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*Research article*

# On concomitants of generalized order statistics arising from bivariate generalized Weibull distribution and its application in estimation

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**Abstract:** In this research, we studied the concomitants of generalized order statistics from the bivariate generalized Weibull distribution. We derived probability density functions and moments of concomitants of generalized order statistics from the bivariate generalized Weibull distribution. Moreover, utilizing the ranked set sample obtained from this distribution, we computed the best linear unbiased (BLU) estimator of the parameter connected with the study variable (variable of primary interest). Also, a real data application was presented.

**Keywords:** BLU estimator; bivariate generalized Weibull distribution; concomitants of generalized order statistics; ranked set sampling
**Mathematics Subject Classification:** 62G30, 62E15, 62H12

## 1. Introduction

In 1995, Kamps presented the notion of generalized order statistics (GOS), which is the unification of different models of ascendingly ordered random variables (RVs). The GOS incorporates significant and well-known concepts that have been discussed individually in the statistical literature. Many models of ascendingly ordered RVs, such as sequential order statistics, progressive Type-II (PT-II) censored order statistics, ordinary order statistics (OOS), record values, and Pfeifer's record model, are theoretically contained in the GOS model.

Assume $F(.)$ to be an arbitrary continuous cumulative distribution function (CDF) with probability density function (PDF) $f(.)$. Assume also $k > 0$, $n \in N$, and $\tilde{m} = (m_1, m_2, \cdots, m_{n-1}) \in \Re^{n-1}$ to be the parameters such that $\gamma_n = k$ and $\gamma_i = k + n - i + M_i$, for $i = 1, \cdots, n-1$, where $M_i = \sum_{\iota=i}^{n-1} m_\iota$. Then, the RVs $X_{i:n,\tilde{m},k}$, $i = 1, \cdots, n$, are said to be GOS, if their joint PDF is given by

$$f_{1,2,\cdots,n:n,\tilde{m},k}(x_1,x_2,\cdots,x_n)=k\left(\prod_{\nu=1}^{n-1}\gamma_\nu\right)\left(\prod_{\nu=1}^{n-1}[\bar{F}(x_\nu)]^{m_\nu}f(x_\nu)\right)[\bar{F}(x_n)]^{k-1}f(x_n), \tag{1.1}$$

where $\bar{F}(.)=1-F(.)$ and $F^{-1}(0)<x_1\leq\cdots\leq x_n<F^{-1}(1)$.

Several models of arranged RVs can be considered special instances of GOS. $m_1=m_2=\cdots=m_{n-1}=m$; $\gamma_i=k+(n-i)(m+1)$, $i=1,\cdots,n$ corresponds to m-generalized order statistics (m-GOS), $\gamma_i=n-i+1$ $(m_i=0,k=1)$ corresponds to OOS, and $m_i=-1;\gamma_i=k$, $i=1,\cdots,n$, $k\in N$ corresponds to k-recored values. Also, for $m_i=R_i$, $n=m_0+\sum_{\nu=1}^{m_0}R_\nu$, $R_\nu\in N$, and $\gamma_i=n-\sum_{\nu=1}^{i-1}R_\nu-i+1$, $1\leq i\leq m_0$, where $m_0$ denotes the fixed number of failure of units to be observed, the model reduces to PT-II censored order statistics.

Under the condition $\gamma_i\neq\gamma_j$, $i,j=1,\cdots,n-1$, $i\neq j$, Kamps and Cramer [23] derived the PDF of $X_{r:n,\tilde{m},k}$, $1\leq r\leq n$ as

$$f_{r:n,\tilde{m},k}(x) = C_{r-1}f(x)\sum_{i=1}^{r}a_{i,r}\left[\bar{F}(x)\right]^{\gamma_i-1}, \tag{1.2}$$

and the joint PDF of $X_{r:n,\tilde{m},k}$ and $X_{s:n,\tilde{m},k}$ , $r,s=1,\cdots,n$, $r<s$ as

$$\begin{aligned} f_{r,s:n,\tilde{m},k}(x,y) &= C_{s-1}\left[\sum_{i=1}^{r}a_{i,r}\left[\bar{F}(x)\right]^{\gamma_i}\right]\left[\sum_{j=r+1}^{s}a_{j,s}^{(r)}\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_j}\right] \\ &\quad\times\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)},\ x<y, \end{aligned} \tag{1.3}$$

where $C_{r-1}=\prod_{i=1}^{r}\gamma_i$, $a_{i,r}=\prod_{\substack{\imath=1\\ \imath\neq i}}^{r}\frac{1}{\gamma_\imath-\gamma_i}$, $1\leq i\leq r\leq n$, and $a_{j,s}^{(r)}=\prod_{\substack{\jmath=r+1\\ \jmath\neq j}}^{s}\frac{1}{\gamma_\jmath-\gamma_j}$, $r+1\leq j\leq s\leq n$.

It can be shown that for $m_1=m_2=\cdots=m_{n-1}=m\neq-1$ (Khan and Khan [24]),

$$a_{i,r}=\frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i},$$

and

$$a_{j,s}^{(r)}=\frac{(-1)^{s-j}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-j}.$$

Therefore, the PDF of $X_{r:n,\tilde{m},k}$ given in (1.2) reduces to

$$f_{r:n,m,k}(x) = \frac{C_{r-1}}{(r-1)!}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left(F(x)\right), \tag{1.4}$$

and the joint PDF of $X_{r:n,\tilde{m},k}$ and $X_{s:n,\tilde{m},k}$ given in (1.3) reduces to

$$\begin{aligned} f_{r,s:n,m,k}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\bar{F}(x)\right]^{m}g_m^{r-1}\left(F(x)\right)\left[h_m\left(F(y)\right)-h_m\left(F(x)\right)\right]^{s-r-1} \\ &\quad\times\left[\bar{F}(y)\right]^{\gamma_s-1}f(x)f(y),\ x<y, \end{aligned} \tag{1.5}$$

where $C_{r-1} = \prod_{i=1}^{r} \gamma_i$, $\gamma_i = k + (n-i)(m+1)$, $h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1, \\ -ln(1-x), & m = -1, \end{cases}$ and $g_m(x) = h_m(x) - h_m(0)$, $x \in [0, 1)$ (see Kamps [22]).

David [8] introduced the concept of concomitants of order statistics (COS), but Yang [47] described the general theory of COS. Concomitants are important in selection and prediction issues, ranked set sampling, parameter estimation, and the characterization of parent bivariate distributions. For a brief overview of the uses of the concomitants of ordered RVs, see Veena and Thomas [46] and the references therein. For a review of fundamental findings on COS, see Daivd and Nagaraja [9]. Furthermore, for some of the recent works on COS, we refer to Philip and Thomas [36–38], Kumar et al. [29], Barakat et al. [3], and Koshti and Kamalja [28].

Several authors have investigated the concomitants of GOS (CGOS), including Ahsanullah and Nevzorov [1], Beg and Ahsanullah [4], El-Din et al. [13], Domma and Giordano [11], Hanif and Shahbaz [15], Shahbaz and Shahbaz [40], Tahmasebi et al. [44], Alawady et al. [2], and Kamal et al. [20]. Let $(X_i, Y_i)$, $i = 1, \cdots, n$ be a random sample from a bivariate distribution function $F_{X,Y}(x, y)$. When the $X$-variates are ordered in ascending order as $X_{1:n,\tilde{m},k} \leq X_{2:n,\tilde{m},k} \leq X_{3:n,\tilde{m},k} \leq \cdots \leq X_{n:n,\tilde{m},k}$, then $Y$-variates paired (not necessarily in ascending order) with these GOS are called the CGOS and are indicated by $Y_{[r:n,\tilde{m},k]}$, $r = 1, \cdots, n$. The PDF of $Y_{[r:n,\tilde{m},k]}$ is given by (Ahsanullah and Nevzorov, [1])

$$h_{[r:n,\tilde{m},k]}(y) = \int_{-\infty}^{\infty} f(y|x) f_{r:n,\tilde{m},k}(x) dx, \tag{1.6}$$

where $f(y|x)$ is the conditional PDF of $Y$ given $X$ and $f_{r:n,\tilde{m},k}(x)$ is defined in (1.2).

