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Research article

Martingale transforms in martingale Hardy spaces with variable exponents

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Abstract: In this paper, we considered the boundedness of Burkholder's martingale transforms for martingale Hardy spaces with variable exponents. In addition, through martingale transforms, some characterizations of predictable variable exponent martingale Hardy spaces were also provided.

Keywords: variable exponent; martingale Hardy space; martingale transform; Doob's maximal inequality; Lipschitz space
Mathematics Subject Classification: 60G42, 60G46

1. Introduction

Variable exponent Lebesgue space $L_{p(\cdot)}(\mathbb{R}^n)$, as an extension of the classical Lebesgue space $L_p(\mathbb{R}^n)$, has attracted more and more attention due to its wide application in calculus of variations, fluid dynamics, partial differential equations, and harmonic analysis; see, for instance, [1, 5, 6, 11, 15, 37, 38, 43]. The essential difficulty in the studying of this type of space is how to find a condition that ensures the boundedness of the Hardy-Littlewood maximal operator. To this end, Diening [13, 14] came up with a great idea, that is, imposing the log-Hölder continuity condition on variable exponents. From then on, the research on Lebesgue spaces with variable exponents has made rapid progress. The interested readers may consult [12,28,29,36,41,42] for more latest developments on variable Lebesgue spaces.

The classical martingale Hardy space, as an indispensable part of martingale theory, has been systematically investigated in monographs [17, 33, 39]. In recent years, martingale Hardy spaces with variable exponents have achieved fruitful results along with the developments of variable Lebesgue spaces mentioned above. Now, consider $(\Omega, \mathcal{F}, \mathbb{P})$ as a probability space and $p(\cdot)$ as a variable exponent on Ω. Jiao et al. first studied variable martingale Hardy spaces $H_{p(\cdot)}(\Omega)$ concerned with $L_{p(\cdot)}(\Omega)$ in [27]. Due to the lack of metrics in abstract probability spaces, instead of the log-Hölder continuity condition, they provided a new condition without metric characterization to describe Doob's maximal inequality on $L_{p(\cdot)}(\Omega)$. Hao [18] established atomic decompositions of predictable martingale Hardy spaces $P_{p(\cdot)}(\Omega)$. The Burkholder-Davis-Gundy inequality in $H_{p(\cdot)}(\Omega)$ was considered by Liu and Wang [32] as well as Weisz [40]. The definitions of relevant martingale spaces are given in §2.3. Also, we refer the readers to [2, 23, 24, 26, 34, 35] for more results about martingale Hardy spaces in the context of variable exponents.

The first rigorous research on martingale transforms is attributed to Burkholder [7]. Specially, the author proved that the martingale transform $T_v f$ of an $L_1(\Omega)$-bounded f converges a.e. on the set $\{M(v) < \infty\}$, where $M(v)$ denotes Doob's maximal function of the adapted process $v = (v_n)_{n\geq 0}$; for the definition of martingale transform $T_v f$, refer to §2.1. Several years later, Chao and Long [8, 9] derived the boundedness of T_v from classical martingale Hardy space $H_q(\Omega)$ to $H_r(\Omega)$, where

$$v \in V_p(\Omega) := \{v : \|M(v)\|_{L_p(\Omega)} < \infty\} \quad \text{and} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Furthermore, they used these boundedness properties to characterize the predictable martingale Hardy spaces via martingale transform T_v. Let X be a Banach function space (see [4]) over $(\Omega, \mathcal{F}, \mathbb{P})$ and wX be the naturally defined weak space associated with X. Recently, Kikuchi [30] and [31], respectively, provided necessary and sufficient conditions for X to have the following properties:

$$\|T_v f\|_X \lesssim \|f\|_X \quad \text{and} \quad \|T_v f\|_{wX} \lesssim \|f\|_{wX},$$

where f is a uniformly integrable martingale and $M(v)$ is bounded by one. Very recently, using extrapolation theory, Ho [22] also established the boundedness of martingale transforms on Banach function spaces. In particular, under the assumption that the filtration $(\mathcal{F}_n)_{n\geq 0}$ is composed of atomic sub-σ-algebras and $p(\cdot)$ satisfies $1 < p_- < p_+ < \infty$ and condition (2.2) (see §2.3 below), the author proved that

$$\|T_v f\|_{L_{p(\cdot)}(\Omega)} \lesssim \|f\|_{L_{p(\cdot)}(\Omega)}$$

holds for any $f \in L_{p(\cdot)}(\Omega)$ and uniformly bounded v. The interested readers can look up literature [3, 10, 16, 19–21, 25] for more results about martingale transforms.

Following the above line of research, especially inspired by Chao and Long [8,9], we are committed to investigating the boundedness of Burkholder's martingale transforms in variable martingale Hardy spaces $H_{p(\cdot)}(\Omega)$. We also use martingale transforms to characterize predictable martingale Hardy spaces with variable exponents. The main results of the present paper are stated in §3. It should be noted that our results extend corresponding results in [8, 9].

At the end of this section, we make some conventions. In this paper, we always use the symbol $\mathbb{N}$ to denote the natural number set. Throughout the present paper, C denotes the absolute positive constant which may differ from line to line, while the positive constant depending only on $p(\cdot)$ is denoted by $C_{p(\cdot)}$. Regardless of $a \leq Cb$ or $a \leq C_{p(\cdot)}b$, we abbreviate it as $a \lesssim b$. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$. For a measurable set A, its characteristic function is denoted by χ_A.

2. Preliminaries

In this section, some preliminary knowledge to be used later is provided.

2.1. Martingales and martingale transforms

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geq 0}$, that is, $(\mathcal{F}_n)_{n\geq 0}$ is a sequence of non-decreasing sub-σ-algebras of $\mathcal{F}$ satisfying $\mathcal{F} = \bigvee_{n\geq 0} \mathcal{F}_n$. The conditional expectation of any integrable function g with respect to $\mathcal{F}_n$ is denoted by $\mathbb{E}(g|\mathcal{F}_n)$. An integrable sequence $f = (f_n)_{n\geq 0}$, which is adapted to $(\mathcal{F}_n)_{n\geq 0}$, is called a martingale, if $\mathbb{E}(f_{n+1}|\mathcal{F}_n) = f_n$ for all $n \in \mathbb{N}$. Denote the set of all martingales relative to $(\mathcal{F}_n)_{n\geq 0}$ by $\mathcal{M}$. For convenience, we always suppose that for any $f \in \mathcal{M}$, $f_0 = 0$. The martingale difference sequence $\{df_n\}_{n\geq 0}$ of $f \in \mathcal{M}$ is defined by $df_n = f_n - f_{n-1}$ (with convention $f_{-1} = 0$).

