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Research article

Bhaskar-Lakshmikantham fixed point theorem vs Ran-Reunrings one and some possible generalizations and applications in matrix equations

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Abstract: We provided a generalization of the existence and uniqueness of fixed points in partially ordered metric spaces for a monotone map. We applied the major results in the investigation of coupled fixed points for ordered pairs of two maps that met various monotone features, which included a mixed monotone property or a total monotone property. To ascertain necessary requirements for the existence and uniqueness of solutions to systems of matrix equations, the results regarding coupled fixed points for ordered pairs of maps were utilized. These results are illustrated with numerical examples. Some of the known results are a consequence of the results we obtained.

Keywords: coupled fixed points; partially ordered metric space; mixed monotone property; matrix equations
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1. Introduction

Let A be a nonempty set. If $x = Tx$, then a map $T : A \to A$ will be considered to have a fixed point $x \in A$.

The Banach fixed point theorem [1], although a century old, has an enormous number of generalizations and applications. There are different types of generalizations that may be classified in four directions: Changing the contractive type condition [2] for Ćirić and Meir-Keeler maps, [3] for Geraghty-Ćirić type maps, [4] for S-Pata-type contractions, [5] for θ contractions; altering the underlying space [6] in modular functions spaces, [2] in super metric spaces, [7, 8] in ultrametrics, b-metrics, w-distances spaces, [9] in b-metric spaces with an application in solving of a system of nonlinear Fredholm integral equations of second-kind, [4] in S-metric spaces, [10] in C^*-algebra valued fuzzy soft metric spaces, [11] in Menger *PbM*-metric spaces; modifying the notion of a fixed point [10] about coupled fixed points, [11, 12] about tripled fixed points, and [13, 14] for cyclic maps, best proximity points, and the three of the mentioned ones simultaneously.

We highlight just a few applications of Banach's fixed point theorem: In [15], the application is in the investigation of a specific type of choice behavior model for the learning process of the paradise fish; [16], where an estimation of the heat source coefficient in a fourth-order time-fractional pseudo-parabolic equation is presented in the application section; [17], new approximate analytical solutions to two nonlinear fractional equations characterizing HIV-1 infection are found and investigated; and [5], where an application to a nonlinear system of matrix equations is presented. Let us point out that the techniques from [5] are different from our approach, and the classes for systems of matrix equations investigated are different.

Let (X, d) be a metric space and $T : X \to X$ be a self map. The contraction mapping theorem and the abstract monotone iterative technique [18] are well known and are applicable to a variety of situations. It is fair to note that the first result for the existence of fixed points in partially ordered spaces was in [19], which is a more sophisticated context, but the results of [18] set the stage for numerous studies in this area. Recently [20–24], there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order, i.e., instead of assuming that the contractive type condition $d(Tx, Ty) \leq \alpha d(x, y)$ must hold for all $x, y \in X$, it is required to hold only of $x \preccurlyeq y$, where $\preccurlyeq$ is a partial order in the metric space (X, d).

A newly considered application of fixed points is in the solving of matrix equations, where the inverted matrix X^{-1} is present in the matrix equation [25]. The authors divert the matrix equation $X + A^*X^{-1}A - B^*X^{-1}B = Q$, where A and B are arbitrary square $N \times N$ matrices and Q is a positive definite $N \times N$ matrix, to a system of two matrix equations

$$\left| \begin{array}{l} X = Q - A^*X^{-1}A + B^*Y^{-1}B, \\ Y = Q - A^*Y^{-1}A + B^*X^{-1}B, \end{array} \right. \tag{1.1}$$

and apply the notion of coupled fixed points in metric spaces with a partial ordering from [26].

The system of Eq (1.1) can be considered as a map $F : \mathcal{X}(N) \times \mathcal{X}(N) \to \mathcal{X}(N)$, where $\mathcal{X}(N)$ is the space of the $N \times N$, matrices and it is defined by $F(X, Y) = Q - A^*X^{-1}A + B^*Y^{-1}B$ and we search for the coupled fixed point (X, Y) [27], i.e., $X = F(X, Y)$ and $Y = F(Y, X)$.

It is well known that when some classical assumptions on the mapping $F : X \times X \to X$ and on the underlying normed space $(X, \|\cdot\|)$ are assumed, then $X = Y$, e.g., [28]. In [28] consideration of two maps $F, G : X \times X \to X$ was initiated, leading to the possibility of solving the system of equations $X = F(X, Y)$, $Y = G(X, Y)$ and ensuring the solutions satisfy $X_0 \neq Y_0$. Whenever $G(X, Y) = F(Y, X)$, we get the well-known result about coupled fixed points [26].

In this paper, following the trend mentioned above [24, 29, 30] and the observations in [28], we extend the notion of mixed monotone maps so that we can enlarge, in a unified manner, the class of problems that can be investigated, and the known results will be a particular case of the new ones. We present adequate criteria to guarantee the uniqueness and existence of solutions for the system of matrix equations

$$\left| \begin{array}{l} X = Q - A^*X^{-1}A + B^*Y^{-1}B, \\ Y = R - C^*X^{-1}C + D^*Y^{-1}D. \end{array} \right. \tag{1.2}$$

The result from [25] will be a particular case when $R = Q$, $C = B$, and $D = A$.

We provide a generalization of the results from [18] of the existence and uniqueness of fixed points in partially ordered metric spaces for a monotone map by removing the assumption on the continuity of the map. We alter the notion of coupled fixed points so that the solution (X, Y) (1.2) does not need to satisfy $X = Y$. We apply the major results in the investigation of coupled fixed points for ordered pairs of two maps that meet various monotone features, including a mixed monotone property or a total monotone property, which generalizes the known investigations on the topic.

2. Materials and methods

2.1. Coupled fixed point

In what follows, let (X, d) be a metric space, $\mathbb{N}$ be the set of natural numbers, and $\mathbb{R}$ be the set of real numbers. Whenever we consider a normed space $(X, \|\cdot\|)$, we will assume that the metric in X is the one endowed by the norm, i.e., $d(x, y) = \|x - y\|$.

Starting with the publications [19, 27], it is assumed that a partially ordered set is a well-known notion. Just for completeness of the presentation, we will recall its definition, following the presentation from [31].

Definition 2.1. *Let X be a set and $p \subseteq X \times X$ be a relation. The relation p is said to be:*

(i) Reflexive, provided that $(x, x) \in p$ for every $x \in X$.

(ii) Antisymmetric, provided that if $(x, y) \in p$ and $(y, x) \in p$, then $x = y$.

(iii) Transitive, provided that $(x, y) \in p$ and $(y, z) \in p$, then $(x, z) \in p$.

Definition 2.2. *Let X be a set. A partial order on X is a relation $p \subseteq X \times X$ that is reflexive, antisymmetric, and transitive. The ordered pair (X, p) is called a partially ordered set.*

For a partially ordered set (X, p), there are various partial orders with their traditional notations. For example, the relation $\delta \subseteq \mathbb{N} \times \mathbb{N}$ that checks if $m \in \mathbb{N}$ is a divisor of $n \in \mathbb{N}$ is denoted by $m|n$. Therefore, we write $m|n$ to denote that $(m, n) \in \delta$. This is the well-known divisibility relation on $\mathbb{N}$. Whenever practical, for generic partially ordered sets, the partial order relation $\preccurlyeq$ is used. Therefore, we will denote $(x, y) \in p$ by $x \preccurlyeq y$, which is easier to perceive.

We will denote by $(X, d, \preccurlyeq)$ a metric space with a partial ordering.

Definition 2.3. *([26, 27]) Let A be a nonempty set and let us define $F : A \times A \to A$. If $x = F(x, y)$ and $y = F(y, x)$, then the ordered pair $(x, y) \in A \times A$ will be considered a coupled fixed point for the mapping F in A.*

The first results on the uniqueness and existence of coupled fixed points were established in [27], where it was assumed that the underlying space is a Banach space with partial ordering by a cone. Later, easier-to-apply results concerning coupled fixed points in a partially ordered complete metric space were obtained in [26], which is a starting point for numerous generalizations.

Definition 2.4. *([26, 27]) Let $(X, \preccurlyeq)$ be a partially ordered set and let us define $F : X \times X \to X$. It is said F satisfies the mixed monotone property, provided that $F(x, y)$ is monotone nondecreasing in x and monotone nonincreasing in y, that is, for all $x, y \in X$,*

$$F(x_1, y) \preccurlyeq F(x_2, y), \text{ provided that } x_1 \preccurlyeq x_2,$$

and

$$F(x, y_1) \succcurlyeq F(x, y_2), \text{ provided that } y_1 \preccurlyeq y_2.$$

This definition coincides with the notion of a mixed monotone function on $\mathbb{R}^2$, and $\leq$ represents the usual total order in $\mathbb{R}$.

In [26], the existence of a coupled fixed point for F under a weak contractivity condition is proven and the uniqueness under an additional assumption on the partial ordering is established. Further, the authors also assert that components of the coupled fixed point are equal.

Let us consider a metric space $(X, d, \preccurlyeq)$ with partial ordering. Relevant to [26], we shall equip $X \times X$ with a particular partial ordering: $(x, y) \preccurlyeq (u, v)$, provided that $x \preccurlyeq u$, $y \succcurlyeq v$. Let $(x, y), (u, v) \in X \times X$ look into the metric $\rho((x, y), (u, v)) = d(x, u) + d(y, v)$.

Proposition 2.1. *([26, 27]) Any two elements $x, y \in X$ have either a lower bound or an upper bound if and only if for any two elements $x, y \in X$ there exists $z \in X$, which is comparable to both x and y, provided that $(X, \preccurlyeq)$ is a partially ordered.*

Theorem 2.1. *([27]) Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space and $F : X \times X \to X$ be a continuous map with the mixed monotone property, such that there is $k \in [0, 1)$ so that the inequality*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v))$$

holds true for all $(x, y), (u, v) \in X \times X$, satisfying $(x, y) \succcurlyeq (u, v)$, i.e., $x \succcurlyeq u$ and $y \preccurlyeq v$. If there exists $x_0, y_0 \in X$ such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$, then there is a coupled fixed point $(\xi, \eta) \in X \times X$ of F, i.e., $\xi = F(\xi, \eta)$ and $\eta = F(\eta, \xi)$.

If, in addition, every couple of elements $x, y \in X$ has a lower bound or an upper bound, then the coupled fixed point (ξ, η) is unique and $\xi = \eta$.

The problem of identifying coupled fixed points can be seen as a problem of solving a system of symmetric equations,

$$\left| \begin{array}{l} x = F(x, y), \\ y = F(y, x). \end{array} \right.$$

In [28], a proposal is made to modify the concept of coupled fixed points so that to solve arbitrary systems of two equations using coupled fixed point results. This allows one to find coupled fixed points (x, y) such that $x \neq y$. The main application of the results from [28] is in altering the problem of finding coupled best proximity points into a task of determining the distance between sets and thus finding the exact solution instead of an approximate one, as proposed by some computer algebra systems.

Definition 2.5. *([28]) Let A be a nonempty set, and $F, G : A \times A \to A$ be two maps. If $x = F(x, y)$ and $y = G(x, y)$, then $(x, y) \in A \times A$ is considered a coupled fixed point for (F, G) in A.*

If $G(x, y) = F(y, x)$, we get the classical definition for coupled fixed points from [26, 27].

