



Research article

Optimizing SNARK networks via double metric dimension

Muhammad Ahmad¹, Muhammad Faheem¹, Sanaa A. Bajri^{2,*}, Zohaib Zahid^{1,*}, Muhammad Javaid¹ and Hamiden Abd El-Wahed Khalifa^{3,4}

¹ Department of Mathematics, University of Management and Technology, Lahore 54000, Pakistan

² Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

³ Department of Mathematics, College of Science, Qassim University, Buraydah 51452, Saudi Arabia

⁴ Department of Operations and Management Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt

* **Correspondence:** Email: sabajri@pnu.edu.sa, zohaib_zahid@hotmail.com.

Abstract: Doubly resolving sets (DRSs) provide a promising approach for source detection. They consist of minimal subsets of nodes with the smallest cardinality, referred to as the double metric dimension (DMD), that can uniquely identify the location of any other node within the network. Utilizing DRSs can improve the accuracy and efficiency of the identification of the origin of a diffusion process. This ability is crucial for early intervention and control in scenarios such as epidemic outbreaks, misinformation spreading in social media, and fault detection in communication networks. In this study, we computed the DMD of flower snarks J_m and quasi-flower snarks G_m by describing their minimal doubly resolving sets (MDRSs). We deduce that the DMD for the flower snarks J_m is finite and depends on the network's order, and the DMD for the quasi-flower snarks G_m is finite and independent of the network's order. Furthermore, our findings offer valuable insights into the structural features of complex networks. This knowledge can offer direction for future studies in network theory and its practical implementations.

Keywords: optimization; complex networks; metric dimension; doubly resolving sets; flower snarks; quasi-flower snarks

Mathematics Subject Classification: 05C12

1. Introduction and preliminaries

Combinatorics is the study of how items are arranged by predetermined principles. Combinatorial approaches are required to count, enumerate, or describe potential solutions. Many difficult real-world

situations require finding an optimal solution in a finite solution space. The literature contains a large number of issues that could be solved in polynomial time, even though some practical problems, such as determining the shortest or cheapest roundtrips, internet data packet routing, planning, scheduling, and timetabling, appear to be NP-complete. Several combinatorial strategies can be used to achieve this, including metric dimension (MD), fault-tolerant MD, and DMD. The networks considered in this study are simple (no multiple loops permitted), finite (both edge and vertex sets are finite), linked (a path exists between any two pairs of vertices), and undirected (the edges are not oriented). Consider a network G of order n with the edge set E_G and vertex set V_G . For a vertex $v \in V_G$, the representation (1-tuple) $r(v) = (d_\tau | 1 \leq \tau \leq l)$, where $d_\tau = d_\tau(v, v_\tau)$, is said to be the distance vector of the vertex v . The distance vector between two vertices $v \neq u$ could be the same. In this case, the question of determining which vertices have unique distance vectors arises. This difficulty led to the definition of the resolving sets, introduced by Slater [1] in 1975 and Harary and Melter [2] in 1976.

- Definition 1.1.**
- 1) The vertex $x \in V_G$ determines (resolves) the vertices $v, w \in V_G$ if $d(v, x) \neq d(w, x)$.
 - 2) Let $W_G \subseteq V_G$, where $W_G = \{y_\tau | 1 \leq \tau \leq l\}$; ($l \leq n$) is an ordered set and $z \in V_G$, then the notation $r(z|W_G)$ is the distance vector of the vertex z in association with W_G is the l -tuple $(d(z, y_\tau))_{\tau=1}^l$, also called the vector of metric coordinates.
 - 3) The set W_G is said to be a resolving set for network G , if the l -tuple $(d(z, y_\tau))_{\tau=1}^l$ is unique with respect to z .
 - 4) A metric basis of network G is a resolving set with minimal number of entries. The number of entries in a metric basis is denoted by $\dim(G)$ and is called the MD of G .

The MD has a wide range of potential applications in science, sociology, and engineering. In this context, we will explore the diverse applications of the MD within different scientific fields. Several disciplines, such as telecommunications [3], establishing routing protocols geographically [4], combinatorial optimization [5], and robotics [6] have benefited from the rise and diversity of MD applications. Another area where the MD has significant implications is network discovery and verification [3]. Many applications in chemistry are derived from the vertex-edge relation in networks and its equivalence to the atom and bond relation [7]. Many strategies for the Mastermind [8] games are built using the resolving sets, and the MD of hamming networks is linked to various coin-weighing difficulties explained in [9, 10]. It is natural to investigate the mathematical features of this parameter due to its significance in other scientific fields. And after that, we look at some studies on the mathematical importance of this graph-theoretic parameter.

Several network families of mathematical importance have been studied from the perspective of MD. The field of MD has seen significant advancements, which we will discuss below. The chorded cycle, kayak paddle networks [11], and Mobius ladders [12] are examples of families for which MD has been estimated. Bailey and others have examined the MD of specific families of distance-regular networks, such as Grassmann networks [13], Kneser networks, and Johnson networks [14].

Cayley networks and Cayley digraphs, which are examples of networks generated by finite groups with group-theoretic importance, have been studied from the MD perspective [15, 16]. There has also been research into the MD and resolving sets of product networks, such as the categorical product of networks [17] and the cartesian product of graphs [18]. Kratica et al. [19] (and, respectively, Imran et al. [20]) investigated the MD of rotationally symmetric convex polytopes (and, respectively, convex polytopes created by wheel-related networks). Siddiqui et al. [21] investigated certain infinite families

derived from wheel networks for the MD problem.

Burdett et al. [22] determined various domination numbers for flower snarks, including independent domination, 2-domination, total domination, connected domination, upper domination, secure domination, and weak Roman domination numbers. In [23], the exact results of the mixed MD on two special classes of networks, namely flower snarks and wheel-related networks, were presented. Another study by Girisha et al. [24] characterized the MD and distance matrix of flower snark and sunflower networks. Additionally, due to computational complexity in finding exact values, Liu et al. provided bounds for the partition dimension of certain convex polytope structures in their work [25]. Lastly, Nawaz et al. [26] discussed the edge MD of various structures of benzodiazepines, which are types of drugs used for the treatment of depression.

Caceres et al. [27] responded to the question of whether the MD is a finite number. They demonstrated this by constructing infinite families of networks having infinite MDs. The computational difficulty of the MD problem was addressed by Garey and Johnson [28], just like many other graph-theoretic characteristics. The bounds of the MD have been examined for generalized Petersen networks by Shao et al. (refer to [29]). All connected networks possessing MDs $n - 1$, $n - 2$, and 1 were determined by Chartrand et al. [7]. Raza et al. [30] examined the fault-tolerant MD of infinite sets of regular networks, categorizing them accordingly. Baca et al. [31] and Tomescu et al. [32] studied the concept of resolving sets for the regular bipartite networks and the family of necklace networks, respectively. For more details, see [33, 34].

MD and its associated parameters are widely used among researchers today because of the relevance of MD in the identification of nodes in networks. The concept of the DRS and DMD of the network, which is a generalization of the MD of the network, is a distinctive and significant parameter. The term “DRS” was defined by Caceres et al. [18] as follows:

Definition 1.2. 1) The vertices $e_1, e_2 \in G$ (where $|G| \geq 2$) are said to doubly resolve the vertices f_1, f_2 of network G if $d(e_1, f_1) - d(e_1, f_2) \neq d(e_2, f_1) - d(e_2, f_2)$.
 2) An ordered set $D_G = \{e_\tau | 1 \leq \tau \leq l\}$ of G is a DRS of G if there does not exist a pair of vertices in G that has the same difference in their associated metric coordinates with respect to D_G .
 3) In other words, all coordinates of the vector $r(f_1|D_G) - r(f_2|D_G)$ cannot be the same for any pair of distinct vertices $f_1, f_2 \in V_G$. i.e. $r(f_1|D_G) - r(f_2|D_G) \neq \gamma I$, where I is the unit vector $(1, 1, \dots, 1)$ and γ is an integer.

A DRS having a minimal cardinal number is called a MDRS. The cardinal number of an MDRS is the DMD of G , denoted by $\psi(G)$. Moreover, a DRS is a resolving set as well, and $\psi(G) \leq \dim(G)$. It was shown in [35] (and, respectively, in [6]) that finding $\psi(G)$ (and, respectively, $\dim(G)$) is an NP-hard task. The task of identifying MDRSs for necklace networks and certain convex polytopes was first addressed in [36] and [19]. Furthermore, authors in [37, 38] demonstrated that the DMD remains constant for certain convex polytope structures. For various families of Harary networks, Ahmad et al. calculated the MDRSs (see [39] for more information).

In [40], it was shown that the $\dim(G)$ and $\psi(G)$ of some families of circulant networks were the same in all cases. The problem of determining the MDRS is discussed for many network families; for example, the MDRSs of prisms and Hamming networks were investigated in [41] and [42], respectively. Chen and Wang [43] conducted a study to identify the MDRSs for trees, cycles, wheels, and k -edge-augmented trees. In another study [44], Chen et al. were the first to present explicit approximations of upper and lower bounds for the MDRS problem.

An investigation into the MDRSs and MD of certain network families was conducted by Liu et al. [45]. In a recent study, Ahmad et al. [46] investigated the MDRSs for the chordal ring networks. Moreover, in [47], the MDRSs of layer-sun networks and their line networks have been explored in detail. The DMD and MDRSs for certain classes of convex polytopes were found by Pan et al. [48]. Jannesari [49] characterized all networks G having the DMD 2 by using 2-connected subgraphs of network G . Recently, the generalized solutions for the DRS problem of several families of networks were found by Ahmad et al. [50].

The reason for studying doubly resolving sets (DRSs) is their potential to improve the accuracy and efficiency of source detection in different network scenarios. Our study aimed to explore the theoretical principles of certain network properties using the flower snark network as a model due to its distinct and well-defined structural characteristics. The results of the study are significant as they present calculated double metric dimensions (DMDs) for complex network models such as flower snarks and quasi-flower snarks. These findings not only demonstrate the real-world application of DRSs in intricate networks but also provide valuable insights into their structural traits. This comprehension can guide future work in the theory of networks and its practical implementations, leading to more effective source detection techniques in various real-world networks.