Let $f \in \mathcal{M}$ and let $v = (v_n)_{n\geq 0}$ be a process adapted to the same filtration $(\mathcal{F}_n)_{n\geq 0}$. Define the martingale transform $T_v f = (T_v f_n)_{n\geq 0}$ as

$$T_v f_0 = 0 \quad \text{and} \quad T_v f_n = \sum_{m=1}^{n} v_{m-1} df_m,\ n \geq 1.$$

It is obvious that $T_v f \in \mathcal{M}$. Now, consider μ to be a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$. Then, the martingale $f \in \mathcal{M}$ stopped at μ, which is denoted by $f^\mu = (f_n^\mu)_{n\geq 0} = (f_{\mu\wedge n})_{n\geq 0}$, is a special martingale transform with $v_{n-1} = \chi_{\{\mu\geq n\}}$ $(n \in \mathbb{N})$.

2.2. Variable exponent Lebesgue spaces $L_{p(\cdot)}(\Omega)$

Consider $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space. The so-called variable exponent is a positive $\mathcal{F}$-measurable function $p(\cdot)$ defined on Ω. For convenience, denote

$$p_-(G) := \inf_{x\in G} p(x), \quad p_+(G) := \sup_{x\in G} p(x),$$

where G is a measurable subset of Ω. If $G = \Omega$, we further abbreviate $p_-(\Omega)$ and $p_+(\Omega)$ as p_- and p_+, respectively. Denote by $\mathcal{P}(\Omega)$ the set of all variable exponents $p(\cdot)$ satisfying $0 < p_- \leq p_+ < \infty$. Let $p(\cdot) \in \mathcal{P}(\Omega)$ be such that $p_- \geq 1$. Define its conjugate exponent $p'(\cdot)$ point-wisely by the equation

$$\frac{1}{p'(\cdot)} = 1 - \frac{1}{p(\cdot)}.$$

Given a variable exponent $p(\cdot)$, the variable Lebesgue space $L_{p(\cdot)}(\Omega)$ consists of all $\mathcal{F}$-measurable functions f for which there exists some $s > 0$ such that $\rho_{p(\cdot)}(f/s) < \infty$, where

$$\rho_{p(\cdot)}(f) := \int_\Omega (|f(x)|)^{p(x)} d\mathbb{P}.$$

It is well-known that the variable Lebesgue space $L_{p(\cdot)}(\Omega)$ becomes a quasi-Banach space if we equip it with the following quasi-norm:

$$\|f\|_{L_{p(\cdot)}(\Omega)} := \inf\{s > 0 : \rho_{p(\cdot)}(f/s) \leq 1\}.$$

Also, it is clear that the variable Lebesgue space $L_{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L_p(\Omega)$ once $p(\cdot)$ degenerates into the constant p.

At the end of this subsection, we collect two results that will be used in the sequel.

Lemma 2.1 ([11, Corollary 2.28]). *Let $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\Omega)$ be such that*

$$\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}.$$

Then, for all $f \in L_{p(\cdot)}(\Omega)$ and $g \in L_{q(\cdot)}(\Omega)$, we have $fg \in L_{r(\cdot)}(\Omega)$ and

$$\|fg\|_{L_{r(\cdot)}(\Omega)} \lesssim \|f\|_{L_{p(\cdot)}(\Omega)}\|g\|_{L_{q(\cdot)}(\Omega)}.$$

Lemma 2.2 ([11, Proposition 2.21 and Corollary 2.22]). *Let $p(\cdot) \in \mathcal{P}(\Omega)$ and $f \in L_{p(\cdot)}(\Omega)$.*

1) If $\|f\|_{L_{p(\cdot)}(\Omega)} > 0$, then

$$\rho_{p(\cdot)}(f/\|f\|_{L_{p(\cdot)}(\Omega)}) = 1;$$

2) $\|f\|_{L_{p(\cdot)}(\Omega)} < 1$ $(= 1, > 1)$ if, and only if, $\rho_{p(\cdot)}(f) < 1$ $(= 1, > 1)$.

2.3. Martingale Hardy spaces with variable exponents

In this subsection, we recall some basic notation and useful results of variable martingale Hardy spaces $H_{p(\cdot)}(\Omega)$, which were first investigated by Jiao et al. [27]. A martingale $f \in \mathcal{M}$ is said to be an $L_{p(\cdot)}(\Omega)$-martingale, if every f_n is in $L_{p(\cdot)}(\Omega)$. Also, we define

$$\|f\|_{L_{p(\cdot)}(\Omega)} := \sup_{n\in\mathbb{N}} \|f_n\|_{L_{p(\cdot)}(\Omega)}$$

in this case. Furthermore, we say that f is a bounded $L_{p(\cdot)}(\Omega)$-martingale when $\|f\|_{L_{p(\cdot)}(\Omega)}$ is finite, and we also abbreviate it as $f \in L_{p(\cdot)}(\Omega)$.

For $f \in \mathcal{M}$, define

$$M(f) := \sup_{n\in\mathbb{N}} |f_n|, \quad S(f) := \left(\sum_{n=1}^{\infty} |df_n|^2\right)^{\frac{1}{2}} \quad \text{and} \quad s(f) := \left(\sum_{n=1}^{\infty} \mathbb{E}(|df_n|^2|\mathcal{F}_{n-1})\right)^{\frac{1}{2}},$$

where $M(f), S(f)$ and $s(f)$ are respectively referred to as the maximal function, the square function, and the conditional square function of f. Moreover, we denote the set, which consists of nonnegative, nondecreasing, and adapted function sequences $\lambda = (\lambda_n)_{n\geq 0}$ with $\lambda_\infty = \lim_{n\to\infty} \lambda_n$, as Γ. Given $p(\cdot) \in \mathcal{P}(\Omega)$, set

$$\Gamma[P_{p(\cdot)}](f) := \{\lambda \in \Gamma : |f_n| \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot)}(\Omega)\},$$
$$\Gamma[Q_{p(\cdot)}](f) := \{\lambda \in \Gamma : S_n(f) \leq \lambda_{n-1}, \lambda_\infty \in L_{p(\cdot)}(\Omega)\},$$

