrix, that is, $(N/\beta)^{-1}$ is an M-matrix. According to the definition of M-matrices, $(N/\beta)^{-1}$ is an Z-matrix. From (11), $[P_\lambda(N/\beta)]^{-1}$ is also an Z-matrix. Together with the previous analysis, $P_\lambda(N/\beta) \leq 0$ and $[P_\lambda(N/\beta)]^{-1}$ is irreducible. According to Lemma 2.3, we deduce that $P_\lambda(N/\beta)$ is an inverse N_0-matrix. In particular, $P(N/\beta)$ is an inverse N_0-matrix.

Next, we demonstrate that $\rho(N)$ is the spectral radius of the Perron complement

$$P(N/\beta) = B - C[\rho(N)I + E]^{-1}D.$$

According to the Perron-Frobenius theorem, for the nonnegative irreducible matrix $-N$, there must exist a positive vector u that satisfies $-Nu = \rho(N)u$. Partition $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ conformally with $-N$. We thus have

$$\begin{pmatrix} -B & -C \\ -D & -E \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \rho(N)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The above equation is equivalent to

$$Bu_1 + Cu_2 = -\rho(N)\,u_1 \tag{12}$$

and

$$Du_1 + Eu_2 = -\rho(N)\,u_2. \tag{13}$$

From (13), we infer that

$$u_2 = -[\rho(N)\,I + E]^{-1} Du_1. \tag{14}$$

Therefore,

$$\begin{aligned} P(N/\beta)\,u_1 &= \left[B - C(\rho(N)\,I + E)^{-1} D\right] u_1 \\ &= Bu_1 - C[\rho(N)\,I + E]^{-1} Du_1 \\ &= Bu_1 + Cu_2 \quad (By(14)) \\ &= -\rho(N)\,u_1. \quad (By(12)) \end{aligned}$$

This shows that $-\rho(N)$ is a characteristic root of the matrix $P(N/\beta)$, which has an associated positive eigenvector given by u_1. Thus, $\rho(N)$ is the spectral radius of $P(N/\beta)$. □

In what follows, we provide a concrete example to verify our conclusions. Consider the following 3×3 nonpositive matrix:

$$N = \begin{pmatrix} -0.2500 & -0.3000 & -0.3500 \\ -1.0000 & -0.4000 & -0.8000 \\ -0.5000 & -0.4000 & -0.3000 \end{pmatrix}.$$

Since

$$N^{-1} = \begin{pmatrix} 4.0000 & -1.0000 & -2.0000 \\ -2.0000 & 2.0000 & -3.0000 \\ -4.0000 & -1.0000 & 4.0000 \end{pmatrix}$$

is an irreducible Z-matrix, according to Lemma 2.3, N is an inverse N_0-matrix.

Suppose $\alpha = \{1,2\}, \beta = \{3\}$. We obtain

$$N[\alpha] = \begin{pmatrix} -0.2500 & -0.3000 \\ -1.0000 & -0.4000 \end{pmatrix}, N[\alpha,\beta] = \begin{pmatrix} -0.3500 \\ -0.8000 \end{pmatrix}, \tag{15}$$

$$N[\beta,\alpha] = \begin{pmatrix} -0.5000 & -0.4000 \end{pmatrix}, N[\beta] = (-0.3000).$$

In addition, we have

$$\rho(N) = 1.3483, \rho(N[\beta]) = 0.3000.$$

By calculating, we obtain

$$\begin{aligned} P_\lambda(N/\beta) &= N[\alpha] - N[\alpha,\beta]\,(\lambda I + N[\beta])^{-1} N[\beta,\alpha] \\ &= \begin{pmatrix} -0.2500 & -0.3000 \\ -1.0000 & -0.4000 \end{pmatrix} - \begin{pmatrix} -0.3500 \\ -0.8000 \end{pmatrix} (\lambda - 0.3000)^{-1} \begin{pmatrix} -0.5000 & -0.4000 \end{pmatrix}. \end{aligned}$$

Note that $\lambda > \rho(N[\beta])$, we may take $\lambda = 0.5$. We acquire

$$P_\lambda(N/\beta) = \begin{pmatrix} -1.1250 & -1.0000 \\ -3.0000 & -2.0000 \end{pmatrix}. \tag{16}$$