MDRSs play a crucial role in identifying diffusion sources in intricate networks. Locating the origins of an outbreak in such networks is an exciting yet challenging task. For instance, consider the scenario where the only available information about the epidemic is from a subset of nodes referred to as sensors. These sensors can indicate if and when they have been infected with the virus. The question then arises: What is the minimum number of sensors required to identify the source of the outbreak accurately? Fortunately, a well-known network feature called DMD offers a solution to this problem (for details, see [51, 52]).

Let us look at a few real-world instances. The DMD in an n -node star network is $n - 1$. DRSs cannot be formed from sets that do not include all of the external nodes. On the other hand, the DMD for a path network is 2. The two nodes, one at each end of the path, are sensors always sufficient to identify the source. Using these instances, it is clear that the DMD and the difficulty in finding the source largely depend on the network topology. A path-like network makes it far easier to find the source of an outbreak than a star-like network [52].

The results on minimal resolving sets for the flower snark J_m and quasi-flower snarks G_m were determined by Imran et al. [53] and Naeem et al. [54] and are given in the following theorems:

Theorem 1.3. [53] *Let J_m be the flower snark. Then, for every odd positive integer $m \geq 5$, we have $\dim(J_m) = 3$.*

Theorem 1.4. [54] *Let G_m be the quasi-flower snark. Then, for every positive integer $m \geq 4$, we have*

$$\dim(G_m) = \begin{cases} 3, & \text{if } m \text{ is odd;} \\ \leq 4, & \text{otherwise.} \end{cases}$$

The results presented above determine the lower bound on the DMD of flower snarks J_m and quasi-flower snarks G_m . Following is how the main contribution of the article is structured:

- We compute the DMD for flower snarks J_m and quasi-flower snarks G_m in Sections 2 and 3.
- The utilization of the research in the context of the source of diffusion in a complex network is presented in Section 4.

- Finally, in Section 5, we demonstrate that the DMD for flower snarks depends on the order of the network, and the DMD for quasi-flower snarks is independent of the order of the network.

2. Double metric dimension for the flower snarks J_m

In this section, we will concentrate on determining the DMD of flower snarks J_m .

The flower snark J_m , depicted in Figure 1, is a cubic network comprising $4m$ vertices and $6m$ edges. The vertex set for the flower snarks J_m is $V_{J_m} = \{c_\tau, f_\tau, g_\tau, h_\tau : \forall 0 \leq \tau \leq m - 1\}$, and the edge set is $E_{J_m} = \{c_\tau c_{\tau+1}, c_\tau f_\tau, f_\tau g_\tau, f_\tau h_\tau, g_\tau g_{\tau+1}, h_\tau h_{\tau+1} : \forall 0 \leq \tau \leq m - 2\} \cup \{c_{m-1} c_0, c_{m-1} f_{m-1}, f_{m-1} g_{m-1}, f_{m-1} h_{m-1}, g_{m-1} g_0, h_{m-1} h_0, g_0 h_{m-1}, g_{m-1} h_0\}$.

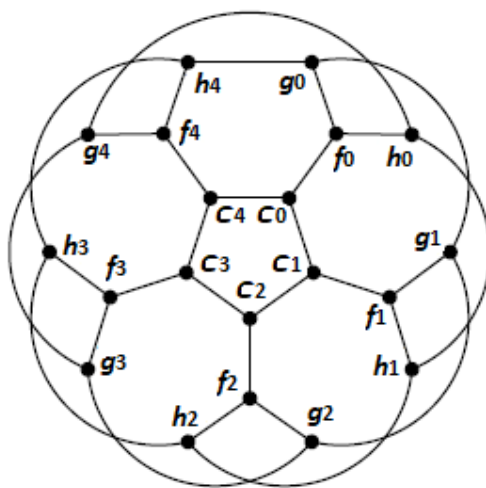


Figure 1. Flower snark J_5 .

As shown in Figure 1, the inner cycle vertices are labeled as $\{c_\tau : \forall 0 \leq \tau \leq m - 1\}$, the set of central vertices are labeled as $\{f_\tau : \forall 0 \leq \tau \leq m - 1\}$, and the outer cycle vertices are labeled as $\{g_\tau : \forall 0 \leq \tau \leq m - 1\} \cup \{h_\tau : \forall 0 \leq \tau \leq m - 1\}$.

MDRSs for J_m are to be found in this section. It follows from Theorem 1.3 that $\psi(J_m) \geq 3$, for all $m \geq 6$. We shall also show, for all $m \geq 6$:

$$\psi(J_m) = \begin{cases} 4, & \text{if } m \text{ is odd;} \\ 5, & \text{if } m \text{ is even.} \end{cases}$$

Define the vertex subset $S_\tau(f_0) = \{g \in V_{J_m} : d(f_0, g) = \tau\} \subseteq V_{J_m}$ having distance τ from f_0 . The Table 1 is a simple formulation for $S_\tau(f_0)$ and will be used to compute the distances between each pair of vertices in V_{J_m} .

Table 1. $S_\tau(f_0)$ for J_m .

m	τ	$S_\tau(f_0)$
	1	$\{c_0, g_0, h_0\}$
	2	$\{c_1, c_{m-1}, g_1, g_{m-1}, h_1, h_{m-1}\}$
	$3 \leq \tau \leq \lfloor \frac{m}{2} \rfloor$	$\{c_{\tau-1}, c_{m-\tau+1}, f_{\tau-2}, f_{m-\tau+2}, g_{\tau-1}, g_{m-\tau+1}, h_{\tau-1}, h_{m-\tau+1}\}$
even	$\frac{m+2}{2}$	$\{c_{\frac{m}{2}}, f_{\frac{m-2}{2}}, f_{\frac{m+2}{2}}, g_{\frac{m}{2}}, h_{\frac{m}{2}}\}$
	$\frac{m+4}{2}$	$\{f_{\frac{m}{2}}\}$
odd	$\frac{m+1}{2}$	$\{c_{\frac{m-1}{2}}, c_{\frac{m+1}{2}}, f_{\frac{m-3}{2}}, f_{\frac{m+3}{2}}, g_{\frac{m-1}{2}}, g_{\frac{m+1}{2}}, h_{\frac{m-1}{2}}, h_{\frac{m+1}{2}}\}$
	$\frac{m+3}{2}$	$\{f_{\frac{m-1}{2}}, f_{\frac{m+1}{2}}\}$

The following facts can be demonstrated as a result of the symmetry of J_m , where $m \geq 6$:

$$d(f_\tau, f_\nu) = d(f_0, f_{|\tau-\nu|}) \text{ if } 0 \leq |\tau - \nu| \leq m - 1; \quad d(c_\tau, c_\nu) = d(f_0, c_{|\tau-\nu|}) - 1 \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(c_\tau, f_\nu) = d(f_0, c_{|\tau-\nu|}) \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(c_\tau, g_\nu) = d(c_\tau, h_\nu) = d(f_0, c_{|\tau-\nu|}) + 1 \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(f_\tau, g_\nu) = d(f_\tau, h_\nu) = d(f_0, g_{|\tau-\nu|}) \text{ if } 0 \leq |\tau - \nu| \leq m - 1.$$

If m is an odd integer, then

$$d(g_\tau, h_\nu) = \begin{cases} d(f_0, c_{|\tau-\nu|}) + 1, & \text{if } 0 \leq |\tau - \nu| \leq \frac{m-1}{2}; \\ d(f_0, c_{|\tau-\nu|}) - 1, & \text{if } \frac{m+1}{2} \leq |\tau - \nu| \leq m - 1. \end{cases}$$

$$d(g_\tau, g_\nu) = d(h_\tau, h_\nu) = \begin{cases} d(f_0, g_{|\tau-\nu|}) - 1, & \text{if } 0 \leq |\tau - \nu| \leq \frac{m-1}{2}; \\ d(f_0, g_{|\tau-\nu|}) + 1, & \text{if } \frac{m+1}{2} \leq |\tau - \nu| \leq m - 1. \end{cases}$$

If m is an even integer, then

$$d(g_\tau, h_\nu) = \begin{cases} d(f_0, c_{|\tau-\nu|}) + 1, & \text{if } 0 \leq |\tau - \nu| \leq \frac{m-2}{2}; \\ d(f_0, c_{|\tau-\nu|}) - 1, & \text{if } \frac{m}{2} \leq |\tau - \nu| \leq m - 1. \end{cases}$$

$$d(g_\tau, g_\nu) = d(h_\tau, h_\nu) = \begin{cases} d(f_0, g_{|\tau-\nu|}) - 1, & \text{if } 0 \leq |\tau - \nu| \leq \frac{m}{2}; \\ d(f_0, g_{|\tau-\nu|}) + 1, & \text{if } \frac{m+2}{2} \leq |\tau - \nu| \leq m - 1. \end{cases}$$

Therefore, we can recalculate the distances between each pair of vertices in V_{J_m} if $d(f_0, g)$ is known for each $g \in V_{J_m}$.

Lemma 2.1. For any even positive integer $m \geq 6$, $\psi(J_m) > 4$.

Proof. To prove the DMD $\psi(J_m) > 4$, we must show that every subset $D_{J_m} \subseteq V_{J_m}$ of cardinality 4 is not a DRS for J_m . Table 2 demonstrates the non-doubly resolved pairs of vertices from the set V_{J_m} as well as all possible forms of subset D_{J_m} . Let us compute that any two vertices of the set $\{c_\tau, c_\nu, c_\omega, f_0; 0 \leq \tau < \nu < \omega \leq m - 1\}$ do not doubly resolve the vertices f_0, g_0 . For example, it is clear that $d(f_0, f_0) = 0, d(f_0, g_0) = 1$. Now, for $0 \leq \tau \leq \frac{m-2}{2}, 1 \leq \nu \leq \frac{m-2}{2}, 2 \leq \omega \leq \frac{m-2}{2}$, we have

- $d(c_\tau, f_0) = d(f_0, c_\tau) = \tau + 1,$
- $d(c_\tau, g_0) = d(f_0, c_\tau) + 1 = \tau + 2,$
- $d(c_\nu, f_0) = d(f_0, c_\nu) = \nu + 1,$
- $d(c_\nu, g_0) = d(f_0, c_\nu) + 1 = \nu + 2,$
- $d(c_\omega, f_0) = d(f_0, c_\omega) = \omega + 1,$

- $d(c_\omega, g_0) = d(f_0, c_\omega) + 1 = \omega + 2$.