where

$$S_n(f) := \left(\sum_{m=1}^{n} |df_m|^2\right)^{\frac{1}{2}}.$$

Now, the variable martingale Hardy spaces tied in with $L_{p(\cdot)}(\Omega)$ are defined as:

$$H^s_{p(\cdot)}(\Omega) := \{f \in \mathcal{M} : \|f\|_{H^s_{p(\cdot)}(\Omega)} = \left\|s(f)\right\|_{L_{p(\cdot)}(\Omega)} < \infty\};$$

$$H^S_{p(\cdot)}(\Omega) := \{f \in \mathcal{M} : \|f\|_{H^S_{p(\cdot)}(\Omega)} = \left\|S(f)\right\|_{L_{p(\cdot)}(\Omega)} < \infty\};$$

$$H^M_{p(\cdot)}(\Omega) := \{f \in \mathcal{M} : \|f\|_{H^M_{p(\cdot)}(\Omega)} = \big\|M(f)\big\|_{L_{p(\cdot)}(\Omega)} < \infty\};$$

$$P_{p(\cdot)}(\Omega) := \Big\{f \in \mathcal{M} : \|f\|_{P_{p(\cdot)}(\Omega)} = \inf_{\lambda\in\Gamma[P_{p(\cdot)}](f)} \|\lambda_\infty\|_{L_{p(\cdot)}(\Omega)} < \infty\Big\};$$

$$Q_{p(\cdot)}(\Omega) := \Big\{f \in \mathcal{M} : \|f\|_{Q_{p(\cdot)}(\Omega)} = \inf_{\lambda\in\Gamma[Q_{p(\cdot)}](f)} \|\lambda_\infty\|_{L_{p(\cdot)}(\Omega)} < \infty\Big\}.$$

Obviously, these variable martingale Hardy spaces $H_{p(\cdot)}(\Omega)$ go back to classical martingale Hardy spaces $H_p(\Omega)$, when $p(\cdot) = p$ is a constant (see [39]).

Different from $\mathbb{R}^n$, the abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ lacks metrics. In order to obtain Doob's maximal inequality on $L_{p(\cdot)}(\Omega)$, Jiao et al. [27] imposed the following condition without metric characterization on variable exponent $p(\cdot)$, that is, for all $B \in \mathcal{F}$,

$$\mathbb{P}(B)^{p_-(B)-p_+(B)} \le C_{p(\cdot)}, \tag{2.1}$$

where $C_{p(\cdot)} \ge 1$ is a constant depending only on $p(\cdot)$. Under this condition, the authors also characterized the dual spaces of $H^s_{p(\cdot)}(\Omega)$ as variable Lipschitz spaces $\Lambda_2(1/p(\cdot) - 1)(\Omega)$ defined below.

Definition 2.3 ([27, Definition 5.5]). *Let $\alpha(\cdot) \in \mathcal{P}(\Omega)$ and let $1 \le r < \infty$. Define $\Lambda_r(\alpha(\cdot))(\Omega)$ to be the space of all functions $f \in L_r(\Omega)$ such that*

$$\|f\|_{\Lambda_r(\alpha(\cdot))(\Omega)} := \sup_{\tau\in\mathcal{T}} \frac{\|f - f^\tau\|_{L_r(\Omega)}}{\|\chi_{\{\tau<\infty\}}\|_{L_{\frac{1}{\alpha(\cdot)}}(\Omega)}\|\chi_{\{\tau<\infty\}}\|_{L_r(\Omega)}} < \infty.$$

Remark 2.4. *If $\alpha(\cdot) = \alpha$ is a constant, then the variable Lipschitz spaces $\Lambda_r(\alpha(\cdot))(\Omega)$ are just the classical Lipschitz spaces $\Lambda_r(\alpha)(\Omega)$ defined by Weisz in [39].*

Lemma 2.5 ([27, Theorem 5.6]). *Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy* (2.1). *If $0 < p_- \le p_+ \le 1$, then*

$$(H^s_{p(\cdot)}(\Omega))^* = \Lambda_2(\alpha(\cdot))(\Omega),$$

where $\alpha(x) = 1/p(x) - 1$.

It should be noted that, to better describe Doob's maximal inequality on variable martingale spaces $L_{p(\cdot)}(\Omega)$, Jiao et al. [24, 27] and Weisz [40] also supposed that the filtration $(\mathcal{F}_n)_{n\ge0}$ is composed of atomic sub-σ-algebras. That is, every $\mathcal{F}_n$ is generated by, at most, countable atoms. Here, we say that the $\mathcal{F}_n$-measurable set B is an $\mathcal{F}_n$-atom if the measure of B is positive, and if for an arbitrary $\mathcal{F}_n$-measurable subset A of B with $\mathbb{P}(A) < \mathbb{P}(B)$, it implies $\mathbb{P}(A) = 0$. Denote $\mathcal{A} = \bigcup_{n\ge0} \mathcal{A}(\mathcal{F}_n)$, where $\mathcal{A}(\mathcal{F}_n)$ is the collection of all $\mathcal{F}_n$-atoms. With this assumption to $(\mathcal{F}_n)_{n\ge0}$, in [24, 40], the variable exponent $p(\cdot)$ only needs to satisfy condition (2.1) for all atoms, namely,

$$\mathbb{P}(A)^{p_-(A)-p_+(A)} \le C_{p(\cdot)}, \quad \forall A \in \mathcal{A}. \tag{2.2}$$

Lemma 2.6 ([24, Theorem 4.5]). *Suppose that $(\mathcal{F}_n)_{n\ge0}$ consists of atomic sub-σ-algebras and $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies* (2.2). *Then,*

1) $\|f\|_{H^M_{p(\cdot)}(\Omega)} \lesssim \|f\|_{H^s_{p(\cdot)}(\Omega)}, \quad$ *if* $\quad 0 < p_- \le p_+ < 2$*;*

2) $\|f\|_{H^M_{p(\cdot)}(\Omega)} \lesssim \|f\|_{P_{p(\cdot)}(\Omega)}, \quad \|f\|_{H^S_{p(\cdot)}(\Omega)} \lesssim \|f\|_{Q_{p(\cdot)}(\Omega)}$*;*

3) $\|f\|_{H^s_{p(\cdot)}(\Omega)} \lesssim \|f\|_{P_{p(\cdot)}(\Omega)}$*;*

4) $\|f\|_{P_{p(\cdot)}(\Omega)} \approx \|f\|_{Q_{p(\cdot)}(\Omega)}$.

