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Research article

A suspension bridges with a fractional time delay: Asymptotic behavior and Blow-up in finite time

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Abstract: In the present paper, we examine a suspension bridges model subject to frictional damping, a fractional delay term, and a source term. First, we prove the existence of global solutions of the problem. Second, for small initial data, we establish the exponential stability of the system by using the energy method. Additionally, we show that if the initial energy assumes a negative value, the solution blows up in finite time.

Keywords: suspension bridges; fractional delay; exponential stability; blow-up
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1. Introduction

Suspension bridges are marvels of engineering, designed to span vast distances and connect areas separated by natural obstacles like rivers, valleys, and urban landscapes. Their design typically consists of a deck suspended from cables, which are anchored at both ends and pass over towering piers. This structure not only allows for impressive spans, but also provides aesthetic appeal and functional efficiency. Despite these advantages, suspension bridges are prone to dynamic instabilities that arise from various factors, such as wind forces, vehicular traffic, and seismic activities. Addressing these instabilities is crucial for ensuring the safety and longevity of the bridge. The dynamic behavior of suspension bridges is complex and influenced by multiple factors. Wind-induced vibrations can lead to significant oscillations. Similarly, traffic loads cause dynamic forces that can resonate with the natural frequencies of the bridge. Seismic activities add another layer of complexity, introducing sudden and powerful forces that can induce oscillations and potential structural damage. These dynamic instabilities, if not properly managed, can compromise the structural integrity and serviceability of the bridge. To mitigate these dynamic instabilities, engineers incorporate various damping mechanisms into the design of suspension bridges. Damping is the process of dissipating the energy of oscillations, thereby reducing their amplitude and preventing resonant vibrations. One effective method is frictional damping, which relies on friction forces to dissipate energy. Frictional dampers can be installed in strategic locations within the bridge structure to absorb vibrational energy, converting it into heat and thus stabilizing the bridge. In addition to frictional damping, advanced control techniques such as fractional delay offer significant potential for enhancing the stability of suspension bridges. Fractional delay, a concept derived from fractional calculus, extends the traditional notion of time delay to fractional orders. This approach allows for more precise and flexible control of dynamic systems. By incorporating fractional delay into the damping system, engineers can achieve better control over the phase and amplitude of vibrations, leading to more effective stabilization. In fact, fractional computing in modeling enhances the ability to capture the complex dynamics of natural systems, and improves control performance beyond what is achievable with integer-order controls. This approach is particularly relevant in fields such as engineering, quantum mechanics, nuclear physics, and biological phenomena like fluid flow (see for example [1–3]). For example, in structural dynamics, fractional-order derivatives provide a more nuanced representation of bridge structures by accurately modeling systems with memory effects and non-local behaviors. These derivatives better reflect the viscoelastic and nonlinear properties inherent in suspension bridges, capturing the distributed nature of forces and displacements and accounting for memory effects that impact the bridge's dynamic response over time. Additionally, the frequency-dependent nature of fractional derivatives allows for damping that varies with oscillation frequency, beneficial for controlling specific vibration modes and enhancing overall stability. This characteristic is especially useful for studying the long-term response of bridges to dynamic loads or environmental conditions (see also [4]).

In this paper, we are interested in the study of the asymptotic behavior and blow-up of suspension bridges in the domain $\Omega = (0,\pi)\times(-d,d)$, where $d << \pi$, and with the presence of frictional damping, fractional delay, and a nonlinear source term. This can be modeled by the following system:

$$\begin{cases} v_{tt}(x,y,t) + \Delta^2 v(x,y,t) + a_0 v_t(x,y,t) & \\ +a_1\partial_t^{\alpha,\beta} v\,(x,y,t-\tau) = v|v|^{p-2}, & \text{in } \Omega\times(0,+\infty), \\ v(0,y,t) = v_{xx}(0,y,t) = v(\pi,y,t) = v_{xx}(\pi,y,t) = 0, & (y,t)\in(-d,d)\times(0,+\infty), \\ v_{yy}(x,\pm d,t) + \mu v_{xx}(x,\pm d,t) = 0, & (x,t)\in(0,\pi)\times(0,+\infty), \\ v_{yyy}(x,\pm d,t) + (2-\mu)v_{xxy}(x,\pm d,t) = 0, & (x,t)\in(0,\pi)\times(0,+\infty), \\ v(x,y,0) = v_0(x,y),\;\; v_t(x,y,0) = v_1(x,y), & \text{in } \Omega, \\ v_t(x,y,t-\tau) = f(x,y,t-\tau), & t\in(0,\tau), \end{cases} \tag{1.1}$$

where $p > 2$ and the damping coefficients a_0 and a_1 are positive reals satisfying

$$2a_1\beta^{\alpha-1} < a_0. \tag{1.2}$$

The constant μ, known as the Poisson ratio, typically falls within the range $(-1, \frac{1}{2})$ for physical reasons (see [5] for more details). It is approximately 0.3 for metals and between 0.1 and 0.2 for concrete. Consequently, we assume that $0 < \mu < \frac{1}{2}$.

The constant $\tau > 0$ represents the time delay. The operator $\partial_t^{\alpha,\beta}$ is the exponential fractional derivative of order α, defined as

$$\partial_t^{\alpha,\beta} v(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\beta(t-s)} \frac{dv}{ds}(s)ds,\ 0 < \alpha < 1,\ \beta > 0.$$