Setting $\lambda = \rho(N) = 1.3483$, we obtain

$$P(N/\beta) = \begin{pmatrix} -0.4169 & -0.4335 \\ -1.3816 & -0.7053 \end{pmatrix}. \tag{17}$$

It is obvious that $P_\lambda(N/\beta)$ and $P(N/\beta)$ are both nonpositive. Moreover,

$$[P_\lambda(N/\beta)]^{-1} = \begin{pmatrix} 2.6667 & -1.3333 \\ -4.0000 & 1.5000 \end{pmatrix} \tag{18}$$

and

$$[P(N/\beta)]^{-1} = \begin{pmatrix} 2.3128 & -1.4218 \\ -4.5308 & 1.3673 \end{pmatrix} \tag{19}$$

are irreducible Z-matrices. We conclude from Lemma 2.3 that $P_\lambda(N/\beta)$ and $P(N/\beta)$ are inverse N_0-matrices. This result is consistent with the conclusion of Theorem 2.1 (1).

In the following, we compute the spectral radius of $P(N/\beta)$. Direct calculation by MATLAB yields that $\rho[P(N/\beta)] = 1.3483 = \rho(N)$. This result complies with Theorem 2.1 (2).

3. Inequalities of inverse N_0-matrices

In this section, we present inequalities on three matrices: $P_\lambda(N/\beta)$, $P(N/\beta)$, and $N[\alpha]$ under certain conditions. In addition, inequalities concerning the inverses of the three matrices are shown later.

Theorem 3.1. *Let N be an inverse N_0-matrix of order n. Then, the following orderings hold for any $\emptyset \neq \beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \backslash \beta$:*
(1) $P_\lambda(N/\beta) \le P(N/\beta) \le N[\alpha]$, for $\rho(N[\beta]) < \lambda \le \rho(N)$;
(2) $P(N/\beta) \le P_\lambda(N/\beta) \le N[\alpha]$, for $\lambda \ge \rho(N)$.

In addition, for $\lambda_2 \ge \lambda_1 > \rho(N[\beta])$, it holds that

$$P_{\lambda_2}(N/\beta) \ge P_{\lambda_1}(N/\beta)$$

and

$$\lim_{\lambda\to\infty} P_\lambda(N/\beta) = N[\alpha].$$

Proof. Without loss of generality, for any $\emptyset \neq \beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \backslash \beta$, we assume that N is simultaneously permuted to the following block matrix:

$$N = \begin{pmatrix} N[\alpha] & N[\alpha,\beta] \\ N[\beta,\alpha] & N[\beta] \end{pmatrix} = \begin{pmatrix} B & C \\ D & E \end{pmatrix}.$$

We have

$$P_\lambda(N/\beta) = B - C(\lambda I + E)^{-1}D, \lambda > \rho(N[\beta]) = \rho(E)$$

and

$$P(N/\beta) = B - C[\rho(N)I + E]^{-1}D.$$

Here it should be mentioned that $C \leq 0\,(C \neq O)$, $D \leq 0\,(D \neq O)$, and $E \leq 0\,(E \neq O)$, and O denotes the zero matrix. This is guaranteed by the fact that N is nonpositive and irreducible. In addition, if the order of E equals one, that is, $E = a_{nn}$, then we must have $a_{nn} < 0$; if the order of E is greater than or equal to two, according to Lemma 2.6, the matrix E, as a submatrix of N, is an inverse N_0-matrix, and inverse N_0-matrices are nonpositive and irreducible. In other words, E is nonpositive and irreducible. Moreover, the irreducibility is independent of the main diagonal elements; therefore, $\lambda I+E$ and $\rho(N)I + E$ are irreducible. To arrive at our conclusions, we divide the proof into two cases.

(i) Consider the case of $\rho(N[\beta]) < \lambda \leq \rho(N)$.

