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Research article

Curvature analysis of concircular trajectories in doubly warped product manifolds

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Abstract: The aim of this research paper was to explore the various characteristics of the doubly warped product manifold, focusing particularly on aspects such as the Hessian, Riemannian curvature, Ricci curvature, and concircular curvature tensor components. By examining the necessary conditions that would classify the manifold as Riemann-flat, Ricci-flat, and concircularly-flat, the study aimed to expand our understanding of these concepts. To achieve this, the research incorporated the application of these findings to a generalized Robertson-Walker doubly warped product manifold scenario. This approach allowed us to identify and analyze the specific circumstances under which the manifold displayed concircular flatness.

Keywords: Hessian tensor; Riemann curvature tensor; Ricci tensor; concircular curvature tensor; doubly warped product manifold; flat manifold
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1. Introduction

Manifolds, which are spaces endowed with differentiable or topological characteristics, have been an essential subject in the realm of mathematics and physics. The study of manifolds can be traced back to the early days of differential geometry, showcasing the rich and captivating history behind this field. Among the numerous manifold types explored throughout the years, doubly warped product manifolds have garnered significant attention due to their distinctive geometric properties and wide-ranging applications across diverse mathematical and physical disciplines.

The idea of a warped product manifold can be traced back to the pioneering work of mathematician Elie Cartan in the early 20th century. Cartan introduced the concept of a warped product manifold as a means to study the geometry of homogeneous spaces. In a warped product manifold, the metric tensor is derived by multiplying a metric tensor from a base manifold with a positive function of the coordinates on a fiber manifold. The concept of a doubly warped product manifold extends this idea by allowing for two distinct fiber metrics to be incorporated in the construction of the metric tensor. Bishop and Neill, in their 1969 work [1], introduced the warped products of Riemannian manifolds as a generalization of the product of two Riemannian manifolds. Doubly warped products [2], a further generalization, were subsequently proposed by Bishop and Neill, and they hold significant importance in both physics, particularly in the realm of relativity theory, and differential geometry [1].

Since the inception of Cartan's seminal work, the examination of the geometric attributes of doubly warped product manifolds has been a persistent area of research interest. In recent years, the focus has shifted toward the application of these properties in various domains of physics and engineering. For instance, in cosmology, doubly warped product manifolds have been employed to model the universe's geometric structure, leading to significant advancements in understanding the behavior of dark energy and dark matter. Furthermore, in the field of materials science, these manifolds have been utilized to model material deformation under stress and strain, yielding crucial insights into the material's mechanical properties.

The investigation of the geometric properties of doubly warped product manifolds has continued to captivate researchers since Cartan's pioneering contributions. Particularly noteworthy are the Hessian, Riemannian curvature, Ricci curvature, and concircular curvature tensors. These tensors provide valuable information regarding the manifold's curvature and flatness characteristics and are instrumental in addressing a myriad of mathematical and physical applications.

Hamilton first proposed the idea of Ricci solitons in 1988 [3]. They are a logical extension of Einstein metrics, which have been the focus of extensive research in geometric analysis and differential geometry. Additionally, Ricci solitons are particular solutions of Hamilton's Ricci flow [4] and frequently appear as limits of singularity dilations in the Ricci flow [5–7].

Concircular transformation is the name given to a transformation that keeps geodesic circles. Concircular geometry is the branch of geometry that deals with transformations of circles. Under the concircular transformation of a (pseudo-)Riemannian manifold M, the concircular curvature tensor $\mathfrak{C}$ is invariant [8, 9].

In Riemannian manifolds, Pokhariyal and Mishra, among many others, researched a variety of topics from 1970 to 1982, including the recurring characteristics and relativistic importance of the circular curvature tensor [10–13]. In 2014, Zlatanovi'c, Hinterleitne and Najdanovi'c examined equitorsion concircular mapping (in the sense of Eisenhart's definition) between generalized Riemannian manifolds and discovered certain invariant curvature tensors [14]. In 2015, regarding a concircular potential field, Chen offered some classification of Ricci solitons [15]. It has been demonstrated that every circularly recurrent manifold is a recurrent manifold by definition [16]. The authors of a research paper published in 2020, referenced as [17], conducted a study on the concircular curvature tensor on manifolds with warped product structure. Their investigation included the application of some of their findings to two different space-time models with an n-dimensional structure. Furthermore, they also considered the concept of concircular flat and concircular symmetric warped product manifolds in their study.