The first results concerning suspension bridges are due to McKenna and Walter [6] and McKenna et al. [7], where the authors provided a model depicting the dynamics of a suspension bridge and established the presence of nonlinear oscillations. Bochicchio et al. [8] and Ma and Zhong [9] examined the asymptotic dynamics and global attractors for coupled suspension bridge equations, respectively. In a recent work, a new suspension bridge model using a plate was proposed by Ferrero and Gazzola [10]. For further details on suspension bridge models, see [11]. The analysis of the bending and stretching energies for the model in [10] was detailed in [12]. Later, in their work, Berchio et al. [13] investigated the structural instability of nonlinear plates used to model suspension bridges. Recently, several studies have addressed the uniform stability and finite time blow-up of suspension bridges. In [14], Wang considered the problem

$$v_{tt} + \Delta^2 v + a_0 v_t + av = v|v|^{p-2},$$

with the same boundary conditions as in (1.1), and where $a = a(x, y, t)$ is a sign-changing and bounded measurable function. The author established necessary and sufficient conditions for the uniqueness and existence of global solutions as well as the finite time blow-up of these solutions. Next, by considering a nonlinear damping (of the form $|v_t|^{m-2}v_t$) instead of a linear one, Liu et al. [15] extended the work of Wang [14]. In [16], the authors considered system (1.1) in the case where $a_1 = 0$, and with a general source term of the form $h(v)$. Through the use of multiplier techniques, the authors proved an exponential decay rate of energy. Following this, Cavalcanti et al. [17] (resp. [18]) analyzed problem (1.1) with $a_1 = 0$ and localized linear (resp. nonlinear) damping in the form $a_0(x, y)v_t$ (resp. $a_0(x, y)g(v_t)$), concentrated around the boundary neighborhood, and established the exponential stability in both cases. It is also worth mentioning the works [19–22], which focus on suspension bridges and present other types of damping, including structural and viscoelastic damping.

On the other hand, in the presence of the time delay, the equation (with the same boundary conditions as in (1.1))

$$v_{tt} + \Delta^2 v + a_0 v_t + a_1 v_t(x, y, t - \tau) + h(v) = f(x, y),$$

has been studied respectively by Messaoudi et al. [23] and Wang [24]. In both works, the authors proved the existence of uniform attractors. Later, in [25], Mukiawa studied the last equation, in the case $f = 0, a_0 = a_0(t), a_1 = a_1(t)$, and $\tau = \tau(t)$, and he established a stability result under some conditions on the delay and weight functions. Let us also mention the works [26–29], which treated the stability of wave equations and the Timoshenko system with fractional damping. Motivated by all these works, in this paper we are interested in the exponential stability and finite time blow-up of solutions of system (1.1). This research appears to be the first to address this particular issue. Inspired by previous studies, our aim is twofold:

- To extend known exponential decay results for suspension bridges with time delay to those with fractional time delay.

- To investigate the blow-up of solutions typically caused by the source term when initial energy is negative.

The paper is organized as follows: In the next section, we study the well-posedness of system (1.1). First, we prove the existence of a unique local solution of (1.1). Second, we show that this solution is global in time. The third section is devoted to establishing the exponential decay of energy of (1.1) for small initial data by constructing a suitable Lyapunov functional. We prove the blow-up of solutions for negative initial energy in the last section.

2. Wellposedness of system (1.1)

2.1. Local existence

In this subsection, we are concerned with reformulating model (1.1) into a first-order system. Here and what follows, we use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the inner product and the usual norm in $L^2(\Omega)$, respectively.

We define the space

$$W = \{v \in H^2(\Omega) : v = 0 \text{ o}n \ \{0, \pi\} \times (-d, d)\},$$

with the scalar product

$$(u, v)_W = \int_\Omega \Big[\Delta u \Delta v + (1 - \mu)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}) \Big] dxdy.$$

We note that $(W, (\cdot, \cdot)_W)$ is a Hilbert space, and we have that the norm $\|.\|_W$ is equivalent to the H^2-norm (see [10, Lemma 4.1]).

We then have the following.

Lemma 2.1. *[10] If $0 < \mu < \frac{1}{2}$ and $f \in L^2(\Omega)$, then there is a unique $u \in W$ such that, for all $v \in W$, we have*

$$(u, v)_W = \int_\Omega fu. \tag{2.1}$$

The function $v \in W$ satisfying (2.1) is known as the weak solution to the stationary problem

$$\begin{cases} \Delta^2 v = f, \\ v(0, y) = v(\pi, y) = v_{xx}(0, y) = v_{xx}(\pi, y) = 0, \\ v_{yy}(x, \pm d) + \mu v_{xx}(x, \pm d) = v_{yyy}(x, \pm d) + (2 - \mu)v_{xxy}(x, \pm d) = 0. \end{cases} \tag{2.2}$$

Lemma 2.2. *[14] Let $v \in W$ and $1 \le r < +\infty$. Then, we have*

$$\|v\|_r^r \le C_e \|v\|_W^r, \tag{2.3}$$

for some positive constant $C_e = C_e(\Omega, r)$, where $\| \cdot \|_r$ is the usual $L^r(\Omega)$-norm.

Lemma 2.3. *[30] Let φ be the function*

$$\varphi(\zeta) = |\zeta|^{\frac{2\alpha-1}{2}}, \ \zeta \in \mathbb{R}, \ 0 < \alpha < 1.$$

Then, the relationship between the 'input' U and the 'output' O of the system

$$\begin{cases} \psi_t(x,y,\zeta,t) + (\zeta^2+\beta)\psi(x,y,\zeta,t) - U(x,y,t)\varphi(\zeta) = 0, \ \zeta \in \mathbb{R}, \ t > 0, \ \beta > 0, \\ \psi(x,y,\zeta,0) = 0, \\ O(t) = \pi^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty} \psi(x,y,\zeta,t)\varphi(\zeta)d\zeta \end{cases} \tag{2.4}$$

is given by

$$O = I^{1-\alpha,\beta}U,$$

where

$$I^{\alpha,\beta}v(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\beta(t-s)}v(s)ds.$$

We need also the next lemma.

