Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 9(8): 22057–22066.
DOI: 10.3934/math.20241072
Received: 26 January 2024
Revised: 31 March 2024
Accepted: 11 April 2024
Published: 15 July 2024

Research article

Solutions of initial and boundary value problems using invariant curves

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Abstract: The purpose of this study is to investigate the solutions of initial and boundary value problems of ordinary differential equations by employing Lie symmetry generators. In this investigation, it shown that invariant curves, which obtained by symmetry generators, also be utilized to find solutions to initial and boundary value problems. A method, involving invariant curves, presented to find solutions to initial and boundary value problems. Solutions to many linear and nonlinear initial and boundary value problems discussed by applying the proposed method.

Keywords: initial and boundary value problems; Lie symmetries; invariant curves; solutions
Mathematics Subject Classification: 22E70, 34B60, 34C45

1. Introduction

For the formulation of engineering problems and physical and mathematical models, initial and boundary value problems are widely used. Initial and boundary value problems of ordinary differential equations (ODEs) play a crucial role in various branches of mathematics and have applications in many fields such as fluid dynamics, astrophysics, quantum mechanics, and other fields of science [1–3]. The search for solutions to initial and boundary value problems has long been a focal concern in mathematics. The analytical and numerical solutions of initial and boundary value problems have been discussed in the past by many researchers using different approaches, such as the decomposition method [4], hybrid block method [5], using green's function [6], considering the positive solutions [7] and Ritz method [8].

Lie (1842–1899) created a new branch in mathematics, group analysis of differential equations. In 1895, Lie remarked on the importance of infinite groups in the theory of invariants of differential equations and many researchers have enhanced his work in different books, such as, Ibragimov in [9, 10], Stephan in [11], Arrigo in [12], Bluman and Kume in [13] and Olver in [14]. In recent years, the Lie group approach has become more popular for finding the solutions of differential equations. Lie group analysis is used to find the solutions in many ways, among which, one of the most important is the application of invariant curves. Many researchers have used Lie groups to find the group invariant solutions of differential equations e.g. optimal system and invariant solutions of Kummer-Schwarz equation [15], particular solutions of ODEs [16], new exact solutions of Date Jimbo Kashiwara Miwa equation [17] and some invariant solutions of modified Kuramoto-Sivashinsky equation [18]. Lie groups are also studied to find solutions for ODEs by employing invariant curves. In this work, it is shown how invariant curves can also be used to find solutions to initial and boundary value problems by following a proposed procedure.

In this article, a method employing the invariant curves to find invariant solutions to initial and boundary value problems is proposed. Many linear and nonlinear initial and boundary value problems are considered and their solutions by applying the proposed method, are discussed. Section 2 consists of some basic definitions and results. The proposed method is explained in Section 3, while applications of this method are discussed in Section 4 by considering many examples.

2. Some basic definitions

2.1. Invariant curve

Let

$$\hat{t} = t + \epsilon\xi(t, w) + O(\epsilon)^2 \tag{2.1}$$

and

$$\hat{w} = w + \epsilon\eta(t, w) + O(\epsilon)^2$$

be a Lie group of transformation [15] of an nth-order ODE written in the following form

$$F(t, w, w', ..., w^{(n)}) = 0. \tag{2.2}$$

A curve defined implicitly by

$$Q(t, w) = 0$$

is said to be an invariant curve under the transformation (2.1) if

$$Q(\hat{t}, \hat{w}) = 0,$$

whenever

$$Q(t, w) = 0.$$

2.2. Invariant solution

A curve [19]

$$Q(t, w) = 0$$

is said to be an invariant solution of (2.2) under the transformation (2.1), if the following two properties are satisfied:

- $Q(t, w) = 0$ is an invariant curve under (1),
- $Q(t, w) = 0$ satisfies (2).

2.3. Characteristic equation

Let
$$X = \xi\partial t + \eta\partial w$$
be a symmetry generator of an ODE (2.2) [15] corresponding to the transformation (2.1). The invariant solution curve corresponding to X of (2.2) is obtained by using the following characteristic equation

$$Q(t, w, w') = \eta - \xi\frac{dw}{dt} = 0. \tag{2.3}$$

3. Method to find invariant solutions of initial and boundary value problems

Let an nth-order ODE (2.2) be an initial or a boundary value problem (with some initial or boundary conditions) with an r-dimensional Lie algebra. Let

$$X = \sum_{i=1}^{k} X_i$$

be a linear combination of some symmetry generators of (2.2). By applying (2.3), the solution of (2.2) can be written in the following form

$$w = c_1 f_1(t) + c_2 f_2(t) + ... + c_n f_n(t) + c_{n+1} \tag{3.1}$$

with two cases:

Case I. By choosing
$$c_1 = c_2 = ... = c_n = 1 \quad \text{and} \quad c_{n+1} = 0$$
in (3.1),
$$w = f_1(t) + f_2(t) + ... + f_n(t) \tag{3.2}$$
a solution of (2.2), is obtained.

Case II. If solution in (3.2) is not satisfied to the considered initial or boundary value problems then find the values of $c_1, c_2, ...c_{n-1}$ and c_n by applying the given initial or boundary conditions. After the substitution of these obtained values in (3.1), a solution of (2.2) is obtained.