In the investigation of coupled fixed points, the iterative sequence $\{(x_n, y_n)\}_{n=0}^{\infty}$ plays a crucial role. We will recall its definition.Choosing an arbitrary $(x_0, y_0) \in A \times A$, let us set $x_1 = F(x_0, y_0)$ and $y_1 = G(x_0, y_0)$. Once we find (x_{n-1}, y_{n-1}), we define $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = G(x_{n-1}, y_{n-1})$. We shall utilize the notations $x_{n+1} = F^{n+1}(x_0, y_0) = F(x_n, y_n) = F(F^n(x_0, y_0), G^n(x_0, y_0))$ and $y_{n+1} = G^{n+1}(x_0, y_0) = G(x_n, y_n) = G(F^n(x_0, y_0), G^n(x_0, y_0))$ just to accommodate some of the formulas within the text box.

A profound observation in [32] connects fixed point results with coupled fixed point ones. Indeed, instead of considering the ordered pair of maps (F, G), we can define the map $T : X \times X \to X \times X$ by $T(x, y) = (F(x, y), G(x, y))$, and (x, y) will be a coupled fixed point for (F, G) if and only if it is a fixed point for T. Using this notation, we can express the iterative sequence $z_n = (x_n, y_n)$ as $z_{n+1} = Tz_n = T^{n+1}z_0$, which simplifies a lot of the formulas.

Thus, instead of using the technique from [27, 33], we will directly apply the results from [18].

For the sake of completeness, let us recall the Schauder fixed point theorem. It proves to be useful in finding upper and lower bounds for the fixed points, as presented in [25].

Theorem 2.2. *(e.g. [34]) Let S be a nonempty, compact, convex subset of a normed vector space. Every continuous function $f : S \to S$ mapping S into itself has a fixed point.*

2.2. Matrix equations

Throughout this section, we will denote by $\mathcal{H}(N)$ the set of all $N \times N$ Hermitian matrices. We will endow $\mathcal{H}(N)$ with a partial ordering $\preccurlyeq$, i.e., $A = (\alpha_{i,j}) \preccurlyeq B = (\beta_{i,j})$ if $\alpha_{i,j} \le \beta_{i,j}$ for all $i, j \in [1, N]$. If $A, B \in \mathcal{H}(N)$ we will write $A \succcurlyeq B$ if $B \preccurlyeq A$. If $A \preccurlyeq B$ and $A \neq B$, we will denote it by $A \prec B$. By $A = 0$, we will indicate the $N \times N$ matrix that satisfies $a_{ij} = 0$ for all $i, j \in [1, N]$. A matrix is positive semi-definite (positive definite) whenever $A \succcurlyeq 0$ ($A \succ 0$). We will signify by $\mathcal{P}(N)$ the set of all $N \times N$ positive definite matrices. For $A, B \in \mathcal{H}(N)$ by $A \succcurlyeq B$ ($A \succ B$), we will assume that $A - B$ is a positive semi-definite (positive definite) matrix. If there are inequalities $A \preccurlyeq X \preccurlyeq B$, we will use the notation $X \in [A, B]$. We will represent by A^* the conjugated transpose, and by $r(A)$ we will denote the spectral radius of A. We will denote by $\|.\|$ the spectral norm, i.e., $\|A\| = \sqrt{\lambda^+(A^*A)}$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A. We will write the $N \times N$ identity matrix as I. We will endow the space $\mathcal{H}(N)$ with the norm $\| \cdot \|_{\mathrm{tr}}$. Let us recall, just for completeness, that $\|A\|_{\mathrm{tr}} = \sum_{j=1}^{N} \sigma_j(A)$, where $\sigma_j(A)$, $j \in [1, N]$ are the singular values of A.

Applying Theorem 3.2 to solve (1.1) and (1.2) will require the following lemmas.

Lemma 2.1. *([18]) Let $A \succcurlyeq 0$ and $B \succcurlyeq 0$ be $N \times N$ matrices, then $0 \le tr(AB) \le \|A\| tr(B)$.*

Lemma 2.2. *([35]) If $0 < \theta \le 1$, and P and Q are positive definite matrices of the same order with $P, Q \ge bI > 0$, then for every unitarily invariant norm $|||\cdot|||$ there hold the inequalities $|||P^\theta - Q^\theta||| \le \theta b^{\theta-1}|||P - Q|||$ and $|||P^{-\theta} - Q^{-\theta}||| \le \theta b^{-(\theta+1)}|||P - Q|||$.*

Lemma 2.3. *([35]) Let $A \in \mathcal{H}(N)$ satisfying $-I \prec A \prec I$, then $\|A\| < 1$.*

Theorem 2.3. *([18]) Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric spaces and $f : X \to X$ be a continuous, monotone (i.e., either order preserving or order reversing) map, such that there is $k \in [0, 1)$ so that the inequality*

$$d(fx, fy) \le kd(x, y)$$

holds true for arbitrary $x, y \in X$, satisfying $x \succcurlyeq y$. A fixed point $\xi \in X$ of f exists if there is $x_0 \in X$ such that either $x_0 \preccurlyeq fx_0$ or $x_0 \succcurlyeq fx_0$.

The fixed point ξ will be unique if each pair of elements $x, y \in X$ possesses a lower bound or an upper bound.

3. Results

3.1. Fixed points for monotone maps in partially ordered metric spaces

In investigations of coupled fixed point results, it has been observed that the overly restrictive assumption regarding the continuity of the map $f : X \to X$ can be replaced by a weaker one [27]. Therefore, we will state a slight generalization of Theorem 2.3.

We will start with some auxiliary results that will help to present a more concise proof of the main result in this section. These findings will also shed light on some properties available in monotone maps on partially ordered sets.

Definition 3.1. *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space. Let $f : X \to X$ be a monotone (i.e., either order preserving or order reversing) map such that there is $k \in [0, 1)$ so that the inequality*

$$d(fx, fy) \leq kd(x, y) \tag{3.1}$$

holds true for all $x, y \in X$, satisfying $x \succcurlyeq y$. We will call these maps monotone contractive maps.

Fact 3.1. *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and $f : X \to X$ be a monotone contractive map. Then, inequality (3.1) is also satisfied for any $x, y \in X$ that are comparable.*

Proof. If $x, y \in X$ satisfy $x \succcurlyeq y$, then it is the definition for a monotone contractive map. Let $x, y \in X$ be such that $x \preccurlyeq y$. Using the symmetry of the metric function $d(\cdot, \cdot)$ we get

$$d(fx, fy) = d(fy, fx) \leq kd(y, x) = kd(x, y)$$

i.e., we can apply inequality (3.1) for any two x, y that are comparable. □

Fact 3.2. *Let $(X, d, \preccurlyeq)$ be a partially ordered metric space and $f : X \to X$ be a monotone contractive map. Let ξ be a fixed point for f and $x_0 \in X$ be comparable with ξ. Then, each element of the iterated sequence $x_n = fx_{n-1} = f^n x_0$, for $n \in \mathbb{N}$ is comparable with ξ.*

Proof. **Case I.** Let there hold $\xi \preccurlyeq x_0$.

Let us assume that f is an increasing map. Then, from the monotone property of f we have $\xi = f\xi \preccurlyeq fx_0 = x_1$. Thus, $\xi = f\xi \preccurlyeq fx_1 = x_2$ and by induction we get that $\xi \preccurlyeq x_n$ for all $n \in \mathbb{N}$.

Now let f be a decreasing map. Then, from the monotone property of f we have $\xi = f\xi \succcurlyeq fx_0 = x_1$. As a result, $\xi = f\xi \preccurlyeq fx_1 = x_2$. Through induction we can deduce that, for every $n \in \mathbb{N}$, $\xi \preccurlyeq x_{2n}$ and $\xi \succcurlyeq x_{2n-1}$, meaning that ξ and x_n are comparable in both scenarios.

Case II. Let there holds $\xi \succcurlyeq x_0$.

Let us assume that f is an increasing map. Then, from the monotone property of f we have $\xi = f\xi \succcurlyeq fx_0 = x_1$. Thus, $\xi = f\xi \succcurlyeq fx_1 = x_2$ and by induction we get that $\xi \succcurlyeq x_n$ for all $n \in \mathbb{N}$.

Thus, $\xi = f\xi \succcurlyeq fx_1 = x_2$, and by induction, we find that $\xi \succcurlyeq x_{2n}$ and $\xi \preccurlyeq x_{2n-1}$ for all $n \in \mathbb{N}$, i.e., in both situations, ξ and x_n are comparable. □

Theorem 3.1. *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space and $f : X \to X$ be a monotone contractive map.*

Let one of the following conditions hold one:

(a) f is continuous.

(b) For any convergent sequence $\lim_{n\to\infty} z_n = z$.

- *If* $z_n \preccurlyeq z_{n+1}$, *then* $z_n \preccurlyeq z$.
- *If* $z_n \succcurlyeq z_{n+1}$, *then* $z_n \succcurlyeq z$.

If there exists $z_0 \in X$ *such that either* $z_0 \preccurlyeq fz_0$ *or* $z_0 \succcurlyeq fz_0$, *then there is a fixed point* $\xi \in X$ *of* f, *which is a limit of the sequence* $\{z_n\}_{n=1}^{\infty}$, *defined by* $z_n = fz_{n-1} = f^n z_0$.

The error estimations are listed below:

(i) A priori error estimate $d(z_n, \xi) \leq \frac{k^n}{1-k} d(z_0, z_1)$.

(ii) A posteriori error estimate $d(z_{n+1}, \xi) \leq \frac{k}{1-k} d(z_n, z_{n+1})$.

(iii) The rate of convergence $d(z_{n+1}, \xi) \leq \frac{k}{1-k} d(z_n, z_{n+1})$.

The fixed point ξ *is unique if each pair of elements* $x, y \in X$ *additionally has a lower bound and an upper bound.*

Proof. If f is continuous, the proof follows from Theorem 2.3, without the error estimates.

Let there hold (b). Let us construct the iterated sequence $\{z_n\}_{n=1}^{\infty}$, starting from an element z_0, such that either $z_0 \preccurlyeq fz_0$ or $z_0 \succcurlyeq fz_0$ holds. Following the proof of Theorem 2.3 it is evident that $\lim_{n\to\infty} z_n = \xi$ and there holds either $z_n \preccurlyeq z_{n+1}$ or $z_n \succcurlyeq z_{n+1}$. The assumption in (b) implies that either $z_n \preccurlyeq \xi$ or $z_n \succcurlyeq \xi$ holds, respectively.

It follows that for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that $d(\xi, z_n) < \varepsilon/2$, because $\xi = \lim_{n\to\infty} z_n$. Thus, using $\xi \preccurlyeq z_n$ or $\xi \succcurlyeq z_n$, i.e., ξ and z_n are comparable, and Fact 3.1, we get

$$\begin{aligned} 0 &\leq d(f\xi, \xi) \leq d(f\xi, z_{n+1}) + d(z_{n+1}, \xi) = d(f\xi, fz_n) + d(z_{n+1}, \xi) \\ &\leq kd(\xi, z_n) + d(z_{n+1}, \xi) =< k\tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

By the arbitrary choice of $\varepsilon > 0$ it follows that $d(f\xi, \xi) = 0$, i.e., ξ is a fixed point of f.