Lemma 2.7 ([40, Theorem 3]). *Suppose that* $(\mathcal{F}_n)_{n\geq 0}$ *consists of atomic sub-*σ*-algebras and* $p(\cdot) \in \mathcal{P}(\Omega)$ *satisfies* (2.2). *If* $1 \leq p_- \leq p_+ < \infty$*, then*

$$\|f\|_{H^M_{p(\cdot)}(\Omega)} \approx \|f\|_{H^S_{p(\cdot)}(\Omega)}.$$

Remark 2.8. *To be precise, Weisz [40] proposed the following new condition on* $p(\cdot)$*:*

$$\mathbb{P}(A)^{1/p_-(A)-1/p_+(A)} \geq K_{p(\cdot)}, \quad \forall A \in \mathcal{A}, \tag{2.3}$$

where $0 < K_{p(\cdot)} < 1$. *However, it is easy to see that* (2.3) *is equivalent to* (2.2) *when* $p(\cdot) \in \mathcal{P}(\Omega)$.

3. Martingale transforms in variable martingale Hardy spaces

In this section, we mainly establish the boundedness of martingale transforms in $H_{p(\cdot)}(\Omega)$. At the end of this section, in some sense, we also provide characterizations of predictable martingale Hardy spaces via martingale transforms.

Let T be an operator from martingale space X to another martingale space Y. If T is bounded from X to Y, i.e., there exists a constant $C > 0$ such that for all $f \in X$,

$$\|Tf\|_Y \leq C\|f\|_X,$$

then we say that T is of type (X, Y). For $p(\cdot) \in \mathcal{P}(\Omega)$, denote by $V_{p(\cdot)}(\Omega)$ the class of all adapted sequences $v = (v_n)_{n\geq 0}$ such that $\|M(v)\|_{L_{p(\cdot)}(\Omega)} < \infty$, namely,

$$V_{p(\cdot)}(\Omega) := \{v : \|v\|_{V_{p(\cdot)}(\Omega)} := \|M(v)\|_{L_{p(\cdot)}(\Omega)} < \infty\}.$$

In [9, Theorem 1], Chao and Long proved that T_v is bounded from $X_q(\Omega)$ to $X_r(\Omega)$ if $X \in \{H^s, H^S\}$, where $0 < p, q < \infty, v \in V_p(\Omega)$, and $1/r = 1/p + 1/q$. Now, we extend their results to variable martingale setting.

Theorem 3.1. *Let* $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$ *and let* $1/r(x) = 1/p(x) + 1/q(x)$. *If* $v \in V_{p(\cdot)}(\Omega)$*, then* T_v *is of types* $(H^s_{q(\cdot)}(\Omega), H^s_{r(\cdot)}(\Omega))$ *and* $(H^S_{q(\cdot)}(\Omega), H^S_{r(\cdot)}(\Omega))$ *with* $\|T_v\| \lesssim \|v\|_{V_{p(\cdot)}(\Omega)}$.

Proof. From the definitions of operators s and S, it is easy to find the following point-wise estimations

$$s(T_v f) \leq M(v)s(f)$$

and

$$S(T_v f) \leq M(v)S(f).$$

Combining these and Lemma 2.1, we obtain the desired results. □

By a duality argument, we further obtain the following theorem.

Theorem 3.2. *Let* $p(\cdot), \alpha(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (2.1) *and* $1/\alpha_- < p_-$. *If* $v \in V_{p(\cdot)}(\Omega)$*, then* T_v *is of type* $(\Lambda_2(\alpha(\cdot))(\Omega), \Lambda_2(\beta(\cdot))(\Omega))$*, where* $\beta(x) = \alpha(x) - 1/p(x)$.

Proof. Define two new variable exponents $r(\cdot)$ and $q(\cdot)$, respectively, by equations

$$r(x) = \frac{1}{1+\alpha(x)} \quad \text{and} \quad \frac{1}{p(x)} + \frac{1}{q(x)} = 1 + \alpha(x).$$

Then, $\alpha(x) = 1/r(x) - 1, \beta(x) = 1/q(x) - 1$ and $1/r(x) = 1/p(x) + 1/q(x)$. Since $1/\alpha_- < p_-$, we have $q(\cdot) \in \mathcal{P}(\Omega)$.

For any given $g \in \Lambda_2(\alpha(\cdot))(\Omega)$, set

$$L_{T_v g}(f) := \mathbb{E}[fT_v g], \quad \forall f \in L_2(\Omega).$$

Note that T_v is self-adjoint, that is,

$$\mathbb{E}[fT_v g] = \sum_{n=1}^{\infty} \mathbb{E}[df_n v_{n-1} dg_n] = \mathbb{E}[gT_v f].$$

From Lemma 2.5 and Theorem 3.1, it follows that

$$|L_{T_v g}(f)| = |\mathbb{E}[gT_v f]| \lesssim \|g\|_{\Lambda_2(\alpha(\cdot))(\Omega)} \|T_v f\|_{H^s_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \|g\|_{\Lambda_2(\alpha(\cdot))(\Omega)} \|f\|_{H^s_{q(\cdot)}(\Omega)}.$$

Note also that $0 < q_- \leq q_+ \leq 1$. It is easy to see from the proof of [27, Theorem 4.2] that $L_2(\Omega)$ is dense in $H^s_{q(\cdot)}(\Omega)$. Therefore, $T_v g \in (H^s_{q(\cdot)}(\Omega))^* = \Lambda_2(\beta(\cdot))(\Omega)$ and

$$\|T_v g\|_{\Lambda_2(\beta(\cdot))(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \|g\|_{\Lambda_2(\alpha(\cdot))(\Omega)}.$$

The proof is complete. □

Remark 3.3. *Let* $p(\cdot) = p, \alpha(\cdot) = \alpha$ *be such that* $0 < 1/\alpha < p < \infty$, *and let* $v \in V_p(\Omega)$. *Then,* T_v *is of type* $(\Lambda_2(\alpha)(\Omega), \Lambda_2(\beta)(\Omega))$, *where* $\beta = \alpha - 1/p$. *This result was first proved by Chao and Long in [9, Theorem 2].*

Applying Theorem 3.1 and martingale inequalities in $H_{p(\cdot)}(\Omega)$, we further prove the boundedness of martingale transform T_v in variable martingale Hardy spaces consisting of predictable martingales.