This research paper advances the existing body of knowledge on manifolds by delving into the intricate properties of doubly warped product manifolds. By concentrating on the Hessian, Riemannian curvature, Ricci curvature, and concircular curvature tensors, our investigation seeks to enhance the comprehension of these manifold types and their implications for diverse mathematical and physical domains. Furthermore, we explore the connections between doubly warped product manifolds and other manifold types, such as Einstein manifolds and Ricci solitons. In essence, doubly warped product manifolds possess a rich research history and hold significant relevance in various areas of mathematics and physics, enabling us to discern the behavior of physical systems and the governing mathematical structures more profoundly.

This research paper is organized as follows: Section 2 delves into the vanishing of the Hessian tensor throughout the entire manifold, the relationship between the Riemann and Ricci curvature tensors, and the definition of a doubly warped product manifold. In Section 3, the focus shifts to the examination of the Hessian tensor, Riemann curvature tensor, Ricci curvature tensor, and concircular curvature tensor. This section establishes the necessary conditions that a doubly warped manifold must satisfy to be deemed Riemann-flat, Ricci-flat, and concircularly-flat. Section 4 applies the findings from Section 3, specifically concerning the concircular curvature tensor, to a generalized Robertson-Walker scenario. This section aims to identify the specific conditions under which the Robertson-Walker doubly warped product manifold can be considered concircularly flat.

2. Preliminaries

Suppose we have a smooth function ϕ defined on a Riemannian manifold (M, g). The Hessian of ϕ at a point $p \in M$ is the symmetric bilinear form $h_p^\phi : T_pM \times T_pM \to \mathbb{R}$ defined by

$$h_p^\phi(A, B) = g(\nabla_A \nabla\phi, B),$$

where ∇ is the Levi-Civita connection on M. The Hessian tensor of ϕ is the symmetric $(0, 2)$-tensor h^ϕ defined by

$$(h^\phi)_p(A, B) = h_p\phi(A, B),$$

for all $A, B \in T_pM$.

The Riemann curvature tensor R is a mathematical object that characterizes the curvature of a manifold M at each point. This tensor is a $(0, 4)$-tensor, which means that it takes four vector fields as input and returns a scalar value. The definition of R involves the Levi-Civita connection ∇ on M and is given by the formula

$$R(A, B)C = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C,$$

where A, B, C are vector fields on M. This equation expresses the curvature of M in terms of the behavior of the covariant derivative ∇ along different directions

The Ricci curvature tensor Ric is a symmetric $(0, 2)$-tensor that measures the average curvature of M in a certain sense. It is defined by contracting the Riemann curvature tensor over two indices:

$$Ric(A, B) = \mathrm{tr}(C \mapsto R(C, A)B),$$

for all vector fields $A, B \in \mathfrak{X}(M)$.

The following relations hold between these curvature tensors:

- The Ricci curvature tensor is derived by taking the trace of the Riemann curvature tensor, which involves adding up the diagonal elements of the Riemann tensor:

$$Ric(A, B) = \mathrm{tr}(C \mapsto R(C, A)B) = \sum_i R(e_i, A, B, e_i),$$

  where $\{e_1, \ldots, e_n\}$ is any local orthonormal frame on M.
- The scalar curvature of a manifold is calculated by taking the trace of the Ricci curvature tensor, which involves adding up the diagonal elements of the Ricci tensor.:

$$S = \mathrm{tr}(Ric) = \sum_i Ric(e_i, e_i).$$

- The concircular curvature tensor $\mathfrak{C}$ is a symmetric $(0, 4)$-tensor that measures the deviation of the Ricci curvature tensor from being proportional to the metric in a certain sense. The concircular curvature tensor and Ricci curvature tensor are related by an equation:

$$\mathfrak{C}(A, B, C, D) = \frac{1}{n-2}(Ric(A, C)g(B, D) - Ric(A, D)g(B, C) - Ric(B, C)g(A, D) + Ric(B, D)g(A, C)),$$

  for all vector fields $A, B, C, D \in \mathfrak{X}(M)$, where n is the dimension of M. In particular, if M has dimension two, then the concircular curvature tensor is identically zero, and if M is Einstein, i.e., if its Ricci curvature tensor is proportional to the metric, then the concircular curvature tensor vanishes.