Lemma 2.4. *[29] If $\lambda \in D_\beta = \mathbb{C}\backslash\,]-\infty,-\beta[$, then*

$$\int\limits_{-\infty}^{+\infty} \frac{\varphi^2(\zeta)}{\lambda+\beta+\zeta^2}d\zeta = \frac{\pi}{\sin(\alpha\pi)}(\lambda+\beta)^{\alpha-1}.$$

As in [31], we introduce the new variable

$$z(x,y,\rho,t) = v_t(x,y,t-\tau\rho), \ (x,y) \in \Omega, \ \rho \in (0,1), \ t \geq 0. \tag{2.5}$$

It is easy to see that

$$\tau z_t(x,y,\rho,t) + z_\rho(x,y,\rho,t) = 0, \ (x,y) \in \Omega, \ \rho \in (0,1), \ t \geq 0. \tag{2.6}$$

Consequently, Lemma 2.3, (2.5), and (2.6) give us the following equivalent (to (1.1)) system:

$$\begin{cases} v_{tt}(x,y,t) + \Delta^2 v(x,y,t) + a_0 v_t(x,y,t) & \\ +\kappa\displaystyle\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta = v|v|^{p-2}, & (x,y)\in\Omega, \ t>0, \\ \psi_t(x,y,\zeta,t) + (\zeta^2+\beta)\psi(x,y,\zeta,t) - z(x,1,t)\varphi(\zeta) = 0, & (x,y)\in\Omega, \ \zeta\in\mathbb{R}, \ t>0, \\ \tau z_t(x,y,\rho,t) + z_\rho(x,y,\rho,t) = 0, & (x,y)\in\Omega, \ \rho\in(0,1), \ t>0, \\ v(0,y,t) = v_{xx}(0,y,t) = v(\pi,y,t) = v_{xx}(\pi,y,t) = 0, & y\in(-d,d), \ t>0, \\ v_{yy}(x,\pm d,t) + \mu v_{xx}(x,\pm d,t) = 0, & (x,t)\in(0,\pi)\times(0,+\infty), \\ v_{yyy}(x,\pm d,t) + (2-\mu)v_{xxy}(x,\pm d,t) = 0, & (x,t)\in(0,\pi)\times(0,+\infty), \\ z(x,y,0,t) = v_t(x,y,t), & (x,y)\in\Omega, \ t>0, \\ v(x,y,0) = v_0(x,y), \ v_t(x,y,0) = v_1(x,y), & (x,y)\in\Omega, \\ \psi(x,y,\zeta,0) = 0, & (x,y)\in\Omega, \ \zeta\in\mathbb{R}, \\ z(x,y,\rho,0) = f(x,y,-\rho\tau), & (x,y)\in\Omega, \ \rho\in(0,1), \end{cases} \tag{2.7}$$

where $\kappa = \frac{a_1 \sin(\alpha\pi)}{\pi}$.

Denote by M the quantity

$$M = \int_{-\infty}^{+\infty} \frac{\varphi^2(\zeta)}{\zeta^2 + \beta} d\zeta,$$

and let δ be a positive constant satisfying

$$\kappa M < \delta < a_0 - \kappa M. \tag{2.8}$$

Remark 2.5. *From* (1.2) *and Lemma 2.4, we can easily check that* $a_0 - \kappa M > \kappa M$.

We define the space energy $\mathcal{H}$ by

$$\mathcal{H} = W \times L^2(\Omega) \times L^2(\Omega \times \mathbb{R}) \times L^2(\Omega \times (0,1)),$$

which is equipped with the inner product

$$(V, \tilde{V})_{\mathcal{H}} = (v, \tilde{v})_W + \langle u, \tilde{u} \rangle + \kappa \int_\Omega \int_{-\infty}^{+\infty} \psi(x,y,\zeta)\tilde{\psi}(x,y,\zeta) d\zeta dx dy + 2\tau\delta \int_\Omega \int_0^1 z(x,y,\rho)\tilde{z}(x,y,\rho) d\rho dx dy,$$

for all $V = (v, u, \psi, z)^T$ and $\tilde{V} = (\tilde{v}, \tilde{u}, \tilde{\psi}, \tilde{z})^T \in \mathcal{H}$. Let us rewrite problem (2.7) as a first-order equation. Let $u = v_t$ and set $V = (v, u, \psi, z)^T$. System (2.7) can be recast as

$$\begin{cases} V_t(t) = \mathcal{A}V(t) + \mathcal{G}(V(t)), \\ V(x,y,0) = V_0 = (v_0, v_1, 0, f(-\rho\tau)), \end{cases} \tag{2.9}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}V := \begin{pmatrix} u \\ -\Delta^2 v - a_0 u - \kappa \displaystyle\int_{-\infty}^{+\infty} \psi(x,y,\zeta)\varphi(\zeta) d\zeta \\ -(\zeta^2 + \beta)\psi(x,y,\zeta) + z(x,y,1)\varphi(\zeta) \\ -\frac{1}{\tau} z_\rho(x,y,\rho) \end{pmatrix}$$

and

$$\begin{aligned} D(\mathcal{A}) \quad = \quad & \Big\{(v, u, \psi, z) \in \mathcal{H} : v \in H^4(\Omega),\ u \in W,\ \zeta\psi \in L^2(\Omega \times \mathbb{R}) \\ & -(\zeta^2 + \beta)\psi(x,y,\zeta) + z(x,1)\varphi(\zeta) \in L^2(\Omega \times \mathbb{R}),\ z_\rho \in L^2(\Omega \times (0,1)),\ u = z(.,0)\Big\}. \end{aligned}$$

The nonlinear operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{G}(V) := \begin{pmatrix} 0 \\ v|v|^{p-2} \\ 0 \\ 0 \end{pmatrix}.$$

We have the following result.

Theorem 2.6. *Assume that* (2.8) *holds true. Therefore, we have:*

(1) If $V_0 \in \mathcal{H}$, then there exists $T_{max} > 0$ such that the system (2.9) *has a unique mild solution $V \in C([0, T_{max}]; \mathcal{H})$.*

(2) If $V_0 \in D(\mathcal{A})$, then V is a regular solution of (2.9).