Also, for $\frac{m}{2} \leq \tau \leq m - 3$, $\frac{m}{2} \leq \nu \leq m - 2$, $\frac{m}{2} \leq \omega \leq m - 1$, we have

- $d(c_\tau, f_0) = d(f_0, c_\tau) = m - \tau + 1$,
- $d(c_\tau, g_0) = d(f_0, c_\tau) + 1 = m - \tau + 2$,
- $d(c_\nu, f_0) = d(f_0, c_\nu) = m - \nu + 1$,
- $d(c_\nu, g_0) = d(f_0, c_\nu) + 1 = m - \nu + 2$,
- $d(c_\omega, f_0) = d(f_0, c_\omega) = m - \omega + 1$,
- $d(c_\omega, g_0) = d(f_0, c_\omega) + 1 = m - \omega + 2$.

So, $d(c_\tau, f_0) - d(c_\tau, g_0) = d(c_\nu, f_0) - d(c_\nu, g_0) = d(c_\omega, f_0) - d(c_\omega, g_0) = d(f_0, f_0) - d(f_0, g_0) = -1$, that is, the set $\{c_\tau, c_\nu, c_\omega, f_0; 0 \leq \tau < \nu < \omega \leq m - 1\}$ is not a DRS of J_m . Accordingly, we can investigate all the possible types of subset D_{J_m} shown in Table 2 and verify their associated non-doubly resolved pairs of vertices.

Table 2. Non-doubly resolving pairs of vertices for J_m , \forall even positive integer $m \geq 6$.

D_{J_m}	Non-doubly resolved pairs
$\{c_\tau, c_\nu, c_\omega, f_0\}, 0 \leq \tau < \nu < \omega \leq m - 1$	$\{f_0, g_0\}$
$\{c_\tau, c_\nu, f_0, f_\omega\}, 0 \leq \tau < \nu \leq m - 1, 1 \leq \omega \leq m - 1$	$\{g_0, h_0\}$
$\{c_\tau, f_0, f_\nu, f_\omega\}, 0 \leq \tau \leq m - 1, 1 \leq \nu < \omega \leq m - 1$	$\{g_0, h_0\}$
$\{c_\tau, f_0, f_\nu, g_\omega\} \cup \{c_\tau, f_0, f_\nu, h_\omega\}, 0 \leq \tau \leq m - 1, 1 \leq \nu \leq m - 1, 0 \leq \omega \leq \frac{m-2}{2}$	$\{g_{\frac{m+2\omega}{2}}, h_{\frac{m+2\omega}{2}}\}$
$\{c_\tau, f_0, f_\nu, g_\omega\} \cup \{c_\tau, f_0, f_\nu, h_\omega\}, 0 \leq \tau \leq m - 1, 1 \leq \nu \leq m - 1, \frac{m}{2} \leq \omega \leq m - 2$	$\{g_{\frac{2\omega-m}{2}}, h_{\frac{2\omega-m}{2}}\}$
$\{c_\tau, f_0, f_\nu, g_\omega\} \cup \{c_\tau, f_0, f_\nu, h_\omega\}, 0 \leq \tau \leq m - 1, 1 \leq \nu \leq m - 1, \omega = m - 1$	$\{g_{\frac{m-2}{2}}, h_{\frac{m-2}{2}}\}$
$\{c_\tau, c_\nu, f_0, g_\omega\}, 0 \leq \tau < \nu \leq m - 1, 0 \leq \omega \leq \frac{m-2}{2}$	$\{f_0, h_0\}$
$\{c_\tau, c_\nu, f_0, g_\omega\}, 0 \leq \tau < \nu \leq m - 1, \frac{m}{2} \leq \omega \leq m - 2$	$\{g_{\frac{2\omega-m}{2}}, h_{\frac{2\omega-m}{2}}\}$
$\{c_\tau, c_\nu, f_0, g_\omega\}, 0 \leq \tau < \nu \leq m - 1, \omega = m - 1$	$\{f_0, g_0\}$
$\{c_\tau, c_\nu, f_0, h_\omega\}, 0 \leq \tau < \nu \leq m - 1, 0 \leq \omega \leq \frac{m-2}{2}$	$\{f_0, g_0\}$
$\{c_\tau, c_\nu, f_0, h_\omega\}, 0 \leq \tau < \nu \leq m - 1, \frac{m}{2} \leq \omega \leq m - 2$	$\{g_{\frac{2\omega-m}{2}}, h_{\frac{2\omega-m}{2}}\}$
$\{c_\tau, c_\nu, f_0, h_\omega\}, 0 \leq \tau < \nu \leq m - 1, \omega = m - 1$	$\{f_0, h_0\}$
$\{f_0, f_\tau, f_\nu, f_\omega\}, 1 \leq \tau < \nu < \omega \leq m - 1$	$\{c_0, g_0\}$
$\{f_0, g_\tau, g_\nu, h_\omega\}, 0 \leq \tau < \nu \leq m - 1, 0 \leq \omega \leq m - 1$	$\{c_0, f_0\}$
$\{f_0, f_\tau, f_\nu, g_\omega\}, 1 \leq \tau < \nu \leq m - 1, 0 \leq \omega \leq \frac{m-2}{2}$	$\{c_0, h_0\}$
$\{f_0, f_\tau, f_\nu, g_\omega\}, 1 \leq \tau < \nu \leq m - 1, \omega = \frac{m}{2}$	$\{g_0, h_0\}$
$\{f_0, f_\tau, f_\nu, g_\omega\}, 1 \leq \tau < \nu \leq m - 1, \frac{m+2}{2} \leq \omega \leq m - 1$	$\{c_0, g_0\}$
$\{f_0, f_\tau, f_\nu, h_\omega\}, 1 \leq \tau < \nu \leq m - 1, 0 \leq \omega \leq \frac{m-2}{2}$	$\{c_0, g_0\}$
$\{f_0, f_\tau, f_\nu, h_\omega\}, 1 \leq \tau < \nu \leq m - 1, \omega = \frac{m}{2}$	$\{g_0, h_0\}$
$\{f_0, f_\tau, f_\nu, h_\omega\}, 1 \leq \tau < \nu \leq m - 1, \frac{m+2}{2} \leq \omega \leq m - 1$	$\{c_0, h_0\}$
$\{f_0, g_\tau, g_\nu, g_\omega\} \cup \{f_0, h_\tau, h_\nu, h_\omega\}, 0 \leq \tau < \nu < \omega \leq m - 1$	$\{c_0, f_0\}$
$\{f_0, g_\tau, h_\nu, h_\omega\}, 0 \leq \tau \leq m - 1, 0 \leq \nu < \omega \leq m - 1$	$\{c_0, f_0\}$

□

Lemma 2.2. For any odd positive integer $m \geq 6$, $\psi(J_m) > 3$.

Proof. To prove the DMD $\psi(J_m) > 3$, we must show that every subset $D_{J_m} \subseteq V_{J_m}$ of cardinality 3 is not a DRS for J_m . Table 3 demonstrates the non-doubly resolved pairs of vertices from the set V_{J_m} as well as all possible forms of subset D_{J_m} . Let us compute that any two vertices of the set $\{f_0, g_\tau, h_\nu; 0 \leq \tau, \nu \leq m-1\}$ do not doubly resolve the vertices c_0, f_0 . For example, it is clear that $d(c_0, f_0) = 1, d(f_0, f_0) = 0$. Now, for $0 \leq \tau, \nu \leq \frac{m-1}{2}$, we have

- $d(c_0, g_\tau) = d(f_0, c_\tau) + 1 = \tau + 2,$
- $d(f_0, g_\tau) = d(f_0, g_\tau) = \tau + 1,$
- $d(c_0, h_\nu) = d(f_0, c_\nu) + 1 = \nu + 2,$
- $d(f_0, h_\nu) = d(f_0, g_\nu) = \nu + 1.$

Also, for $\frac{m+1}{2} \leq \tau, \nu \leq m-1$, we have

- $d(c_0, g_\tau) = d(f_0, c_\tau) + 1 = m - \tau + 2,$
- $d(f_0, g_\tau) = d(f_0, g_\tau) = m - \tau + 1,$
- $d(c_0, h_\nu) = d(f_0, c_\nu) + 1 = m - \nu + 2,$
- $d(f_0, h_\nu) = d(f_0, g_\nu) = m - \nu + 1.$

So, $d(c_0, g_\tau) - d(f_0, g_\tau) = d(c_0, h_\nu) - d(f_0, h_\nu) = d(c_0, f_0) - d(f_0, f_0) = 1$, that is, the set $\{f_0, g_\tau, h_\nu; 0 \leq \tau, \nu \leq m-1\}$ is not a DRS of J_m . Accordingly, we can investigate all the possible types of subset D_{J_m} shown in Table 3 and verify their associated non-doubly resolved pairs of vertices.

Table 3. Non-doubly resolving pairs of vertices for J_m, \forall odd positive integer $m \geq 6$.