To establish the error estimates let us recall that in both cases we end up with the convergent sequence $\{z_n\}_{n=0}^{\infty}$, $z_n = fz_{n-1} = f^n z_0$ for any arbitrarily chosen z_0, satisfying either $z_0 \preccurlyeq fz_0$ or $z_0 \succcurlyeq fz_0$. The constructed sequence is the same in the classical version of the Banach fixed point theorem and therefore, there holds

(i) a priori error estimate $\rho(z_n, \xi) \leq \frac{k^n}{1-k} \rho(z_0, z_1)$,

(ii) a posteriori error estimate $\rho(z_{n+1}, \xi) \leq \frac{k}{1-k} \rho(z_n, z_{n+1})$,

(iii) the rate of convergence $\rho(z_n, \xi) \leq \frac{k}{1-k} \rho(z_{n-1}, \xi)$.

Proposition 2.1 states that if each pair of elements $x, y \in X$ additionally has a lower bound and an upper bound, then for any two elements $x, y \in X$ there exists $z \in X$, comparable with both of them.

Let us assume the contrary, i.e., there are two fixed points for f, say ξ and η.

If they are comparable, then by employing Fact 3.1, we can apply (3.1) to any pair of elements that are comparable. Thus, we get

$$d(\xi, \eta) = d(f\xi, f\eta) \leq kd(\xi, \eta) < d(\xi, \eta),$$

which is a contradiction.

Let us assume for the moment that ξ and η are not comparable. Then, there is x_0, which is comparable with both of them. From Fact 3.2 it follows that any $x_n = f^n x_0$ for any $n \in \mathbb{N}$ is comparable with both ξ and η and by Fact 3.1 we can write the chain of inequalities

$$d(\xi, x_n) = d(f\xi, fx_{n-1}) \le kd(\xi, x_{n-1}) \le \cdots \le k^n d(\xi, x_0),$$

and

$$d(\eta, x_n) = d(f\eta, fx_{n-1}) \le kd(\eta, x_{n-1}) \le \cdots \le k^n d(\eta, x_0).$$

Thus, there holds

$$d(\xi, \eta) \le d(\xi, x_{n+1}) + d(x_{n+1}, \eta) \le k^n (d(\xi, x_0) + d(\eta, x_0)),$$

and from $k \in [0, 1)$, resulting in the equality $d(\xi, \eta) = 0$. □

Let us note that condition (b) does not imply continuity at the fixed point.

3.2. Coupled fixed points for maps in partially ordered metric spaces

3.2.1. Coupled fixed points for maps with the mixed monotone property

The investigations in [28] were not in a partially ordered set X. For an ordered pair of maps (F, G), we shall therefore expand the idea of maps with the mixed monotone condition.

Definition 3.2. *[36] Let $(X, \preccurlyeq)$ be a partially ordered set and $F, G : X \times X \to X$ be two maps. We say that the ordered pair of maps (F, G) has the mixed monotone property if for any $x, y \in X$ there holds*

$$x_1, x_2 \in X \text{ if } x_1 \preccurlyeq x_2 \text{ then } F(x_1, y) \preccurlyeq F(x_2, y), G(x_1, y) \preccurlyeq G(x_2, y),$$

and

$$y_1, y_2 \in X \text{ if } y_1 \preccurlyeq y_2 \text{ then } G(x, y_1) \succcurlyeq G(x, y_2), F(x, y_1) \succcurlyeq F(x, y_2).$$

Definition 3.2, introduced in [36], unfortunately contains a misprint. It fails to specify that G is a monotone increasing of its first variable and F is decreasing of its second variable, but for the proof of the results in [36] these assumptions are used.

Fact 3.3. *Let $(X, d, \preccurlyeq)$ be a metric space with a partial ordering, $F, G : X \times X \to X$ be such that (F, G) has the mixed monotone property. If $(X \times X, \preccurlyeq)$ is endowed with the partial ordering $z = (x, y) \preccurlyeq (u, v) = w$, provided that $x \preccurlyeq u$ and $y \succcurlyeq v$, then $Tz = (F(z), G(z))$ is monotone increasing map.*

Proof. From the mixed monotone property of (F, G) for any $z = (x, y) \preccurlyeq (u, v) = w$, i.e., $x \preccurlyeq u$ and $y \succcurlyeq v$ it follows

$$F(x, y) \preccurlyeq F(u, y) \preccurlyeq F(u, v),$$

and

$$G(x, y) \succcurlyeq G(u, y) \succcurlyeq G(u, v).$$

Thus,

$$Tz = T(x, y) = (F(x, y), G(x, y)) \preccurlyeq (F(u, v), G(u, v)) = T(u, v) = Tw.$$

Consequently, T is a monotone increasing with respect to the ordering $\preccurlyeq$ in the Cartesian product space $(X \times X, \preccurlyeq)$. □

Fact 3.4. *Let $(X, \preccurlyeq)$ be a partially ordered set and $(X \times X, \preccurlyeq)$ be endowed with the partial ordering $z = (x, y) \preccurlyeq (u, v) = w$, provided that $x \preccurlyeq u$ and $y \succcurlyeq v$. If any two elements $x, y \in X$ have both a lower and upper bound, then for any two elements $(x, y), (u, v) \in X \times X$ there exists an element in $X \times X$ compared with both of them.*

Proof. Now, let any two elements in X have bot ah lower and upper bound and let $x, y, u, v \in X$ be arbitrary. Let now any two of the elements $x, y, u, v \in X$ have both a lower and upper bound. Then, there exist $z_1, z_2 \in X$ so that $z_1 \preccurlyeq x, u \preccurlyeq z_2$ and there exist $w_1, w_2 \in X$ so that $w_1 \preccurlyeq y, v \preccurlyeq w_2$. Thus,

$$(z_1, w_2) \preccurlyeq (x, y), (u, v) \preccurlyeq (z_2, w_1).$$

Consequently, any two ordered pairs $(x, y), (u, v) \in X \times X$ have both a lower and upper bound and thus, according to Proposition 2.1 for any $(x, y), (u, v) \in X \times X$ there is an element compared with both of them. □

We can state a generalization of the main result from [28] in a partially ordered complete metric space. The existence and uniqueness of the coupled fixed point in the case where the maps F and G are continuous is proven in the next theorem with the assistance of a variational technique in [36].

Theorem 3.2. *Let $(X, d, \preccurlyeq)$ be a metric space with a partial ordering, let $F, G : X \times X \to X$ be such that (F, G) has the mixed monotone property, and there is $k \in [0, 1)$ such that*

$$d(F(x, y), F(u, v)) + d(G(x, y), G(u, v)) \leq k(d(x, u) + d(y, v)) \tag{3.2}$$

holds for every $x \succcurlyeq u$, $y \preccurlyeq v$.

Let one of the following hold:

(a) F and G are continuous maps.

(b) For any convergent sequence $\lim_{n\to\infty}(x_n, y_n) = (x, y)$, $(x_n, y_n) \in X \times X$.

- *If $(x_n, y_n) \preccurlyeq (x_{n+1}, y_{n+1})$, then $(x_n, y_n) \preccurlyeq (x, y)$.*
- *If $(x_n, y_n) \succcurlyeq (x_{n+1}, y_{n+1})$, then $(x_n, y_n) \succcurlyeq (x, y)$.*

If there are $x_0, y_0 \in X$ so that one of the following holds

- *$x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq G(x_0, y_0)$,*
- *$x_0 \succcurlyeq F(x_0, y_0)$ and $y_0 \preccurlyeq G(x_0, y_0)$,*

then, a coupled fixed point $(\xi, \eta) \in X \times X$ exists.

The following error estimates hold:

(i) A priori error estimate $\max\{d(x_n, \xi), d(y_n, \eta)\} \leq \frac{k^n}{1-k}(d(x_0, x_1) + d(y_0, y_1))$.

(ii) A posteriori error estimate $\max\{d(x_{n+1}, \xi), d(y_{n+1}, \eta)\} \leq \frac{k}{1-k}(d(x_n, x_{n+1}) + d(y_n, y_{n+1}))$.

(iii) The rate of convergence $\max\{d(x_{n+1}, \xi), d(y_{n+1}, \eta)\} \leq \frac{k}{1-k}(d(x_n, \xi) + d(y_n, \eta))$.

If each pair of components $x, y \in X$ also has a lower bound or an upper bound, then

- *(ξ, η) is a unique coupled fixed point,*

- *if $G(\xi,\eta) = F(\eta,\xi)$, then $\xi = \eta$.*

Proof. Let us assign the partial ordering $w = (u, v) \preccurlyeq (x, y) = z$ to $(X \times X, \preccurlyeq)$, given that $u \preccurlyeq x$ and $y \preccurlyeq v$. Let us define the function $T(x, y) = (F(x, y), G(x, y))$. From Fact 3.3, if $X \times X$ is endowed with the partial ordering $(x, y) \preccurlyeq (u, v)$, then T is an increasing map.

We will endow the partially ordered Cartesian product $(X \times X, \leq)$ with the metric $\rho((x, y), (u, v)) = d(x, u)+d(y, v)$, that satisfies the property $\lim_{n\to\infty} \rho((x_n, y_n), (x, y)) = 0$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$.

We will rewrite (3.2) for the map T and the metric $\rho(\cdot, \cdot)$. Let us denote $z = (x, y), w = (u, v) \in X\times X$ and $z \succcurlyeq w$. Then,

$$\begin{aligned} \rho(Tz, Tw) &= \rho((F(x, y), G(x, y)), (F(u, v), G(u, v))) \\ &= d(F(x, y), F(u, v)) + d(G(x, y), G(u, v)) \leq k(d(x, u) + d(y, v)) \\ &= k\rho((x, y), (u, v)) = k\rho(z, w). \end{aligned} \tag{3.3}$$

Consequently, $T : X \times X \to X \times X$ satisfies (3.1), i.e., T is a monotone contractive map.

If $F$and G are continuous in (X, d), then T is continuous in $(X \times X, \rho)$.

Assumption (b) in Theorem 3.2 is (b) in Theorem 3.1.

If $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq G(x_0, y_0)$, then $(x_0, y_0) \preccurlyeq (F(x_0, y_0), G(x_0, y_0)) = T(x_0, y_0)$.

If $x_0 \succcurlyeq F(x_0, y_0)$ and $y_0 \preccurlyeq G(x_0, y_0)$, then $(x_0, y_0) \succcurlyeq (F(x_0, y_0), G(x_0, y_0)) = T(x_0, y_0)$.

As a result, we can conclude that there is a coupled fixed point for the ordered pair of maps (F, G), or that there is a fixed point $(\xi, \eta) \in X\times X$ of T, because all the conditions in Theorem 3.1 are fulfilled.

It remains to establish the error estimates.

(i) A priori error estimate

$$\max\{d(x_n, \xi), d(y_n, \eta)\} \leq \rho(z_n, \zeta) \leq \frac{k^n}{1-k}\rho(z_0, z_1) = \frac{k^n}{1-k}(d(x_1, x_0) + d(y_1, y_0)).$$

(ii) A posteriori error estimate

$$\begin{aligned} \max\{d(x_{n+1}, \xi), d(y_{n+1}, \eta)\} &\leq \rho(z_{n+1}, \zeta) \leq \tfrac{k}{1-k}\rho(z_n, \zeta) \\ &= \tfrac{k}{1-k}(d(x_{n+1}, x_n) + d(y_{n+1}, y_n)). \end{aligned}$$

(iii) The rate of convergence

$$\max\{d(x_{n+1}, \xi), d(y_{n+1}, \eta)\} \leq \rho(z_{n+1}, \zeta) \leq \frac{k}{1-k}\rho(z_n, \zeta) = \frac{k}{1-k}(d(x_n, \xi) + d(y_n, \eta)).$$

In the following corollary, we will weaken the contractive requirement in Theorem 2.1.