Moreover, the joint PDF of $Y_{[r:n,\tilde{m},k]}$ and $Y_{[s:n,\tilde{m},k]}$, $r, s = 1, \cdots, n$, $r < s$ is given by

$$h_{[r,s:n,\tilde{m},k]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{x_1}^{\infty} f(y_1|x_1) f(y_2|x_2) f_{r,s:n,\tilde{m},k}(x_1, x_2) dx_2 dx_1, \tag{1.7}$$

where $f_{r,s:n,\tilde{m},k}(x_1, x_2)$ is given in (1.3).

One of the most notable applications of COS is in ranked set sampling (RSS). RSS is considered a beneficial sampling strategy for improving estimation efficiency and precision if the variable under study is expensive to measure or difficult to get, yet inexpensive and simple to rank. RSS was proposed by McIntyre [31] and then supported by Takahasi and Wakimoto [45] through mathematical theory. The procedure for RSS is described as follows:

1) Randomly choose $n^2$ units from the population under study, then divide them into $n$ sets of $n$ units.
2) Order the elements of each set without making actual measurements.
3) Choose and quantify the $i^{th}$ minimum from the $i^{th}$ set, $i = 1, \cdots, n$, to create a new set of size $n$, known as the RSS.
4) If a large sample size is required, repeat the above three steps $d$ times (cycles) until a sample of size $nd$ is obtained.

For a comprehensive review of the theory and applications of RSS, see Chen et al. [7]. In some practical applications, the study variable, say, $Y$, is more difficult to measure, whereas an auxiliary variable $X$ associated with $Y$ is easily quantifiable and may be precisely arranged. In this situation, Stokes [42] created another RSS technique, which is as follows:

1) At random, choose $n$ independent bivariate sets of size $n$.
2) Take note of the value of the auxiliary variable on each of these units.
3) From the $i^{th}$ set of size $n$, choose the variable $Y$ associated with the $i^{th}$ smallest $X$, $i = 1, \cdots, n$.

The resulting set of $n$ units is known as the RSS. Consider $(X_{(i:n)_i}, Y_{[i:n]_i})$, $i = 1, \cdots, n$ to be the pair chosen from the $i^{th}$ set, where $X_{(i:n)_i}$ is the $i^{th}$ order statistics of the auxiliary variate in the $i^{th}$ set and $Y_{[i:n]_i}$ is the measurement made on the $Y$ variate associated with $X_{(i:n)_i}$. $Y_{[i:n]_i}$ is obviously the concomitant of the $i^{th}$ order statistics resulting from the $i^{th}$ sample. Numerous authors in the literature have considered the estimation of parameters of the various bivariate distributions using RSS and its modifications. Some work in this area is by Chacko and Thomas [5], Philip and Thomas [36, 37], Koshti and Kamalja [26], Irshad et al. [16, 17], and Dong et al. [12].

COS and higher moments of multivariate distributions have received a lot of attention in recent years. Most of the literature on concomitants is concentrated on symmetric distributions such as multivariate normal (Sheikhi et al., [41]; Chaumette and Vincent, [6]) or multivariate elliptical (Jamalizadeh and Balakrishnan, [18]). Skewed distributions have gained a lot of interest recently in the literature since many datasets encountered in reality have some degree of skewness. In this regard, the distribution theory of COS from skew distributions has been investigated by several authors, including Hanif and Shahbaz [15], Shahbaz and Shahbaz [40], Tahmasebi et al. [44], Shahbaz et al. [39], and Kamal et al. [20]. In this article, we consider the bivariate generalized Weibull (BGW) distribution and the CGOS arising from it. There are numerous reasons for considering this particular bivariate distribution. Due to the presence of four parameters, the joint PDF of the BGW distribution is quite flexible and can take on various shapes depending on the shape parameter. The joint PDF, joint CDF, and conditional PDF for the BGW distribution are all in closed forms, making them appropriate for usage in practice. The univariate marginals of this distribution are able to analyze various types of hazard rates. In addition, it can be utilized for modeling bivariate lifetime data in a variety of scenarios. So far, no results on CGOS arising from the BGW distribution have been found in the literature. Thus, the current study aims to develop the distribution theory of CGOS originating from the BGW distribution and apply it to associated inference problems.

The article is structured as follows: In Section 2, we provide a brief overview of the BGW distribution and some of its properties. In Section 3, we present the marginal PDF as well as the explicit expressions for the single moments of CGOS from the BGW distribution. The joint PDF of CGOS from the BGW distribution is also obtained in Section 3. Furthermore, the explicit expressions for the product moments of CGOS are derived. Section 4 presents the best linear unbiased (BLU) estimator of the parameter of the study variable contained in the BGW distribution using Stokes's RSS and some of the other modified RSS schemes. In Section 5, we apply the results to a real dataset. In Section 6, conclusions are provided.

## 2. BGW distribution

A bivariate RV $(X, Y)$ is said to follow a BGW distribution if its PDF is given by (Pathak et al. [34])

$$f(x, y) = \theta\alpha^2(\beta_1\beta_2)^{-1}x^{\alpha-1}y^{\alpha-1}e^{-\omega(x,y;\phi)}(1 - e^{-\omega(x,y;\phi)})^{\theta-2}(1 - \theta e^{-\omega(x,y;\phi)}), \tag{2.1}$$

where $x, y \geq 0$, $\alpha, \beta_1, \beta_2 > 0$, $0 < \theta \leq 1$, $\omega(x, y; \phi) = \frac{x^\alpha}{\beta_1} + \frac{y^\alpha}{\beta_2}$, and $\phi = (\alpha, \beta_1, \beta_2)$. The BGW distribution includes the bivariate generalized exponential distribution (refer to Mirhosseini et al. [32]) and the bivariate generalized Rayleigh distribution (refer to Pathak and Vellaisamy [35]) as sub-models. The conditional PDF of $Y$ given $X = x$ is (Pathak et al. [34])

$$f(y|x) = \alpha(\beta_2)^{-1}y^{\alpha-1}e^{-\frac{y^\alpha}{\beta_2}}(1-e^{-\omega(x,y;\phi)})^{\theta-2}(1-\theta e^{-\omega(x,y;\phi)})(1-e^{-\frac{x^\alpha}{\beta_1}})^{1-\theta}. \quad (2.2)$$

The RV $X \sim EW(\alpha, \beta_1, \theta)$ is a member of the exponentiated Weibull (EW) distribution with PDF

$$f(x) = \theta\alpha(\beta_1)^{-1}x^{\alpha-1}e^{-\frac{x^\alpha}{\beta_1}}(1-e^{-\frac{x^\alpha}{\beta_1}})^{\theta-1}, x \geq 0, \quad (2.3)$$

and CDF

$$F(x) = (1-e^{-\frac{x^\alpha}{\beta_1}})^{\theta}, x \geq 0. \quad (2.4)$$

Similarly, $Y \sim EW(\alpha, \beta_2, \theta)$. A series expansion of the PDF of the BGW distribution is given by

$$f(x,y) = \alpha^2(\beta_1\beta_2)^{-1}x^{\alpha-1}y^{\alpha-1}\sum_{j=1}^{\infty}\binom{\theta}{j}(-1)^{j+1}j^2e^{-j\omega(x,y;\phi)}. \quad (2.5)$$

Pathak et al. [34] showed that the product moments of the BGW distribution are

$$E(x^py^q) = \Gamma(1+\frac{p}{\alpha})\Gamma(1+\frac{q}{\alpha})\beta_1^{\frac{p}{\alpha}}\beta_2^{\frac{q}{\alpha}}\sum_{j=1}^{\infty}\binom{\theta}{j}(-1)^{j+1}\frac{1}{j^{(p+q)/\alpha}}. \quad (2.6)$$

If we make the transformation, $U = \frac{X}{\beta_1^\star}$ and $V = \frac{Y}{\beta_2^\star}, \beta_i^\star = \beta_i^{1/\alpha}, i = 1, 2$, the standard BGW distribution has the joint PDF as

$$f^\star(u,v) = \alpha^2u^{\alpha-1}v^{\alpha-1}\sum_{j=1}^{\infty}\binom{\theta}{j}(-1)^{j+1}j^2e^{-j(u^\alpha+v^\alpha)}. \quad (2.7)$$

It is clear that the variables $U$ and $V$ have the standard EW distribution as marginal functions with PDFs are, respectively, given by

$$f^\star(u) = \theta\alpha u^{\alpha-1}e^{-u^\alpha}(1-e^{-u^\alpha})^{\theta-1}, u \geq 0, \quad (2.8)$$

$$f^\star(v) = \theta\alpha v^{\alpha-1}e^{-v^\alpha}(1-e^{-v^\alpha})^{\theta-1}, v \geq 0. \quad (2.9)$$

## 3. Distribution theory of CGOS from BGW distribution

In this part, we obtain the distributions and moments of CGOS arising from the BGW distribution.