Proof. We start by proving that the operator $\mathcal{A}$ is maximal dissipative. Indeed, let $V \in D(\mathcal{A})$. Then we infer that

$$\begin{aligned}(\mathcal{A}V, V)_{\mathcal{H}} &= -(a_0 - \delta)\int_\Omega z^2(x, y, 0, t)dxdy - \delta\int_\Omega z^2(x, y, 1, t)dxdy \\ &\quad -\kappa\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2 + \beta)\psi^2(x, y, \zeta, t)d\zeta dxdy \\ &\quad -\kappa\int_\Omega z(x, y, 0, t)\int_{-\infty}^{+\infty}\psi(x, y, \zeta, t)\varphi(\zeta)d\zeta dxdy \\ &\quad +\kappa\int_\Omega z(x, y, 1, t)\int_{-\infty}^{+\infty}\psi(x, y, \zeta, t)\varphi(\zeta)d\zeta dxdy. \end{aligned} \tag{2.10}$$

By combining Young's inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_\Omega z(x, y, 1, t)\int_{-\infty}^{+\infty}\psi(x, y, \zeta, t)\varphi(\zeta)d\zeta dxdy \\ &\le M\int_\Omega z^2(x, y, 1, t)dxdy + \frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2 + \beta)\psi^2(x, y, \zeta, t)d\zeta dxdy, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\int_\Omega z(x, y, 0, t)\int_{-\infty}^{+\infty}\psi(x, y, \zeta, t)\varphi(\zeta)d\zeta dxdy \\ &\le M\int_\Omega z^2(x, y, 0, t)dxdy + \frac{1}{4}\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2 + \beta)\psi^2(x, y, \zeta, t)d\zeta dxdy. \end{aligned} \tag{2.12}$$

Inserting (2.11) and (2.12) into (2.10), it holds that

$$\begin{aligned}(\mathcal{A}V, V)_{\mathcal{H}} &\le -(a_0 - \kappa M - \delta)\int_\Omega z^2(x, y, 0, t)dxdy - (\delta - \kappa M)\int_\Omega z^2(x, y, 1, t)dxdy \\ &\quad -\frac{\kappa}{2}\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2 + \beta)\psi^2(x, y, \zeta, t)d\zeta dxdy, \end{aligned} \tag{2.13}$$

which shows that $\mathcal{A}$ is dissipative by means of (2.8). Next, it is easy to prove that the equation

$$-\mathcal{A}V = F, \ \forall\, F \in \mathcal{H}, \tag{2.14}$$

where $F = (f_1, f_2, f_3, f_4)$, has a unique solution $V \in D(\mathcal{A})$. In fact, (2.14) gives us:

$$\begin{cases} -u = f_1, \\ \Delta^2 v + a_0 u + \kappa \displaystyle\int_{-\infty}^{+\infty} \psi(x,y,\zeta)\varphi(\zeta)d\zeta = f_2, \\ (\zeta^2+\beta)\psi(x,y,\zeta) - z(x,y,1)\varphi(\zeta) = f_3, \\ z_\rho(x,y,\rho) = \tau f_4. \end{cases} \tag{2.15}$$

The first and last equations in (2.15) give us

$$u = -f_1, \quad z(x,y,\rho) = -f_1 + \tau \int_0^\rho f_4(x,y,s)ds. \tag{2.16}$$

Using (2.16) and the third equation in (2.15), we obtain

$$\psi(x,y,\zeta) = \frac{\left(-f_1 + \tau \displaystyle\int_0^1 f_4(x,y,s)ds\right)\varphi(\zeta) + f_3}{\zeta^2+\beta}. \tag{2.17}$$

Using (2.16) and (2.17) in (2.15), integrating by parts over Ω, we find that

$$c(v,\phi) = L(\phi), \ \forall \ \phi \in W, \tag{2.18}$$

where

$$c(v,\phi) = (v,\phi)_W,$$

and

$$L(\phi) = \int_\Omega (a_0 f_1 + f_2)\phi dxdy + \kappa \int_\Omega \phi \int_{-\infty}^{+\infty} \frac{f_1\varphi^2(\zeta) - f_3\varphi(\zeta)}{\zeta^2+\beta} d\zeta dxdy - \kappa\tau M \int_\Omega \phi \int_0^1 f_4(x,y,s)dsdxdy.$$

We can check that c is a continuous bilinear coercive form on $W \times W$ and L is a continuous linear form on W. Then, by the Lax-Milgram Theorem, problem (2.18) has a unique solution $v \in W$. Lemma 2.1 ensures that $v \in H^4(\Omega)$. Thus, $0 \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$. By the resolvent identity, we deduce that $\mathcal{A}$ is maximal (see Theorem 1.2.4 in [32]). Therefore, from the Lumer-Phillips Theorem (see e.g., [33]), $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions in $\mathcal{H}$.

It remains to prove that $\mathcal{G}$ is locally Lipschitz continuous in $\mathcal{H}$, that is, we need to prove that, for a given $R > 0$, $V = (v,u,\psi,z)^T$, and $\tilde{V} = (\tilde{v},\tilde{u},\tilde{\psi},\tilde{z})^T \in \mathcal{H}$, there exists $C_R > 0$ such that

$$\|\mathcal{G}(V) - \mathcal{G}(\tilde{V})\|_{\mathcal{H}} \leq C_R\|V - \tilde{V}\|_{\mathcal{H}}, \text{ provided that } \|V\|_{\mathcal{H}}, \ \|\tilde{V}\|_{\mathcal{H}} \leq R. \tag{2.19}$$

From the definition of $\mathcal{G}$, we have

$$\begin{aligned} \|\mathcal{G}(V) - \mathcal{G}(\tilde{V})\|_{\mathcal{H}}^2 &= \|v|v|^{p-2} - \tilde{v}|\tilde{v}|^{p-2}\|^2 \\ &\leq C\|v - \tilde{v}\|_W^2 \\ &\leq C\|V - \tilde{V}\|_{\mathcal{H}}^2, \end{aligned}$$

which implies that $\mathcal{G}$ is locally Lipschitz in $\mathcal{H}$. Hence, by Theorem 1.4 and Theorem 1.6 in [33, Chapter 6], the Cauchy problem (2.9) has a unique mild solution on the interval $[0, T_{max})$ for some $T_{max} > 0$. Furthermore, if $V_0 \in D(\mathcal{A})$, the solution is regular. □