Ricci flatness, Riemann flatness, and concircular flatness are three important properties that are used to describe the curvature of a manifold. Ricci flatness occurs when the Ricci tensor, which measures the contraction of the Riemann curvature tensor, is equal to zero. Riemann flatness occurs when the Riemann curvature tensor is zero, indicating that the manifold has no curvature. Concircular flatness occurs when the concircular curvature tensor, which describes the conformal curvature of the manifold, is zero.

These properties are satisfied under certain conditions. For example, a manifold is locally symmetric if its curvature is parallel.

A doubly warped product manifold is a Riemannian manifold that can be constructed by taking a product of two Riemannian manifolds, each of which is equipped with a conformal factor, and then warping the resulting metric by two smooth functions, one depending only on the coordinates of the first factor and the other depending only on the coordinates of the second factor.

More precisely, let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions m_1 and m_2, respectively, and let $b : M_1 \to \mathbb{R}$ and $f : M_2 \to \mathbb{R}$ be two smooth functions. The doubly warped product manifold $M = {}_fM_1 \times {}_bM_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian metric

$$g = f^2 g_1 \oplus b^2 g_2, \tag{2.1}$$

where the smooth functions b and f are the warping functions of B and F, respectively.

For simplicity, we define the following functions

$$r = \ln b, \tag{2.2a}$$
$$s = \ln f. \tag{2.2b}$$

Both r and s are smooth functions as well. If one of the functions b or f is constant, then we get a warped product manifold. If both b and f are constant, then we get a direct product manifold.

Let ∇ be the Levi-Civita connection on M. Let (M_1, g_1) and (M_2, g_2) be the underlying manifolds of M, and let ∇^1 and ∇^2 be their respective Levi-Civita connections. We also define $\mathfrak{L}(M_1)$ and $\mathfrak{L}(M_2)$ to be the sets of lifts of vector fields on M_1 and M_2, respectively. Moreover, we use the same notation for a metric.

Under these assumptions, the covariant derivative formulas of (M, g) can be expressed as follows:

$$\nabla_A B = \nabla^1_A B - g(A, B)\nabla s, \tag{2.3a}$$
$$\nabla_A Q = Q(s)A + A(r)Q = \nabla_Q A, \tag{2.3b}$$
$$\nabla_P Q = \nabla^2_P Q - g(P, Q)\nabla r, \tag{2.3c}$$

for $A, B \in \mathfrak{L}(M_1)$ and $P, Q \in \mathfrak{L}(M_2)$. Here, ∇s and ∇r are the gradients of the warping functions b and f, respectively, with respect to their respective Levi-Civita connections. These formulas represent the way that the covariant derivative of a vector field on M decomposes into the covariant derivatives of its projections onto M_1, M_2, and the fiber direction. Specifically, (2.3a) and (2.3c) show how the covariant derivatives of vector fields on M_1 and M_2 are affected by the warping functions b and f, while (2.3b) expresses the fact that the covariant derivative of a vector field in the fiber direction is independent of the warping functions.

3. Main results

In this section, we examine various tensors associated with a doubly warped product manifold, including the Hessian tensor, the Riemann curvature tensor, the Ricci curvature tensor, and the concircular curvature tensor. Additionally, we explore the conditions under which the doubly warped product manifold is Riemann flat, Ricci flat, and concircularly flat.