Theorem 3.4. *Suppose that* $(\mathcal{F}_n)_{n\geq 0}$ *is composed of atomic sub-*σ*-algebras. If* $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (2.2), $v \in V_{p(\cdot)}(\Omega)$, *and* $1/r(x) = 1/p(x) + 1/q(x)$, *then* T_v *is of types* $(P_{q(\cdot)}(\Omega), P_{r(\cdot)}(\Omega))$ *and* $(Q_{q(\cdot)}(\Omega), Q_{r(\cdot)}(\Omega))$ *with* $\|T_v\| \lesssim \|v\|_{V_{p(\cdot)}(\Omega)}$.

Proof. First, we assume that $f \in P_{q(\cdot)}(\Omega)$. By the definition of $\|f\|_{P_{q(\cdot)}(\Omega)}$, there exists a sequence $\lambda = (\lambda_n)_{n\geq 0} \in \Gamma[P_{q(\cdot)}](f)$ satisfying

$$|f_n| \leq \lambda_{n-1} \quad \text{and} \quad \|\lambda_\infty\|_{L_{q(\cdot)}(\Omega)} \leq 2\|f\|_{P_{q(\cdot)}(\Omega)}.$$

Thus, $|df_n| \leq |f_n| + |f_{n-1}| \leq 2\lambda_{n-1}$ and

$$|d(T_v f)_n| = |v_{n-1} df_n| \leq 2M_{n-1}(v)\lambda_{n-1} =: \rho_{n-1},$$

where $M_{n-1}(v) = \sup_{0\leq m\leq n-1} |v_m|$. From Lemma 2.6, as well as Theorem 3.1, we find that

$$\|T_v f\|_{H^M_{r(\cdot)}(\Omega)} \lesssim \|T_v f\|_{H^s_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \|f\|_{H^s_{q(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \|f\|_{P_{q(\cdot)}(\Omega)}$$

is valid for $0 < r_- \le r_+ < 2$. On the other hand, for $1 \le r_- \le r_+ < \infty$, combining Lemma 2.7 with Theorem 3.1 and Lemma 2.6, we obtain

$$\begin{aligned}\|T_v f\|_{H^M_{r(\cdot)}(\Omega)} \approx \|T_v f\|_{H^S_{r(\cdot)}(\Omega)} &\lesssim \|v\|_{V_{p(\cdot)}(\Omega)}\|f\|_{H^S_{q(\cdot)}(\Omega)} \\ &\lesssim \|v\|_{V_{p(\cdot)}(\Omega)}\|f\|_{Q_{q(\cdot)}(\Omega)} \approx \|v\|_{V_{p(\cdot)}(\Omega)}\|f\|_{P_{q(\cdot)}(\Omega)}.\end{aligned}$$

Hence, for all $r(\cdot) \in \mathcal{P}(\Omega)$,

$$\|T_v f\|_{H^M_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)}\|f\|_{P_{q(\cdot)}(\Omega)}.$$

Note that

$$|T_v f_n| \le |T_v f_{n-1}| + |d(T_v f)_n| \le M_{n-1}(T_v f) + \rho_{n-1},$$

where $M_{n-1}(T_v f) = \sup_{0\le m\le n-1} |T_v f_m|$ and

$$\|\rho_\infty\|_{L_{r(\cdot)}(\Omega)} \lesssim \|M(v)\|_{L_{p(\cdot)}(\Omega)}\|\lambda_\infty\|_{L_{q(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)}\|f\|_{P_{q(\cdot)}(\Omega)}.$$

Consequently,

$$\|T_v f\|_{P_{r(\cdot)}(\Omega)} \lesssim \|T_v f\|_{H^M_{r(\cdot)}(\Omega)} + \|\rho_\infty\|_{L_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)}\|f\|_{P_{q(\cdot)}(\Omega)}.$$

Taking a similar argument for $Q_{q(\cdot)}(\Omega)$, we complete the proof. □

Further on, let us consider the boundedness of T_v on $H^M_{p(\cdot)}(\Omega)$. Our method is the following Davis's decomposition for $H^M_{p(\cdot)}(\Omega)$, which was given by Weisz [40] without proof. For the convenience of readers, we provide a detailed proof here.

Lemma 3.5. *Assume that $(\mathcal{F}_n)_{n\ge0}$ is composed of atomic sub-σ-algebras. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy* (2.2) *and $1 \le p_- \le p_+ < \infty$. If $f \in H^M_{p(\cdot)}(\Omega)$, then there exist $h \in \mathcal{G}_{p(\cdot)}(\Omega)$ and $g \in P_{p(\cdot)}(\Omega)$ such that $f = g + h$ with*

$$\|h\|_{\mathcal{G}_{p(\cdot)}(\Omega)} \lesssim \|f\|_{H^M_{p(\cdot)}(\Omega)} \quad \text{and} \quad \|g\|_{P_{p(\cdot)}(\Omega)} \lesssim \|f\|_{H^M_{p(\cdot)}(\Omega)},$$

where we say that $h \in \mathcal{G}_{p(\cdot)}(\Omega)$, if

$$\|h\|_{\mathcal{G}_{p(\cdot)}(\Omega)} := \left\| \sum_{n=1}^{\infty} |dh_n| \right\|_{L_{p(\cdot)}(\Omega)} < \infty.$$

Proof. Let $f \in H^M_{p(\cdot)}(\Omega)$. Take a nonnegative, nondecreasing, and adapted function sequence $\lambda = (\lambda_n)_{n\ge0}$ with $\lambda_\infty \in L_{p(\cdot)}(\Omega)$ and $M_n(f) \le \lambda_n$ ($\forall n \in \mathbb{N}$). Now, we consider the following two sequences:

$$h_0 = 0, \quad h_n := \sum_{i=1}^{n} [df_i\chi_{\{\lambda_i>2\lambda_{i-1}\}} - \mathbb{E}(df_i\chi_{\{\lambda_i>2\lambda_{i-1}\}}|\mathcal{F}_{i-1})], \; n \ge 1$$

and

$$g_0 = 0, \quad g_n := \sum_{i=1}^{n} [df_i\chi_{\{\lambda_i\le2\lambda_{i-1}\}} - \mathbb{E}(df_i\chi_{\{\lambda_i\le2\lambda_{i-1}\}}|\mathcal{F}_{i-1})], \; n \ge 1.$$