2.2. Global existence

Let us define

$$h(t)=\kappa\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy+\|v\|_W^2-\int_\Omega|v|^p dxdy+\delta\tau\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy,$$

and

$$k(t)=\frac{\kappa}{2}\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy+\frac{1}{2}\|v\|_W^2-\frac{1}{p}\int_\Omega|v|^p dxdy+\delta\tau\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy.$$

Define the energy of system (2.7) by

$$E(t)=\frac{1}{2}\|v_t\|^2+k(t),$$

which verifies

$$\begin{aligned}E'(t) &\leq -(a_0-\kappa M-\delta)\|v_t\|^2-(\delta-\kappa M)\int_\Omega z^2(x,y,1,t)dxdy\\ &\quad -\frac{\kappa}{2}\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\beta)\psi^2(x,y,\zeta,t)d\zeta dxdy. \end{aligned}\tag{2.20}$$

Hence, system (2.7) is dissipative and we have $E(t)\leq E(0)$.

We have the following result.

Proposition 2.7. *Assume that* (2.8) *holds true. Then, for* $V_0\in\mathcal{H}$ *satisfying*

$$\begin{cases}\chi=C_e\left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}}<1,\\ h(0)>0,\end{cases}\tag{2.21}$$

we have $h(t)>0,\ \forall\ t>0$. *Moreover, the solution of* (2.9) *is bounded and global in time.*

Proof. Using the continuity of v and the fact that $h(0)>0$, we get the existence of $t_0<T_{max}$ such that $h(t)\geq 0,\ \forall\ t\in[0,t_0]$.

From the definition of $h(t)$ and $k(t)$, it is easy to see that

$$\begin{aligned}\frac{2p}{p-2}k(t) &= \kappa\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy\\ &\quad+\frac{2}{p-2}h(t)+\|v\|_W^2+\frac{2(p-1)\delta\tau}{p-2}\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy\\ &\geq \|v\|_W^2.\end{aligned}$$

Hence, we obtain

$$\|v\|_W^2\leq\frac{2p}{p-2}k(t)\leq\frac{2p}{p-2}E(t)\leq\frac{2p}{p-2}E(0),\ \forall\ t\in[0,t_0].$$

Now, using Lemma 2.2, it holds that

$$\int_\Omega |v|^p dxdy \le C_e \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} \|v\|_W^2 < \|v\|_W^2, \ \forall\ t \in [0, t_0].$$

Therefore, one has $h(t) > 0, \ \forall\ t \in [0, t_0]$. By repeating this procedure and using the fact that

$$\lim_{t \to t_0^-} C_e \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} < 1,$$

we can take $t_0 = T_{max}$. Besides, we find that

$$\frac{1}{2}\|v_t\|^2 + \frac{p-2}{2p}\|v\|_W^2 \le \frac{1}{2}\|v_t\|^2 + k(t) = E(t) \le E(0),$$

which means that the solution of system (2.9) is bounded and global (in time). This completes the proof. □

3. Exponential stability

In this section, we will prove that system (2.9), with initial data satisfying (2.21), is exponentially stable. To do this, we need the following lemmas.

Lemma 3.1. *Let $V = (v, u, \psi, z)^T$ be a solution of system* (2.9)*. The functional*

$$\phi_1(t) = \int_\Omega v v_t dxdy + \frac{\kappa}{2}\int_\Omega \int_{-\infty}^{+\infty} (\zeta^2 + \beta)\Gamma(x, y, \zeta, t) d\zeta dxdy$$

satisfies

$$\begin{aligned} \phi_1'(t) \ &\le \ \left(1 + \frac{a_0^2 C_e}{2}\right)\|v_t\|^2 - \frac{1}{2}\|v\|_W^2 + \delta\tau^2 \int_\Omega z^2(x, y, \rho, t) dxdy + \frac{\kappa}{4}\int_\Omega \int_{-\infty}^{+\infty} (\zeta^2 + \beta)\psi^2(x, y, \zeta, t) d\zeta dxdy \\ &\quad + \|v\|_p^p - \kappa \int_\Omega \int_{-\infty}^{+\infty} \psi^2(x, y, \zeta, t) d\zeta dxdy, \end{aligned} \tag{3.1}$$

where $\Gamma(x, y, \zeta, t) = \displaystyle\int_0^t \psi(x, y, \zeta, s) ds - \frac{\tau\varphi(\zeta)}{\zeta^2 + \beta}\int_0^1 f(x, y, -\rho\tau) d\rho + \frac{v_0(x, y)\varphi(\zeta)}{\zeta^2 + \beta}.$

Proof. By differentiating ϕ_1 and using $(2.7)_1$, we infer that

$$\begin{aligned} \phi_1'(t) \ &= \ \|v_t\|^2 - \|v\|_W^2 - a_0 \int_\Omega v v_t dxdy - \kappa \int_\Omega v(x, y, t) \int_{-\infty}^{+\infty} \psi(x, y, \zeta, t)\varphi(\zeta) d\zeta dxdy \\ &\quad + \|v\|_p^p + \kappa \int_\Omega \int_{-\infty}^{+\infty} (\zeta^2 + \beta)\psi(x, y, \zeta, t)\Gamma(x, y, \zeta, t) d\zeta dxdy. \end{aligned} \tag{3.2}$$

According to Lemma 6 in [28], we have

$$
\begin{aligned}
&\int_{\Omega}\int_{-\infty}^{+\infty}\left(\zeta^2+\beta\right)\psi(x,y,\zeta,t)\Gamma(x,y,\zeta,t)d\zeta dxdy\\
=\;&\int_{\Omega}v(x,y,t)\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta dxdy-\tau\int_{\Omega}\int_0^1 z(x,y,\rho,t)\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta d\rho dxdy\\
&-\int_{\Omega}\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy. \qquad (3.3)
\end{aligned}
$$