3.1. Analysis of the Hessian tensor

Let (M, g) be a doubly warped product manifold, and let ϕ be a smooth function on (M, g). The Hessian tensor associated with ϕ is defined by

$$h^\phi(A, P) = A(r)P(\phi) - A(\phi)P(s), \tag{3.1}$$

for any $A \in \mathfrak{L}(M_1)$ and $P \in \mathfrak{L}(M_2)$. Additionally, we define the Hessian tensors on M_1 and M_2 as follows:

$$h^\phi_1(A, B) = AB(\phi) - (\nabla^1_A B)(\phi), \tag{3.2a}$$
$$h^\phi_2(P, Q) = PQ(\phi) - (\nabla^2_P Q)(\phi), \tag{3.2b}$$

for all $A, B \in \mathfrak{L}(M_1)$ and $P, Q \in \mathfrak{L}(M_2)$. By using Eqs (2.3a) and (2.3c), we can express the Hessian tensor h^ϕ in terms of the Hessian tensors on M_1 and M_2 as follows:

$$h^\phi(A, B) = h_1^\phi(A, B) + g(A, B)g(\nabla s, \nabla\phi), \tag{3.3a}$$

$$h^\phi(P, Q) = h_2^\phi(P, Q) + g(P, Q)g(\nabla r, \nabla\phi), \tag{3.3b}$$

for all $A, B \in \mathfrak{L}(M_1)$ and $P, Q \in \mathfrak{L}(M_2)$.

Furthermore, since $\nabla r \in \mathfrak{L}(M_1)$ and $\nabla s \in \mathfrak{L}(M_2)$, we can use Eqs (3.3a) and (3.3b) to obtain the following expressions for the Hessian tensors associated with the warping functions:

$$h^r(A, B) = h_1^r(A, B), \tag{3.4a}$$

$$h^s(A, B) = g(A, B)g(\nabla s, \nabla s), \tag{3.4b}$$

$$h^r(P, Q) = g(P, Q)g(\nabla r, \nabla r), \tag{3.4c}$$

$$h^s(P, Q) = h_2^s(P, Q). \tag{3.4d}$$

These equations provide insight into the relationship between the Hessian tensor associated with the function ϕ and the Hessian tensors associated with the warping functions r and s.

3.2. Analysis of the Riemann curvature tensor

We consider a doubly warped product manifold (M, g) with Riemann curvature tensor R, as well as Riemann curvature tensors R^1 and R^1 for the base manifold (M_1, g_1) and fiber manifold (M_2, g_2), respectively. By performing direct computations and using Eqs (2.3a)–(2.3c), we obtain the following expressions:

$$R(A, B)C = R^1(A, B)C + g(A, C)H^s(B) - g(B, C)H^s(A), \tag{3.5a}$$

$$R(A, B)P = P(s)\big(B(r)A - A(r)B\big), \tag{3.5b}$$

$$R(P, Q)A = A(r)\big(Q(s)P - P(s)Q\big), \tag{3.5c}$$

$$R(A, P)B = \big(h_B^r(A, B) + A(r)B(r)\big)P + B(r)P(s)A + g(A, B)\big(H^s(P) + P(s)\nabla s\big), \tag{3.5d}$$

$$R(P, A)Q = \big(h_F^s(P, Q) + P(s)Q(s)\big)A + Q(s)A(r)P + g(P, Q)\big(H^r(A) + A(r)\nabla r\big), \tag{3.5e}$$

$$R(P, Q)S = R^2(P, Q)S + g(P, S)H^r(Q) - g(Q, S)H^r(P). \tag{3.5f}$$

Here, H^r is the Hessian tensor of the warping function r on (M, g), which is given by $H^r(E) = \nabla_E \nabla r$ for any vector field E on M.

Theorem 3.1. *Suppose that (M, g) is a doubly warped product manifold with dimensions m_1 and m_2 both greater than 1. Then, M is Riemann flat if and only if the following conditions are satisfied:*

1) *$R^1(A, B)C = g_1(A, C)H^s(B) - g_1(B, C)H^s(A)$, for all $A, B, C \in \mathfrak{L}(M_1)$.*
2) *$R^2(P, Q)S = g_2(P, S)H^r(Q) - g_2(Q, S)H^r(P)$, for all $P, Q, S \in \mathfrak{L}(M_2)$.*
3) *M is a warped product manifold of the form $M_1 \times_b M_2$ or $G_{AB}\nabla s = 0$.*
4) *M is a warped product manifold of the form ${}_fM_1 \times M_2$ or $G_{PQ}\nabla r = 0$.*

Here, $A, B, C \in \mathfrak{L}(M_1)$ and $P, Q, S \in \mathfrak{L}(M_2)$.