### *3.1. Marginal PDF and single moments of CGOS*

Suppose $(X_i, Y_i)$ and $(U_i, V_i)$ are random samples of size $n$ each originating from the BGW distribution and the standard BGW distribution, with PDFs provided by (2.1) and (2.7), respectively. Let $V_{[r:n,\tilde{m},k]}$ be the concomitant of the $r^{th}$ GOS $U_{r:n,\tilde{m},k}$. Then, the PDF and the $p^{th}$ moments of $V_{[r:n,\tilde{m},k]}$, $r = 1, \cdots, n$ are given by the following two theorems:

**Theorem 1.** *If $V_{[r:n,\tilde{m},k]}$ is the concomitant of the $r^{th}$ GOS from the standard BGW distribution, then the PDF of $V_{[r:n,\tilde{m},k]}$, for $r = 1, \cdots, n$, is given by*

$$h_{[r:n,\tilde{m},k]}(v) \;=\; \alpha\, C_{r-1} \sum_{i=1}^{r} \sum_{j=1}^{\infty} a_{i,r}\, j^2 \delta_\theta(j, \gamma_i) v^{\alpha-1} e^{-jv^\alpha}, v \geq 0, \tag{3.1}$$

*where $\delta_\theta(j,\gamma_i) = \binom{\theta}{j} \sum_{\tau=0}^{\gamma_i-1} (-1)^{j+\tau+1} \binom{\gamma_i-1}{\tau} B(j, \theta\tau + 1)$ and $B(.,.)$ is the complete beta function.*

*Proof.* Using the PDF of $U_{r:n,\tilde{m},k}$ (1.2) in (1.6), the PDF of the $r^{th}$ CGOS $V_{[r:n,\tilde{m},k]}$ is given as

$$h_{[r:n,\tilde{m},k]}(v) \;=\; C_{r-1} \sum_{i=1}^{r} a_{i,r} \int_0^\infty f(v|u) \left[\bar{F}(u)\right]^{\gamma_i - 1} f(u) du. \tag{3.2}$$

In view of (2.7) and (2.8), we get

$$\begin{aligned} h_{[r:n,\tilde{m},k]}(v) \;=\; & \alpha\, C_{r-1} \sum_{i=1}^{r} \sum_{j=1}^{\infty} \sum_{\tau=0}^{\gamma_i-1} a_{i,r} (-1)^{j+\tau+1} j^2 \binom{\theta}{j} \\ & \times \binom{\gamma_i - 1}{\tau} v^{\alpha-1} e^{-jv^\alpha} \int_0^\infty e^{-jz} (1 - e^{-z})^{\theta\tau} dz \\ & \text{[provided that } \gamma_i \text{ is an integer]}, \end{aligned} \tag{3.3}$$

where $z = u^\alpha$. Now, by using Eq (3.3121) in Gradshteyn and Ryzhik [14] to compute the integral in (3.3), we obtain the result given in (3.1). □

**Corollary 1.** *Taking $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ in (3.1), the PDF of the $r^{th}$ concomitant of m-GOS from the standard BGW distribution is given by*

$$h_{[r:n,m,k]}(v) \;=\; \frac{\alpha C_{r-1}}{(r-1)!} \sum_{i=1}^{r} \sum_{j=1}^{\infty} (-1)^{r-i} \frac{j^2}{(m+1)^{r-1}} \binom{r-1}{r-i} \delta_\theta(j,\gamma_i)\, v^{\alpha-1} e^{-jv^\alpha}, v \geq 0, \tag{3.4}$$

*where $\gamma_i = k + (n-i)(m+1)$.*

**Remark 1.** *When $m = 0$ and $k = 1$ in (3.4), we get the PDF of the $r^{th}$ COS from the standard BGW distribution as*

$$h_{[r:n]}(v) \;=\; \alpha\, C_{r:n} \sum_{i=1}^{r} \sum_{j=1}^{\infty} (-1)^{r-i} j^2 \binom{r-1}{r-i} \delta_\theta(j, n-i+1)\, v^{\alpha-1} e^{-jv^\alpha}, \tag{3.5}$$

*where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.*

**Theorem 2.** *Under the conditions of Theorem 1, the $p^{th}$ moment of $V_{[r:n,\tilde{m},k]}$ is*

$$\mu^{(p)}_{[r:n,\tilde{m},k]} \;=\; C_{r-1} \sum_{i=1}^{r} \sum_{j=1}^{\infty} a_{i,r}\, \delta_\theta(j,\gamma_i) \frac{\Gamma(\frac{p}{\alpha}+1)}{j^{\frac{p}{\alpha}-1}}. \tag{3.6}$$

*Proof.* Using (3.1), the $p^{th}$ moment of $V_{[r:n,\tilde{m},k]}$ is given as

$$\begin{aligned}\mu^{(p)}_{[r:n,\tilde{m},k]} &= E[V^p_{[r:n,\tilde{m},k]}] = \int_0^\infty v^p h_{[r:n,\tilde{m},k]}(v)dv \\ &= C_{r-1}\sum_{i=1}^{r}\sum_{j=1}^{\infty} a_{i,r}\, j^2\, \delta_\theta(j,\gamma_i)\int_0^\infty z^{\frac{p}{\alpha}} e^{-jz}dz, \end{aligned} \tag{3.7}$$

where $z = v^\alpha$. Then, after integration, we get (3.6). □

**Corollary 2.** *Taking $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ in (3.6), the $p^{th}$ moment of the concomitant of m-GOS is given by*

$$\mu^{(p)}_{[r:n,m,k]} = \frac{C_{r-1}}{(r-1)!}\sum_{i=1}^{r}\sum_{j=1}^{\infty}\frac{(-1)^{r-i}}{(m+1)^{r-1}}\binom{r-1}{r-i}\delta_\theta(j,\gamma_i)\frac{\Gamma(\frac{p}{\alpha}+1)}{j^{\frac{p}{\alpha}-1}}. \tag{3.8}$$

**Remark 2.** *Let $m = 0$ and $k = 1$ in (3.8), then the $p^{th}$ moment of COS is*

$$\begin{aligned}\mu^{(p)}_{[r:n]} &= E[V^p_{[r:n]}] \\ &= C_{r:n}\sum_{i=1}^{r}\sum_{j=1}^{\infty}(-1)^{r-i}\binom{r-1}{r-i}\delta_\theta(j,n-i+1)\frac{\Gamma(\frac{p}{\alpha}+1)}{j^{\frac{p}{\alpha}-1}}.\end{aligned} \tag{3.9}$$

### *3.2. Joint PDF and product moments of CGOS*

Let $V_{[r:n,\tilde{m},k]}$ and $V_{[s:n,\tilde{m},k]}$, $r, s = 1,\cdots,n, r < s$ be the concomitants of the $r^{th}$ and $s^{th}$ GOS from the standard BGW distribution. Then, the joint PDF and the product moments of $V_{[r:n,\tilde{m},k]}$ and $V_{[s:n,\tilde{m},k]}$ are given by the following two theorems:

**Theorem 3.** *The joint PDF of concomitants $V_{[r:n,\tilde{m},k]}$ and $V_{[s:n,\tilde{m},k]}$, $r, s = 1,\cdots,n, r < s$ is given by*

$$\begin{aligned}h_{[r,s:n,\tilde{m},k]}(v_1,v_2) &= \alpha^2 C_{s-1}\sum_{i=1}^{r}\sum_{j=r+1}^{s}\sum_{\kappa_1=1}^{\infty}\sum_{\kappa_2=1}^{\infty} a_{i,r}\, a^{(r)}_{j,s}\,\kappa_1^2\kappa_2\delta_\theta(\kappa_1,\kappa_2,\gamma_i,\gamma_j) \\ &\quad \times v_1^{\alpha-1}v_2^{\alpha-1}e^{-(\kappa_1 v_1^\alpha+\kappa_2 v_2^\alpha)}, v_1,\ v_2 > 0,\end{aligned} \tag{3.10}$$

*where*

$$\delta_\theta(\kappa_1,\kappa_2,\gamma_i,\gamma_j) = \binom{\theta}{\kappa_1}\binom{\theta}{\kappa_2}\sum_{\tau_1=0}^{\gamma_j-1}\sum_{\tau_2=0}^{\gamma_i-\gamma_j-1}(-1)^{\kappa_1+\kappa_2+\tau_1+\tau_2+2}\binom{\gamma_j-1}{\tau_1}\binom{\gamma_i-\gamma_j-1}{\tau_2}$$
$$\times B(\kappa_1+\kappa_2,\theta\tau_2+1)\,{}_3F_2\left(\kappa_2,-\theta\tau_1,\kappa_1+\kappa_2;\kappa_2+1,\kappa_1+\kappa_2+\theta\tau_2+1;1\right),$$

*where ${}_3F_2(a_1,\ a_2,\ a_3;\ b_1,\ b_2;\ x)$ denotes the hypergeometric function defined by*

$${}_3F_2(a_1,\ a_2,\ a_3;\ b_1,\ b_2;\ x) = \sum_{\ell=0}^{\infty}\frac{(a_1)_\ell(a_2)_\ell(a_3)_\ell}{(b_1)_\ell(b_2)_\ell}\frac{x^\ell}{\ell!},$$

*and $(c)_\ell = c(c+1)\cdots(c+\ell-1)$ is the ascending factorial.*

*Proof.* Using (1.3) in (1.7), the joint PDF of the $r^{th}$ and $s^{th}$ CGOS $V_{[r:n,\tilde{m},k]}$ and $V_{[s:n,\tilde{m},k]}$ is given as

$$\begin{aligned} h_{[r,s:n,\tilde{m},k]}(v_1, v_2) &= C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} a_{i,r}\, a_{j,s}^{(r)} \int_0^\infty \int_{u_1}^\infty f(v_1|u_1) f(v_2|u_2) \\ &\quad \times \left[\bar{F}(u_1)\right]^{\gamma_i-\gamma_j-1} \left[\bar{F}(u_2)\right]^{\gamma_j-1} f(u_1) f(u_2) du_2\, du_1. \end{aligned} \tag{3.11}$$

In view of (2.7) and (2.8), we get

$$\begin{aligned} h_{[r,s:n,\tilde{m},k]}(v_1, v_2) &= \alpha^4 C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} a_{i,r}\, a_{j,s}^{(r)} (-1)^{\kappa_1+\kappa_2+2} \\ &\quad \times \kappa_1^2 \kappa_2^2 \binom{\theta}{\kappa_1}\binom{\theta}{\kappa_2} v_1^{\alpha-1} v_2^{\alpha-1} e^{-\kappa_1 v_1^\alpha} e^{-\kappa_2 v_2^\alpha} \\ &\quad \times \int_0^\infty u_1^{\alpha-1} e^{-\kappa_1 u_1^\alpha} [1-(1-e^{-u_1^\alpha})^\theta]^{\gamma_i-\gamma_j-1} I(u_1) du_1, \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} I(u_1) &= \sum_{\tau_1=0}^{\gamma_j-1} (-1)^{\tau_1} \binom{\gamma_j-1}{\tau_1} \int_{u_1}^\infty u_2^{\alpha-1} e^{-\kappa_2 u_2^\alpha} (1-e^{-u_2^\alpha})^{\tau_1\theta} du_2 \\ &= \alpha^{-1} \sum_{\tau_1=0}^{\gamma_j-1} (-1)^{\tau_1} \binom{\gamma_j-1}{\tau_1} B_w(\kappa_2, \tau_1\theta+1), \end{aligned} \tag{3.13}$$

where $w = e^{-u_1^\alpha}$ and $B_w(.,.)$ denotes the incomplete beta function defined by $B_w(a_1, a_2) = \int_0^w x^{a_1-1}(1-x)^{a_2-1} dx$.

Now, putting the value of $I(u_1)$ in (3.12), we get

$$\begin{aligned} h_{[r,s:n,\tilde{m},k]}(v_1, v_2) &= \alpha^2 C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} \sum_{\tau_1=0}^{\gamma_j-1} \sum_{\tau_2=0}^{\gamma_i-\gamma_j-1} a_{i,r}\, a_{j,s}^{(r)} (-1)^{\kappa_1+\kappa_2+\tau_1+\tau_2+2} \\ &\quad \times \kappa_1^2 \kappa_2^2 \binom{\theta}{\kappa_1}\binom{\theta}{\kappa_2}\binom{\gamma_j-1}{\tau_1}\binom{\gamma_i-\gamma_j-1}{\tau_2} v_1^{\alpha-1} v_2^{\alpha-1} e^{-\kappa_1 v_1^\alpha} e^{-\kappa_2 v_2^\alpha} \\ &\quad \times \int_0^\infty e^{-\kappa_1 z} (1-e^{-z})^{\tau_2\theta} B_{e^{-z}}(\kappa_2, \tau_1\theta+1) dz \\ &\quad [\text{provided that } \gamma_i - \gamma_j \text{ is an integer}], \end{aligned} \tag{3.14}$$

where $z = u_1^\alpha$. We know that $B_w(a_1, a_2) = \frac{w^{a_1}}{a_1} {}_2F_1(a_1,\ 1-a_2;\ a_1+1;\ w)$ (see Mathai and Saxena, [30]), and

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, {}_2F_1(c,\ d;\ e;\ x) dx = B(a,b)\, {}_3F_2(c,\ d,\ a;\ e,\ a+b;\ 1). \tag{3.15}$$

Therefore,

$$
\begin{aligned}
h_{[r,s:n,\tilde{m},k]}(v_1,v_2) &= \alpha^2 C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} \sum_{\tau_1=0}^{\gamma_j-1} \sum_{\tau_2=0}^{\gamma_i-\gamma_j-1} a_{i,r}\, a_{j,s}^{(r)} (-1)^{\kappa_1+\kappa_2+\tau_1+\tau_2+2} \\
&\times \kappa_1^2 \kappa_2 \binom{\theta}{\kappa_1}\binom{\theta}{\kappa_2}\binom{\gamma_j-1}{\tau_1}\binom{\gamma_i-\gamma_j-1}{\tau_2} v_1^{\alpha-1} v_2^{\alpha-1} e^{-\kappa_1 v_1^{\alpha}} e^{-\kappa_2 v_2^{\alpha}} \\
&\times \int_0^1 t^{\kappa_1+\kappa_2-1}(1-t)^{\tau_2\theta}\, {}_2F_1(\kappa_2,\, -\tau_1\theta;\, \kappa_2+1;\, t)dt,
\end{aligned}
$$

where $t = e^{-z}$. Now, using (3.15), we get the result of (3.10). □

**Corollary 3.** *At $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ in (3.10), the joint PDF of concomitants $V_{[r:n,m,k]}$ and $V_{[s:n,m,k]}$ of the $r^{th}$ and $s^{th}$ m-GOS for the standard BGW distribution is given by*

$$
\begin{aligned}
h_{[r,s:n,m,k]}(v_1,v_2) &= \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} \frac{(-1)^{r-i+s-j}}{(m+1)^{s-2}} \binom{r-1}{r-i}\binom{s-r-1}{s-j} \kappa_1^2 \kappa_2 \\
&\times \delta_\theta(\kappa_1,\kappa_2,\gamma_i,\gamma_j)\, v_1^{\alpha-1} v_2^{\alpha-1} e^{-(\kappa_1 v_1^{\alpha}+\kappa_2 v_2^{\alpha})}, v_1,\ v_2 > 0, \qquad (3.16)
\end{aligned}
$$