Inserting (3.3) in (3.2), one finds that

$$
\begin{aligned}
\phi_1'(t) \;=\;& \|v_t\|^2-\|v\|_W^2-a_0\int_{\Omega}vv_t dxdy-\kappa\tau\int_{\Omega}\int_0^1 z(x,y,\rho,t)\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta d\rho dxdy\\
&+\|v\|_p^p-\kappa\int_{\Omega}\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy. \qquad (3.4)
\end{aligned}
$$

Similarly to (2.11) and (2.12), we have

$$
\begin{aligned}
&\int_{\Omega}z(x,y,\rho,t)\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta dxdy\\
&\le M\int_{\Omega}z^2(x,y,\rho,t)dxdy+\frac{1}{4}\int_{\Omega}\int_{-\infty}^{+\infty}(\zeta^2+\beta)\psi^2(x,y,\zeta,t)d\zeta dxdy. \qquad (3.5)
\end{aligned}
$$

Using Young's inequality and Lemma 2.2, it is easy to see that

$$
-a_0\int_{\Omega}vv_t dxdy\le a_0\eta\|v_t\|^2+\frac{a_0C_e}{4\eta}\|v\|_W^2, \qquad (3.6)
$$

for any $\eta>0$.

Inserting (3.5) and (3.6) into (3.4) and choosing $\eta=\frac{a_0C_e}{2}$, we get the desired inequality (3.1). □

Lemma 3.2. *Let $V=(v,u,\psi,z)^T$ be a solution of system* (2.9). *The functional*

$$
\phi_2(t)=\tau\int_{\Omega}\int_0^1 e^{-\tau\rho}z^2(x,y,\rho,t)d\rho dxdy,
$$

satisfies

$$
\phi_2'(t)\le\|v_t\|^2-\tau e^{-\tau}\int_{\Omega}\int_0^1 z^2(x,y,\rho,t)d\rho dxdy. \qquad (3.7)
$$

Proof. By differentiating ϕ_2 and using $(2.7)_3$ and the fact that $z(x,y,0,t)=v_t$, we get

$$
\begin{aligned}
\phi_2'(t) \;=\;& 2\tau\int_{\Omega}\int_0^1 e^{-\tau\rho}z(x,y,\rho,t)z_t(x,y,\rho,t)d\rho dxdy\\
=\;& -2\int_{\Omega}\int_0^1 e^{-\tau\rho}z(x,y,\rho,t)z_\rho(x,y,\rho,t)d\rho dxdy
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_\Omega\int_0^1 \frac{d}{d\rho}[e^{-\tau\rho}z^2(x,y,\rho,t)]d\rho dxdy - \tau\int_\Omega\int_0^1 e^{-\tau\rho}z^2(x,y,\rho,t)d\rho dxdy \\
&= -\tau\int_\Omega\int_0^1 e^{-\tau\rho}z^2(x,y,\rho,t)d\rho dxdy - e^{-\tau}\int_\Omega z^2(x,y,1,t)d\rho dxdy + \|v_t\|^2 \\
&\le -\tau e^{-\tau}\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy + \|v_t\|^2.
\end{aligned}
$$

□

The main result of this section reads as follows.

Theorem 3.3. *Suppose* (2.8) *and* (2.21). *Then, there exist positive constants a and b such that any solution of* (2.9) *verifies*

$$E(t) \le aE(0)e^{-bt}, \quad t \ge 0. \tag{3.8}$$

Proof. Define the functional

$$\mathcal{E}(t) = NE(t) + N_1\phi_1(t) + \phi_2(t),$$

where N and N_1 are positive constants that will be chosen later. It is not difficult to prove, for N large enough, that

$$s_1E(t) \le \mathcal{E}(t) \le s_2E(t), \tag{3.9}$$

for some positive constants s_1 and s_2.

Using (2.20), (3.1), and (3.7), we find

$$
\begin{aligned}
\mathcal{E}'(t) \le\ & -\left(N(a_0-\kappa M-\delta) - N_1(1+\frac{a_0^2C_e}{2}) - 1\right)\|v_t\|^2 - \frac{N_1}{2}\|v\|_W^2 - \kappa N_1\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy \\
& -\frac{\kappa}{2}\left(N-\frac{N_1}{2}\right)\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\beta)\psi^2(x,y,\zeta,t)d\zeta dxdy + N_1\|v\|_p^p \\
& -\tau\left(e^{-\tau} - N_1\delta\tau\right)\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy.
\end{aligned}
\tag{3.10}
$$

At this point, we choose N_1 small enough so that

$$e^{-\tau} - N_1\delta\tau > 0.$$

Next, we choose N large enough so that

$$N > \max\left\{\frac{N_1(1+\frac{a_0^2C_e}{2})+1}{a_0-\kappa M-\delta}, \frac{N_1}{2}\right\}.$$

Consequently, we obtain the existence of a positive constant s_3 such that

$$\mathcal{E}'(t) \le -s_3E(t).$$

Using (3.9), we find

$$\mathcal{E}'(t) \le -\frac{s_3}{s_2}\mathcal{E}(t).$$

Hence,

$$\mathcal{E}(t) \le \mathcal{E}(0)e^{-\frac{s_3}{s_2}t}.$$

Again using (3.9), one has

$$E(t) \le \frac{\mathcal{E}(t)}{s_1} \le \frac{1}{s_1}\mathcal{E}(0)e^{-\frac{s_3}{s_2}t} \le \frac{s_2}{s_1}E(0)e^{-\frac{s_3}{s_2}t}.$$

Consequently, (3.8) holds true with $a = \frac{s_2}{s_1}$ and $b = \frac{s_3}{s_2}$. □

Remark 3.4. *Asymptotic analysis helps engineers understand how the dynamic responses of a suspension bridge evolve over time. Furthermore, asymptotic results can identify the natural frequencies of the bridge and how they change with varying load conditions. This information is crucial for avoiding resonance, which can lead to catastrophic failures, as seen in historical bridge collapses. By understanding the asymptotic behavior, engineers can design bridges to avoid critical frequencies that match environmental or traffic-induced vibrations.*

4. Blow-up

In this section, we will prove that the solution of (2.7) blows up in finite time when the initial energy $E(0)$ is negative.