D_{J_m}	Non-doubly resolved pairs
$\{c_\tau, c_\nu, f_0\}, 0 \leq \tau < \nu \leq m-1$	$\{f_0, g_0\}$
$\{c_\tau, f_0, f_\nu\}, 0 \leq \tau \leq m-1, 1 \leq \nu \leq m-1$	$\{g_0, h_0\}$
$\{f_0, f_\tau, f_\nu\}, 1 \leq \tau < \nu \leq m-1$	$\{c_0, g_0\}$
$\{f_0, f_\tau, g_\nu\}, 1 \leq \tau \leq m-1, 0 \leq \nu \leq \frac{m-3}{2}$	$\{c_0, h_0\}$
$\{f_0, f_\tau, g_\nu\}, 1 \leq \tau \leq m-1, \frac{m-1}{2} \leq \nu \leq m-1$	$\{c_{\nu-1}, h_{\nu-1}\}$
$\{f_0, f_\tau, h_\nu\}, 1 \leq \tau \leq m-1, 0 \leq \nu \leq \frac{m-3}{2}$	$\{c_0, g_0\}$
$\{f_0, f_\tau, h_\nu\}, 1 \leq \tau \leq m-1, \frac{m-1}{2} \leq \nu \leq m-1$	$\{c_{\nu-1}, g_{\nu-1}\}$
$\{f_0, g_\tau, g_\nu\} \cup \{f_0, h_\tau, h_\nu\}, 0 \leq \tau < \nu \leq m-1$	$\{c_0, f_0\}$
$\{f_0, g_\tau, h_\nu\}, 0 \leq \tau, \nu \leq m-1$	$\{c_0, f_0\}$

□

Lemma 2.3. $\psi(J_m) = 5, \forall$ even integers $m \geq 6$.

Proof. It suffices to identify a DRS having order 5 to demonstrate that $\psi(J_m) = 5$, if $m \geq 6$ is an even integer. Now, by using the sets $S_\tau(f_0)$ from Table 1, the metric coordinate vectors are demonstrated in Table 4 for each vertex of J_m with respect to the set $D_{J_m} = \{c_0, c_1, c_{\frac{m}{2}}, g_0, g_{m-1}\}$.

Table 4. Metric coordinate vectors for J_m if $m \geq 6$ is even.

τ	$S_\tau(f_0)$	$D_{J_m} = \{c_0, c_1, c_{\frac{m}{2}}, g_0, g_{m-1}\}$	
0	f_0	$(1, 2, \frac{m+2}{2}, 1, 2)$	
1	c_0	$(0, 1, \frac{m}{2}, 2, 3)$	
	g_0	$(2, 3, \frac{m+4}{2}, 0, 3)$	
	h_0	$(2, 3, \frac{m+4}{2}, 2, 1)$	
2	c_1	$(1, 0, \frac{m-2}{2}, 3, 4)$	
	c_{m-1}	$(1, 2, \frac{m-2}{2}, 3, 2)$	
	g_1	$(3, 2, \frac{m+2}{2}, 1, 4)$	
	g_{m-1}	$(3, 4, \frac{m+2}{2}, 3, 0)$	
	h_1	$(3, 2, \frac{m+2}{2}, 3, 2)$	
	h_{m-1}	$(3, 4, \frac{m+2}{2}, 1, 2)$	
$3 \leq \tau \leq \frac{m}{2}$	$c_{\tau-1}$	$(\tau - 1, \tau - 2, \frac{m-2\tau+2}{2}, \tau + 1, \tau + 2)$	
	$c_{m-\tau+1}$	$(\tau - 1, \tau, \frac{m-2\tau+2}{2}, \tau + 1, \tau)$	
	$f_{\tau-2}$	$(\tau - 1, \tau - 2, \frac{m-2\tau+6}{2}, \tau - 1, \tau)$	
	$f_{m-\tau+2}$	$(\tau - 1, \tau, \frac{m-2\tau+6}{2}, \tau - 1, \tau - 2)$	
	$g_{\tau-1}$	$= \begin{cases} (\tau + 1, \tau, \frac{m-2\tau+6}{2}, \tau - 1, \tau + 2), & \text{if } \tau < \frac{m}{2}; \\ (\frac{m+2}{2}, \frac{m}{2}, 3, \frac{m-2}{2}, \frac{m}{2}), & \text{if } \tau = \frac{m}{2}. \end{cases}$	
	$g_{m-\tau+1}$	$(\tau + 1, \tau + 2, \frac{m-2\tau+6}{2}, \tau + 1, \tau - 2)$	
	$h_{\tau-1}$	$(\tau + 1, \tau, \frac{m-2\tau+6}{2}, \tau + 1, \tau)$	
	$h_{m-\tau+1}$	$(\tau + 1, \tau + 2, \frac{m-2\tau+6}{2}, \tau - 1, \tau)$	
	$\frac{m+2}{2}$	$c_{\frac{m}{2}}$	$(\frac{m}{2}, \frac{m-2}{2}, 0, \frac{m+4}{2}, \frac{m+2}{2})$
		$f_{\frac{m-2}{2}}$	$(\frac{m}{2}, \frac{m-2}{2}, 2, \frac{m}{2}, \frac{m+2}{2})$
$f_{\frac{m+2}{2}}$		$(\frac{m}{2}, \frac{m+2}{2}, 2, \frac{m}{2}, \frac{m-2}{2})$	
$g_{\frac{m}{2}}$		$(\frac{m+4}{2}, \frac{m+2}{2}, 2, \frac{m}{2}, \frac{m-2}{2})$	
$h_{\frac{m}{2}}$		$(\frac{m+4}{2}, \frac{m+2}{2}, 2, \frac{m}{2}, \frac{m+2}{2})$	
$\frac{m+4}{2}$	$f_{\frac{m}{2}}$	$(\frac{m+2}{2}, \frac{m}{2}, 1, \frac{m+2}{2}, \frac{m}{2})$	

Now, consider Table 4, where the vector of $f_0 \in S_\tau(f_0)$ has its first metric coordinate equal to 1. We may also check that no two vertices $a_1, a_2 \in S_\tau(f_0)$ exist such as $r(a_1, D_{J_m}) - r(a_2, D_{J_m}) = 0I$, where I is the unit vector and $\tau \in \{1, 2, \dots, \frac{m+4}{2}\}$. Furthermore, no two vertices $a_1 \in S_\tau(f_0)$ and $a_2 \in S_\nu(f_0)$ exist such as $r(a_1, D_{J_m}) - r(a_2, D_{J_m}) = \tau - \nu$, for some $\tau, \nu \in \{1, 2, \dots, \frac{m+4}{2}\}$ and $\tau \neq \nu$. Finally, the set $D_{J_m} = \{c_0, c_1, c_{\frac{m}{2}}, g_0, g_{m-1}\}$ is the MDRS. As a result, Lemma 2.3 holds true. \square

Lemma 2.4. $\psi(J_m) = 4, \forall$ odd integers $m \geq 6$.

Proof. To demonstrate that $\psi(J_m) = 4$, where $m \geq 6$ is an odd integer, it is enough to find a DRS of order 4. Now, by using the sets $S_\tau(f_0)$ from Table 1, the metric coordinate vectors are demonstrated in Table 5 for each vertex of J_m with respect to the set $D_{J_m} = \{c_0, c_{\frac{m-3}{2}}, f_{\frac{m-1}{2}}, g_0\}$.

Table 5. Metric coordinate vectors for J_m if $m \geq 6$ is odd.

τ	$S_\tau(f_0)$	$D_{J_m} = \{c_0, c_{\frac{m-3}{2}}, f_{\frac{m-1}{2}}, g_0\}$	
0	f_0	$(1, \frac{m-1}{2}, \frac{m+3}{2}, 1)$	
1	c_0	$(0, \frac{m-3}{2}, \frac{m+1}{2}, 2)$	
	g_0	$(2, \frac{m+1}{2}, \frac{m+1}{2}, 0)$	
	h_0	$(2, \frac{m+1}{2}, \frac{m+1}{2}, 2)$	
2	c_1	$(1, \frac{m-5}{2}, \frac{m-1}{2}, 3)$	
	c_{m-1}	$(1, \frac{m-1}{2}, \frac{m+1}{2}, 3)$	
	g_1	$(3, \frac{m-1}{2}, \frac{m-1}{2}, 1)$	
	g_{m-1}	$(3, \frac{m+3}{2}, \frac{m+1}{2}, 3)$	
	h_1	$(3, \frac{m-1}{2}, \frac{m-1}{2}, 3)$	
	h_{m-1}	$(3, \frac{m+3}{2}, \frac{m+1}{2}, 1)$	
$3 \leq \tau \leq \frac{m-1}{2}$	$c_{\tau-1}$	$(\tau - 1, \frac{m-2\tau-1}{2}, \frac{m-2\tau+3}{2}, \tau + 1)$	
	$c_{m-\tau+1}$	$(\tau - 1, \frac{m-2\tau+5}{2}, \frac{m-2\tau+5}{2}, \tau + 1)$	
	$f_{\tau-2}$	$(\tau - 1, \frac{m-2\tau+3}{2}, \frac{m-2\tau+7}{2}, \tau - 1)$	
	$f_{m-\tau+2}$	=	$(2, \frac{m+1}{2}, \frac{m+3}{2}, 2),$ if $\tau = 3;$
			$(\tau - 1, \frac{m-2\tau+9}{2}, \frac{m-2\tau+9}{2}, \tau - 1),$ if $\tau > 3.$
	$g_{\tau-1}$	$(\tau + 1, \frac{m-2\tau+3}{2}, \frac{m-2\tau+3}{2}, \tau - 1)$	
	$g_{m-\tau+1}$	$(\tau + 1, \frac{m-2\tau+9}{2}, \frac{m-2\tau+5}{2}, \tau + 1)$	
	$h_{\tau-1}$	$(\tau + 1, \frac{m-2\tau+3}{2}, \frac{m-2\tau+3}{2}, \tau + 1)$	
	$h_{m-\tau+1}$	$(\tau + 1, \frac{m-2\tau+9}{2}, \frac{m-2\tau+5}{2}, \tau - 1)$	
	$\frac{m+1}{2}$	$c_{\frac{m-1}{2}}$	$(\frac{m-1}{2}, 1, 1, \frac{m+3}{2})$
$c_{\frac{m+1}{2}}$		$(\frac{m-1}{2}, 2, 2, \frac{m+3}{2})$	
$f_{\frac{m-3}{2}}$		$(\frac{m-1}{2}, 1, 3, \frac{m-1}{2})$	
$f_{\frac{m+3}{2}}$		$(\frac{m-1}{2}, 4, 4, \frac{m-1}{2})$	
$g_{\frac{m-1}{2}}$		$(\frac{m+3}{2}, 3, 1, \frac{m-1}{2})$	
$g_{\frac{m+1}{2}}$		$(\frac{m+3}{2}, 4, 2, \frac{m+1}{2})$	
$h_{\frac{m-1}{2}}$		$(\frac{m+3}{2}, 3, 1, \frac{m+1}{2})$	
$h_{\frac{m+1}{2}}$		$(\frac{m+3}{2}, 4, 2, \frac{m-1}{2})$	
$\frac{m+3}{2}$	$f_{\frac{m-1}{2}}$	$(\frac{m+1}{2}, 2, 0, \frac{m+1}{2})$	
	$f_{\frac{m+1}{2}}$	$(\frac{m+1}{2}, 3, 3, \frac{m+1}{2})$	