*where $\gamma_i = k + (n-i)(m+1)$.*

**Remark 3.** *For $m = 0$ and $k = 1$ in (3.16), we obtain the joint PDF of the $r^{th}$ and $s^{th}$ COS from the standard BGW distribution as*

$$
\begin{aligned}
h_{[r,s:n]}(v_1,v_2) &= \alpha^2 C_{r,s:n} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} (-1)^{r-i+s-j} \binom{r-1}{r-i}\binom{s-r-1}{s-j} \kappa_1^2 \kappa_2 \\
&\times \delta_\theta(\kappa_1,\kappa_2,n-i+1,n-j+1)\, v_1^{\alpha-1} v_2^{\alpha-1} e^{-(\kappa_1 v_1^{\alpha}+\kappa_2 v_2^{\alpha})}, v_1,\ v_2 > 0, \qquad (3.17)
\end{aligned}
$$

*where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.*

**Theorem 4.** *The product moments of two concomitants $V_{[r:n,\tilde{m},k]}$ and $V_{[s:n,\tilde{m},k]}$ are given by*

$$
\begin{aligned}
\mu_{[r,s:n,\tilde{m},k]}^{(p,q)} &= C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} a_{i,r}\, a_{j,s}^{(r)}\, \delta_\theta(\kappa_1,\kappa_2,\gamma_i,\gamma_j) \\
&\times \frac{\Gamma(\frac{p}{\alpha}+1)}{\kappa_1^{\frac{p}{\alpha}-1}} \frac{\Gamma(\frac{q}{\alpha}+1)}{\kappa_2^{\frac{q}{\alpha}}}. \qquad (3.18)
\end{aligned}
$$

*Proof.* Using (3.10), the $p^{th}$ and $q^{th}$ moments of $V_{[r:n,\tilde{m},k]}$ and $V_{[s:n,\tilde{m},k]}$ are given as

$$
\begin{aligned}
\mu_{[r,s:n,\tilde{m},k]}^{(p,q)} &= E[V_{[r:n,\tilde{m},k]}^{p} V_{[s:n,\tilde{m},k]}^{q}] = \int_0^\infty \int_0^\infty v_1^p v_2^q h_{[r,s:n,\tilde{m},k]}(v_1,v_2) dv_1 dv_2 \\
&= C_{s-1} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} a_{i,r}\, a_{j,s}^{(r)}\, \kappa_1^2 \kappa_2\, \delta_\theta(\kappa_1,\kappa_2,\gamma_i,\gamma_j) \\
&\times \int_0^\infty \int_0^\infty z_1^{\frac{p}{\alpha}} z_2^{\frac{q}{\alpha}} e^{-(\kappa_1 z_1+\kappa_2 z_2)} dz_1 dz_2, \qquad (3.19)
\end{aligned}
$$

where $z_i = v_i^{\alpha}, i = 1, 2$. Then, after integration, we get (3.18). □

**Corollary 4.** *Setting $m_1 = m_2 = \cdots = m_{n-1} = m \neq -1$ in (3.18), we can get the product moments of two concomitants of m-GOS of the standard BGW distribution.*

**Remark 4.** *When $m = 0$ and $k = 1$ in (3.18), we get the product moments of COS as*

$$\begin{aligned} \mu_{[r,s:n]}^{(p,q)} &= E[V_{[r:n]}^{p} V_{[s:n]}^{q}] \\ &= C_{r,s:n} \sum_{i=1}^{r} \sum_{j=r+1}^{s} \sum_{\kappa_1=1}^{\infty} \sum_{\kappa_2=1}^{\infty} (-1)^{r-i+s-j} \binom{r-1}{r-i} \binom{s-r-1}{s-j} \\ &\times \delta_{\theta}(\kappa_1, \kappa_2, n-i+1, n-j+1) \frac{\Gamma(\frac{p}{\alpha}+1)}{\kappa_1^{\frac{p}{\alpha}-1}} \frac{\Gamma(\frac{q}{\alpha}+1)}{\kappa_2^{\frac{q}{\alpha}}}. \end{aligned} \tag{3.20}$$

**Remark 5.** *At $m_i = R_i$, $n = m_0 + \sum_{i=1}^{m_0} R_i$, and $\gamma_i = n - \sum_{\nu=1}^{i-1} R_\nu - i + 1$, $1 \leq i \leq m_0$ in Theorems 1–4, the results for PT-II censored order statistics can be obtained.*

**Remark 6.** *From (3.9), the expressions of means and variances of the COS $Y_{[i:n]}$, $i = 1, \cdots, n$, arising from the BGW distribution, are obtained as follows:*

$$E[Y_{[i:n]}] = \beta_2^{\star} E[V_{[i:n]}]$$

$$= \beta_2^{\star} \mu_{[i:n]},$$

$$Var[Y_{[i:n]}] = {\beta_2^{\star}}^2 Var[V_{[i:n]}]$$

$$= {\beta_2^{\star}}^2 \delta_{i,i:n},$$

*where $Var[V_{[i:n]}] = \mu_{[i:n]}^{(2)} - (\mu_{[i:n]})^2$. The expression of the covariances between $Y_{[i:n]}$ and $Y_{[j:n]}$ is given, using (3.9) and (3.20), by*

$$Cov[Y_{[i:n]}, Y_{[j:n]}] = {\beta_2^{\star}}^2 Cov[V_{[i:n]}, V_{[j:n]}]$$

$$= {\beta_2^{\star}}^2 \delta_{i,j:n},$$

*where $Cov(V_{[i:n]}, V_{[j:n]}) = \mu_{[i,j:n]} - \mu_{[i:n]}\mu_{[j:n]}$, $1 \leq i < j \leq n$.*

The means and variances of the COS of the standard BGW distribution for $n = 1, \cdots, 5$ and different values of the parameters $\alpha$ and $\theta$ are calculated in Tables 1 and 2. It can be noted that the condition $\sum_{r=1}^{n} \mu_{[r:n]}^{j} = n\mu_{1:1}^{j}$, $j = 1, 2$ is satisfied (see David and Nagaraja, [10]). In Tables 3–6, we have computed the means and variances of the concomitants of PT-II censored order statistics. From Tables 1–6, one can observe that the variances are decreasing with respect to $\alpha$.

**Table 1.** Means and variances of the COS for the standard BGW distribution with $\theta = 0.50$.

| | | Mean | | Variance | |
|---|---|---|---|---|---|
| $n$ | $r$ | $\alpha = 1$ | $\alpha = 2$ | $\alpha = 1$ | $\alpha = 2$ |
| 1 | 1 | 0.613519 | 0.626050 | 0.710359 | 0.221580 |
| 2 | 1 | 0.454473 | 0.503844 | 0.546719 | 0.200615 |
| | 2 | 0.772564 | 0.748256 | 0.823408 | 0.212676 |
| 3 | 1 | 0.363977 | 0.427653 | 0.445089 | 0.181090 |
| | 2 | 0.635465 | 0.656226 | 0.700844 | 0.204833 |
| | 3 | 0.841114 | 0.794272 | 0.870593 | 0.210246 |
| 4 | 1 | 0.304634 | 0.374043 | 0.375757 | 0.164726 |
| | 2 | 0.542006 | 0.588481 | 0.610825 | 0.195696 |
| | 3 | 0.728923 | 0.723970 | 0.773394 | 0.204791 |
| | 4 | 0.878510 | 0.817706 | 0.897399 | 0.209868 |
| 5 | 1 | 0.262399 | 0.333637 | 0.325339 | 0.151085 |
| | 2 | 0.473577 | 0.535668 | 0.541753 | 0.186636 |
| | 3 | 0.644650 | 0.667700 | 0.696872 | 0.198827 |
| | 4 | 0.785106 | 0.761484 | 0.816517 | 0.205248 |
| | 5 | 0.901861 | 0.831761 | 0.914894 | 0.210035 |