Because E is nonpositive and irreducible and $\rho(N) \geq \lambda > \rho(N[\beta])$, according to the definition of M-matrices, both $\lambda I + E$ and $\rho(N)I + E$ are irreducible M-matrices. Note that $\lambda I + E \leq \rho(N)I + E$, by Lemma 2.8, we obtain

$$(\lambda I + E)^{-1} \geq [\rho(N)I + E]^{-1} \geq 0.$$

Considering that $C \leq 0$ and $D \leq 0$, we further have that

$$C(\lambda I + E)^{-1}D \geq C[\rho(N)I + E]^{-1}D \geq 0.$$

By the definitions of $P_\lambda(N/\beta)$ and $P(N/\beta)$, we have

$$P_\lambda(N/\beta) \leq P(N/\beta) \leq B = N[\alpha].$$

(ii) Consider the case of $\lambda \geq \rho(N)$.

Because N is an inverse N_0-matrix, N is clearly irreducible. Thus, $\rho(N) > \rho(N[\beta])$. As in case (i), when $\lambda \geq \rho(N) > \rho(N[\beta])$, we can easily deduce that both $\lambda I + E$ and $\rho(N)I + E$ are irreducible M-matrices. Note that $\lambda I + E \geq \rho(N)I + E$, by Lemma 2.8, we obtain

$$[\rho(N)I + E]^{-1} \geq (\lambda I + E)^{-1} \geq 0.$$

Combining $C \leq 0$ and $D \leq 0$, we obtain

$$C[\rho(N)I + E]^{-1}D \geq C(\lambda I + E)^{-1}D \geq 0.$$

From the above inequality, we further obtain

$$B - C[\rho(N)I + E]^{-1}D \leq B - C(\lambda I + E)^{-1}D \leq B.$$

According to the definitions of $P_\lambda(N/\beta)$ and $P(N/\beta)$, we get

$$P(N/\beta) \leq P_\lambda(N/\beta) \leq B = N[\alpha].$$

Now, suppose $\lambda_2 \geq \lambda_1 > \rho(N[\beta])$, we have

$$P_{\lambda_2}(N/\beta) - P_{\lambda_1}(N/\beta) = C(\lambda_1 I + E)^{-1}D - C(\lambda_2 I + E)^{-1}D$$

$$= C\left[(\lambda_1 I + E)^{-1} - (\lambda_2 I + E)^{-1}\right] D.$$

Because $\lambda_1 I+E$, $\lambda_2 I+E$ are irreducible nonsingular M-matrices and $\lambda_1 I+E \le \lambda_2 I+E$, from Lemma 2.8, it holds that

$$(\lambda_1 I + E)^{-1} \ge (\lambda_2 I + E)^{-1},$$

that is,

$$(\lambda_1 I + E)^{-1} - (\lambda_2 I + E)^{-1} \ge 0.$$

As $C \le 0$ and $D \le 0$, we obtain

$$C\left[(\lambda_1 I + E)^{-1} - (\lambda_2 I + E)^{-1}\right] D \ge 0.$$

This means that $P_{\lambda_2}(N/\beta) \ge P_{\lambda_1}(N/\beta)$.

In addition,

$$\begin{aligned}
\lim_{\lambda\to\infty} P_\lambda(N/\beta) &= \lim_{\lambda\to\infty}\left[B - C(\lambda I + E)^{-1} D\right] \\
&= \lim_{\lambda\to\infty}\left[B - \frac{1}{\lambda} C\left(I + \frac{1}{\lambda}E\right)^{-1} D\right] \\
&= \lim_{\lambda\to\infty}\left[B - \frac{1}{\lambda} C\left(I - \frac{1}{\lambda}E + \frac{1}{\lambda^2}E^2 + \cdots\right) D\right] \\
&= B \\
&= N[\alpha].
\end{aligned}$$

Therefore, the proof of Theorem 3.1 has been completed. □

In the following section, we compare $[P_\lambda(N/\beta)]^{-1}$, $[P(N/\beta)]^{-1}$, and $(N[\alpha])^{-1}$. We recall the following result from Johnson [14].

Lemma 3.1. *[14] If N_1, N_2 are N_0-matrices of order $n \ge 2$ such that $N_1 \ge N_2$, then ${N_1}^{-1} \le {N_2}^{-1}$.*

We have a similar conclusion, as follows:

Corollary 3.1. *If N_1, N_2 are inverse N_0-matrices of order $n \ge 2$ such that $N_1 \ge N_2$, then ${N_1}^{-1} \le {N_2}^{-1}$.*

Proof. To illustrate this issue, two aspects are considered.