4. Invariant solutions of some initial and boundary value problems

In this section, many solutions to initial and boundary value problems are obtained by applying the presented method.

(1) Consider a nonlinear initial value problem [20]

$$\frac{dw}{dt} = 1 + (t - w)^2, \quad 2 \le t \le 3, \quad w(2) = 1 \tag{4.1}$$

with Lie algebra

$$X_1 = (t - w)^2\partial_w,$$

$$X_2 = (t-w)(t^2 - tw - 1)\partial_w.$$

The invariant curve (3) for $X_1 + X_2$ is

$$((t-w) + (t-w)(t^2 - tw - 1))dt = 0.$$

Consider w in the form

$$w = c_1 t - c_2 \frac{1}{1+t} + c_3.$$

By substituting

$$c_1 = c_2 = 1 \quad \text{and} \quad c_3 = 0,$$

$$w = t - \frac{1}{1+t}$$

is a solution of (4.1).

(2) Consider a second order initial value problem [20]

$$\frac{d^3w}{dt^3} + \frac{d^2w}{dt^2} - 4\frac{dw}{dt} = 4w, \quad 0 \le t \le 2, \quad y(0) = 3, \tag{4.2}$$

$$\frac{dw}{dt}(0) = -1,$$
$$\frac{d^2w}{dt^2}(0) = 9$$

with

$$\begin{aligned} X_1 &= \exp(2t)\partial_w, \\ X_2 &= w\partial_w, \\ X_3 &= \exp(-t)\partial_w, \\ X_4 &= \exp(2t)\partial_w, \\ X_5 &= \exp -2t\partial_w. \end{aligned}$$

From $X_2 + X_3 + X_4 + X_5$, invariant curve is

$$w = -(c_1 \exp(-t) + c_2 \exp(2t) + c_3 \exp(-2t)) + c_4.$$

$$w = -(\exp(-t) + \exp(2t) + \exp(-2t))$$

is a solution of (4.2).

(3) Consider a boundary value problem [20]

$$\frac{d^2w}{dt^2} - \frac{dw}{dt} - 2w = \cos x, \quad 0 \le t \le \frac{\pi}{2}, \quad y(0) = -0.3, \quad y(\frac{\pi}{2}) = -0.1 \tag{4.3}$$

with

$$X_1 = \exp(2t)\partial_w,$$

$$X_2 = (3\cos t + \sin t + 10w)\partial_w,$$
$$X_3 = \partial_t - (\cos t + 3w)\partial_w.$$

Using X_2, the invariant curve is

$$(3\cos t + \sin t + 10w)dt = 0,$$

or

$$w = -\frac{1}{10}(c_1 3\cos t + c_2 \sin t) + c_3.$$

$$w = -\frac{1}{10}(3\cos t + \sin t)$$

is a solution of (4.3).

(4) Consider a fourth-order boundary value problem [21]

$$\frac{d^4w}{dt^4} - w - 4\exp(t) = 0, \quad 0 < t < 1 \tag{4.4}$$

having first derivative boundary conditions

$$w(0) = 1, \quad \frac{dw}{dt}(0) = 2,$$
$$w(1) = 2e, \quad \frac{dw}{dt}(1) = 3e,$$

with Lie algebra

$$X_1 = \exp t\partial_w,$$
$$X_2 = \exp(-t)\partial_w,$$
$$X_3 = \partial_t + w\partial_w,$$
$$X_4 = (t\exp t - w)\partial_w.$$

Using $X_3 + X_4$, the invariant curve is

$$w = c_1 t\exp(t) - c_2\exp(t) + c_3 + c_4.$$

By applying boundary conditions, we have

$$c_1 = 1, \quad c_2 = -1 \quad \text{and} \quad c_3 = c_4 = 0.$$

$$w = t\exp(t) + \exp(t)$$

is a solution of (4.4).

(5) Consider a nonlinear boundary value problem [20]

$$\frac{d^2w}{dt^2} - w\frac{dw}{dt} = w^3, \quad 1 \le t \le 2, \quad w(1) = \frac{1}{2}, \quad w(2) = \frac{1}{3} \tag{4.5}$$

with

$$X_1 = \partial_t,$$
$$X_2 = t\partial_t - w\partial_w.$$

Using $X_1 + X_2$, solution is

$$w = \frac{c_1}{1+t} + c_2.$$

By applying the given boundary conditions, we have

$$c_1 = 1, \quad c_2 = 0$$

with

$$w = \frac{1}{1+t},$$

a solution of (4.5).

The sixth-order boundary value problems arise in the narrow convecting layers bounded by stable are believed to surround A-type stars [22, 23]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in, when this instability is as ordinary convection the ODE is of sixth order [24].