It is obvious that $h = (h_n)_{n\ge0}$ and $g = (g_n)_{n\ge0}$ are both martingales such that $f = h + g$. Moreover, $\lambda_i < 2(\lambda_i - \lambda_{i-1})$ on the set $\{\lambda_i > 2\lambda_{i-1}\}$ and, thus,

$$|df_i|\chi_{\{\lambda_i>2\lambda_{i-1}\}} \le 2M_i(f)\chi_{\{\lambda_i>2\lambda_{i-1}\}} \le 2\lambda_i\chi_{\{\lambda_i>2\lambda_{i-1}\}} \le 4(\lambda_i - \lambda_{i-1}).$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{n}|dh_i| &\leq \sum_{i=1}^{n}|df_i|\chi_{\{\lambda_i>2\lambda_{i-1}\}} + \sum_{i=1}^{n}\mathbb{E}(|df_i|\chi_{\{\lambda_i>2\lambda_{i-1}\}}\big|\mathcal{F}_{i-1}) \\ &\leq 4\lambda_n + 4\sum_{i=1}^{n}\mathbb{E}(\lambda_i-\lambda_{i-1}|\mathcal{F}_{i-1}).\end{aligned}$$

From this and [40, Theorem 2], we conclude that

$$\|h\|_{\mathcal{G}_{p(\cdot)}(\Omega)} \lesssim \|\lambda_\infty\|_{L_{p(\cdot)}(\Omega)} + \left\|\sum_{i=1}^{\infty}(\lambda_i-\lambda_{i-1})\right\|_{L_{p(\cdot)}(\Omega)} \approx \|\lambda_\infty\|_{L_{p(\cdot)}(\Omega)}.$$

Furthermore, it is easy to see that

$$|df_i|\chi_{\{\lambda_i\leq 2\lambda_{i-1}\}} \leq 2\lambda_i\chi_{\{\lambda_i\leq 2\lambda_{i-1}\}} \leq 4\lambda_{i-1},$$

which implies that

$$|dg_n| \leq 8\lambda_{n-1}.$$

Consequently,

$$\begin{aligned}M_n(g) &\leq M_{n-1}(f) + M_{n-1}(h) + |dg_n| \\ &\leq 13\lambda_{n-1} + 4\sum_{i=1}^{n-1}\mathbb{E}(\lambda_i-\lambda_{i-1}|\mathcal{F}_{i-1}).\end{aligned}$$

Combining this and [40, Theorem 2], we deduce that

$$\|g\|_{P_{p(\cdot)}(\Omega)} \lesssim \|\lambda_\infty\|_{L_{p(\cdot)}(\Omega)}.$$

Finally, we just need to take $\lambda_n = M_n(f)$. The proof is complete. □

Theorem 3.6. *Suppose that* $(\mathcal{F}_n)_{n\geq 0}$ *is composed of atomic sub-*σ*-algebras. If* $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$ *satisfy* (2.2), $1 \leq q_- \leq q_+ < \infty$, $v \in V_{p(\cdot)}(\Omega)$, *and* $1/r(x) = 1/p(x) + 1/q(x)$, *then* T_v *is of type* $(H^M_{q(\cdot)}(\Omega), H^M_{r(\cdot)}(\Omega))$ *with* $\|T_v\| \lesssim \|v\|_{V_{p(\cdot)}(\Omega)}$.

Proof. Let $f \in H^M_{q(\cdot)}(\Omega)$. By Lemma 3.5, we can find $g \in P_{q(\cdot)}(\Omega)$ and $h \in \mathcal{G}_{q(\cdot)}(\Omega)$ such that $f = g + h$ and

$$\|g\|_{P_{q(\cdot)}(\Omega)} \lesssim \|f\|_{H^M_{q(\cdot)}(\Omega)}, \quad \|h\|_{\mathcal{G}_{q(\cdot)}(\Omega)} \lesssim \|f\|_{H^M_{q(\cdot)}(\Omega)}.$$

Hence,

$$\|T_vf\|_{H^M_{r(\cdot)}(\Omega)} \lesssim \|T_vh\|_{H^M_{r(\cdot)}(\Omega)} + \|T_vg\|_{H^M_{r(\cdot)}(\Omega)} \leq \|T_vh\|_{\mathcal{G}_{r(\cdot)}(\Omega)} + \|T_vg\|_{P_{r(\cdot)}(\Omega)},$$

where the last inequality is due to the following fact and Lemma 2.6:

$$M(T_vh) \leq \sum_{n=1}^{\infty}|d(T_vh)_n|.$$

Now, we claim that

$$\|T_v h\|_{\mathcal{G}_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \|h\|_{\mathcal{G}_{q(\cdot)}(\Omega)}. \tag{3.1}$$

In fact, we only need to note that $|d(T_v h)_n| \leq M(v)|dh_n|$, then Lemma 2.1 yields (3.1). According to Theorem 3.4 and (3.1), we conclude that

$$\|T_v f\|_{H^M_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \big(\|h\|_{\mathcal{G}_{q(\cdot)}(\Omega)} + \|g\|_{P_{q(\cdot)}(\Omega)}\big).$$

Consequently,

$$\|T_v f\|_{H^M_{r(\cdot)}(\Omega)} \lesssim \|v\|_{V_{p(\cdot)}(\Omega)} \|f\|_{H^M_{q(\cdot)}(\Omega)}.$$

This completes the proof. □

Remark 3.7. *If we take all the variable exponents in Theorem 3.6 as constant exponents, then we obtain [33, Theorem 5.2.6] in the framework of atomic σ-algebras.*

In [17, 33, 39], authors proved that the elements of $X_r(\Omega)$ are martingale transforms of the ones in $X_q(\Omega)$, where $X \in \{H^s, P, Q\}$ and $r < q < \infty$. Inspired by this, we study the same problem in the context of variable exponents. To our surprise, we obtain the same result without the assumptions of atomic σ-algebras and condition (2.2), which are important for Theorems 3.4 and 3.6. Instead, we only need $r(\cdot)$ and $q(\cdot)$ to satisfy certain measurability. Our results are stated as follows:

Theorem 3.8. *Let $q(\cdot), r(\cdot) \in \mathcal{P}(\Omega)$ be such that $r_+ < q_-$ and $1/r(x) = 1/p(x) + 1/q(x)$. If $r(\cdot)$ and $q(\cdot)$ are both $\mathcal{F}_0$-measurable, then for any $f \in X_{r(\cdot)}(\Omega)$, there exist a martingale $g \in X_{q(\cdot)}(\Omega)$ and a sequence $v \in V_{p(\cdot)}(\Omega)$ such that $f = T_v g$ with*

$$\|v\|_{V_{p(\cdot)}(\Omega)} \leq \max\{\|f\|^{r_+/p_-}_{X_{r(\cdot)}(\Omega)}, \|f\|^{r_-/p_+}_{X_{r(\cdot)}(\Omega)}\}$$

and

$$\|g\|_{X_{q(\cdot)}(\Omega)} \leq \left(\frac{q_+}{r_-}\right)^{1/2} \max\{\|f\|^{r_+/q_-}_{X_{r(\cdot)}(\Omega)}, \|f\|^{r_-/q_+}_{X_{r(\cdot)}(\Omega)}\},$$

where $X \in \{H^s, P, Q\}$.

Remark 3.9. *Indeed, such measurability assumption to variable exponents was also considered by Aoyama [2] to characterize Doob's maximal inequality on $L_{p(\cdot)}(\Omega)$. According to the following proof of Theorem 3.8, one can see that the measurability assumption of $r(\cdot)$ and $q(\cdot)$ is used to ensure that v is an adapted process with respect to $(\mathcal{F}_n)_{n\geq 0}$, and so T_v is a martingale transform.*

Before proving Theorem 3.8, we first state and prove the following simple but useful result.

Lemma 3.10. *Let $p(\cdot), r(\cdot) \in \mathcal{P}(\Omega)$. If $f \in L_{r(\cdot)}(\Omega)$, then $|f(\cdot)|^{r(\cdot)/p(\cdot)} \in L_{p(\cdot)}(\Omega)$ and*

$$\left\||f(\cdot)|^{r(\cdot)/p(\cdot)}\right\|_{L_{p(\cdot)}(\Omega)} \leq \max\{\|f\|^{r_+/p_-}_{L_{r(\cdot)}(\Omega)}, \|f\|^{r_-/p_+}_{L_{r(\cdot)}(\Omega)}\}.$$

Proof. If $\|f\|_{L_{r(\cdot)}(\Omega)} = 0$, then $f = 0$ a.e., so there is nothing to prove. Therefore, we may assume that $\|f\|_{L_{r(\cdot)}(\Omega)} > 0$. To begin, we observe that

$$\rho_{p(\cdot)}(|f(\cdot)|^{r(\cdot)/p(\cdot)}) = \int_\Omega |f(x)|^{r(x)} dx = \rho_{r(\cdot)}(f).$$

If $\|f\|_{L_{r(\cdot)}(\Omega)} = 1$, by Lemma 2.2, we have $\left\| |f(\cdot)|^{r(\cdot)/p(\cdot)} \right\|_{L_{p(\cdot)}(\Omega)} = 1$. If $\|f\|_{L_{r(\cdot)}(\Omega)} = a > 1$, denote $\lambda = a^{r_+/p_-}$. Then,

$$\begin{aligned}\rho_{p(\cdot)}(|f(\cdot)|^{r(\cdot)/p(\cdot)}/\lambda) &= \int_\Omega |f(x)|^{r(x)} \lambda^{-p(x)} dx \\ &= \int_\Omega \left(\frac{|f(x)|}{a} \right)^{r(x)} \frac{a^{r(x)}}{\lambda^{p(x)}} dx \\ &\leq \frac{a^{r_+}}{\lambda^{p_-}} \rho_{r(\cdot)}(f/a) = 1,\end{aligned}$$

where the last equality is because of Lemma 2.2. This implies that

$$\left\| |f(\cdot)|^{r(\cdot)/p(\cdot)} \right\|_{L_{p(\cdot)}(\Omega)} \leq \|f\|^{r_+/p_-}_{L_{r(\cdot)}(\Omega)}.$$

If $\|f\|_{L_{r(\cdot)}(\Omega)} = b < 1$, denote $\mu = b^{r_-/p_+}$. Similarly, we can deduce that

$$\rho_{p(\cdot)}(|f(\cdot)|^{r(\cdot)/p(\cdot)}/\mu) \leq 1,$$

that is,

$$\left\| |f(\cdot)|^{r(\cdot)/p(\cdot)} \right\|_{L_{p(\cdot)}(\Omega)} \leq \|f\|^{r_-/p_+}_{L_{r(\cdot)}(\Omega)}.$$

□

Applying the lemma above, we can now prove Theorem 3.8.

Proof of Theorem 3.8. We shall prove this theorem in three cases: $X = H^s$, $X = Q$, and $X = P$.

(i) **Case 1:** $X = H^s$**.** Since $r_+ < q_-$, it is easy to see that $p(\cdot) \in \mathcal{P}(\Omega)$. Given $f \in H^s_{r(\cdot)}(\Omega)$, set

$$v_{n-1}(x) := [s_n(f)(x)]^{r(x)/p(x)} \quad \text{and} \quad g_n(x) := \sum_{m=1}^n [s_m(f)(x)]^{-r(x)/p(x)} df_m(x),$$

where $s_n(f)(x) := \left(\sum_{m=1}^n \mathbb{E}(|df_m|^2 | \mathcal{F}_{m-1})(x) \right)^{1/2}$. Since $r(\cdot)$ and $q(\cdot)$ are $\mathcal{F}_0$-measurable, $v = (v_n)_{n \geq 0}$ is an adapted process and $g = (g_n)_{n \geq 0}$ is a martingale such that $f = T_v g$. Moreover, by Lemma 3.10, we get

$$\|v\|_{V_{p(\cdot)}(\Omega)} = \left\| [s(f)(\cdot)]^{r(\cdot)/p(\cdot)} \right\|_{L_{p(\cdot)}(\Omega)} \leq \max \{ \|f\|^{r_+/p_-}_{H^s_{r(\cdot)}(\Omega)}, \|f\|^{r_-/p_+}_{H^s_{r(\cdot)}(\Omega)} \}.$$