Now, consider Table 5, where the vector of $f_0 \in S_\tau(f_0)$ has its first metric coordinate equal to 1. We may also check that no two vertices $a_1, a_2 \in S_\tau(f_0)$ exist such as $r(a_1, D_{J_m}) - r(a_2, D_{J_m}) = 0I$, where I is the unit vector and $\tau \in \{1, 2, \dots, \frac{m+3}{2}\}$. Furthermore, no two vertices $a_1 \in S_\tau(f_0)$ and $a_2 \in S_\nu(f_0)$ exist such as $r(a_1, D_{J_m}) - r(a_2, D_{J_m}) = \tau - \nu$, for some $\tau, \nu \in \{1, 2, \dots, \frac{m+3}{2}\}$ and $\tau \neq \nu$. Finally, the set $D_{J_m} = \{c_0, c_{\frac{m-3}{2}}, f_{\frac{m-1}{2}}, g_0\}$ is the MDRS. As a result, Lemma 2.4 holds true. \square

The combination of Lemmas 2.3 and 2.4 yields the following main theorem:

Theorem 2.5. Let J_m be the flower snark, then for $m \geq 6$,

$$\psi(J_m) = \begin{cases} 4, & \text{if } m \text{ is odd;} \\ 5, & \text{if } m \text{ is even.} \end{cases}$$

3. Double metric dimension for the quasi-flower snarks G_m

In this section, we will concentrate on determining the DMD of quasi-flower snarks G_m .

Figure 2 demonstrates the network of quasi-flower snark G_m , which has $4m$ vertices and $6m$ edges. The vertex set for the quasi-flower snarks is $V_{G_m} = \{c_\tau, f_\tau, g_\tau, h_\tau : \forall 0 \leq \tau \leq m - 1\}$, and the edge set is $E_{G_m} = \{c_\tau c_{\tau+1}, c_\tau f_\tau, f_\tau g_\tau, f_\tau h_\tau, g_\tau g_{\tau+1}, h_\tau h_{\tau+1} : \forall 0 \leq \tau \leq m - 2\} \cup \{c_{m-1} c_0, c_{m-1} f_{m-1}, f_{m-1} g_{m-1}, f_{m-1} h_{m-1}, g_{m-1} g_0, h_{m-1} h_0\}$.

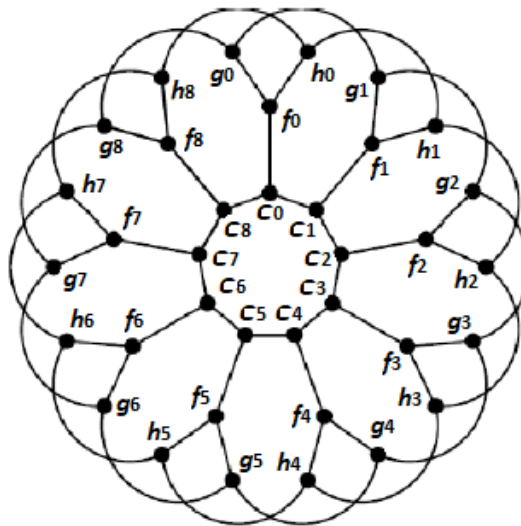


Figure 2. Quasi-flower snark G_9 .

As shown in Figure 2, the inner cycle vertices are labeled as $\{c_\tau : \forall 0 \leq \tau \leq m - 1\}$, the set of central vertices are labeled as $\{f_\tau : \forall 0 \leq \tau \leq m - 1\}$, and the outer cycle vertices are labeled as $\{g_\tau : \forall 0 \leq \tau \leq m - 1\} \cup \{h_\tau : \forall 0 \leq \tau \leq m - 1\}$.

MDRSs for G_m are to be found in this section. It follows from Theorem 1.4 that

$$\psi(G_m) \geq \begin{cases} 3, & \text{if } m \text{ is odd;} \\ \leq 4, & \text{otherwise,} \end{cases}$$

for all $m \geq 6$. We shall also show that $\psi(G_m) = 4$, for all $m \geq 6$.

Define the vertex subset $S_\tau(f_0) = \{g \in V_{G_m} : d(f_0, g) = \tau\} \subseteq V_{G_m}$ having distance τ from f_0 . The Table 6 is a simple formulation for $S_\tau(f_0)$ and will be used to compute the distances between each pair of vertices in V_{G_m} .

Table 6. $S_\tau(f_0)$ for G_m .

m	τ	$S_\tau(f_0)$
	1	$\{c_0, g_0, h_0\}$
	2	$\{c_1, c_{m-1}, g_1, g_{m-1}, h_1, h_{m-1}\}$
	$3 \leq \tau \leq \lfloor \frac{m}{2} \rfloor$	$\{c_{\tau-1}, c_{m-\tau+1}, f_{\tau-2}, f_{m-\tau+2}, g_{\tau-1}, g_{m-\tau+1}, h_{\tau-1}, h_{m-\tau+1}\}$
<i>even</i>	$\frac{m+2}{2}$	$\{c_{\frac{m}{2}}, f_{\frac{m-2}{2}}, f_{\frac{m+2}{2}}, g_{\frac{m}{2}}, h_{\frac{m}{2}}\}$
	$\frac{m+4}{2}$	$\{f_{\frac{m}{2}}\}$
<i>odd</i>	$\frac{m+1}{2}$	$\{c_{\frac{m-1}{2}}, c_{\frac{m+1}{2}}, f_{\frac{m-3}{2}}, f_{\frac{m+3}{2}}, g_{\frac{m-1}{2}}, g_{\frac{m+1}{2}}, h_{\frac{m-1}{2}}, h_{\frac{m+1}{2}}\}$
	$\frac{m+3}{2}$	$\{f_{\frac{m-1}{2}}, f_{\frac{m+1}{2}}\}$

The following facts can be demonstrated as a result of the symmetry of G_m , where $m \geq 6$:

$$d(f_\tau, g_\nu) = d(f_\tau, h_\nu) = d(f_0, g_{|\tau-\nu|}) \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(c_\tau, c_\nu) = d(f_0, c_{|\tau-\nu|}) - 1 \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(c_\tau, f_\nu) = d(f_0, c_{|\tau-\nu|}) \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(c_\tau, g_\nu) = d(c_\tau, h_\nu) = d(f_0, c_{|\tau-\nu|}) + 1 \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(g_\tau, h_\nu) = d(f_0, c_{|\tau-\nu|}) + 1 \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(f_\tau, f_\nu) = d(f_0, f_{|\tau-\nu|}) \text{ if } 0 \leq |\tau - \nu| \leq m - 1;$$

$$d(g_\tau, g_\nu) = d(h_\tau, h_\nu) = d(f_0, g_{|\tau-\nu|}) - 1 \text{ if } 0 \leq |\tau - \nu| \leq m - 1.$$

Therefore, we can recalculate the distances between each pair of vertices in V_{G_m} if $d(f_0, g)$ is known for each $g \in V_{G_m}$.

Lemma 3.1. For any positive integer $m \geq 6$, $\psi(G_m) > 3$.

Proof. To prove the DMD $\psi(G_m) > 3$, we must show that every subset $D_{G_m} \subseteq V_{G_m}$ of cardinality 3 is not a DRS for G_m . Table 7 demonstrates the non-doubly resolved pairs of vertices from the set V_{G_m} as well as all possible forms of subset D_{G_m} . Let us compute that any two vertices of the set $\{c_\tau, c_\nu, f_0; 0 \leq \tau < \nu \leq m - 1\}$ do not doubly resolve the vertices f_0, g_0 . For example, it is clear that $d(f_0, f_0) = 0, d(f_0, g_0) = 1$. Now, for $0 \leq \tau \leq \frac{m}{2}, 1 \leq \nu \leq \frac{m}{2}$, we have

- $d(c_\tau, f_0) = d(f_0, c_\tau) = \tau + 1;$
- $d(c_\tau, g_0) = d(f_0, c_\tau) + 1 = \tau + 2;$
- $d(c_\nu, f_0) = d(f_0, c_\nu) = \nu + 1;$
- $d(c_\nu, g_0) = d(f_0, c_\nu) + 1 = \nu + 2.$

Also, for $\frac{m+2}{2} \leq \tau \leq m - 2, \frac{m+2}{2} \leq \nu \leq m - 1$, we have

- $d(c_\tau, f_0) = d(f_0, c_\tau) = m - \tau + 1;$
- $d(c_\tau, g_0) = d(f_0, c_\tau) + 1 = m - \tau + 2;$
- $d(c_\nu, f_0) = d(f_0, c_\nu) = m - \nu + 1;$
- $d(c_\nu, g_0) = d(f_0, c_\nu) + 1 = m - \nu + 2.$

So, $d(c_\tau, f_0) - d(c_\tau, g_0) = d(c_\nu, f_0) - d(c_\nu, g_0) = d(f_0, f_0) - d(f_0, g_0) = -1$, that is, the set $\{c_\tau, c_\nu, f_0; 0 \leq \tau < \nu \leq m - 1\}$ is not a DRS of G_m . Accordingly, we can investigate all the possible types of subset D_{G_m} shown in Table 7 and verify their associated non-doubly resolved pairs of vertices.

Table 7. Non-doubly resolving pairs of vertices for G_m , where $m \geq 6$.