(6) Consider a sixth order boundary value problem [25]

$$\frac{d^6w}{dt^6} + 6\exp(t) - w = 0, \quad 0 < t < 1 \tag{4.6}$$

subject to boundary conditions

$$w(0) = 1, \quad w(1) = 0,$$
$$\frac{d^2w}{dt^2}(0) = -1,$$
$$\frac{d^2w}{dt^2}(1) = -2e,$$
$$\frac{d^4w}{dt^4}(0) = -3,$$
$$\frac{d^4w}{dt^4}(1) = -4e,$$

with symmetry generators

$$X_1 = \partial_t + w\partial_w,$$
$$X_2 = \exp(t)\partial_w,$$
$$X_3 = \exp(-t)\partial_w,$$
$$X_4 = (t\exp(t) + w)\partial_w.$$

Using $X_1 - X_4$, the obtained invariant curve is

$$w = c_1\exp(t) - c_2 t\exp(t) + c_3.$$

By substituting

$$c_1 = c_2 = 1 \quad \text{and} \quad c_3 = 0,$$

we have

$$w = (1 - t)\exp(t)),$$

a solution of (4.6).

5. Graphs

This section represents the visual depictions of the solutions to the boundary value problems under consideration. Figures 1–5 show only those parts of the solution functions that adhere to the prescribed boundary conditions. These graphs are helpful to study the behavior of the solutions of the considered boundary value problems.

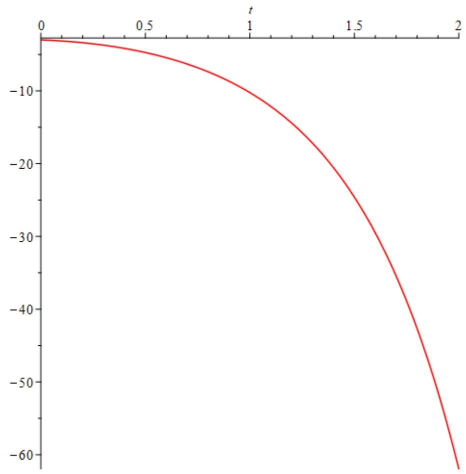

Figure 1. Graphically representation of the solution of (4.2).

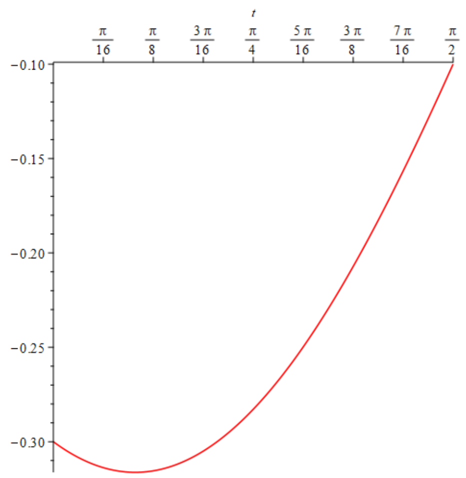

Figure 2. Representation of the solution of (4.3).

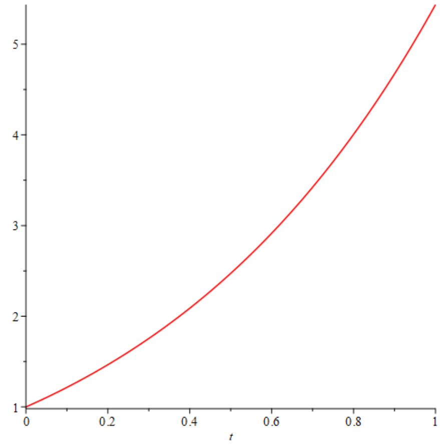

Figure 3. Representation of the solution of (4.4).

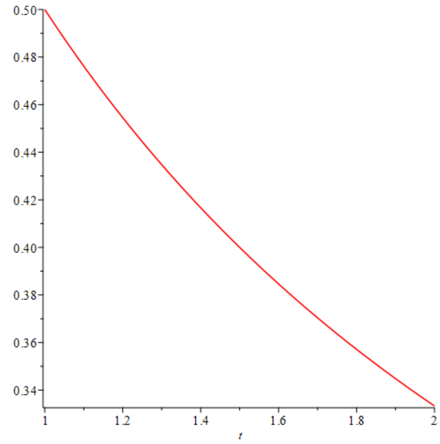

Figure 4. Representation of the solution of (4.5).

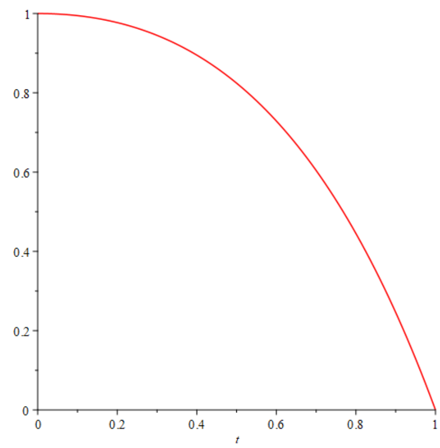

Figure 5. Representation of the solution of (4.6).

6. Conclusions

Solutions of ODEs with initial or boundary conditions have received much attention in recent years. Group analysis of an underlying ODE is a powerful tool for finding solutions for ODEs. A newly presented method employing invariant curves provides an effective framework for the solutions of initial and boundary value problems. Using this method, one can easily find the solutions of initial and boundary value problems.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no competing interest.

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