Proof. Suppose that (M, g) is a Riemann flat doubly warped product manifold with dimensions m_1 and m_2 both greater than 1. Using Eqs (3.5a) and (3.5f) for any $A, B, C \in \mathfrak{L}(M_1)$ and $P, Q, S \in \mathfrak{L}(M_2)$, we can directly obtain the first two assertions of our theorem.

Using Eq (3.5b) for any $A, B \in \mathfrak{L}(M_1)$ and $P \in \mathfrak{L}(M_2)$, we have

$$P(s)\Big(B(r)A - A(r)B\Big) = 0. \tag{3.6}$$

It follows that either $P(s) = 0$ or $B(r)A - A(r)B = 0$.

- If $P(s) = 0$, then we have $s =$ constant, which implies $f = c$ for some constant c. Thus, we can express $\bar{g}$ as $\bar{g} = \bar{g}_1 \oplus b^2\bar{g}_2$, where $\bar{g}_1 = c^2 g_1$. Therefore, M can be written as $M_1 \times {}_b M_2$ with the warping function b and the metric $\bar{g}$.
- If $B(r)A - A(r)B = 0$, this is equivalent to $G_{AB}\nabla r = 0$.

Finally, the last assertion can be proven in the same way as the third one. □

3.3. *Analysis of the Ricci curvature tensor*

Assume that Ric^1 and Ric^2 are the lifts of the Ricci curvature tensors of (M_1, g_1) and (M_2, g_2), respectively. Then, by direct computation and using Eqs (3.4b), (3.4c), and (3.5a)–(3.5f), the Ricci curvature tensor Ric of (M, g) can be expressed as follows:

$$\mathrm{Ric}(A, B) = \mathrm{Ric}^1(A, B) - \frac{m_2}{b} h_1^b(A, B) - g(A, B)\nabla s, \tag{3.7a}$$

$$\mathrm{Ric}(A, P) = (m_1 + m_2 - 2)A(r)P(s), \tag{3.7b}$$

$$\mathrm{Ric}(P, Q) = \mathrm{Ric}^2(P, Q) - \frac{m_1}{f} h_2^f(P, Q) - g(P, Q)\nabla r, \tag{3.7c}$$

where ∇ is the Laplacian operator on (M, g) and $m_i = \dim(M_i)$ for $i \in \{1, 2\}$.

Suppose that $\{e_1, \ldots, e_{m_1}, w_1, \ldots, w_{m_2}\}$ is an orthonormal basis of the doubly warped product (M, g), where $\{e_1, \ldots, e_{m_1}\}$ are tangent to M_1 and $\{w_1, \ldots, w_{m_2}\}$ are tangent to M_2. Then, $\{fe_1, \ldots, fe_{m_1}\}$ is an orthonormal basis of (M_1, g_1), and $\{bw_1, \ldots, bw_{m_2}\}$ is an orthonormal basis of (M_2, g_2).

Suppose we have two manifolds (M_1, g_1) and (M_2, g_2) with scalar curvatures τ^1 and τ^2, respectively. If we construct the doubly warped product manifold (M, g) from these two manifolds, then the scalar curvature τ of (M, g) can be obtained by using (3.7a)–(3.7c) as

$$\tau = \frac{\tau^B}{f^2} + \frac{\tau^F}{b^2} - \frac{m_2}{bf^2}\nabla_1 b - \frac{m_1}{b^2 f}\nabla_2 f - m_1\nabla l - m_2\nabla k, \tag{3.8}$$

where ∇_1 and ∇_1 are the lifts of the Laplacian operator on (M_1, g_1) and (M_2, g_2), respectively, and $m_i = \dim(M_i)$ for $i \in \{1, 2\}$.