Corollary 3.1. *Assume in Theorem 3.2 that $G(x, y) = F(y, x)$, and replace inequality (3.2) with*

$$d(F(x, y), F(u, v)) \leq kd(x, u) \tag{3.4}$$

for every $x \succcurlyeq u$, $y \preccurlyeq v$ and some $k \in [0, 1)$. Then, all statements of Theorem 3.2 hold true.

Proof. We need only to show that the from (3.4) follows (3.2). Indeed, let $(x, y) \preccurlyeq (u, v)$, then using the symmetry of the metric function, we can write

$$d(F(y, x), F(v, u)) = d(F(v, u), F(y, x)) \leq kd(v, y).$$

Consequently, we find

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k(d(x, u) + d(y, v)),$$

which is (3.2), and we can apply Theorem 3.2. □

3.2.2. Application of Theorem 3.2

We will use the notation $\Omega_{aI} = \{X \in \mathcal{H}(N) : X \succcurlyeq aI\}$. Let us consider the Eq (3.5)

$$\left| \begin{array}{l} X = Q - A^*X^{-1}A + B^*Y^{-1}B, \\ Y = R - C^*X^{-1}C + D^*Y^{-1}D, \end{array} \right. \tag{3.5}$$

where $A, B, C, D \in \mathcal{X}(N)$, $Q, R \in \mathcal{H}(N)$ and $Q, R, A^*A, B^*B, C^*C, D^*D \in \mathcal{P}(N)$.

Theorem 3.3. *Let there be* $0 < a < b$*, so that*

(i) $a^{-1}A^*A + aI \preccurlyeq Q$ *and* $a^{-1}C^*C + aI \preccurlyeq R$*,*

(ii) $bA^*A - aB^*B \preccurlyeq ab(Q - aI)$ *and* $bC^*C - aD^*D \preccurlyeq ab(R - aI)$*,*

(iii) where $\delta = \max\left\{\frac{\|A^*A+C^*C\|}{a^2}, \frac{\|B^*B+D^*D\|}{a^2}\right\} < 1$*,*

are satisfied. Then, the system (3.5) has a unique solution $\widetilde{X}, \widetilde{Y} \in \Omega_{aI}$*. Let* $\{X_n\}_{n=0}^{\infty}$ *and* $\{Y_n\}_{n=0}^{\infty}$ *be defined as follows*

$$\left\{ \begin{array}{l} X_0 = aI, \ Y_0 = bI, \\ X_{n+1} = Q - A^*X_n^{-1}A + B^*Y_n^{-1}B, \\ Y_{n+1} = R - C^*X_n^{-1}C + D^*Y_n^{-1}D, \end{array} \right. \tag{3.6}$$

converge to $\widetilde{X}$ *and* $\widetilde{Y}$ *in* $(\mathcal{H}(N), \|\cdot\|_{\mathrm{tr}}, \preccurlyeq)$*, respectively. The following error estimations are listed below:*

1) A priori error estimate

$$\max\{\|X_n - \tilde{X}\|_{\mathrm{tr}}, \|Y_n - \tilde{Y}\|_{\mathrm{tr}}\} \leq \frac{\delta^n}{1-\delta}(\|X_1 - X_0\|_{\mathrm{tr}} + \|Y_1 - Y_0\|_{\mathrm{tr}}).$$

2) A posteriori error estimate

$$\max\{\|X_{n+1} - \tilde{X}\|_{\mathrm{tr}}, \|Y_{n+1} - \tilde{Y}\|_{\mathrm{tr}}\} \leq \frac{\delta}{1-\delta}(\|X_{n+1} - X_n\|_{\mathrm{tr}} + \|Y_{n+1} - Y_n\|_{\mathrm{tr}}).$$

3) The convergence rate

$$\max\{\|X_{n+1} - \tilde{X}\|_{\mathrm{tr}}, \|Y_{n+1} - \tilde{Y}\|_{\mathrm{tr}}\} \leq \frac{\delta}{1-\delta}(\|X_n - \widetilde{X}\|_{\mathrm{tr}} + \|Y_n - \widetilde{Y}\|_{\mathrm{tr}}).$$

If, in addition, $Q, P, Q - a^{-1}A^*A, R - a^{-1}C^*C \in \mathcal{P}(N)$ *and* F *and* G *are continuous, then there is* $n_0 \in \mathbb{N}$ *so that* $(\widetilde{X}, \widetilde{Y}) \in [aI, n_0Q] \times [aI, n_0R]$.

Proof. Let us define the maps $F, G : \mathcal{H}(N) \times \mathcal{H}(N) \to \mathcal{H}(N)$ by $F(X, Y) = Q - A^*X^{-1}A + B^*Y^{-1}B$ and $G(X, Y) = R - C^*X^{-1}C + D^*Y^{-1}D$.

Let $X, Y \in \Omega_{aI}$, i.e., $X \geqslant aI$ and $Y \geqslant aI$. Then, from $B^*B, D^*D \in \mathcal{P}(N)$ and (i) we get

$$F(X, Y) = Q - A^*X^{-1}A + B^*Y^{-1}B \geqslant Q - A^*X^{-1}A \geqslant Q - a^{-1}A^*A \geqslant aI,$$

and

$$G(X, Y) = R - C^*X^{-1}C + D^*Y^{-1}D \geqslant R - C^*X^{-1}C \geqslant R - a^{-1}C^*C \geqslant aI.$$

Thus, $F, G : \Omega_{aI} \times \Omega_{aI} \to \Omega_{aI}$.

We will show that the ordered pair of maps (F, G) satisfies the mixed monotone property. Let $X, Y, U, V \in \Omega_{aI}$ be such that $X \geqslant U$ and $Y \leqslant V$, then $X^{-1} \leqslant U^{-1}$ and $Y^{-1} \geqslant V^{-1}$. Therefore, by $A^*A, C^*C \in \mathcal{P}(N)$ we get

$$F(X, Y) - F(U, Y) = -A^*(X^{-1} - U^{-1})A = A^*(U^{-1} - X^{-1})A \geqslant 0,$$

and

$$G(X, Y) - G(U, Y) = -C^*(X^{-1} - U^{-1})C = C^*(U^{-1} - X^{-1})C \geqslant 0,$$

i.e., F and G are increasing maps on their first variable. From the inequalities

$$F(X, Y) - F(X, V) = B^*(Y^{-1} - V^{-1})B \geqslant 0,$$

and

$$G(X, Y) - G(X, V) = D^*(Y^{-1} - V^{-1})D \geqslant 0.$$

F and G are decreasing maps on their second variable. Thus, the ordered pair of maps satisfies the mixed monotone property.

If we set $X_0 = aI$ and $Y_0 = bI$ using (ii), we can easily show that $X_0 \leqslant F(X_0, Y_0)$ and $Y_0 \geqslant G(X_0, Y_0)$. Indeed $F(X_0, Y_0) = Q - a^{-1}A^*A + b^{-1}B^*B \geqslant aI = X_0$ is equivalent to $Q - aI \geqslant a^{-1}A^*A - b^{-1}B^*B$, i.e., $ab(Q - aI) \geqslant bA^*A - aB^*B$. The proof that $Y_0 \geqslant G(X_0, Y_0)$ is done in a similar fashion $G(X_0, Y_0) = R - a^{-1}C^*C + b^{-1}D^*D \leqslant aI = Y_0$ is equivalent to $R - aI \leqslant a^{-1}C^*C - b^{-1}D^*D$, i.e., $ab(R - aI) \leqslant bC^*C - aD^*D$.

There exists a greatest lower bound and a least upper bound for each $X, Y \in \mathcal{H}(N)$.

The maps F and G are continuous.

Let us put $T(x, y) = (F(x, y), G(x, y)) : \Omega_{aI} \times \Omega_{aI} \to \Omega_{aI} \times \Omega_{aI}$ and $(\Omega_{aI} \times \Omega_{aI}, ||| \cdot |||, \preccurlyeq)$, where

$|||(x,y)||| = \|x\|_{\rm tr} + \|y\|_{\rm tr}$ and $(x,y) \leqslant (u,v)$ if $x \leqslant u$ and $y \geqslant v$. We can write down the chain of inequalities

$$\begin{array}{rcl} S & = & |||T(X,Y) - T(U,V)||| = |||(F(X,Y) - F(U,V), G(X,Y) - G(U,V))||| \\ & = & \|(F(X,Y) - F(U,V)\|_{\rm tr} + \|G(X,Y) - G(U,V))\|_{\rm tr} \\ & = & \|A^*(U^{-1} - X^{-1})A + B^*(Y^{-1} - V^{-1})B\|_{\rm tr} + \|C^*(U^{-1} - X^{-1})C + D^*(Y^{-1} - V^{-1})D\|_{\rm tr} \\ & = & {\rm tr}(A^*A(U^{-1} - X^{-1})) + {\rm tr}(B^*B(Y^{-1} - V^{-1})) \\ & & +{\rm tr}(C^*C(U^{-1} - X^{-1})) + {\rm tr}(D^*D(Y^{-1} - V^{-1})) \\ & = & {\rm tr}((A^*A + C^*C)(U^{-1} - X^{-1})) + {\rm tr}((B^*B + D^*D)(Y^{-1} - V^{-1})) \\ & \leq & \|A^*A + C^*C\|.{\rm tr}(U^{-1} - X^{-1}) + \|B^*B + D^*D\|.{\rm tr}(Y^{-1} - V^{-1}) \\ & \leq & \|A^*A + C^*C\|\frac{1}{a^2}{\rm tr}(X - U) + \|B^*B + D^*D\|\frac{1}{a^2}{\rm tr}(V - Y) \\ & \leq & \max\{\|A^*A + C^*C\|, \|B^*B + D^*D\|\}(\frac{1}{a^2}{\rm tr}(X - U) + \frac{1}{a^2}{\rm tr}(V - Y)) \\ & = & \max\{\frac{\|A^*A+C^*C\|}{a^2}, \frac{\|B^*B+D^*D\|}{2}\}({\rm tr}(X - U) + {\rm tr}(V - Y)) \\ & = & \delta(\|X - U)\|_{\rm tr} + \|V - Y\|_{\rm tr}) \\ & = & \delta|||(X,Y) - (U,V)|||, \end{array}$$

where $\delta = \max\left\{\frac{\|A^*A+C^*C\|}{a^2}, \frac{\|B^*B+D^*D\|}{a^2}\right\} < 1$.

Consequently, the ordered pair of maps (F, G) satisfies the assumption in Theorem 3.2, and thus a unique ordered pair $(\widetilde{X}, \widetilde{Y}) \in \Omega_{aI} \times \Omega_{aI}$ exists, which is a solution to system (1.1).

We have proven the existence of unique solutions $\widetilde{X}, \widetilde{Y} \geqslant aI$. We will give a shorter interval where the solutions are situated.

Let us assume $Q, P, Q - a^{-1}A^*A, R - a^{-1}C^*C \in \mathcal{P}(N)$ and F and G are continuous.