**Table 2.** Means and variances of the COS for the standard BGW distribution with $\theta = 0.90$.

| | | Mean | | Variance | |
|---|---|---|---|---|---|
| $n$ | $r$ | $\alpha = 1$ | $\alpha = 2$ | $\alpha = 1$ | $\alpha = 2$ |
| 1 | 1 | 0.933392 | 0.846157 | 0.956977 | 0.217410 |
| 2 | 1 | 0.898058 | 0.823580 | 0.935961 | 0.219775 |
| | 2 | 0.968725 | 0.868735 | 0.975495 | 0.214025 |
| 3 | 1 | 0.873819 | 0.807493 | 0.922204 | 0.221774 |
| | 2 | 0.946537 | 0.855752 | 0.959950 | 0.214226 |
| | 3 | 0.979820 | 0.875226 | 0.982898 | 0.213799 |
| 4 | 1 | 0.855345 | 0.794891 | 0.912004 | 0.223493 |
| | 2 | 0.929241 | 0.845300 | 0.948711 | 0.214709 |
| | 3 | 0.963834 | 0.866204 | 0.970591 | 0.213524 |
| | 4 | 0.985148 | 0.878234 | 0.986887 | 0.213854 |
| 5 | 1 | 0.840423 | 0.784489 | 0.903900 | 0.224999 |
| | 2 | 0.915036 | 0.836498 | 0.939966 | 0.215307 |
| | 3 | 0.950548 | 0.858502 | 0.961071 | 0.213522 |
| | 4 | 0.972691 | 0.871339 | 0.976741 | 0.213459 |
| | 5 | 0.988263 | 0.879957 | 0.989375 | 0.213938 |

**Table 3.** Means of the concomitants of PT-II censored order statistics for the standard BGW distribution with $\theta = 0.50$.

| $\alpha$ | $m_0, n$ | Scheme | Mean | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2, 10 | $8, 0$ | 0.155927 | 0.664362 | | | |
| 1 | 2, 10 | $0, 8$ | 0.155927 | 0.292953 | | | |
| 1 | 3, 10 | $7, 0, 0$ | 0.155927 | 0.52911 | 0.799614 | | |
| 1 | 3, 10 | $0, 0, 7$ | 0.155927 | 0.292953 | 0.412797 | | |
| 1 | 4, 10 | $6, 0, 0, 0$ | 0.155927 | 0.453142 | 0.681045 | 0.858899 | |
| 1 | 4, 10 | $0, 0, 0, 6$ | 0.155927 | 0.292953 | 0.412797 | 0.51814 | |
| 1 | 5, 10 | $5, 0, 0, 0, 0$ | 0.155927 | 0.403773 | 0.601249 | 0.760842 | 0.891585 |
| 1 | 5, 10 | $0, 0, 0, 0, 5$ | 0.155927 | 0.292953 | 0.412797 | 0.51814 | 0.61135 |
| 2 | 2, 10 | $8, 0$ | 0.220176 | 0.671147 | | | |
| 2 | 2, 10 | $0, 8$ | 0.220176 | 0.377969 | | | |
| 2 | 3, 10 | $7, 0, 0$ | 0.220176 | 0.574761 | 0.767534 | | |
| 2 | 3, 10 | $0, 0, 7$ | 0.220176 | 0.377969 | 0.492427 | | |
| 2 | 4, 10 | $6, 0, 0, 0$ | 0.220176 | 0.516571 | 0.691139 | 0.805731 | |
| 2 | 4, 10 | $0, 0, 0, 6$ | 0.220176 | 0.377969 | 0.492427 | 0.579475 | |
| 2 | 5, 10 | $5, 0, 0, 0, 0$ | 0.220176 | 0.476621 | 0.636421 | 0.745858 | 0.825689 |
| 2 | 5, 10 | $0, 0, 0, 0, 5$ | 0.220176 | 0.377969 | 0.492427 | 0.579475 | 0.648751 |

**Table 4.** Means of the concomitants of PT-II censored order statistics for the standard BGW distribution with $\theta = 0.90$.

| $\alpha$ | $m_0, n$ | Scheme | Mean | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2, 10 | $8, 0$ | 0.791751 | 0.94913 | | | |
| 1 | 2, 10 | $0, 8$ | 0.791751 | 0.867371 | | | |
| 1 | 3, 10 | $7, 0, 0$ | 0.791751 | 0.924635 | 0.973624 | | |
| 1 | 3, 10 | $0, 0, 7$ | 0.791751 | 0.867371 | 0.905027 | | |
| 1 | 4, 10 | $6, 0, 0, 0$ | 0.791751 | 0.908991 | 0.955924 | 0.982475 | |
| 1 | 4, 10 | $0, 0, 0, 6$ | 0.791751 | 0.867371 | 0.905027 | 0.929301 | |
| 1 | 5, 10 | $5, 0, 0, 0, 0$ | 0.791751 | 0.897741 | 0.94274 | 0.969107 | 0.986931 |
| 1 | 5, 10 | $0, 0, 0, 0, 5$ | 0.791751 | 0.867371 | 0.905027 | 0.929301 | 0.946824 |
| 2 | 2, 10 | $8, 0$ | 0.749137 | 0.856937 | | | |
| 2 | 2, 10 | $0, 8$ | 0.749137 | 0.805526 | | | |
| 2 | 3, 10 | $7, 0, 0$ | 0.749137 | 0.84219 | 0.871684 | | |
| 2 | 3, 10 | $0, 0, 7$ | 0.749137 | 0.805526 | 0.830734 | | |
| 2 | 4, 10 | $6, 0, 0, 0$ | 0.749137 | 0.832503 | 0.861564 | 0.876744 | |
| 2 | 4, 10 | $0, 0, 0, 6$ | 0.749137 | 0.805526 | 0.830734 | 0.846014 | |
| 2 | 5, 10 | $5, 0, 0, 0, 0$ | 0.749137 | 0.825394 | 0.853831 | 0.869298 | 0.879226 |
| 2 | 5, 10 | $0, 0, 0, 0, 5$ | 0.749137 | 0.805526 | 0.830734 | 0.846014 | 0.85658 |

**Table 5.** Variances of the concomitants of PT-II censored order statistics for the standard BGW distribution with $\theta = 0.50$.

| $\alpha$ | $m_0, n$ | Scheme | Variance | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2, 10 | 8, 0 | 0.195285 | 0.741739 | | | |
| 1 | 2, 10 | 0, 8 | 0.195285 | 0.347562 | | | |
| 1 | 3, 10 | 7, 0, 0 | 0.195285 | 0.606725 | 0.840167 | | |
| 1 | 3, 10 | 0, 0, 7 | 0.195285 | 0.347562 | 0.469548 | | |
| 1 | 4, 10 | 6, 0, 0, 0 | 0.195285 | 0.525646 | 0.734255 | 0.882579 | |
| 1 | 4, 10 | 0, 0, 0, 6 | 0.195285 | 0.347562 | 0.469548 | 0.570046 | |
| 1 | 5, 10 | 5, 0, 0, 0, 0 | 0.195285 | 0.47150 | 0.658836 | 0.796939 | 0.906852 |
| 1 | 5, 10 | 0, 0, 0, 0, 5 | 0.195285 | 0.347562 | 0.469548 | 0.570046 | 0.654972 |
| 2 | 2, 10 | 8, 0 | 0.107449 | 0.213924 | | | |
| 2 | 2, 10 | 0, 8 | 0.107449 | 0.150092 | | | |
| 2 | 3, 10 | 7, 0, 0 | 0.107449 | 0.19876 | 0.210507 | | |
| 2 | 3, 10 | 0, 0, 7 | 0.107449 | 0.150092 | 0.170312 | | |
| 2 | 4, 10 | 6, 0, 0, 0 | 0.107449 | 0.186296 | 0.203372 | 0.209697 | |
| 2 | 4, 10 | 0, 0, 0, 6 | 0.107449 | 0.150092 | 0.170312 | 0.182349 | |
| 2 | 5, 10 | 5, 0, 0, 0, 0 | 0.107449 | 0.176605 | 0.196217 | 0.204539 | 0.209823 |
| 2 | 5, 10 | 0, 0, 0, 0, 5 | 0.107449 | 0.150092 | 0.170312 | 0.182349 | 0.190473 |