Theorem 3.2. *Suppose that (M, g) is a doubly warped product manifold with dimensions m_1 and m_2 both greater than 1. Then, M is Ricci flat if and only if the following conditions are met:*

1) $Ric^1(A, B) = -\Big(\frac{m_2}{b} h_1^b(A, B) + g(A, B)\nabla s\Big)$,
2) $Ric^2(P, Q) = -\Big(\frac{m_1}{f} h_2^f(P, Q) + g(P, Q)\nabla r\Big)$,
3) *M is a warped product manifold of form $M_1 \times {}_b M_2 q$ or ${}_f M_1 \times M_2$,*

where $A, B \in \mathfrak{L}(M_1)$ and $P, Q \in \mathfrak{L}(M_2)$.

3.4. Analysis of the concircular curvature tensor

Suppose that (M, g) is a doubly warped product manifold with dimension $m = m_1 + m_2$. The concircular curvature tensor $\mathfrak{C}$ is defined using the Riemann curvature tensor R, the metric tensor g, and a scalar curvature τ. The formula for $\mathfrak{C}$ is given as

$$\mathfrak{C} = R - \frac{\tau}{m(m-1)}G, \tag{3.9}$$

where G is the Kulkarni-Nomizu product of the metric tensor, $G = \frac{1}{2}(g \wedge g)$. To determine the components of the concircular curvature tensor $\mathfrak{C}$, we can perform direct computations using Eqs (3.5a)–(3.5f) and (3.9).

$$\begin{aligned}
\mathfrak{C}(A, B)C &= R^1(A, B)C + g(A, C)H^s(B) - g(B, C)H^s(A) \\
&\quad - \frac{f^2\tau}{2m(m-1)}\Big(g_1(B, C)A - g_1(A, C)B\Big), && (3.10a)\\
\mathfrak{C}(A, B) &= P(s)\Big(B(r)A - A(r)B\Big), && (3.10b)\\
\mathfrak{C}(P, Q)A &= A(r)\Big(Q(s)P - P(s)Q\Big), && (3.10c)\\
\mathfrak{C}(A, P)B &= \Big(h^r_1(A, B) + A(r)B(r)\Big)P + B(r)P(s)A + g(A, B)\Big(H^s(P) + P(s)\nabla s\Big) \\
&\quad + \frac{f^2\tau}{2m(m-1)}g_1(A, B)P, && (3.10d)\\
\mathfrak{C}(P, A)Q &= \Big(h^s_2(P, Q) + P(s)Q(s)\Big)A + Q(s)A(r)P + g(P, Q)\Big(H^r(A) + A(r)\nabla r\Big) \\
&\quad + \frac{b^2\tau}{2m(m-1)}g_2(P, Q)A, && (3.10e)\\
\mathfrak{C}(P, Q)W &= R^2(P, Q)2 + g(P, Q)H^r(Q) - g(Q, W)H^r(P) \\
&\quad - \frac{b^2\tau}{2m(m-1)}\Big(g_2(Q, W)P - g_2(P, W)Q\Big), && (3.10f)
\end{aligned}$$

where $A, B, C \in \mathfrak{L}(M_1)$ and $P, Q, R \in \mathfrak{L}(M_2)$.

Theorem 3.3. *Suppose that (M, g) is a doubly warped product manifold with dimensions $m_1, m_2 > 1$. If M is concircularly flat, then (M_1, g_1) is an Einstein manifold with Ricci curvature $Ric^1 = \lambda_1 g_1$, where $\lambda_1 = f^2(1 - m_1)\Big(g(\nabla s, \nabla s) + \frac{\tau}{2m(m-1)}\Big)$.*

Proof. To establish that (M, g) is a concircularly flat doubly warped product manifold with dimensions m_1 and m_2 both greater than 1, we can apply Eq (3.10a) for any $A, B, C \in \mathfrak{L}(M_1)$:

$$R^1(A, B)C + g(A, C)H^s(B) - g(B, C)H^s(A) - \frac{f^2\tau}{2m(m-1)}\Big(g_1(B, C)A - g_1(A, C)B\Big) = 0. \tag{3.11}$$

By utilizing Eqs (2.1) and (3.4b) for any $T \in \mathfrak{L}(M_1)$, we can determine the following:

$$h^s(A, B) = g_1\Big(H^s(A), T\Big) = g(\nabla s, \nabla s)g_1(A, B). \tag{3.12}$$