Note that

$$s_n^2(g)(x) = \sum_{m=1}^n [s_m^2(f)(x)]^{-r(x)/p(x)} \mathbb{E}(|df_m|^2 \big| \mathcal{F}_{m-1})(x).$$

We further deduce that

$$\begin{aligned}s_n^2(g)(x) &= \sum_{m=1}^n \frac{s_m^2(f)(x) - s_{m-1}^2(f)(x)}{[s_m^2(f)(x)]^{r(x)/p(x)}} \\ &\leq \int_0^{s_n^2(f)(x)} \frac{d\alpha}{\alpha^{r(x)/p(x)}} = \frac{[s_n^2(f)(x)]^{1-r(x)/p(x)}}{1 - r(x)/p(x)} = \frac{q(x)}{r(x)} [s_n^2(f)(x)]^{r(x)/q(x)},\end{aligned}$$

which means that

$$s(g)(x) \le \left(\frac{q_+}{r_-}\right)^{1/2} [s(f)(x)]^{r(x)/q(x)}.$$

By Lemma 3.10, we obtain

$$\begin{aligned}\|g\|_{H^s_{q(\cdot)}(\Omega)} &\le \left(\frac{q_+}{r_-}\right)^{1/2} \left\|[s(f)(\cdot)]^{r(\cdot)/q(\cdot)}\right\|_{L_{q(\cdot)}(\Omega)}\\ &\le \left(\frac{q_+}{r_-}\right)^{1/2} \max\{\|f\|^{r_+/q_-}_{H^s_{r(\cdot)}(\Omega)}, \|f\|^{r_-/q_+}_{H^s_{r(\cdot)}(\Omega)}\}.\end{aligned}$$

(ii) **Case 2: $X = Q$.** Let $f \in Q_{r(\cdot)}(\Omega)$. Consider $\lambda = (\lambda_n)_{n\ge0}$ to be the least majorant of $(S_n(f))_{n\ge0}$ with convention $S_0(f) = 0$. Set

$$v_n(x) := \lambda_n(x)^{r(x)/p(x)} \quad \text{and} \quad g_n(x) := \sum_{m=1}^{n} \lambda_{m-1}(x)^{-r(x)/p(x)} df_m(x).$$

Then, $f = T_v g$, and Lemma 3.10 gives that

$$\begin{aligned}\|v\|_{V_{p(\cdot)}(\Omega)} = \left\|\lambda_\infty(\cdot)^{r(\cdot)/p(\cdot)}\right\|_{L_{p(\cdot)}(\Omega)} &\le \max\{\|\lambda_\infty\|^{r_+/p_-}_{L_{r(\cdot)}(\Omega)}, \|\lambda_\infty\|^{r_-/p_+}_{L_{r(\cdot)}(\Omega)}\}\\ &= \max\{\|f\|^{r_+/p_-}_{Q_{r(\cdot)}(\Omega)}, \|f\|^{r_-/p_+}_{Q_{r(\cdot)}(\Omega)}\}.\end{aligned}$$

Furthermore, applying twice Abel rearrangement, we have

$$\begin{aligned}S_n^2(g)(x) &= \sum_{m=1}^{n} \lambda_{m-1}(x)^{-2r(x)/p(x)} |df_m(x)|^2\\ &= \sum_{m=1}^{n} \frac{S_m^2(f)(x) - S_{m-1}^2(f)(x)}{(\lambda_{m-1}^2(x))^{r(x)/p(x)}}\\ &= \frac{S_n^2(f)(x)}{(\lambda_{n-1}^2(x))^{r(x)/p(x)}} + \sum_{m=1}^{n-1} S_m^2(f)(x)\left[\frac{1}{(\lambda_{m-1}^2(x))^{r(x)/p(x)}} - \frac{1}{(\lambda_m^2(x))^{r(x)/p(x)}}\right]\\ &\le \frac{\lambda_{n-1}^2(x)}{(\lambda_{n-1}^2(x))^{r(x)/p(x)}} + \sum_{m=1}^{n-1} \lambda_{m-1}^2(x)\left[\frac{1}{(\lambda_{m-1}^2(x))^{r(x)/p(x)}} - \frac{1}{(\lambda_m^2(x))^{r(x)/p(x)}}\right]\\ &= \sum_{m=1}^{n} \frac{\lambda_{m-1}^2(x) - \lambda_{m-2}^2(x)}{(\lambda_{m-1}^2(x))^{r(x)/p(x)}} \qquad \text{(with convention } \lambda_{-1}(x) = 0)\\ &\le \int_0^{\lambda_{n-1}^2(x)} \frac{d\alpha}{\alpha^{r(x)/p(x)}} = \frac{(\lambda_{n-1}^2(x))^{1-r(x)/p(x)}}{1 - r(x)/p(x)} \le \frac{q(x)}{r(x)} (\lambda_{n-1}^2(x))^{r(x)/q(x)}.\end{aligned}$$

Hence,

$$S_n(g)(x) \le \left(\frac{q_+}{r_-}\right)^{1/2} (\lambda_{n-1}(x))^{r(x)/q(x)}.$$

Consequently,

$$\|g\|_{Q_{q(\cdot)}(\Omega)} \le \left(\frac{q_+}{r_-}\right)^{1/2} \left\|(\lambda_\infty(\cdot))^{r(\cdot)/q(\cdot)}\right\|_{L_{q(\cdot)}(\Omega)}$$

$$= \left(\frac{q_+}{r_-}\right)^{1/2} \max\left\{\|f\|_{Q_{r(\cdot)}(\Omega)}^{r_+/q_-}, \|f\|_{Q_{r(\cdot)}(\Omega)}^{r_-/q_+}\right\}.$$

(iii) **Case 3:** $X = P$. In this case, we can take a similar argument as Case 2. □

Author contributions

Tao Ma: Conceptualization, Methodology, Validation, Formal analysis, Writing-original draft, Project administration, Funding acquisition; Jianzhong Lu: Conceptualization, Methodology, Validation, Formal analysis, Writing-original draft, Writing-review and editing; Xia Wu: Conceptualization, Methodology, Validation, Formal analysis, Writing-original draft, supervision. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that they have no conflicts of interest in this paper.

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