(i) Consider the case of $N_1 = N_2$.

Obviously, we have ${N_1}^{-1} = {N_2}^{-1}$.

(ii) Consider the case of $N_1 > N_2$.

Because N_1 and N_2 are inverse N_0-matrices, we know that ${N_1}^{-1}$ and ${N_2}^{-1}$ are N_0-matrices. Suppose ${N_1}^{-1} \ge {N_2}^{-1}$. According to Lemma 3.1, one can obtain $N_1 \le N_2$. This contradicts $N_1 > N_2$. Thus, we have ${N_1}^{-1} < {N_2}^{-1}$.

In summary, for two inverse N_0-matrices N_1, N_2 with $N_1 \ge N_2$, we have ${N_1}^{-1} \le {N_2}^{-1}$. □

The second conclusion of this section is presented below.

Theorem 3.2. *Let N be an inverse N_0-matrix of order n. Then, the following orderings hold for any $\varnothing \neq \beta \subset \langle n\rangle$ and $\alpha = \langle n\rangle \backslash \beta$:*
(1) $[P_\lambda(N/\beta)]^{-1} \geq [P(N/\beta)]^{-1} \geq (N[\alpha])^{-1}$, for $\rho(N[\beta]) < \lambda \leq \rho(N)$;
(2) $[P(N/\beta)]^{-1} \geq [P_\lambda(N/\beta)]^{-1} \geq (N[\alpha])^{-1}$, for $\lambda \geq \rho(N)$.

In addition, when $\lambda_2 \geq \lambda_1 > \rho(N[\beta])$, it holds that

$$[P_{\lambda_2}(N/\beta)]^{-1} \leq [P_{\lambda_1}(N/\beta)]^{-1}$$

and

$$\lim_{\lambda\to\infty}[P_\lambda(N/\beta)]^{-1} = (N[\alpha])^{-1}.$$

Proof. According to Lemma 2.6 and Theorem 2.1, we find that the matrices $P_\lambda(N/\beta)$, $P(N/\beta)$, and $N[\alpha]$ are all inverse N_0-matrices. From Theorem 3.1, when $\rho(N[\beta]) < \lambda \leq \rho(N)$, we have that:

$$P_\lambda(N/\beta) \leq P(N/\beta) \leq N[\alpha].$$

By Corollary 3.1, it holds that

$$[P_\lambda(N/\beta)]^{-1} \geq [P(N/\beta)]^{-1} \geq (N[\alpha])^{-1}.$$

Theorem 3.2 (2) is similarly proven. Moreover, from (8), we obtain

$$\lim_{\lambda\to\infty}[P_\lambda(N/\beta)]^{-1} = \lim_{\lambda\to\infty}\left[B^{-1} + B^{-1}C(\lambda I + N/\alpha)^{-1}DB^{-1}\right] = B^{-1} = (N[\alpha])^{-1}.$$

This completes the proof of Theorem 3.2. □

Next, we examine the correctness of Theorems 3.1 and 3.2. We again consider the example in Section 2 and validate our conclusions in three cases.

(i) For $\rho(N[\beta]) < \lambda \leq \rho(N)$, that is, $0.3 < \lambda \leq 1.3483$, we may take $\lambda = 0.5$, as in Section 2. From (15)–(17), we obtain

$$\begin{pmatrix}-1.1250 & -1.0000\\ -3.0000 & -2.0000\end{pmatrix} \leq \begin{pmatrix}-0.4169 & -0.4355\\ -1.3816 & -0.7053\end{pmatrix} \leq \begin{pmatrix}-0.2500 & -0.3000\\ -1.0000 & -0.4000\end{pmatrix}.$$

This means that

$$P_\lambda(N/\beta) \leq P(N/\beta) \leq N[\alpha].$$

In addition,

$$(N[\alpha])^{-1} = \begin{pmatrix}2.0000 & -1.5000\\ -5.0000 & 1.2500\end{pmatrix}. \tag{20}$$

From (18)–(20), we obtain

$$\begin{pmatrix}2.6667 & -1.3333\\ -4.0000 & 1.5000\end{pmatrix} \geq \begin{pmatrix}2.3128 & -1.4218\\ -4.5308 & 1.3673\end{pmatrix} \geq \begin{pmatrix}2.0000 & -1.5000\\ -5.0000 & 1.2500\end{pmatrix}.$$