For any $Q, R \in \mathcal{P}(N)$ there is $n_0 \in \mathbb{N}$ so that

$$-n_0^{-N}A^*Q^{-1}A + a^{-1}B^*B \leqslant (n_0 - 1)Q,$$

and

$$-n_0^{-N}C^*Q^{-1}C + a^{-1}D^*D \leqslant (n_0 - 1)R.$$

The last two inequalities are equivalent to

$$\begin{array}{rcl} F(n_0Q, aI) & = & Q - A^*(n_0Q)^{-1}A + B^*(aI)^{-1}B \\ & = & Q - n_0^{-N}A^*A + a^{-1}B^*B \leqslant n_0Q, \end{array} \tag{3.7}$$

and

$$\begin{array}{rcl} G(n_0Q, aI) & = & R - C^*(n_0Q)^{-1}C + D^*(aI)^{-1}D \\ & = & R - n_0^{-N}C^*Q^{-1}C + a^{-1}D^*D \leqslant n_0R. \end{array} \tag{3.8}$$

Using the additional assumption that $Q - a^{-1}A^*A, R - a^{-1}C^*C \in \mathcal{P}(N)$, we obtain

$$\begin{array}{rcl} F(aI, n_0R) & = & Q - A^*(aI)^{-1}A + B^*(n_0R)^{-1}B \\ & = & Q - a^{-1}A^*A + n_0^{-N}B^*R^{-1}B \geqslant aI, \end{array} \tag{3.9}$$

and

$$\begin{array}{rcl} G(aI, n_0R) & = & R - C^*(aI)^{-1}C + D^*(n_0R)^{-1}D \\ & = & R - a^{-1}C^*C + n_0^{-N}D^*R^{-1}D \geqslant aI. \end{array} \tag{3.10}$$

Let $X \in [aI, n_0Q]$ and $Y \in [aI, n_0R]$, i.e.,

$$aI \leqslant X \leqslant n_0Q \text{ and } aI \leqslant Y \leqslant n_0R.$$

Using the mixed monotone property of (F, G) and (3.7)–(3.10), we get

$$aI \preccurlyeq F(aI, n_0R) \preccurlyeq F(X, Y) \preccurlyeq F(n_0Q, aI) \preccurlyeq n_0Q, \tag{3.11}$$

and

$$aI \preccurlyeq G(aI, n_0R) \preccurlyeq G(X, Y) \preccurlyeq G(n_0Q, aI) \preccurlyeq n_0R. \tag{3.12}$$

Consequently, from (3.11) and (3.12), it follows that $T : [aI, n_0Q] \times [aI, n_0R] \to [aI, n_0Q] \times [aI, n_0R]$, where $T(X, Y) = (F(X, Y), G(X, Y))$. As far as T is a continuous map and $[aI, n_0Q]\times[aI, n_0R]$ is compact and convex according to Theorem 2.2, it follows that T has at least one fixed point in $[aI, n_0Q] \times [aI, n_0R]$. □

If $C = B$, $D = A$, and $R = Q$, we get the results from [25].

Let us say that if F and G are continuous maps and $Q - a^{-1}A^*A, R - a^{-1}C^*C \in \mathcal{H}(N)$ and $Q, R \in \mathcal{H}(N)$, then the system (3.5) has a solution. If the assumption of Theorem 3.3 holds, without $Q - a^{-1}A^*A$ and $R - a^{-1}C^*C$ being positive definite, then the system (3.5) has a unique solution (X, Y). However, we only know that $(X, Y) \in \Omega_{aI} \times \Omega_{aI}$.

Thus, we can state an existence result without uniqueness of the solution for the system (3.5).

Theorem 3.4. *Let $Q, R, B^*B, D^*D, Q - a^{-1}A^*A, R - a^{-1}C^*C \in \mathcal{P}(N)$, $A, C \in \mathcal{X}(\mathcal{N})$, and suppose there are $0 < a < b$, so that*

$$a^{-1}A^*A + aI \preccurlyeq Q,$$

and

$$a^{-1}C^*C + aI \preccurlyeq R,$$

are satisfied. Then, the system (3.5) has a solution $(\widetilde{X}, \widetilde{Y})$ such that for some $n_0 \in \mathbb{N}$ there holds $(\widetilde{X}, \widetilde{Y}) \in [aI, n_0Q] \times [aI, n_0R]$.

3.2.3. Illustrative examples

Example 3.1. *Let*

$$A = \begin{pmatrix} 0.2 & 1 & 1 \\ 1 & 0.3 & 1 \\ 1 & 1 & 0.2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -0.2 & -0.2 \\ 0.3 & 1 & -0.3 \\ 0.4 & 0.4 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.5 & 0.5 & 1 \\ 1 & 0.4 & 0.5 \\ 0.5 & 1 & 0.5 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0.3 & -0.3 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.5 & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 2 \\ 1 & 2 & 8 \end{pmatrix}, \; R = \begin{pmatrix} 10 & 5 & 3.4 \\ 5 & 10 & 6.7 \\ 3.4 & 6.7 & 10 \end{pmatrix},$$

and $a = 3$, $b = 4$.

It is easy to calculate

$$A^*A \succcurlyeq \begin{pmatrix} 1.9 & 1.5 & 1.4 \\ 1.5 & 1.9 & 1.5 \\ 1.4 & 1.5 & 1.9 \end{pmatrix} \succ 0, \; B^*B \succcurlyeq \begin{pmatrix} 1.2 & 0.2 & 0.1 \\ 0.2 & 1.2 & 0.1 \\ 0.1 & 0.1 & 1.1 \end{pmatrix} \succ 0,$$

$$C^*C \geqslant \begin{pmatrix} 1.5 & 1.1 & 1.2 \\ 1.1 & 1.4 & 1.2 \\ 1.2 & 1.2 & 2.5 \end{pmatrix} \succ 0, \; D^*D \geqslant \begin{pmatrix} 1.4 & 0.9 & 0.3 \\ 0.9 & 1.3 & 0.8 \\ 0.3 & 0.81 & 1.2 \end{pmatrix} \succ 0,$$

and thus $Q, R, A^*A, B^*B, C^*C, D^*D \in \mathcal{P}(N)$. We get

$$a^{-1}A^*A + aI - Q \preccurlyeq \begin{pmatrix} -3.3 & -0.5 & -0.5 \\ -0.5 & -3.3 & -1.5 \\ -1.5 & -1.5 & -4.3 \end{pmatrix} \prec 0,$$

and

$$a^{-1}C^*C + aI - R \preccurlyeq \begin{pmatrix} -6.5 & -4.6 & -2.9 \\ -4.6 & -6.5 & -6.3 \\ -2.9 & -6.3 & -6.5 \end{pmatrix} \prec 0.$$

We calculate

$$bA^*A - aB^*B - ab(Q - aI) \preccurlyeq \begin{pmatrix} -43.5 & -6.7 & -6.7 \\ -6.7 & -43.2 & -18.4 \\ -6.7 & -18.4 & -55.2 \end{pmatrix} \prec 0,$$

and

$$bC^*C - aD^*D - ab(R - aI) \preccurlyeq \begin{pmatrix} -82.2 & -58.2 & -36.8 \\ -58.2 & -82.3 & -78.0 \\ -36.8 & -78.0 & -81.7 \end{pmatrix} \prec 0.$$

Where $\lambda^+(A^*A + C^*C) < 8.87$ and $\lambda^+(B^*B + D^*D) < 4.34$, and consequently we get

$$\frac{\|A^*A + C^*C\|}{a^2} < 0.331 \text{ and } \frac{\|B^*B + D^*D\|}{a^2} < 0.232.$$

After calculations, we get

$$Q - a^{-1}A^*A \geqslant \begin{pmatrix} 6.3 & 0.5 & 0.5 \\ 0.5 & 6.3 & 1.5 \\ 0.5 & 1.5 & 7.3 \end{pmatrix} \succ 0,$$

and

$$R - a^{-1}C^*C \geqslant \begin{pmatrix} 9.5 & 4.6 & 2.9 \\ 4.6 & 9.5 & 6.3 \\ 2.9 & 6.3 & 9.5 \end{pmatrix} \succ 0.$$

If we take $n_0 = 2$ we get

$$\frac{A^*Q^{-1}A}{n_0^9} + \frac{B^*B}{a} - (n_0 - 1)Q \preccurlyeq \begin{pmatrix} -6.5 & -0.9 & -0.9 \\ -0.9 & -6.6 & -1.9 \\ 0.9 & -1.9 & -7.6 \end{pmatrix} \prec 0,$$

and

$$\frac{C^*Q^{-1}C}{n_0^9} + \frac{D^*D}{a} - (n_0 - 1)R \preccurlyeq \begin{pmatrix} -9.5 & -4.6 & -3.2 \\ -4.6 & -9.5 & -6.4 \\ -3.2 & -6.4 & -9.5 \end{pmatrix} \prec 0.$$

Thus, the assumptions of Theorem 3.3 are satisfied. It remains to find the interval, in which the coupled fixed point is located. From

$$Q - a^{-1}A^*A + b^{-1}B^*B \geqslant \begin{pmatrix} 6.63 & 0.56 & 0.56 \\ 0.56 & 6.60 & 1.53 \\ 0.56 & 1.53 & 7.60 \end{pmatrix},$$

$$n_0(Q - b^{-1}A^*A + a^{-1}B^*B) \leqslant \begin{pmatrix} 13.81 & 1.42 & 1.37 \\ 1.42 & 13.75 & 3.34 \\ 1.37 & 3.34 & 15.73 \end{pmatrix},$$

and

$$R - a^{-1}C^*C + b^{-1}D^*D \geqslant \begin{pmatrix} 9.85 & 4.85 & 3.07 \\ 4.85 & 9.86 & 6.50 \\ 3.07 & 6.50 & 9.81 \end{pmatrix},$$

$$n_0(R - b^{-1}C^*C + a^{-1}D^*D) \leqslant \begin{pmatrix} 20.19 & 10.06 & 6.42 \\ 10.06 & 20.19 & 13.34 \\ 6.42 & 13.34 & 20.09 \end{pmatrix},$$

it follows that

$$\widetilde{X} \in \left[\begin{pmatrix} 6.63 & 0.56 & 0.56 \\ 0.56 & 6.60 & 1.53 \\ 0.56 & 1.53 & 7.60 \end{pmatrix}, \begin{pmatrix} 13.81 & 1.42 & 1.37 \\ 1.42 & 13.75 & 3.34 \\ 1.37 & 3.34 & 15.73 \end{pmatrix}\right],$$

and

$$\widetilde{Y} \in \left[\begin{pmatrix} 9.85 & 4.85 & 3.07 \\ 4.85 & 9.86 & 6.50 \\ 3.07 & 6.50 & 9.81 \end{pmatrix}, \begin{pmatrix} 20.19 & 10.06 & 6.42 \\ 10.06 & 20.19 & 13.34 \\ 6.42 & 13.34 & 20.09 \end{pmatrix}\right].$$

We calculate the first iteration by setting $X_0 = aI$ and $Y_0 = aI$.

$$X_1 = \begin{pmatrix} 6.737 & 0.587 & 0.57 \\ 0.587 & 6.703 & 1.547 \\ 0.57 & 1.547 & 7.697 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 9.97 & 4.9(3) & 3.10(3) \\ 4.9(3) & 9.977 & 6.57 \\ 3.10(3) & 6.57 & 9.917 \end{pmatrix}.$$

We calculate,

$$\|X_0 - X_1\|_{\mathrm{tr}} + \|Y_0 - Y_1\|_{\mathrm{tr}} \leq 33.$$

From $\delta \leq 0.331$ we get the a priori error estimate

$$\max\{\|X_n - \tilde{X}\|_{\mathrm{tr}}, \|Y_n - \tilde{Y}\|_{\mathrm{tr}}\} \leq \frac{0.331^n}{0.669} 33.$$

Thus, we find that for an approximation of 0.001 we will need at least 10 iterations.