Define the functions

$$\begin{aligned}
\mathcal{R}(t) &= -E(t) \\
&= -\frac{1}{2}\|v_t\|^2 - \frac{1}{2}\|v\|_W^2 - \frac{\kappa}{2}\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy \\
&\quad +\frac{1}{p}\int_\Omega |v|^p dxdy - \delta\tau\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy,
\end{aligned}$$

and

$$\mathcal{L}(t) = \mathcal{R}^{1-\nu}(t) + \varepsilon\int_\Omega vv_t dxdy + \frac{a_0\varepsilon}{2}\|v\|^2,$$

where $0 < \nu < \frac{p-2}{2p}$ and ε will be chosen later.

Proposition 4.1. *Suppose that* (2.8) *and* $E(0) < 0$ *hold true. Then, there exists a positive constant* M_1 *such that*

$$\begin{aligned}
\mathcal{L}'(t) \ge\ & M_1\Big[\mathcal{R}(t) + \|v_t\|^2 + \|v\|_W^2 + \|v\|_p^p + \kappa\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy \\
& +\delta\tau\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy\Big]. \qquad (4.1)
\end{aligned}$$

Proof. First, we can easily see that

$$\|v\|_p^q \le C_1\left(\|v\|_W^2 + \|v\|_p^p\right),\ \forall\ v\in W,\ 2\le q\le p, \qquad (4.2)$$

for some positive constant C_1. Indeed, if $\|v\|_p \le 1$, it follows from (2.3) that

$$\|v\|_p^q \le \|v\|_p^2 \le C_e^{\frac{2}{p}}\|v\|_W^2,\ 2\le q\le p.$$

Now, if $\|v\|_p > 1$, then one has

$$\|v\|_p^q \le \|v\|_p^p,\ 2 \le q \le p.$$

Combining these last two inequalities, we get (4.2). Now, from (2.20), we have

$$\mathcal{R}'(t) = -E'(t) \ge \frac{\kappa}{2}\int_\Omega \int_{-\infty}^{+\infty} (\zeta^2+\beta)\psi^2(x,y,\zeta,t)d\zeta dxdy \ge 0, \tag{4.3}$$

which implies that

$$0 < \mathcal{R}(0) \le \mathcal{R}(t) \le \frac{1}{p}\|v\|_p^p. \tag{4.4}$$

Differentiating $\mathcal{L}(t)$ and using $(2.7)_1$, we get

$$\mathcal{L}'(t) = (1-\nu)\mathcal{R}'(t)\mathcal{R}^{-\nu}(t) + \varepsilon\|v_t\|^2 - \varepsilon\|v\|_W^2 - \varepsilon\kappa\int_\Omega v\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta dxdy + \varepsilon\|v\|_p^p. \tag{4.5}$$

The penultimate term (on the righthand side of (4.5)) can be estimated as

$$-\int_\Omega v\int_{-\infty}^{+\infty}\psi(x,y,\zeta,t)\varphi(\zeta)d\zeta dxdy \ge -\theta M\|v\|^2 - \frac{1}{4\theta}\int_\Omega\int_{-\infty}^{+\infty}(\zeta^2+\beta)\psi^2(x,y,\zeta,t)d\zeta dxdy,$$

for any $\theta > 0$.

In the last inequality, choose $\theta = \frac{1}{2\lambda\mathcal{R}^{-\nu}(t)}$, for some λ to be specified later, and combine the resulting inequality with (4.3), (4.5), and the fact that $-\kappa M > -\delta$ yields to

$$\mathcal{L}'(t) \ge (1-\nu-\varepsilon\lambda)\mathcal{R}'(t)\mathcal{R}^{-\nu}(t) + \varepsilon\|v_t\|^2 - \varepsilon\|v\|_W^2 - \frac{\delta\varepsilon}{2\lambda}\mathcal{R}^{\nu}(t)\|v\|^2 + \varepsilon\|v\|_p^p. \tag{4.6}$$

Therefore, by using (4.1), we obtain for some $0 < m < 1$

$$\begin{aligned}\mathcal{L}'(t) \ \ge\ & (1-\nu-\varepsilon\lambda)\mathcal{R}'(t)\mathcal{R}^{-\nu}(t) + \varepsilon\frac{p(1-m)+2}{2}\|v_t\|^2 + \varepsilon\frac{p(1-m)-2}{2}\|v\|_W^2 - \frac{\delta\varepsilon}{2\lambda}\mathcal{R}^{\nu}(t)\|v\|^2 + \varepsilon m\|v\|_p^p \\ & + p(1-m)\varepsilon\mathcal{R}(t) + \varepsilon\frac{p(1-m)\kappa}{2}\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy \\ & + \varepsilon p(1-m)\delta\tau\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy. \end{aligned} \tag{4.7}$$