This implies that

$$[P_\lambda(N/\beta)]^{-1} \geq [P(N/\beta)]^{-1} \geq (N[\alpha])^{-1}.$$

(ii) For $\lambda \geq \rho(N) = 1.3483$, we may take $\lambda = 2.3$. One can obtain

$$P_\lambda(N/\beta) = \begin{pmatrix} -0.3375 & -0.3700 \\ -1.2000 & -0.5600 \end{pmatrix} \tag{21}$$

and

$$[P_\lambda(N/\beta)]^{-1} = \begin{pmatrix} 2.1961 & -1.4510 \\ -4.7059 & 1.3235 \end{pmatrix}. \tag{22}$$

According to (15), (17), and (21), we obtain

$$\begin{pmatrix} -0.4169 & -0.4355 \\ -1.3816 & -0.7053 \end{pmatrix} \leq \begin{pmatrix} -0.3375 & -0.3700 \\ -1.2000 & -0.5600 \end{pmatrix} \leq \begin{pmatrix} -0.2500 & -0.3000 \\ -1.0000 & -0.4000 \end{pmatrix}.$$

This shows that

$$P(N/\beta) \leq P_\lambda(N/\beta) \leq N[\alpha].$$

From (19), (20), and (22), we obtain

$$\begin{pmatrix} 2.3128 & -1.4218 \\ -4.5308 & 1.3673 \end{pmatrix} \geq \begin{pmatrix} 2.1961 & -1.4510 \\ -4.7059 & 1.3235 \end{pmatrix} \geq \begin{pmatrix} 2.0000 & -1.5000 \\ -5.0000 & 1.2500 \end{pmatrix}.$$

It could be seen that

$$[P(N/\beta)]^{-1} \geq [P_\lambda(N/\beta)]^{-1} \geq (N[\alpha])^{-1}.$$

(iii) For $\lambda_2 \geq \lambda_1 > \rho(N[\beta]) = 0.3$, we set $\lambda_2 = 0.5$, $\lambda_1 = 0.4$. By calculation, we obtain

$$P_{\lambda_2}(N/\beta) = \begin{pmatrix} -1.1250 & -1.0000 \\ -3.0000 & -2.0000 \end{pmatrix} \geq P_{\lambda_1}(N/\beta) = \begin{pmatrix} -2.0000 & -1.7000 \\ -5.0000 & -3.6000 \end{pmatrix}$$

and

$$[P_{\lambda_2}(N/\beta)]^{-1} = \begin{pmatrix} 2.6667 & -1.3333 \\ -4.0000 & 1.5000 \end{pmatrix} \leq [P_{\lambda_1}(N/\beta)]^{-1} = \begin{pmatrix} 2.7692 & -1.3077 \\ -3.8462 & 1.5385 \end{pmatrix}.$$

In addition,

$$P_\lambda(N/\beta) = \begin{pmatrix} -0.2500 & -0.3000 \\ -1.0000 & -0.4000 \end{pmatrix} - (\lambda - 0.3000)^{-1} \begin{pmatrix} 0.1750 & 0.1400 \\ 0.4000 & 0.3200 \end{pmatrix}$$

and

$$\begin{aligned} [P_\lambda(N/\beta)]^{-1} &= (N[\alpha])^{-1} + (N[\alpha])^{-1} N[\alpha,\beta] (\lambda I + N/\alpha)^{-1} N[\beta,\alpha] (N[\alpha])^{-1} \\ &= \begin{pmatrix} 2.0000 & -1.5000 \\ -5.0000 & 1.25000 \end{pmatrix} + (\lambda + 0.2500)^{-1} \begin{pmatrix} 0.5000 & 0.1250 \\ 0.7500 & 0.1875 \end{pmatrix}. \end{aligned}$$

It is simple to see

$$\lim_{\lambda \to \infty} P_\lambda(N/\beta) = \begin{pmatrix} -0.2500 & -0.3000 \\ -1.0000 & -0.4000 \end{pmatrix} = N[\alpha]$$

and

$$\lim_{\lambda \to \infty} [P_\lambda(N/\beta)]^{-1} = \begin{pmatrix} 2.0000 & -1.5000 \\ -5.0000 & 1.2500 \end{pmatrix} = (N[\alpha])^{-1}.$$

The above calculations are consistent with Theorems 3.1 and 3.2.