Using the a posteriori error estimate we find that 6 iterations satisfy

$$\max\{\|X_n - \tilde{X}\|_{\mathrm{tr}}, \|Y_n - \tilde{Y}\|_{\mathrm{tr}}\} \leq 0.001.$$

The approximate solutions with $\varepsilon = 0.0000001$, using the a posteriori error estimate, will be

$$X_9 \approx \begin{pmatrix} 6.89498 & 0.80569 & 0.88706 \\ 0.80569 & 6.93529 & 1.76536 \\ 0.88706 & 1.76536 & 8.00508 \end{pmatrix},$$

$$Y_9 \approx \begin{pmatrix} 9.94000 & 4.90592 & 3.26087 \\ 4.90592 & 9.95969 & 6.60805 \\ 3.26087 & 6.60805 & 9.96880 \end{pmatrix}.$$

Example 3.2. *Let*

$$A = \begin{pmatrix} 0.6 & 1 & -2 \\ -1 & -1.9 & 2 \\ -1 & -1.1 & 0.2 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -0.2 & -0.2 \\ 0.3 & 1 & -0.3 \\ 0.4 & 0.4 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} -0.5 & -0.5 & 1 \\ 1 & 0.4 & 0.5 \\ -0.5 & 1 & 0.5 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0.3 & -0.3 \\ 0.4 & 1 & 0.4 \\ 0.5 & 0.5 & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 7 & 5 & 1 \\ 1 & 7 & 2 \\ 1 & 2 & 18 \end{pmatrix}, \; R = \begin{pmatrix} 10 & 5 & 3.4 \\ 5 & 10 & 6.7 \\ 3.4 & 6.7 & 10 \end{pmatrix},$$

and $a = 1$*,* $b = 4$*.*

It is easy to calculate

$$A^*A \not> 0, \; B^*B > 0, \; C^*C \not> 0, \; D^*D > 0,$$

and thus $Q, R, B^*B, D^*D \in \mathcal{P}(N)$. There hold

$$a^{-1}A^*A + aI - Q \leqslant 0, \; a^{-1}C^*C + aI - R \leqslant 0.$$

We get

$$\frac{\|A^*A + C^*C\|}{a^2} > 3.7 \text{ and } \frac{\|B^*B + D^*D\|}{a^2} < 2.08.$$

After calculations, we get

$$Q - a^{-1}A^*A > 0, \; R - a^{-1}C^*C > 0.$$

If we take $n_0 = 2$ we get

$$\frac{A^*Q^{-1}A}{n_0^9} + \frac{B^*B}{a} - (n_0 - 1)Q > 0,$$

and

$$\frac{C^*Q^{-1}C}{n_0^9} + \frac{D^*D}{a} - (n_0 - 1)R > 0.$$

Thus, the assumptions of Theorem 3.4 are satisfied.

It follows that

$$\widetilde{X} \in [I, 2Q], \; \widetilde{Y} \in [I, 2R].$$

It is not difficult to construct examples with higher dimensions.

Example 3.3. *Let*

$$A=\begin{pmatrix} 0.2 & 1 & 1 & 1 & -1\\ 1 & 0.3 & 1 & 1 & 1\\ 1 & 1 & 0.2 & 1 & 1\\ 1 & 1 & 1 & 0.3 & 1\\ -2 & 1 & 1 & 1 & 0.2 \end{pmatrix},\ C=\begin{pmatrix} 0.5 & 0.5 & 1 & 1 & 1\\ 1 & 0.4 & 0.5 & 1 & -1.5\\ 0.5 & 1 & 0.5 & 1 & 1\\ 0.5 & 1 & 0.5 & 0.5 & 1\\ 0.5 & 1 & 0.5 & -1.2 & 0.4 \end{pmatrix},$$

$$B=\begin{pmatrix} 1 & -0.2 & -0.2 & 1 & 1\\ 0.3 & 1 & -0.3 & 1 & 1\\ 0.4 & 0.4 & 1 & 1 & 1\\ 0.4 & 0.4 & 0.5 & 1 & 1\\ 0.4 & 0.4 & 1 & 1 & -0.7 \end{pmatrix},\ D=\begin{pmatrix} 1 & 0.3 & -0.3 & 1 & 1\\ 0.4 & 1 & -0.4 & 1 & 1\\ 0.5 & 0.5 & 1 & 1 & 1\\ 0.5 & 0.5 & 1 & 0.2 & 0.4\\ 0.5 & 0.5 & 1 & 0.2 & -0.3 \end{pmatrix},$$

$$Q=\begin{pmatrix} 7 & 2 & 2 & 2 & 3\\ 2 & 6 & 2 & 3 & 2\\ 2 & 2 & 7 & 2 & 2\\ 2 & 3 & 2 & 5 & 2\\ 3 & 2 & 2 & 2 & 7 \end{pmatrix},\ R=\begin{pmatrix} 10 & 5 & 3.4 & 1 & 1\\ 5 & 10 & 6.7 & 2 & 2\\ 3.4 & 6.7 & 10 & 1 & 1\\ 3.4 & 6.7 & 10 & 20 & 2\\ 3.4 & 6.7 & 10 & 2 & 10 \end{pmatrix},$$

and $a = 3$*,* $b = 4$*.*

It is easy to calculate $A^*A \succ 0$, $B^*B \succ 0$, $C^*C \succ 0$, $D^*D \succ 0$ and thus $Q, R, A^*A, B^*B, C^*C, D^*D \in \mathcal{P}(N)$. We get successively $a^{-1}A^*A + aI - Q \prec 0$, $a^{-1}C^*C + aI - R \prec 0$, $bA^*A - aB^*B - ab(Q - aI) \prec 0$, and $bC^*C - aD^*D - ab(R - aI) \prec 0$. There hold $\lambda^+(A^*A + C^*C) < 4.69$ and $\lambda^+(B^*B + D^*D) < 4.43$ and consequently, we get

$$\frac{\|A^*A + C^*C\|}{a^2} < 0.521 \text{ and } \frac{\|B^*B + D^*D\|}{a^2} < 0.493.$$

After calculations, we get $Q - a^{-1}A^*A \succ 0$ and $R - a^{-1}C^*C \succ 0$.

If we take $n_0 = 2$, we get $\frac{A^*Q^{-1}A}{n_0^{25}} + \frac{B^*B}{a} - (n_0 - 1)Q \prec 0$ and $\frac{C^*Q^{-1}C}{n_0^{25}} + \frac{D^*D}{a} - (n_0 - 1)R \prec 0$. Thus, the assumptions of Theorem 3.3 are satisfied. It remains to find the interval, in which the coupled fixed point is located. We get

$$\widetilde{X} \in \left[\begin{pmatrix} 5.045 & 1.978 & 2.044 & 2.458 & 2.655\\ 1.978 & 5.016 & 1.018 & 2.300 & 1.830\\ 2.044 & 1.018 & 6.248 & 1.333 & 1.608\\ 2.458 & 2.300 & 1.333 & 4.886 & 2.325\\ 2.655 & 1.830 & 1.608 & 2.325 & 6.775 \end{pmatrix}, \begin{pmatrix} 11.527 & 4.137 & 4.274 & 5.417 & 6.014\\ 4.137 & 10.969 & 2.744 & 5.534 & 4.131\\ 4.274 & 2.744 & 13.567 & 3.584 & 3.501\\ 5.417 & 5.534 & 3.584 & 11.289 & 5.451\\ 6.014 & 4.131 & 3.501 & 5.451 & 14.974 \end{pmatrix}\right],$$

and

$$\widetilde{Y} \in \left[\begin{pmatrix} 9.810 & 4.645 & 3.076 & 0.975 & 1.420\\ 4.645 & 9.323 & 6.219 & 2.100 & 1.695\\ 3.076 & 6.219 & 10.145 & 0.625 & 0.616\\ 3.375 & 6.800 & 9.625 & 19.206 & 2.581\\ 3.820 & 6.395 & 9.616 & 2.581 & 9.009 \end{pmatrix}, \begin{pmatrix} 20.274 & 9.892 & 6.619 & 2.576 & 3.201\\ 9.892 & 19.522 & 12.974 & 4.734 & 4.084\\ 6.619 & 12.974 & 21.167 & 1.642 & 1.542\\ 7.376 & 14.134 & 19.642 & 39.709 & 5.754\\ 8.001 & 13.484 & 19.542 & 5.754 & 19.462 \end{pmatrix}\right].$$

We calculate the first iteration by setting $X_0 = aI$ and $Y_0 = aI$ and we get

$$\|X_0 - X_1\|_{\mathrm{tr}} + \|Y_0 - Y_1\|_{\mathrm{tr}} \leq 59.$$

From $\delta \leq 0.521$, we obtain the a priori error estimate

$$\max\{\|X_n - \tilde{X}\|_{\mathrm{tr}}, \|Y_n - \tilde{Y}\|_{\mathrm{tr}}\} \leq \frac{0.521^n}{0.521} 59.$$

Thus, we find that for an approximation of 10^{-7} we will need at least 33 iterations.

Using the a posteriori error estimate, we find that only 17 iterations are needed to satisfy

$$\max\{\|X_n - \tilde{X}\|_{\mathrm{tr}}, \|Y_n - \tilde{Y}\|_{\mathrm{tr}}\} \leq 10^{-7}.$$

The approximate solutions with $\varepsilon = 10^{-7}$, using the a posteriori error estimate will be

$$X_9 \approx \begin{pmatrix} 5.50774839 & 1.90890894 & 2.01845984 & 2.14359822 & 2.65195866 \\ 1.93888125 & 5.79977968 & 1.67775243 & 2.85685407 & 1.91533469 \\ 2.03276995 & 1.68595570 & 6.90256733 & 1.81603011 & 1.76168238 \\ 2.15282595 & 2.86339386 & 1.80659862 & 4.75642586 & 1.98923557 \\ 2.74862639 & 1.95702800 & 1.97145213 & 2.15606464 & 6.43488883 \end{pmatrix},$$

$$Y_9 \approx \begin{pmatrix} 9.92486564 & 4.91150599 & 3.31195213 & 0.87654074 & 1.23845753 \\ 4.91112567 & 9.76950418 & 6.51518577 & 2.10729725 & 1.62870585 \\ 3.30240536 & 6.50948347 & 10.11928530 & 0.65294312 & 0.65429868 \\ 3.32473396 & 6.82321405 & 9.80003774 & 19.11726951 & 2.16377258 \\ 3.71281771 & 6.37525327 & 9.86364387 & 2.23549931 & 8.67442281 \end{pmatrix}.$$

3.2.4. Coupled fixed points for maps without the mixed monotone property

Let us mention that whenever we have a partially ordered set $(X, \preccurlyeq)$, we can introduce different partial orderings in the Cartesian product set $X \times X$. The widely used partial ordering in $(X \times X)$ is that $(x, y) \preccurlyeq (u, v)$, provided that $x \preccurlyeq u$ and$y \succcurlyeq v$. We will define a different partial ordering in $X \times X$, which will help us apply Theorem 3.1 in solving the matrix equation $X = Q + A^* X^{-1} A + B^* X^{-1} B$. Let us say that $(x, y) \preceq (u, v)$ if there holds $x \preccurlyeq u$ and $y \preccurlyeq v$. Let us consider an ordered pair of maps (F, G), so that $F : X \times X \to X$ and $G : X \times X \to X$ are decreasing maps of each of its variables.