**Table 6.** Variances of the concomitants of PT-II censored order statistics for the standard BGW distribution with $\theta = 0.90$.

| $\alpha$ | $m_0, n$ | Scheme | Variance | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2, 10 | 8, 0 | 0.877894 | 0.963287 | | | |
| 1 | 2, 10 | 0, 8 | 0.877894 | 0.913014 | | | |
| 1 | 3, 10 | 7, 0, 0 | 0.877894 | 0.946946 | 0.978427 | | |
| 1 | 3, 10 | 0, 0, 7 | 0.877894 | 0.913014 | 0.932212 | | |
| 1 | 4, 10 | 6, 0, 0, 0 | 0.877894 | 0.937071 | 0.965229 | 0.984791 | |
| 1 | 4, 10 | 0, 0, 0, 6 | 0.877894 | 0.913014 | 0.932212 | 0.946122 | |
| 1 | 5, 10 | 5, 0, 0, 0, 0 | 0.877894 | 0.93025 | 0.956015 | 0.974096 | 0.988277 |
| 1 | 5, 10 | 0, 0, 0, 0, 5 | 0.877894 | 0.913014 | 0.932212 | 0.946122 | 0.957275 |
| 2 | 2, 10 | 8, 0 | 0.230545 | 0.214788 | | | |
| 2 | 2, 10 | 0, 8 | 0.230545 | 0.218499 | | | |
| 2 | 3, 10 | 7, 0, 0 | 0.230545 | , 0.215351 | 0.213791 | | |
| 2 | 3, 10 | 0, 0, 7 | 0.230545 | 0.218499 | 0.214908 | | |
| 2 | 4, 10 | 6, 0, 0, 0 | 0.230545 | 0.21593 | 0.213631 | 0.213794 | |
| 2 | 4, 10 | 0, 0, 0, 6 | 0.230545 | 0.218499 | 0.214908 | 0.213561 | |
| 2 | 5, 10 | 5, 0, 0, 0, 0 | 0.230545 | 0.216466 | 0.213713 | 0.213429 | 0.213892 |
| 2 | 5, 10 | 0, 0, 0, 0, 5 | 0.230545 | 0.218499 | 0.214908 | 0.213561 | 0.213094 |

## 4. BLU estimator of the parameter $\beta_2^\star$ of BGW distribution

In this part, we obtain the BLU estimator of $\beta_2^\star$ involved in the BGW distribution using Stoke's RSS. Assume that $n$ sets of units, each of size $n$, are taken from the BGW distribution with the PDF given in (2.1). Let $X_{(i:n)_i}$, $i = 1, \cdots, n$ represent the observation made on the auxiliary variable $X$ in the $i^{th}$ unit of the RSS, and $Y_{[i:n]_i}$ represent the measurement performed on the $Y$ variable in the same unit. It is obvious that $Y_{[i:n]_i}$ has the same distribution as $Y_{[i:n]}$, the concomitant of the $i^{th}$ order statistics (see David and Nagraja, [10], p. 145). From Remark 6, the mean and the variance of $Y_{[i:n]_i}$ are given as $E[Y_{[i:n]_i}] = \beta_2^\star \mu_{[i:n]}$, and $Var[Y_{[i:n]_i}] = {\beta_2^\star}^2 \delta_{i,i:n}$, $1 \leq i \leq n$. Because the two measurements $Y_{[i:n]_i}$ and $Y_{[j:n]_j}$ $(i \neq j)$ of $Y$ are based on two independent samples, we have $Cov[Y_{[i:n]_i}, Y_{[j:n]_j}] = 0$.

Let $\mathbf{Y}_{[n]} = (Y_{[1:n]_1}, Y_{[2:n]_2}, \cdots, Y_{[n:n]_n})'$ denote the column vector of COS. Then, the mean vector and the variance-covariance matrix of $\mathbf{Y}_{[n]}$ can be written as

$$E[\mathbf{Y}_{[n]}] = \beta_2^\star \boldsymbol{\mu}, \tag{4.1}$$

and

$$D[\mathbf{Y}_{[n]}] = {\beta_2^\star}^2 \Lambda, \tag{4.2}$$

where $\boldsymbol{\mu} = (\mu_{[1:n]}, \cdots, \mu_{[n:n]})'$ and $\Lambda = diag(\delta_{1,1:n}, \delta_{2,2:n}, \cdots, \delta_{n,n:n})$. If the parameters $\alpha$ and $\theta$ are known, then the combination of (4.1) and (4.2) allow us to apply the generalized Gauss-Markov theorem (see David and Nagraja, [10], p. 185). Hence, the BLU estimator $\hat{\beta_2^\star}$ of $\beta_2^\star$ is given as

$$\begin{aligned} \hat{\beta_2^\star} &= (\boldsymbol{\mu}' \Lambda^{-1} \boldsymbol{\mu})^{-1} \boldsymbol{\mu}' \Lambda^{-1} \mathbf{Y}_{[n]} \\ &= \sum_{i=1}^{n} a_i Y_{[i:n]_i}, \end{aligned} \tag{4.3}$$

where $a_i = \frac{\mu_{[i:n]}/\delta_{i,i:n}}{\sum_{i=1}^{n} \mu_{[i:n]}^2/\delta_{i,i:n}}$, and the variance of $\hat{\beta_2^\star}$ is given by

$$\begin{aligned} Var[\hat{\beta_2^\star}] &= (\boldsymbol{\mu}' \Lambda^{-1} \boldsymbol{\mu})^{-1} {\beta_2^\star}^2 \\ &= \left( \sum_{i=1}^{n} \mu_{[i:n]}^2 / \delta_{i,i:n} \right)^{-1} {\beta_2^\star}^2. \end{aligned} \tag{4.4}$$

We have calculated the coefficients $a_i$ of $Y_{[i:n]_i}, i = 1, \cdots, n$ in $\hat{\beta_2^\star}$ and $Var[\hat{\beta_2^\star}]/{\beta_2^\star}^2$ for $n = 1, \cdots, 5$, and different values of the parameters $\alpha$ and $\theta$ are presented in Tables 7 and 8.

A modified RSS approach is presented by Stokes [43], wherein only the largest or smallest judgment ranked unit is selected for quantification. Let $n$ random samples each of size $n$ be drawn from the BGW distribution. From each of the $n$ samples, choose the unit for which the measurement on the auxiliary variable $X$ is the smallest (largest) and measure the $Y$ variable associated with it. Then, we call the collection of observations $Y_{[1:n]_1}, Y_{[1:n]_2}, \cdots, Y_{[1:n]_n}$ $(Y_{[n:n]_1}, Y_{[n:n]_2}, \cdots, Y_{[n:n]_n})$ as the lower RSS (LRSS) (upper RSS (URSS)).

Based on LRSS and URSS, the BLU estimators $\tilde{\beta}_{2,LRSS}^\star$ and $\tilde{\beta}_{2,URSS}^\star$ of $\beta_2^\star$ are

$$\tilde{\beta}_{2,LRSS}^\star = \frac{1}{n\mu_{[1:n]}} \sum_{i=1}^{n} Y_{[1:n]_i}, \tag{4.5}$$

$$\tilde{\beta}^{\star}_{2,URSS} = \frac{1}{n\mu_{[n:n]}} \sum_{i=1}^{n} Y_{[n:n]_i}, \tag{4.6}$$

and their variances are

$$Var[\tilde{\beta}^{\star}_{2,LRSS}] = \left(n\mu^2_{[1:n]}/\delta_{1,1:n}\right)^{-1} {\beta^{\star}_2}^2, \tag{4.7}$$

$$Var[\tilde{\beta}^{\star}_{2,URSS}] = \left(n\mu^2_{[n:n]}/\delta_{n,n:n}\right)^{-1} {\beta^{\star}_2}^2. \tag{4.8}$$