D_{G_m}	Non-doubly resolving pairs
$\{c_\tau, c_\nu, f_0\}, 0 \leq \tau < \nu \leq m - 1$	$\{f_0, g_0\}$
$\{c_\tau, f_0, g_\nu\}, 0 \leq \tau, \nu \leq m - 1$	$\{f_0, h_0\}$
$\{c_\tau, f_0, f_\nu\}, 0 \leq \tau \leq m - 1, 1 \leq \nu \leq m - 1$	$\{g_0, h_0\}$
$\{c_\tau, f_0, h_\nu\}, 0 \leq \tau, \nu \leq m - 1$	$\{f_0, g_0\}$
$\{f_0, f_\tau, g_\nu\}, 1 \leq \tau \leq m - 1, 0 \leq \nu \leq m - 1$	$\{c_0, h_0\}$
$\{f_0, f_\tau, f_\nu\}, 1 \leq \tau < \nu \leq m - 1$	$\{c_0, g_0\}$
$\{f_0, g_\tau, h_\nu\}, 0 \leq \tau, \nu \leq m - 1$	$\{c_0, f_0\}$
$\{f_0, f_\tau, h_\nu\}, 1 \leq \tau \leq m - 1, 0 \leq \nu \leq m - 1$	$\{c_0, g_0\}$
$\{f_0, h_\tau, h_\nu\}, 0 \leq \tau < \nu \leq m - 1$	$\{f_0, g_0\}$
$\{f_0, g_\tau, g_\nu\}, 0 \leq \tau < \nu \leq m - 1$	$= \begin{cases} \{c_0, h_0\}, & \text{if } m \text{ is even;} \\ \{c_0, f_0\}, & \text{if } m \text{ is odd.} \end{cases}$

□

Lemma 3.2. $\psi(G_m) = 4, \forall$ even integers $m \geq 6$.

Proof. To demonstrate that $\psi(G_m) = 4$, where $m \geq 6$ is an even integer, it is enough to find a DRS of order 4. Now, by using the sets $S_\tau(f_0)$ from Table 6, the metric coordinate vectors are demonstrated in Table 8 for every vertex of G_m with respect to the set $D_{G_m} = \{c_0, c_{\frac{m}{2}}, f_1, g_{\frac{m+2}{2}}\}$.

Now, consider Table 8, where the vector of $f_0 \in S_\tau(f_0)$ has its first metric coordinate equal to 1. We may also check that no two vertices $b_1, b_2 \in S_\tau(f_0)$ exist such as $r(b_1, D_{G_m}) - r(b_2, D_{G_m}) = 0I$, where I is the unit vector and $\tau \in \{1, 2, \dots, \frac{m+4}{2}\}$. Furthermore, no two vertices $b_1 \in S_\tau(f_0)$ and $b_2 \in S_\nu(f_0)$ exist such as $r(b_1, D_{G_m}) - r(b_2, D_{G_m}) = \tau - \nu$, for some $\tau, \nu \in \{1, 2, \dots, \frac{m+4}{2}\}$ and $\tau \neq \nu$. Finally, the set $D_{G_m} = \{c_0, c_{\frac{m}{2}}, f_1, g_{\frac{m+2}{2}}\}$ is the MDRS. As a result, Lemma 3.2 holds true. □

Lemma 3.3. $\psi(G_m) = 4, \forall$ odd integers $m \geq 6$.

Proof. To demonstrate that $\psi(G_m) = 4$, where $m \geq 6$ is an odd integer, it is enough to find a DRS of order 4. Now, by using the sets $S_\tau(f_0)$ from Table 6, the metric coordinate vectors are demonstrated in Table 9 for every vertex of G_m with respect to the set $D_{G_m} = \{c_0, c_{\frac{m-3}{2}}, f_{\frac{m-1}{2}}, g_0\}$.

Now, consider Table 9, where the vector of $f_0 \in S_\tau(f_0)$ has its first metric coordinate equal to 1. We may also check that no two vertices $b_1, b_2 \in S_\tau(f_0)$ exist such as $r(b_1, D_{G_m}) - r(b_2, D_{G_m}) = 0I$, where I denotes the unit vector and $\tau \in \{1, 2, \dots, \frac{m+3}{2}\}$. Furthermore, no two vertices $b_1 \in S_\tau(f_0)$ and $b_2 \in S_\nu(f_0)$ exist such as $r(b_1, D_{G_m}) - r(b_2, D_{G_m}) = \tau - \nu$, for some $\tau, \nu \in \{1, 2, \dots, \frac{m+3}{2}\}$ and $\tau \neq \nu$. Finally, the set $D_{G_m} = \{c_0, c_{\frac{m-3}{2}}, f_{\frac{m-1}{2}}, g_0\}$ is the MDRS. As a result, Lemma 3.3 holds true. □

Table 8. Metric coordinate vectors for G_m if $m \geq 6$ is even.

τ	$S_\tau(f_0)$	$D_{G_m} = \{c_0, c_{\frac{m}{2}}, f_1, g_{\frac{m+2}{2}}\}$	
0	f_0	$(1, \frac{m+2}{2}, 3, \frac{m}{2})$	
1	c_0	$(0, \frac{m}{2}, 2, \frac{m+2}{2})$	
	g_0	$(2, \frac{m+4}{2}, 2, \frac{m-2}{2})$	
	h_0	$(2, \frac{m+4}{2}, 2, \frac{m+2}{2})$	
2	c_1	$(1, \frac{m-2}{2}, 1, \frac{m+4}{2})$	
	c_{m-1}	$(1, \frac{m-2}{2}, 3, \frac{m}{2})$	
	g_1	$(3, \frac{m+2}{2}, 1, \frac{m}{2})$	
	g_{m-1}	$(3, \frac{m+2}{2}, 3, \frac{m-4}{2})$	
	h_1	$(3, \frac{m+2}{2}, 1, \frac{m+4}{2})$	
	h_{m-1}	$(3, \frac{m+2}{2}, 3, \frac{m}{2})$	
$3 \leq \tau \leq \frac{m}{2}$	$c_{\tau-1}$	$(\tau - 1, \frac{m-2\tau+2}{2}, \tau - 1, \frac{m-2\tau+8}{2})$	
	$c_{m-\tau+1}$	$(\tau - 1, \frac{m-2\tau+2}{2}, \tau + 1, \frac{m-2\tau+4}{2})$	
	$f_{\tau-2}$	=	$(2, \frac{m}{2}, 0, \frac{m+2}{2}),$ if $\tau = 3;$
			$(\tau - 1, \frac{m-2\tau+6}{2}, \tau - 1, \frac{m-2\tau+8}{2}),$ if $\tau > 3.$
	$f_{m-\tau+2}$	$(\tau - 1, \frac{m-2\tau+6}{2}, \tau + 1, \frac{m-2\tau+4}{2})$	
	$g_{\tau-1}$	$(\tau + 1, \frac{m-2\tau+6}{2}, \tau - 1, \frac{m-2\tau+4}{2})$	
	$g_{m-\tau+1}$	$(\tau + 1, \frac{m-2\tau+6}{2}, \tau + 1, \frac{m-2\tau}{2})$	
	$h_{\tau-1}$	$(\tau + 1, \frac{m-2\tau+6}{2}, \tau - 1, \frac{m-2\tau+8}{2})$	
	$h_{m-\tau+1}$	$(\tau + 1, \frac{m-2\tau+6}{2}, \tau + 1, \frac{m-2\tau+4}{2})$	
	$\frac{m+2}{2}$	$c_{\frac{m}{2}}$	$(\frac{m}{2}, 0, \frac{m}{2}, 3)$
$f_{\frac{m-2}{2}}$		$(\frac{m}{2}, 2, \frac{m}{2}, 3)$	
$f_{\frac{m+2}{2}}$		$(\frac{m}{2}, 2, \frac{m+4}{2}, 1)$	
$g_{\frac{m}{2}}$		$(\frac{m+4}{2}, 2, \frac{m}{2}, 1)$	
$h_{\frac{m}{2}}$		$(\frac{m+4}{2}, 2, \frac{m}{2}, 3)$	
$\frac{m+4}{2}$	$f_{\frac{m}{2}}$	$(\frac{m+2}{2}, 1, \frac{m+2}{2}, 2)$	

Table 9. Metric coordinate vectors for G_m if $m \geq 6$ is odd.