Definition 3.3. *Let $(X, \preccurlyeq)$ be a partially ordered set and $F, G : X \times X \to X$ be two maps. We say that the ordered pair of maps (F, G) has the total decreasing monotone property if for any $x, y \in X$ there holds*

$$x_1, x_2 \in X \textit{ if } x_1 \preccurlyeq x_2 \textit{ then } F(x_1, y) \succcurlyeq F(x_2, y), G(x_1, y) \succcurlyeq G(x_2, y),$$

and

$$y_1, y_2 \in X \textit{ if } y_1 \preccurlyeq y_2 \textit{ then } G(x, y_1) \succcurlyeq G(x, y_2), F(x, y_1) \succcurlyeq F(x, y_2).$$

Let us introduce a partial order $\preceq$ in $X \times X$ by $z = (x, y) \preceq (u, v) = w$, provided that $x \preccurlyeq u$ and $y \preccurlyeq v$. We will denote this partial order with $\preceq$ in order to distinguish it from the previously defined partial order $\preccurlyeq$.

Fact 3.5. *Let $(X, \preccurlyeq)$ be a partially ordered set, and $F, G : X \times X \to X$. If the ordered pair of maps (F, G) has the total decreasing monotone property, then $Tz = (F(z), G(z))$ is a monotone decreasing map in $(X \times X, \preceq)$.*

Proof. Let $z = (x, y) \preceq (u, v) = w$, i.e., $x \preccurlyeq u$ and $y \preccurlyeq v$. Then, $F(x, y) \succcurlyeq F(u, y) \succcurlyeq F(u, v)$ and $G(x, y) \succcurlyeq G(u, y) \succcurlyeq G(u, v)$. Thus,

$$Tz = T(x, y) = (F(x, y), G(x, y)) \succeq (F(u, v), G(u, v)) = T(u, v) = Tw.$$

Consequently, T is monotone decreasing with respect to the ordering $\preceq$ in the Cartesian product space $(X \times X, \preceq)$. □

We can state a version of the well known coupled fixed point result from [27] for maps (F, G) that have the total decreasing monotone property.

Theorem 3.5. *Let $(X, d, \preccurlyeq)$ be a partially ordered complete metric space and $F, G : X \times X \to X$ be maps, so that the ordered pair (F, G) has the total decreasing monotone property and there is $k \in [0, 1)$ so that the inequality*

$$d(F(x, y), F(u, v)) + d(G(x, y), G(u, v)) \leq k(d(x, u) + d(y, v)) \tag{3.13}$$

holds for every $x \preccurlyeq u$, $y \preccurlyeq v$.

Let one of the following hold:

(a) F and G are continuous maps.

(b) For any convergent sequence $\lim_{n\to\infty}(x_n, y_n) = (x, y)$, $(x_n, y_n) \in X \times X$.

- *if $(x_n, y_n) \preceq (x_{n+1}, y_{n+1})$, then $(x_n, y_n) \preceq (x, y)$.*
- *if $(x_n, y_n) \succeq (x_{n+1}, y_{n+1})$, then $(x_n, y_n) \succeq (x, y)$.*

If there exists $x_0, y_0 \in X$ such that there holds one of the following

- *$x_0 \succeq F(x_0, y_0)$ and $y_0 \succeq G(x_0, y_0)$, i.e., $(x_0, y_0) \preceq (F(x_0, y_0), G(x_0, y_0))$.*
- *$x_0 \preceq F(x_0, y_0)$ and $y_0 \preceq G(x_0, y_0)$, i.e., $(x_0, y_0) \succeq (F(x_0, y_0), G(x_0, y_0))$.*

Then, a coupled fixed point $(\xi, \eta) \in X \times X$ exists for (F, G).

The following error estimations are listed below:

(i) A priori error estimate $\max\{d(x_n, \xi), d(y_n, \eta)\} \leq \frac{k^n}{1-k}(d(x_0, x_1) + d(y_0, y_1))$.

(ii) A posteriori error estimate $\max\{d(x_{n+1}, \xi), d(y_{n+1}, \eta)\} \leq \frac{k}{1-k}(d(x_n, x_{n+1}) + d(y_n, y_{n+1}))$.

(iii) The rate of convergence $\max\{d(x_{n+1}, \xi), d(y_{n+1}, \eta)\} \leq \frac{k}{1-k}(d(x_n, \xi) + d(y_n, \eta))$.

Furthermore, if each pair of components $x, y \in X$ has a lower bound or an upper bound, then

- *(ξ, η) is a unique coupled fixed point.*
- *If $G(x, y) = F(y, x)$, then $\xi = \eta$.*

Proof. The proof can be done in a similar fashion to that of Theorem 3.2 by considering the map $T(x, y) = (F(x, y), G(x, y))$. We have seen that if $X\times X$ is endowed with the partial ordering $(x, y) \preceq (u, v)$ if and only if $x \preccurlyeq u$ and $y \preccurlyeq v$, then T is a decreasing map.

We will endow the partially ordered Cartesian product $(X \times X, \preceq)$ with the metric $\rho((x, y), (u, v)) = d(x, u)+d(y, v)$ that satisfies the property $\lim_{n\to\infty} \rho((x_n, y_n), (x, y)) = 0$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$.

We will rewrite (3.13) for the map T and the metric $\rho(\cdot, \cdot)$. Let us denote $z = (x, y), w = (u, v) \in X\times X$ and $z \succeq w$. Then,

$$\begin{aligned} \rho(Tz, Tw) &= \rho((F(x, y), G(x, y)), (F(u, v), G(u, v))) \\ &= d(F(x, y), F(u, v)) + d(G(x, y), G(u, v)) \leq k(d(x, u) + d(y, v)) \\ &= k\rho((x, y), (u, v)) = k\rho(z, w). \end{aligned} \tag{3.14}$$

Consequently, $T : X \times X \to X \times X$ satisfies (3.1).

There exists $z_0 = (x_0, y_0) \in X \times X$, so that there holds one of the following $(x_0, y_0) \preceq (F(x_0, y_0), G(x_0, y_0))$, i.e., $z_0 = (x_0, y_0) \succeq T(x_0, y_0) = T(z_0)$ or $(x_0, y_0) \succeq (F(x_0, y_0), G(x_0, y_0))$, i.e., $z_0 = (x_0, y_0) \preceq T(x_0, y_0) = T(z_0)$.

From Theorem 3.1 we can conclude the existence of a fixed point $(\xi, \eta) \in X\times X$ of T, i.e., a coupled fixed point of (F, G), and the error estimates hold.

Following the proof of Fact 3.4, if any $x, y \in X$ have both lower and upper bounds, then for any $(x, y), (u, v) \in X \times X$ there is (z, w) comparable with both (x, y) and (u, v). From Fact 3.2 it follows that if (ξ, η) is a fixed point for T, (z_0, w_0) is comparable with (ξ, η), then (ξ, η) is comparable with any $(z_n, w_n) = T(z_{n-1}, w_{n-1}) = T^n(z_0, w_0)$.

Let us assume that $x, y \in X$ have both lower and upper bounds, but there are two coupled fixed points, (ξ, η) and (x, y). If (ξ, η) and (x, y) are comparable, then from (3.13), using the comment that due to the symmetry of the metric function we need only both elements to be comparable, we get

$$\rho((\xi, \eta), (x, y)) = \rho(T(\xi, \eta), T(x, y)) \leq k\rho((\xi, \eta), (x, y)),$$

which is a contradiction, because $k < 1$.

If (ξ, η) and (x, y) are not comparable, there exists (z_0, w_0) that is comparable with (ξ, η) and (x, y). Then, any $(z_n, w_n) = T^n(z_0, w_0)$ is comparable with (ξ, η) and (x, y). There exists $N \in \mathbb{N}$ such that for any $n \geq N$ there holds $k^n\rho((\xi, \eta), (z_0, w_0)) + k^n\rho((z_0, w_0), (x, y)) < \rho((\xi, \eta), (x, y))$. The series of inequalities is obtained.

$$\begin{aligned} \rho((\xi, \eta), (x, y)) &= \rho(T(\xi, \eta), T(x, y)) \\ &\leq \rho(T(\xi, \eta), T(z_n, w_n)) + \rho(T(z_n, w_n), T(x, y)) \\ &\leq k\rho((\xi, \eta), (z_n, w_n)) + k\rho((z_n, w_n), (x, y)) \\ &= k\rho(T(\xi, \eta), T(z_{n-1}, w_{n-1})) + k\rho(T(z_{n-1}, w_{n-1}), T(x, y)) \\ &\leq k^2\rho((\xi, \eta), (z_{n-1}, w_{n-1})) + k^2\rho((z_{n-1}, w_{n-1}), (x, y)) \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq k^n\rho((\xi, \eta), (z_0, w_0)) + k^n\rho((z_0, w_0), (x, y)) \\ &\leq \rho((\xi, \eta), (x, y)), \end{aligned}$$

which is a contradiction if we have chosen $n \geq N$.

If $G(x, y) = F(x, y)$, the proof is done in a similar fashion to that in ([26], Theorem 2.5). □

3.2.5. Application of Theorem 3.4

The matrix equation $X = Q \pm \sum_{k=1}^{n} A_k^* \mathcal{F}(X) A_k$, where $\mathcal{F}$ is a monotone map, was investigated in [18]. If we simplify this equation by setting $n = 2$ to the equation $X = Q \pm \sum_{k=1}^{2} A_k^* \mathcal{F}(X) A_k$, it is not possible to apply the technique proposed in [25], as long as the map $F(X,Y) = Q \pm (A_1^* \mathcal{F}(X) A_1 + A_2^* \mathcal{F}(X) A_2)$ does not satisfy the mixed monotone property. Fortunately, it is either a totally monotone decreasing or totally monotone increasing map, and we will be able to apply Theorem 3.4.

Theorem 3.6. *Let us look at the system of two matrix equations:*

$$\left| \begin{array}{l} X = Q + A^* X^{-1} A + B^* Y^{-1} B, \\ Y = R + C^* X^{-1} C + D^* Y^{-1} D. \end{array} \right. \tag{3.15}$$

Let Q, R, A, B, C, D be $N \times N$ matrices, $Q, R \in \mathcal{H}(N)$ and $Q, R, A^*A, B^*B, C^*C, D^*D \in P(N)$. Let there exist $0 < b < a$, such that the following conditions

(1) $Q, R \in \Omega_{aI}$,

(2) where $\delta = \max\left\{\frac{\|A^*A + C^*C\|}{a^2}, \frac{\|B^*B + D^*D\|}{a^2}\right\} < 1$,

are satisfied. Then,

I) the system (3.15) has a solution $(\tilde{X}, \tilde{Y}) \in \Omega_Q \times \Omega_R$, where $\Omega_W = \{X \in \mathcal{H}(N) : X \succcurlyeq W\}$,

II) the sequences defined by

$$\left\{ \begin{array}{l} X_0 = aI, \ Y_0 = bI, \\ X_{n+1} = Q + A^* X_n^{-1} A + B^* Y_n^{-1} B, \\ Y_{n+1} = R + C^* X_n^{-1} C + D^* Y_n^{-1} D, \end{array} \right. \tag{3.16}$$

converge to $\widetilde{X}$ and $\widetilde{Y}$ in $(\mathcal{H}(N), \|\cdot\|_{\mathrm{tr}}, \preccurlyeq)$.