The efficiencies $e_1$ of $\tilde{\beta}^{\star}_{2,LRSS}$ and $e_2$ of $\tilde{\beta}^{\star}_{2,URSS}$ relative to $\hat{\beta}^{\star}_2$ are given by

$$e_1 = \frac{Var[\hat{\beta}^{\star}_2]}{Var[\tilde{\beta}^{\star}_{2,LRSS}]}, \quad e_2 = \frac{Var[\hat{\beta}^{\star}_2]}{Var[\tilde{\beta}^{\star}_{2,URSS}]},$$

see, for example, Koshti and Kamalja [27] and Philip and Thomas [37]. We have computed the efficiencies $e_1$ and $e_2$ for $n = 2, \cdots, 5$, $\alpha = 1, 2$, and $\theta = 0.50, 0.90$, which are presented in Table 9. From Table 9, it can be observed that:

- The efficiency $e_1$ is less than one for all selected values of $\alpha, \theta$, and $n$. So, $\hat{\beta}^{\star}_2$ is relatively more efficient than $\tilde{\beta}^{\star}_{2,LRSS}$.
- The efficiency $e_1$ decreases as $\alpha$ increases, and for a fixed pair $(n, \alpha)$, $e_1$ increases as $\theta$ increases.
- The efficiency $e_2$ is greater than one for all selected values of $\alpha, \theta$, and $n$. Thus, $\tilde{\beta}^{\star}_{2,URSS}$ is relatively more efficient than $\hat{\beta}^{\star}_2$.
- The efficiency $e_2$ increases as $\alpha$ increases, and for a fixed pair $(n, \alpha)$, $e_2$ decreases as $\theta$ increases.

**Table 7.** The coefficients $a_i$ in the BLUE $\hat{\beta}^{\star}_2$ and $Var[\hat{\beta}^{\star}_2]/{\beta^{\star}_2}^2$ for $\theta = 0.50$.

| $\alpha$ | $n$ | Coefficients ($a_i$) | | | | | $Var[\hat{\beta}^{\star}_2]/{\beta^{\star}_2}^2$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1.62994 | | | | | 1.88722 |
| | 2 | 0.75389 | 0.85091 | | | | 0.90691 |
| | 3 | 0.48490 | 0.53764 | 0.57288 | | | 0.59296 |
| | 4 | 0.35637 | 0.39005 | 0.41430 | 0.43032 | | 0.43957 |
| | 5 | 0.28143 | 0.30502 | 0.32279 | 0.33551 | 0.34396 | 0.34893 |
| 2 | 1 | 1.59732 | | | | | 0.56534 |
| | 2 | 0.64431 | 0.90259 | | | | 0.25654 |
| | 3 | 0.38632 | 0.52409 | 0.61801 | | | 0.16359 |
| | 4 | 0.27147 | 0.35952 | 0.42265 | 0.46582 | | 0.11956 |
| | 5 | 0.20763 | 0.26986 | 0.31575 | 0.34884 | 0.37235 | 0.09402 |

**Table 8.** The coefficients $a_i$ in the BLUE $\hat{\beta_2^\star}$ and $Var[\hat{\beta_2^\star}]/\beta_2^{\star 2}$ for $\theta = 0.90$.

| $\alpha$ | $n$ | Coefficients ($a_i$) | | | | | $Var[\hat{\beta_2^\star}]/\beta_2^{\star 2}$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1.07136 | | | | | 1.09843 |
| | 2 | 0.52613 | 0.54453 | | | | 0.54834 |
| | 3 | 0.34606 | 0.36012 | 0.36408 | | | 0.36523 |
| | 4 | 0.25675 | 0.26814 | 0.27185 | 0.27327 | | 0.27375 |
| | 5 | 0.20354 | 0.21310 | 0.21651 | 0.21800 | 0.21866 | 0.21891 |
| 2 | 1 | 1.18181 | | | | | 0.30365 |
| | 2 | 0.56671 | 0.61384 | | | | 0.15123 |
| | 3 | 0.36625 | 0.40182 | 0.41178 | | | 0.10059 |
| | 4 | 0.26791 | 0.29656 | 0.30558 | 0.30934 | | 0.07533 |
| | 5 | 0.20987 | 0.23386 | 0.24202 | 0.24571 | 0.24758 | 0.06019 |

**Table 9.** Efficiencies of the estimators $\tilde{\beta}^\star_{2,LRSS}$ and $\tilde{\beta}^\star_{2,URSS}$ relative to $\hat{\beta_2^\star}$.

| | | $e_1$ | | $e_2$ | |
|---|---|---|---|---|---|
| $n$ | $\theta$ | $\alpha = 1$ | $\alpha = 2$ | $\alpha = 1$ | $\alpha = 2$ |
| 2 | 0.50 | 0.68524 | 0.64926 | 1.31476 | 1.35074 |
| 3 | 0.50 | 0.52948 | 0.49563 | 1.44557 | 1.47260 |
| 4 | 0.50 | 0.43425 | 0.40617 | 1.51216 | 1.52362 |
| 5 | 0.50 | 0.36923 | 0.34637 | 1.55104 | 1.54852 |
| 2 | 0.90 | 0.94500 | 0.93347 | 1.05501 | 1.06654 |
| 3 | 0.90 | 0.90719 | 0.88724 | 1.07020 | 1.08120 |
| 4 | 0.90 | 0.87843 | 0.85183 | 1.07685 | 1.08669 |
| 5 | 0.90 | 0.85528 | 0.82321 | 1.08048 | 1.08932 |

## 5. Real data application

For illustration purposes, we have considered the American Football League dataset given in Jamalizadeh and Kundu [19]. The bivariate dataset represents the game time to the first points scored by kicking the ball between goal posts ($X$) and the 'game time' by moving the ball into the end zone ($Y$). Pathak et al. [34] demonstrated that the BGW distribution fits this data better than other real-life time models. Here, we generate random samples of size five using forty-two pairs of observations. The samples under RSS schemes are displayed in Table 10.

**Table 10.** Samples of size $n = 5$ under various RSS schemes.

| Scheme | Sample values for Y-variable | | | | |
|---|---|---|---|---|---|
| RSS | 0.75 | 7.78 | 38.07 | 49.75 | 20.57 |
| LRSS | 0.75 | 2.9 | 2.9 | 6.42 | 3.98 |
| URSS | 49.88 | 15.53 | 49.75 | 42.35 | 20.57 |

The estimator of $\beta_2^\star$ under various RSS schemes is a function of $\alpha$ and $\theta$, which are unknown in this case. Thus, the method of moment estimation can be taken (see Kamalja and Koshti [21], for example). To obtain the moment estimators of $\alpha$ and $\theta$, we use the moment equations based on the moments of $Y$-observations and the moment equation based on the correlation between $(X, Y)$. These give $\hat{\alpha} = 3.39821$ and $\hat{\theta} = 0.24259$. Table 11 shows the estimates of $\beta_2^\star$ under the RSS, LRSS, and URSS schemes. The results show that $\tilde{\beta}^\star_{2,URSS}$ has the smallest variance. This is consistent with the findings of the efficiency performance study in Section 4.

**Table 11.** The estimates of $\beta_2^\star$ under various RSS schemes.

| Scheme | Estimator of $\beta_2^\star$ | Estimate of $\beta_2^\star$ | $Variance/\beta_2^{\star 2}$ |
|---|---|---|---|
| RSS | $\hat{\beta_2^\star}$ | 48.4195 | 0.07167 |
| LRSS | $\tilde{\beta}^\star_{2,LRSS}$ | 33.1903 | 0.89618 |
| URSS | $\tilde{\beta}^\star_{2,URSS}$ | 45.0528 | 0.03010 |

## 6. Conclusions

In this paper, we have considered the CGOS from the BGW distribution. We have derived the PDFs and moments of CGOS from the BGW distribution. Similar results for order statistics and PT-II censored order statistics are presented as special instances. Finally, we have obtained the BLU estimator of the parameter associated with the study variable based on Stoke's RSS. Moreover, a real dataset is used for illustration purposes. The results for higher joint moments can be used to create skewness or kurtosis matrices (Kollo, [25]), which have important applications in both independent component analysis and invariant coordinate selection. This could be an interesting topic for future research. It will also be interesting to discuss the problem of predicting intervals for future order statistics and record values using concomitants of order statistics and record values arising from BGW distribution; see, for example, Muraleedharan and Chacko [33]. In addition, some information measures, such as the Shannon entropy and extropy, for CGOS can also be investigated.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declares that no conflict of interest exist.

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