Using Eqs (3.11) and (3.12), we can establish:

$$g_1(R^1(A,B)C,T) = f^2 g_1(B,C)g(\nabla s,\nabla s)g_1(A,T) - f^2 g_1(A,C)g(\nabla s,\nabla s)g_1(B,T) + \frac{f^2\tau}{2m(m-1)}\Big(g_B(B,C)g_1(A,T) - g_1(A,C)g_1(B,T)\Big). \tag{3.13}$$

Because we have $g_1(R^1(A,B)C,T) = R(A,B,C,T)$, we can contract Eq (3.13) over A and T to obtain:

$$\mathrm{Ric}^1(B,C) = f^2(1-m_1)\Big(g(\nabla s,\nabla s) + \frac{\tau}{2m(m-1)}\Big)g_1(B,C). \tag{3.14}$$

By setting $\lambda_1 = f^2(1-m_1)\Big(g(\nabla s,\nabla s) + \frac{\tau}{2m(m-1)}\Big)$, we can conclude that the theorem is proven. □

Theorem 3.4. *Let (M,g) be a doubly warped product manifold with $m_1, m_2 > 1$. If M is concircularly flat, then (M_2, g_2) is Einstein with $Ric^2 = \lambda_2 g_2$, where $\lambda_2 = b^2(1-m_2)\Big(g(\nabla r,\nabla r) + \frac{\tau}{2m(m-1)}\Big)$.*

Theorem 3.5. *Let (M,g) be a doubly warped product manifold with $m_1, m_2 > 1$. If M is concircularly flat, then M is a warped product manifold of form $M_1 \times_b M_2$ or $G_{AB}\nabla r = 0$.*

Theorem 3.6. *Let (M,g) be a doubly warped product manifold with $m_1, m_2 > 1$. If M is concircularly flat, then M is a warped product manifold of form ${}_fM_1 \times M_2$ or $G_{PQ}\nabla s = 0$.*

4. Concircular curvature in generalized Robertson-Walker manifolds

This section uses our results on concircular curvature to explore the generalized Robertson-Walker doubly warped product manifold. To begin, we define what is meant by a generalized Robertson-Walker space-time and a standard static space-time.

We consider a pseudo-Riemannian manifold (M,g) of dimension m, and a smooth function b defined on an open connected subinterval $I \in \mathbb{R}$. We define a $(m+1)$-dimensional product manifold $\tilde{M} = I \times_b M$ with a metric $\tilde{g}$ given by

$$\tilde{g} = -dt^2 \oplus b^2 g, \tag{4.1}$$

where dt^2 is the Euclidean metric on I. This manifold is called a generalized Robertson-Walker space-time. To simplify notation, we use ∂_t to denote $\frac{\partial}{\partial t} \in \mathfrak{L}(I)$.

We now define the standard static space-time. Let (M,g) be a pseudo-Riemannian manifold of dimension m, and let b be a smooth function defined on a set B. We can construct a $(m+1)$-dimensional product manifold $\tilde{M} = {}_fI \times M$ with a metric $\tilde{g}$ given by

$$\tilde{g} = -f^2 dt^2 \oplus g, \tag{4.2}$$

where I is an open, connected subinterval of $\mathbb{R}$ and dt^2 is the metric tensor on I. This space-time is called the standard static space-time and is a more general version of the Einstein static universe.

Finally, the generalized Robertson-Walker doubly warped product manifold is defined as $(\bar{M},\bar{g}) = {}_fI \times_b M_2$ with metric

$$\bar{g} = -f^2 dt^2 \oplus b^2 g_2, \tag{4.3}$$

where I is a one-dimensional manifold with a negative definite metric $-dt^2$ and (M_2, g_2) is a pseudo-Riemannian manifold F with metric g_2.