The following error estimations are listed below:

a) A priori error estimate

$$\max\{\|X_n - \tilde{X}\|_{\mathrm{tr}}, \|Y_n - \tilde{Y}\|_{\mathrm{tr}}\} \le \frac{\delta^n}{1-\delta}(\|X_1 - X_0\|_{\mathrm{tr}} + \|Y_1 - Y_0\|_{\mathrm{tr}}).$$

b) A posteriori error estimate

$$\max\{\|X_{n+1} - \tilde{X}\|_{\mathrm{tr}}, \|Y_{n+1} - \tilde{Y}\|_{\mathrm{tr}}\} \le \frac{\delta}{1-\delta}(\|X_{n+1} - X_n\|_{\mathrm{tr}} + \|Y_{n+1} - Y_n\|_{\mathrm{tr}}).$$

c) The rate of convergence

$$\max\{\|X_{n+1} - \tilde{X}\|_{\mathrm{tr}}, \|Y_{n+1} - \tilde{Y}\|_{\mathrm{tr}}\} \le \frac{\delta}{1-\delta}(\|X_n - \tilde{X}\|_{\mathrm{tr}} + \|Y_n - \tilde{Y}\|_{\mathrm{tr}}).$$

There holds $(\tilde{X}, \tilde{Y}) \in [Q, F(Q,R)] \times [R, G(Q,R)]$.

Proof. Let us define the maps $F, G : \mathcal{H}(N) \times \mathcal{H}(N) \to \mathcal{H}(N)$ by $F(X, Y) = Q + A^*X^{-1}A + B^*Y^{-1}B$ and $G(X, Y) = R + C^*X^{-1}C + D^*Y^{-1}D$. It is easy to observe using the assumption A^*A,B^*B,C^*C, $D^*D \in P(N)$ that for any $X \in \Omega_Q$ and $X \in \Omega_R$, there holds $F(X, Y) \geqslant Q$ and $G(X, Y) \geqslant R$, as long as $A^*X^{-1}A + B^*Y^{-1}B \geqslant 0$ and $C^*X^{-1}C + D^*Y^{-1}D \geqslant 0$, i.e., $F : \Omega_Q \times \Omega_R \to \Omega_Q$ and $G : \Omega_Q \times \Omega_R \to \Omega_R$.

We will show that the ordered pair of maps (F, G) satisfies the total decreasing monotone property. Let $X, U \in \Omega_Q$ and $Y, V \in \Omega_R$ be such that $X \leqslant U$ and $Y \leqslant V$, then $X^{-1} \geqslant U^{-1}$ and $Y^{-1} \geqslant V^{-1}$. Therefore,

$$F(X, Y) - F(U, Y) = A^*(X^{-1} - U^{-1})A \geqslant 0,$$

and

$$G(X, Y) - G(U, Y) = C^*(X^{-1} - U^{-1})C \geqslant 0,$$

i.e., F and G are decreasing maps on their first variable. From the inequalities

$$F(X, Y) - F(X, V) = B^*(Y^{-1} - V^{-1})B \geqslant 0,$$

and

$$G(X, Y) - G(X, V) = D^*(Y^{-1} - V^{-1})D \geqslant 0.$$

F and G are decreasing maps on their second variable. Thus, the ordered pair of maps satisfies the total decreasing monotone property.

Let us consider $T(x, y) = (F(x, y), G(x, y)) : \Omega_Q \times \Omega_R \to \Omega_Q \times \Omega_R$ and $(\Omega_R \times \Omega_R, |||\cdot|||)$, where $|||(x, y)||| = \|x\|_{\mathrm{tr}} + \|y\|_{\mathrm{tr}}$.

$$\begin{array}{rcl}
S & = & |||T(X, Y) - T(U, V)||| = |||(F(X, Y) - F(U, V), G(X, Y) - G(U, V))||| \\
& = & \|(F(X, Y) - F(U, V)\|_{\mathrm{tr}} + \|G(X, Y) - G(U, V))\|_{\mathrm{tr}} \\
& = & \|A^*(X^{-1} - U^{-1})A + B^*(Y^{-1} - V^{-1})B\|_{\mathrm{tr}} + \|C^*(X^{-1} - U^{-1})C + D^*(Y^{-1} - V^{-1})D\|_{\mathrm{tr}} \\
& = & \mathrm{tr}(A^*A(X^{-1} - U^{-1})) + \mathrm{tr}(B^*B(Y^{-1} - V^{-1})) \\
& & +\mathrm{tr}(C^*C(X^{-1} - U^{-1})) + \mathrm{tr}(D^*D(Y^{-1} - V^{-1})) \\
& = & \mathrm{tr}((A^*A + C^*C)(X^{-1} - U^{-1})) + \mathrm{tr}((B^*B + D^*D)(Y^{-1} - V^{-1})) \\
& \leq & \|A^*A + C^*C\|\mathrm{tr}(X^{-1} - U^{-1}) + \|B^*B + D^*D\|\mathrm{tr}(Y^{-1} - V^{-1}) \\
& \leq & \|A^*A + C^*C\|\frac{1}{a^2}\mathrm{tr}(X - U) + \|B^*B + D^*D\|\frac{1}{a^2}\mathrm{tr}(V - Y) \\
& \leq & \delta(\frac{1}{a^2}\mathrm{tr}(X - U) + \frac{1}{a^2}\mathrm{tr}(V - Y)) \\
& = & \delta(\mathrm{tr}(X - U) + \mathrm{tr}(V - Y)) \\
& = & \delta(\|X - U)\|_{\mathrm{tr}} + \|V - Y\|_{\mathrm{tr}}) \\
& = & \delta|||(X, Y) - (U, V)|||.
\end{array}$$

Consequently, to apply Theorem 3.1 for (F, G), it remains to show that there are $Q_0 \in \Omega_Q$ and $R_0 \in \Omega_R$ so that one of the following holds:

- $Q_0 \geqslant F(Q_0, R_0)$ and $R_0 \geqslant G(Q_0, R_0)$.
- $Q_0 \leqslant F(Q_0, R_0)$ and $R_0 \leqslant G(Q_0, R_0)$.

From 1, it follows that $aI \in \Omega_Q \cap \Omega_R$.

If we set $Q_0 = Q$ and $R_0 = R$, we can easily show that $Q_0 \leqslant F(Q_0, R_0) = Q_1$ and $R_0 \leqslant G(Q_0, R_0) = R_1$.

For any $X, Y \in \mathcal{H}(N)$, there are a least upper bound and a greatest lower one.

The maps F and G are continuous.

Therefore, by Theorem 3.1, it follows that there exists a unique ordered pair $(\widetilde{X}, \widetilde{Y}) \in \Omega_Q \times \Omega_R$, which is the solution to system (3.15).

We have proven that the unique solutions are in the set $\Omega_Q \times \Omega_R$. We will give a shorter interval where the solutions are situated. For any $(X, Y) \in \Omega_Q \times \Omega_R$, using the total decreasing property of the ordered pair (F, G), we get $F(X, Y) \preccurlyeq F(Q, R)$ and $G(X, Y) \preccurlyeq G(Q, R)$. Therefore for any Let $X \in [Q, F(Q, R)]$ and $Y \in [R, G(Q, R)]$ there hold $F : [Q, F(Q, R)] \times [R, G(Q, R)] \to [Q, F(Q, R)]$ and $G : [Q, F(Q, R)] \times [R, G(Q, R)] \to [R, G(Q, R)]$.

Consequently, $T : [Q, F(Q, R)] \times [R, G(Q, R)] \to [Q, F(Q, R)] \times [R, G(Q, R)]$, where $T(X, Y) = (F(X, Y), G(X, Y))$. As long as T is a continuous map and $[Q, F(Q, R)] \times [R, G(Q, R)]$ is compact and convex according to Theorem 2.2, it follows that T has at least one fixed point in $[Q, F(Q, R)] \times [R, G(Q, R)]$. □

3.2.6. Examples

Example 3.4. *Let us consider the matrices from Example 3.1. The restrictions in Theorem 3.6 are included in those of Theorem 3.3. Therefore, the system of equations*

$$\left| \begin{array}{l} X = Q + A^*X^{-1}A + B^*Y^{-1}B, \\ Y = R + C^*X^{-1}C + D^*Y^{-1}D, \end{array} \right.$$

has a unique solution $(\widetilde{X}, \widetilde{Y}) \in [Q, F(Q, R)] \times [R, G(Q, R)]$.

We get that

$$F(Q,R) \leq \begin{pmatrix} 7.32 & 1.07 & 1.16 \\ 1.07 & 7.41 & 2.05 \\ 1.16 & 2.05 & 8.52 \end{pmatrix}, \ G(Q,R) \leq \begin{pmatrix} 10.27 & 5.11 & 3.51 \\ 5.11 & 10.25 & 6.84 \\ 3.51 & 6.84 & 10.31 \end{pmatrix}.$$

The approximate solutions with $\varepsilon = 0.0000001$, using the a posteriori error estimate, will be

$$X_8 \approx \begin{pmatrix} 7.30967 & 1.06516 & 1.15057 \\ 1.06516 & 7.39480 & 2.05766 \\ 1.15057 & 2.05766 & 8.50237 \end{pmatrix},$$

$$Y_8 \approx \begin{pmatrix} 10.26432 & 5.10729 & 3.51199 \\ 5.10729 & 10.24571 & 6.84171 \\ 3.51199 & 6.84171 & 10.30540 \end{pmatrix}.$$

3.3. Connection of Theorem 3.6 with the results in [18]

We will use the technique of [25] to solve the system of equations

$$\left| \begin{array}{l} X = Q + A^*X^{-1}A + B^*Y^{-1}B, \\ Y = Q + A^*Y^{-1}A + B^*X^{-1}B. \end{array} \right. \tag{3.17}$$

The system (3.17) is (3.15) with $C = B$ and $D = A$. Therefore, from Theorem 3.6, it follows that (3.17) has a solution $(\widetilde{X}, \widetilde{Y}) \in \Omega_Q \times \Omega_Q$ such that $\widetilde{X} = \widetilde{Y}$, provided that $0 < b < a$, $Q \geqslant aI$ and $\max\left\{\frac{\|A^*A+B^*B\|}{a^2}\right\} < 1$. Therefore, $\widetilde{X} \in \Omega_Q$ is the unique solution of (3.17).

In [18] a more general case is investigated, compared to $X = Q + A^*X^{-1}A + B^*X^{-1}B$. It is proven that if $A^*A + B^*B \preccurlyeq mI$ for some constant m, then the equation $X = Q + A^*X^{-1}A + B^*Y^{-1}B$ has a solution. We have demonstrated in Theorem 3.6 that the constant m from [18] is a^2.

4. Conclusions

In the present paper, with the help of the results in [18, 32], we constructed a generalization of the coupled fixed points introduced in [26, 27]. We have demonstrated that there is a close connection between the type of monotonicity of the maps under study and the definition of partial ordering in the proposed idea in [27] of studying coupled fixed points for maps with the mixed monotone property in partially ordered metric spaces. This allows us to solve various classes of systems of matrix equations. The presented idea of considering an ordered pair of maps instead of one map generalizes the technique proposed by [25] for solving only symmetric systems of matrix equations.

Author contributions

Aynur Ali, Cvetelina Dinkova, Atanas Ilchev, Boyan Zlatanov: Conceptualization, Methodology, Investigation, Writing-original draft, Writing-review & editing, and are listed in alphabetical order. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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