Using (4.2), (4.4), and the fact that $\|v\| \le C_2\|v\|_p$, we deduce that

$$\mathcal{R}^{\nu}(t)\|v\|^2 \le \left(\frac{1}{p}\right)^{\nu}\|v\|_p^{p\nu}\|v\|^2 \le C_2\|v\|_p^{p\nu+2} \le C_3\left(\|v\|_W^2 + \|v\|_p^p\right),$$

where C_2 and C_3 are positive constants. Note that in the last inequality we have used the fact that $2 < p\nu + 2 \le p$. Hence, (4.7) becomes

$$\mathcal{L}'(t) \ \ge\ (1-\nu-\varepsilon\lambda)\mathcal{R}'(t)\mathcal{R}^{-\nu}(t) + \varepsilon\frac{p(1-m)+2}{2}\|v_t\|^2 + \varepsilon\left(\frac{p(1-m)-2}{2} - \frac{C_3\delta}{2\lambda}\right)\|v\|_W^2$$

$$+\varepsilon\left(m-\frac{C_3\delta}{2\lambda}\right)\|v\|_p^p+p(1-m)\varepsilon\mathcal{R}(t)+\varepsilon\frac{p(1-m)\kappa}{2}\int_\Omega\int_{-\infty}^{+\infty}\psi^2(x,y,\zeta,t)d\zeta dxdy$$
$$+\varepsilon p(1-m)\delta\tau\int_\Omega\int_0^1 z^2(x,y,\rho,t)d\rho dxdy. \tag{4.8}$$

Now, we choose our parameters carefully. First, we select m small enough such that

$$p(1-m)-2>0.$$

Second we choose λ large enough so that

$$\frac{p(1-m)-2}{2}-\frac{C_3\delta}{2\lambda}>0 \text{ and } m-\frac{C_3\delta}{2\lambda}.$$

Now, we choose ε small enough such that

$$1-\nu-\varepsilon\lambda>0 \text{ and } \mathcal{R}^{1-\nu}(0)-\frac{\varepsilon}{2a_0}\|v_1\|^2>0.$$

With these choices, (4.1) holds true. This ends the proof. □

Remark 4.2. *By the above choice of* ε *and* (4.1)*, it is clear that* $\mathcal{L}(t)\geq\mathcal{L}(0)>0,\ \forall\ t\geq 0$.

Theorem 4.3. *Under the conditions of Proposition 4.1, the solution of* (2.7) *blows up in finite time.*

Proof. By the definition of $\mathcal{L}(t)$, we infer that

$$\begin{aligned}\mathcal{L}^{\frac{1}{1-\nu}}(t) &= \left(\mathcal{R}^{1-\nu}(t)+\varepsilon\int_\Omega vv_t dxdy+\frac{a_0\varepsilon}{2}\|v\|^2\right)^{\frac{1}{1-\nu}}\\ &\leq C_4\left(\mathcal{R}(t)+\left|\int_\Omega vv_t dxdy\right|^{\frac{1}{1-\nu}}+\|v\|^{\frac{2}{1-\nu}}\right),\end{aligned} \tag{4.9}$$

for some constant $C_4>0$.

Since $2<\frac{2}{1-2\nu}<p$, then using Hölder and Young's inequalities and (4.2), we infer that

$$\begin{aligned}\left|\int_\Omega vv_t dxdy\right|^{\frac{1}{1-\nu}} &\leq \|v\|^{\frac{1}{1-\nu}}\|v_t\|^{\frac{1}{1-\nu}}\\ &\leq C_2^{\frac{1}{1-\nu}}\|v\|_p^{\frac{1}{1-\nu}}\|v_t\|^{\frac{1}{1-\nu}}\\ &\leq C_5\left(\|v\|_p^{\frac{2}{1-2\nu}}+\|v_t\|^2\right)\\ &\leq C_6\left(\|v\|_W^2+\|v\|_p^p+\|v_t\|^2\right),\end{aligned} \tag{4.10}$$

where $C_5, C_6>0$.

Since $2<\frac{2}{1-\nu}<p$, then by (4.2), we get

$$\|v\|^{\frac{2}{1-\nu}}\leq C_2^{\frac{2}{1-\nu}}\|v\|_p^{\frac{2}{1-\nu}}\leq C_2^{\frac{2}{1-\nu}}C_1\left(\|v\|_W^2+\|v\|_p^p\right). \tag{4.11}$$

Inserting (4.10) and (4.11) into (4.9), we obtain the existence of a positive constant C_7 such that

$$\mathcal{L}^{\frac{1}{1-\nu}}(t)\leq C_7\left(\mathcal{R}(t)+\|v\|_W^2+\|v\|_p^p+\|v_t\|^2\right),\ \forall\ t\geq 0.$$

The last inequality combined with (4.1) gives us

$$\mathcal{L}'(t) \geq C_8 \mathcal{L}^{\frac{1}{1-\nu}}(t), \ \forall \ t \geq 0, \tag{4.12}$$

where $C_8 > 0$. Integrating (4.12) over $(0, t)$, we arrive at

$$\mathcal{L}(t) \geq \left(\frac{1}{\mathcal{L}^{\frac{-\nu}{1-\nu}}(0) - \frac{C_8 \nu}{1-\nu} t} \right)^{\frac{1-\nu}{\nu}},$$

and, consequently, we deduce that $\mathcal{L}(t)$ blows up in a finite time

$$T \leq T_0 = \frac{1-\nu}{C_8 \nu \mathcal{L}^{\frac{\nu}{1-\nu}}(0)}.$$

This ends the proof. □

Remark 4.4. *Finite time blow-up is the occurrence when solutions become unbounded within a finite amount of time. While not desirable in practical applications due to the potential for structural failure, studying the factors that cause blow-up can provide useful insights into the limits of safe functioning. Engineers can apply this knowledge to develop damping systems that efficiently prevent or mitigate such catastrophic events.*

5. Conclusions

This paper examines suspension bridges subjected to both frictional and fractional damping, along with an external force source. First, we proved the existence of weak solutions and regular ones by using the semigroup method. Second, by constructing a suitable Lyapunov functional, we proved the exponential decay of energy for small initial data. Finally, we also showed that if the initial energy is negative, then the solution of our model blows-up in a finite time. Regarding future works, we can consider other types of damping, for example structural damping (of the form $-\Delta v_t$), strong damping (of the form $\Delta^2 v_t$), or a viscoelastic term (of the form $-\int_0^t g(t-s)\Delta^2 v(s)ds$ where g is the relaxation function), combined with a fractional time delay term and study the asymptotic behavior of these new models.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there are no conflicts of interest.

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