τ	$S_\tau(f_0)$	$D_{G_m} = \{c_0, c_{\frac{m-3}{2}}, f_{\frac{m-1}{2}}, g_0\}$	
0	f_0	$(1, \frac{m-1}{2}, \frac{m+3}{2}, 1)$	
1	c_0	$(0, \frac{m-3}{2}, \frac{m+1}{2}, 2)$	
	g_0	$(2, \frac{m+1}{2}, \frac{m+1}{2}, 0)$	
	h_0	$(2, \frac{m+1}{2}, \frac{m+1}{2}, 2)$	
2	c_1	$(1, \frac{m-5}{2}, \frac{m-1}{2}, 3)$	
	c_{m-1}	$(1, \frac{m-1}{2}, \frac{m+1}{2}, 3)$	
	g_1	$(3, \frac{m-1}{2}, \frac{m-1}{2}, 1)$	
	g_{m-1}	$(3, \frac{m+3}{2}, \frac{m+1}{2}, 1)$	
	h_1	$(3, \frac{m-1}{2}, \frac{m-1}{2}, 3)$	
	h_{m-1}	$(3, \frac{m+3}{2}, \frac{m+1}{2}, 3)$	
$3 \leq \tau \leq \frac{m-1}{2}$	$c_{\tau-1}$	$(\tau - 1, \frac{m-2\tau-1}{2}, \frac{m-2\tau+3}{2}, \tau + 1)$	
	$c_{m-\tau+1}$	$(\tau - 1, \frac{m-2\tau+5}{2}, \frac{m-2\tau+5}{2}, \tau + 1)$	
	$f_{\tau-2}$	$(\tau - 1, \frac{m-2\tau+3}{2}, \frac{m-2\tau+7}{2}, \tau - 1)$	
	$f_{m-\tau+2}$	=	$(2, \frac{m+1}{2}, \frac{m+3}{2}, 2),$ if $\tau = 3;$
			$(\tau - 1, \frac{m-2\tau+9}{2}, \frac{m-2\tau+9}{2}, \tau - 1),$ if $\tau > 3.$
	$g_{\tau-1}$	$(\tau + 1, \frac{m-2\tau+3}{2}, \frac{m-2\tau+3}{2}, \tau - 1)$	
	$g_{m-\tau+1}$	$(\tau + 1, \frac{m-2\tau+9}{2}, \frac{m-2\tau+5}{2}, \tau - 1)$	
	$h_{\tau-1}$	$(\tau + 1, \frac{m-2\tau+3}{2}, \frac{m-2\tau+3}{2}, \tau + 1)$	
	$h_{m-\tau+1}$	$(\tau + 1, \frac{m-2\tau+9}{2}, \frac{m-2\tau+5}{2}, \tau + 1)$	
	$\frac{m+1}{2}$	$c_{\frac{m-1}{2}}$	$(\frac{m-1}{2}, 1, 1, \frac{m+3}{2})$
$c_{\frac{m+1}{2}}$		$(\frac{m-1}{2}, 2, 2, \frac{m+3}{2})$	
$f_{\frac{m-3}{2}}$		$(\frac{m-1}{2}, 1, 3, \frac{m-1}{2})$	
$f_{\frac{m+3}{2}}$		$(\frac{m-1}{2}, 4, 4, \frac{m-1}{2})$	
$g_{\frac{m-1}{2}}$		$(\frac{m+3}{2}, 3, 1, \frac{m-1}{2})$	
$g_{\frac{m+1}{2}}$		$(\frac{m+3}{2}, 4, 2, \frac{m-1}{2})$	
$h_{\frac{m-1}{2}}$		$(\frac{m+3}{2}, 3, 1, \frac{m+3}{2})$	
$h_{\frac{m+1}{2}}$		$(\frac{m+3}{2}, 4, 2, \frac{m+3}{2})$	
$\frac{m+3}{2}$	$f_{\frac{m-1}{2}}$	$(\frac{m+1}{2}, 2, 0, \frac{m+1}{2})$	
	$f_{\frac{m+1}{2}}$	$(\frac{m+1}{2}, 3, 3, \frac{m+1}{2})$	

The combination of Lemmas 3.2 and 3.3 yields the following main theorem:

Theorem 3.4. *Let G_m be the quasi-flower snark for $m \geq 6$. Then, $\psi(G_m) = 4$.*

4. Application

DRSs have recently been employed to find the origin of epidemics in several complex networks, including social networks, human epidemics, the cause of a disease outbreak, etc. (see [51, 52]). To

get the best possible detection of an epidemic source, we specifically take into account a network that uses DRSs to lower the number of observers.

Assume a social network set up as a flower snark network J_5 , where the node set $V_{J_5} = \{c_\tau, f_\tau, g_\tau, h_\tau : \forall 0 \leq \tau \leq 4\}$ stands in for the individuals and the edge set $E_{J_5} = \{c_\tau c_{\tau+1}, c_\tau f_\tau, f_\tau g_\tau, g_\tau h_\tau, h_\tau c_{\tau+1}, g_\tau g_{\tau+1}, h_\tau h_{\tau+1}, g_0 h_{n-1}, g_{n-1} h_0 : \forall 0 \leq \tau \leq 4\}$ between the nodes stands in for the connections between the individuals. An immediate solution can be found when observers are placed throughout the network, and the inter-node distances are reliable and well-known. However, the entire process would be very costly and time-consuming.

Then, what number of observers is necessary to account for epidemic sources with an unknown starting time and node-to-node transmission delays? Each node has a unique representation depending upon the minimum distance from the observer nodes. We employed the MDRS $D_{J_5} = \{c_0, c_1, f_2, g_0\}$ of the flower snark network J_5 to reduce the number of observers.

We only placed the observers on the nodes c_0, c_1, f_2, g_0 by applying Lemma 2.4 and Theorem 2.5. It is clear from Table 5 and Figure 3 that each node has a unique representation and the epidemic propagation will be optimal by placing observers at the nodes c_0, c_1, f_2, g_0 . This hypothetical situation clarifies how MDRSs can be used to solve source localization issues.

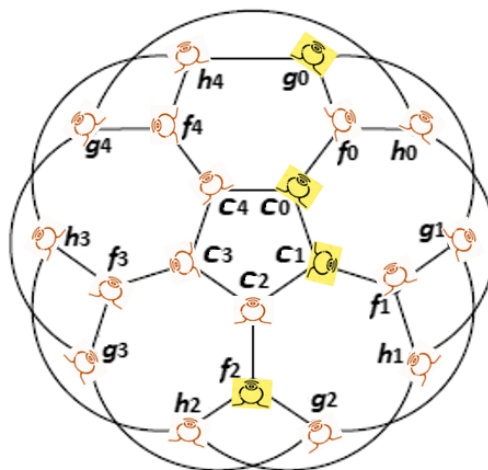


Figure 3. Source detection in social network.

Moreover, the DRSs are valuable for identifying faults in communication networks, assisting administrators in promptly locating and repairing issues for reliable operations. Additionally, they are beneficial in cybersecurity to detect and localize cyberattacks, allowing for quick responses to breaches. In environmental monitoring, DRSs aid in identifying sources of pollution, facilitating swift actions to address environmental hazards. The findings of this study provide a fundamental understanding of DRSs and their potential to enhance source detection in various real-world scenarios.

5. Conclusions

In this article, the DMD for the flower snarks J_m and quasi-flower snarks G_m is computed by describing their MDRSs. We conclude that the DMD for the flower snarks J_m is finite and depends upon the order of the network. Also, the DMD for the quasi-flower snarks G_m is finite and free from

the order of the network. An algorithm is created to calculate the MDRSs and DMD of a network with the help of Matlab or any other simulation tool. Further, an application of the work done in the context of source localization is also provided. It is also clearly observed that computing MDRSs for a network is an NP-hard problem (see [35]), which explains the limitations of computing this parameter and the importance of the obtained results. Exploring future directions for DRSs in various families of networks can help in the development of combinatorial designs, error-correcting codes, network analysis, and optimization problems, among other fields. Following are a few points that address the limitations of the study:

- Constructing DRSs with desirable properties can be a challenging task.
- The problem of finding MDRSs is known to be NP-hard, which means that there is no known efficient algorithm to solve it for all cases.
- As a result, finding optimal DRSs can require significant computational effort.

In conclusion, DRSs have limitations in terms of size, optimality, ambiguity, construction difficulty, practical implementation, and generalization to various problem domains. These restrictions emphasize the importance of carefully taking into account the particular problem requirements and limits and illustrate the difficulties and complexities involved with using DRSs in different scenarios.

Open Problem 5.1. Investigate the DRSs for the power paths P_n^r for all integers r , and n , where $r < \frac{n}{2}$.

Author contributions

Muhammad Ahmad was an engaged participant in initiating the problem, and ensuring the validation of results. Muhammad Faheem contributed to the methodology and conducting the final review. Sanaa A. Bajri played a crucial role in analyzing and computing the results, as well as initiating the drafting process of the study. Zohaib Zahid played a pivotal role in problem discussion, gathering essential data, and providing continuous supervision. Muhammad Javaid participated in characterizing and calculating the results and assisted with software in drafting the study. Hamiden Abd El-Wahed Khalifa contributed to the discussing the problem, and meticulously proofread the final version. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is funded by Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2024R527), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

1. P. J. Slater, Leaves of trees, In: *Proceeding of the 6th southeastern conference on combinatorics, graph theory, and computing, congressus numerantium*, **14** (1975), 549–559.
2. F. Harary, R. A. Melter, On the metric dimension of a graph, *Ars Comb.*, **2** (1976), 191–195.
3. Z. Beerliova, F. Eberhard, T. Erlabach, A. Hall, M. Hoffmann, M. Mihalak, et al., Network discovery and verification, *IEEE J. Sel. Area. Comm.*, **24** (2006), 2168–2181. <https://doi.org/10.1109/JSAC.2006.884015>
4. K. Liu, N. Abu-Ghazaleh, *Virtual coordinate with backtracking for void transversal in geographic routing*, Berlin, Heidelberg: Springer, 2006, 46–59. https://doi.org/10.1007/11814764_6
5. A. Sebo, E. Tannier, On metric generators of graphs, *Math. Oper. Res.*, **29** (2004), 191–406. <https://doi.org/10.1287/moor.1030.0070>
6. S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.*, **70** (1996), 217–229. [https://doi.org/10.1016/0166-218X\(95\)00106-2](https://doi.org/10.1016/0166-218X(95)00106-2)
7. G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, *Discrete Appl. Math.*, **105** (2000), 99–113. [https://doi.org/10.1016/S0166-218X\(00\)00198-0](https://doi.org/10.1016/S0166-218X(00)00198-0)
8. V. Chvtal, Mastermind, *Combinatorica*, **3** (1983), 325–329. <https://doi.org/10.1007/BF02579188>
9. P. Erdos, A. Renyi, On two problems of information theory, *Publ. Math. Inst. Hung. Acad. Sci.*, **8** (1963), 241–254.
10. B. Lindstrom, On a combinatory detection problem I, *Publ. Math. Institute Hung.*, **9** (1964), 195–207.
11. A. Ahmad, M. Baca, S. Sultan, Computing the metric dimension of kayak paddles graph and cycles with chord, *Proyecciones*, **39** (2020), 287–300. <http://dx.doi.org/10.22199/issn.0717-6279-2020-02-0018>
12. M. Ali, G. Ali, M. Imran, A. Q. Baig, M. K. Shafiq, On the metric dimension of Mobius ladders, *Ars Comb.*, **105** (2012), 403–410.
13. R. F. Bailey, K. Meagher, On the metric dimension of Grassmann graphs, *Discrete Math. Theoret. Comput. Sci.*, **13** (2011), 97–104. <https://doi.org/10.46298/dmtcs.532>
14. R. F. Bailey, P. J. Cameron, Basic size, metric dimension and other invariants of groups and graphs, *Bull. Lond. Math. Soc.*, **43** (2011), 209–242. <https://doi.org/10.1112/blms/bdq096>
15. M. Fehr, S. Gosselin, O. R. Oellermann, The metric dimension of Cayley digraphs, *Discrete Math.*, **306** (2006), 31–41. <https://doi.org/10.1016/j.disc.2005.09.015>
16. A. Ahmad, M. Imran, O. Al-Mushayt, S. A. U. H. Bokhary, On the metric dimension of barycentric subdivision of Cayley graphs $Cay(Z_n \oplus Z_m)$, *Miskolc Math. Notes*, **16** (2015), 637–646. <https://doi.org/10.18514/MMN.2015.1192>
17. T. Vetrik, A. Ahmad, Computing the metric dimension of the categorial product of some graphs, *Int. J. Comput. Math.*, **94** (2017), 363–371. <https://doi.org/10.1080/00207160.2015.1109081>