Finally, we present conclusions concerning the spectral radius of an inverse N_0-matrix and its generalized Perron complement, which are similar to the results proposed by Lu [7].

Theorem 3.3. *Let N be an inverse N_0-matrix of order n. Then, the following conclusions hold for any $\emptyset \neq \beta \subset \langle n \rangle$ and $\alpha = \langle n \rangle \backslash \beta$:*
(1) $\rho\left[P_\lambda\left(N/\beta\right)\right] \geq \rho\left(N\right)$, for $\rho\left(N\left[\beta\right]\right) < \lambda \leq \rho\left(N\right)$;
(2) $\rho\left[P_\lambda\left(N/\beta\right)\right] \leq \rho\left(N\right)$, for $\lambda \geq \rho\left(N\right)$.

Proof. By Theorem 3.1, for $\rho\left(N\right) \geq \lambda > \rho\left(N\left[\beta\right]\right)$, we obtain

$$P\left(N/\beta\right) = P_{\rho(N)}\left(N/\beta\right) \geq P_\lambda\left(N/\beta\right).$$

Considering that $P\left(N/\beta\right)$ and $P_\lambda\left(N/\beta\right)$ are inverse N_0-matrices and that inverse N_0-matrices are nonpositive, we have

$$-P_\lambda\left(N/\beta\right) \geq -P\left(N/\beta\right) \geq 0.$$

According to Lemma 2.9, we obtain

$$\rho\left[-P_\lambda\left(N/\beta\right)\right] \geq \rho\left[-P\left(N/\beta\right)\right].$$

It is simple that

$$\rho\left[-P_\lambda\left(N/\beta\right)\right] = \rho\left[P_\lambda\left(N/\beta\right)\right], \rho\left[-P\left(N/\beta\right)\right] = \rho\left[P\left(N/\beta\right)\right] = \rho\left(N\right).$$

Therefore, we conclude Theorem 3.3 (1) immediately. Theorem 3.3 (2) can be similarly proven. □

We end this section by verifying Theorem 3.3 through the previous example.

For $\rho\left(N\left[\beta\right]\right) < \lambda \leq \rho\left(N\right)$, as before, we take $\lambda = 0.5$. By calculating, we obtain

$$\rho\left[P_\lambda\left(N/\beta\right)\right] = 3.3490 \geq \rho\left(N\right) = 1.3483.$$

For $\lambda \geq \rho\left(N\right)$, we set $\lambda = 2.3$. Through computation, one can obtain

$$\rho\left[P_\lambda\left(N/\beta\right)\right] = 1.1243 \leq \rho\left(N\right) = 1.3483.$$

These comply with Theorem 3.3.

4. Conclusions

For the collection of inverse N_0-matrices, we introduced the notion of a generalized Perron complement. By utilizing the properties of M-matrices, nonnegative matrices, and inverse N_0-matrices, we proved the closure of the generalized Perron complement. Furthermore, we have rigorously proven that, as an inverse N_0-matrix N, the Perron complement $P\left(N/\beta\right)$ and the matrix N possess the same spectral radius.

In addition, we presented some general inequalities concerning the inverse N_0-matrices. The generalized Perron complement $P_\lambda\left(N/\beta\right)$, Perron complement $P\left(N/\beta\right)$, and submatrix $N\left[\alpha\right]$ are closely related to the original inverse N_0-matrix N. We compared the three types of matrices under certain conditions. The inequalities concerning the inverses of the three types of matrices are also shown. We also discussed the monotonicity and limitations of the generalized Perron complement.

In conclusion, we investigated the relationship between the spectral radius of the generalized Perron complement and that of the inverse N_0-matrices.

In summary, our results offer informative perspectives on inverse N_0-matrices, which can be considered a useful addition to the current body of research.

Author contributions

Qin Zhong: Conceptualization, Data curation, Writing-original draft, Methodology; Ling Li: Formal analysis, Writing-review & editing, Investigation, Data curation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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