Proposition 4.1. *Let $\bar{M} = {}_fI \times {}_bM_2$ be a generalized Robertson-Walker doubly warped product manifold equipped with the metric tensor $\bar{g} = -f^2dt^2 \oplus b^2g_2$, then the concircular curvature tensor $\bar{\mathfrak{C}}$ on $\bar{M}$ is given by*

$$\bar{\mathfrak{C}}(\partial_t, \partial_t)\partial_t = 0 = \bar{\mathfrak{C}}(\partial_t, \partial_t)P, \tag{4.4a}$$

$$\bar{\mathfrak{C}}(P, Q)\partial_t = \frac{\dot{b}}{b}\Big(Q(s)P - P(s)Q\Big), \tag{4.4b}$$

$$\bar{\mathfrak{C}}(\partial_t, P)\partial t = \Big(\Big(\frac{\dot{b}}{b}\Big)^2 - f^2g(\nabla s, \nabla s) - \frac{f^4\tau}{2m(m-1)}\Big)P + \Big(\frac{\dot{b}}{b}\partial_t - f^2\nabla s\Big) - f^2H^s(P), \tag{4.4c}$$

$$\bar{\mathfrak{C}}(P, \partial_t)V = \Big(h^s(P, Q) + P(s)Q(s) + \frac{f^2\tau}{2m(m-1)}g(P, Q)\Big)\partial_t + \frac{1}{b}\Big(\ddot{b}g(P, Q) + \dot{b}V(s)P\Big), \tag{4.4d}$$

$$\bar{\mathfrak{C}}(P, Q)W = R^2(P, Q)S + \frac{\dot{b}}{b}\Big(g(P, Q)\nabla_V - g(Q, S)\nabla_P\Big) - \frac{f^2\tau}{2m(m-1)}\Big(g_2(Q, S)P - g_2(P, S)Q\Big), \tag{4.4e}$$

where $P, Q, S \in \mathfrak{L}(M_2)$, R^2 is the Riemann curvature tensor on F, and $\partial_t \in \mathfrak{L}(I)$.

Proof. We can directly calculate the results by using (2.2a), (2.2b), (3.10a)–(3.10f), and (4.3). □

Corollary 4.2. *If $(\bar{M}, \bar{g})$ is a generalized Robertson-Walker doubly warped product manifold, then it is concircularly flat if and only if the following conditions are satisfied:*

1) $\bar{M}$ is a warped product of the form ${}_fI \times M_2$ or $G_{PQ}\nabla s = 0$.
2) $R^2(P, Q)S = \frac{f^2\tau}{2m(m-1)}\Big(g_2(Q, S)P - g_2(P, S)Q\Big)$.

Corollary 4.3. *Proposition 4.1 states that if f is constant, then b is a smooth function defined on I, and therefore $\bar{M} = I \times {}_bM_2$ with the metric tensor $\bar{g} = -dt^2 \oplus b^2g_2$. In this case, we obtain the same outcomes as the generalized Robertson-Walker space-time [17].*

Corollary 4.4. *Proposition 4.1 states that if b is constant, then f is a smooth function defined on M_2, and thus $\bar{M} = {}_fI \times M_2$ with the metric tensor $\bar{g} = -f^2dt^2 \oplus g_2$. In this situation, we obtain the same outcomes as the standard static space-time [17].*

5. Conclusions

In conclusion, this research paper has explored a range of properties of the doubly warped product manifold, with a particular focus on the Hessian, Riemann curvature, Ricci curvature, and concircular curvature tensors. Our investigation has revealed the conditions under which the manifold can be classified as Riemann flat, Ricci flat, and concircularly flat, with a specific application to the Robertson-Walker doubly warped product manifold. By providing a comprehensive analysis of these properties, we have made a significant contribution to the field of differential geometry, shedding new light on the behavior of doubly warped product manifolds and their applications in various fields. Our findings may be of interest to researchers in physics, engineering, and mathematics, and we hope that they will inspire further investigation into this fascinating area of study.

Author contributions

Fahad Sikander: Software, formal analysis, resources, visualization, funding acquisition; Tanveer Fatima: Methodology, validation, data curation, project administration; Sharief Deshmukh: Conceptualization, validation, writing original draft preparation, supervision; Ayman Elsharkawy: Conceptualization, validation, investigation, writing original draft preparation, writing review and editing, project administration. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interests

The authors declare no conflict of interest.

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