18. J. Caceres, C. Hernado, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, et al., On the metric dimension of cartesian products of graphs, *SIAM J. Discrete Math.*, **21** (2007), 423–441. <https://doi.org/10.1137/050641867>
19. J. Kratica, V. Kovacevic-Vujcic, M. Cangalovic, M. Stojanovic, Minimal doubly resolving sets and the strong metric dimension of some convex polytope, *Appl. Math. Comput.*, **218** (2012), 9790–9801. <https://doi.org/10.1016/j.amc.2012.03.047>
20. M. Imran, H. M. A. Siddiqui, Computing the metric dimension of convex polytopes generated by the wheel related graphs, *Acta Math. Hungar.*, **149** (2016), 10–30. <https://doi.org/10.1007/s10474-016-0606-1>
21. H. M. A. Siddiqui, M. Imran, Computing the metric dimension of wheel related graphs, *Appl. Math. Comput.*, **242** (2014), 624–632. <https://doi.org/10.1016/j.amc.2014.06.006>
22. R. Burdett, M. Haythorpe, A. Newcombe, Variants of the domination number for flower snarks, *Ars Math. Contemp.*, **24** (2024), 1–26. <https://doi.org/10.26493/1855-3974.2710.f3d>
23. M. M. Danas, The mixed metric dimension of flower snarks and wheels, *Open Math.*, **19** (2021), 629–640. <https://doi.org/10.1515/math-2021-0065>
24. A. Girisha, P. Rajendra, U. V. C. Kumar, S. Pushpa, On metric dimensions of flower graphs, *J. Phys. Conf. Ser.*, **2571** (2023), 012019. <https://doi.org/10.1088/1742-6596/2571/1/012019>
25. J. B. Liu, M. F. Nadeem, M. Azeem, Bounds on the partition dimension of convex polytopes, *Comb. Chem. High Throughput Scr.*, **25** (2022), 547–553. <https://doi.org/10.2174/1386207323666201204144422>
26. R. Nawaz, M. K. Jamil, M. Azeem, Edge-based metric resolvability of anti-depression molecular structures and its application, *Results Chem.*, **7** (2024), 101458. <https://doi.org/10.1016/j.rechem.2024.101458>
27. J. Caceres, C. Hernando, M. Mora, I. M. Pelayoe, M. L. Puertas, On the metric dimension of infinite graphs, *Electron. Notes Discrete Math.*, **35** (2012), 15–20. <https://doi.org/10.1016/j.endm.2009.11.004>
28. M. R. Garey, D. S. Johnson, *Computers and intractability: A guide to the theory of NP-completeness*, 1 Eds., W. H. Freeman, 1979.
29. Z. Shao, S. M. Sheikholeslami, P. Wu, J. B. Liu, The metric dimension of some generalized Petersen graphs, *Discrete Dyn. Nature Soc.*, **2018** (2018), 531958. <https://doi.org/10.1155/2018/4531958>
30. H. Raza, S. Hayat, M. Imran, X. F. Pan, Fault-tolerant resolvability and extremal structures of graphs, *Mathematics*, **7** (2019), 78. <https://doi.org/10.3390/math7010078>
31. M. Baca, E. T. Baskoro, A. N. M. Salman, S. W. Saputro, D. Suprijanto, The metric dimension of regular bipartite graphs, *Bull. Math. Soc. Sci. Math. Roumanie*, **54** (2011), 15–28.
32. I. Tomescu, M. Imran, R-Sets and metric dimension of necklace graphs, *Appl. Math. Inf. Sci.*, **9** (2015), 63–67. <http://dx.doi.org/10.12785/amis/010109>
33. H. Raza, S. Hayat, X. F. Pan, On the fault-tolerant metric dimension of certain interconnection networks, *J. Appl. Math. Comput.*, **60** (2019), 517–535, <https://doi.org/10.1007/s12190-018-01225-y>

34. S. Hayat, A. Khan, M. Y. H. Malik, M. Imran, M. K. Siddiqui, Fault-tolerant metric dimension of interconnection networks, *IEEE Access*, **8** (2020), 145435–145445. <https://doi.org/10.1109/ACCESS.2020.3014883>
35. J. Kratica, M. Cangalovic, V. Kovacevic-Vujcic, Computing minimal doubly resolving sets of graphs, *Comput. Oper. Res.*, **36** (2009), 2149–2159. <https://doi.org/10.1016/j.cor.2008.08.002>
36. A. Ahmad, M. Baca, S. Sultan, Minimal doubly resolving sets of necklace graph, *Math. Rep.*, **20** (2018), 123–129.
37. M. Ahmad, D. Alrowaili, R. Ali, Z. Zahid, I. Siddique, Double metric resolvability in convex polytopes, *J. Math.*, **2022** (2022), 884924. <https://doi.org/10.1155/2022/5884924>
38. M. Ahmad, D. Alrowaili, Z. Zahid, I. Siddique, A. Iampan, Minimal doubly resolving sets of some classes of convex polytopes, *J. Math.*, **2022** (2022), 1818734. <https://doi.org/10.1155/2022/1818734>
39. A. Ahmad, M. Baca, S. Sultan, On the minimal doubly resolving sets of Harary graph, *Math. Univ. Comenian.*, **89** (2020), 123–129.
40. A. Ahmad, S. Sultan, On minimal doubly resolving sets of circulant graphs, *Acta Mech. Slovaca*, **21** (2017), 6–11. <https://doi.org/10.21496/ams.002>.
41. M. Cangalovic, J. Kratica, V. Kovacevic-Vujcic, M. Stojanovic, Minimal doubly resolving sets of prism graphs, *Optimization*, **62** (2013), 1037–1043. <https://doi.org/10.1080/02331934.2013.772999>
42. J. Kratica, V. Kovacevic-Vujcic, M. Cangalovic, M. Stojanovic, Minimal doubly resolving sets and the strong metric dimension of Hamming graphs, *Appl. Anal. Discrete Math.*, **6** (2012), 63–71. <https://doi.org/10.2298/AADM111116023K>
43. X. Chen, C. Wang, Approximability of the minimum weighted doubly resolving set problem, In: *Lecture notes in computer science*, Springer, **8591** (2014), 357–368. https://doi.org/10.1007/978-3-319-08783-2_31
44. X. Chen, X. Hu, C. Wang, Approximation for the minimum cost doubly resolving set problem, *Theor. Comput. Sci.*, **609** (2016), 526–543. <https://doi.org/10.1016/j.tcs.2015.03.048>
45. J. B. Liu, A. Zafari, H. Zarei, Metric dimension, minimal doubly resolving sets, and the strong metric dimension for jellyfish graph and cocktail party graph, *Complexity*, **2020** (2020), 407456. <https://doi.org/10.1155/2020/9407456>
46. M. Ahmad, Z. Zahid, M. Javaid, E. Bonyah, Studies of chordal ring networks via double metric dimensions, *Math. Probl. Eng.*, **2022** (2022), 303242. <https://doi.org/10.1155/2022/8303242>
47. J. B. Liu, A. Zafari, Computing minimal doubly resolving sets and the strong metric dimension of the layer sun graph and the line graph of the layer sun graph, *Complexity*, **2020** (2020), 267072. <https://doi.org/10.1155/2020/6267072>
48. L. Pan, M. Ahmad, Z. Zahid, S. Zafar, Computation of the double metric dimension in convex polytopes, *J. Math.*, **2021** (2021), 958969. <https://doi.org/10.1155/2021/9958969>
49. M. Jannesari, Graphs with doubly resolving number 2, *Discrete Appl. Math.*, **339** (2023), 178–183. <https://doi.org/10.1016/j.dam.2023.06.017>

50. M. Ahmad, Z. Zahid, M. Javaid, M. A. Ashebo, A study on minimal doubly resolving sets of certain families of networks, *IEEE Access*, **11** (2023), 56344–56352. <https://doi.org/10.1109/ACCESS.2023.3282701>
51. B. Spinelli, L. E. Celis, P. Thiran, The effect of transmission variance on observer placement for source-localization, *Appl. Netw. Sci.*, **2** (2017), 20. <https://doi.org/10.1007/s41109-017-0040-5>
52. B. Spinelli, L. E. Celis, P. Thiran, How many sensors to localize the source? The double metric dimension of random networks, In: *2018 56th Annual Allerton conference on communication, control, and computing (Allerton)*, 2018, 1036–1043. <https://doi.org/10.1109/ALLERTON.2018.8635897>
53. M. Imran, S. A. H. Bokhary, A. Ahmad, A. Semnicova-Fenovcikova, On classes of regular graphs with constant metric dimension, *Acta Math. Sci.*, **33** (2013), 187–206. [https://doi.org/10.1016/S0252-9602\(12\)60204-5](https://doi.org/10.1016/S0252-9602(12)60204-5)
54. R. Naeem, M. Imran, Metric dimension and exchange property for resolving sets in rotationally-symmetric graphs, *Appl. Math. Inf. Sci.*, **8** (2014), 1665–1674. <https://doi.org/10.12785/amis/080422>



AIMS Press

©2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<https://creativecommons.org/licenses/by/4.0>)