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Research article

The constant in asymptotic expansions for a cubic recurrence

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Abstract: Some properties of the constant in asymptotic expansion of iterates of a cubic function were investigated. This paper analyzed the monotonicity, differentiability of the constant with respect to the initial value and the functional equation that is satisfied.

Keywords: cubic function; iterate; asymptotic expansion; Euler Maclaurin formula
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1. Introduction

Asymptotic analysis has a wide range of applications in the complexity of algorithms [1], solutions of all kinds of equations (e.g., [2–4]), combinatorial mathematics [5], some economic, biological or physical models [6].

Let I be an interval of $\mathbb{R}$ and $C(I, I)$ consist of all continuous functions $f\colon I \to I$. The n-th iterate f^n of $f \in C(I, I)$ is defined by

$$f^n(x) = f\left(f^{n-1}(x)\right)$$

and

$$f^0(x) = x$$

for all $x \in I$ recursively. Some researchers have given nice asymptotic expansions of iterates of elementary functions (e.g., [7, 8]), some special functions (e.g., [9, 10]), and some recursive sequences (e.g., [11, 12]). It is known from [13–15] that

Lemma 1. *Let*

$$x_{n+1} = f(x_n)$$

for all $n \geq 1$*, where* f*:* $(0,1) \to (0,1)$*,*

$$f(x) = x - ax^{p+1} + o(x^{p+1}),$$

$a > 0$*,* $p > 0$*. Then*

$$\lim_{n\to\infty} nx_n^p = \frac{1}{pa}.$$

Furthermore, Stević [16] obtained the first two terms of the asymptotic expansion of x_n for three cases [17]. In particular, Ionascu and Stanica [17] obtained the first six terms of the asymptotic expansion of

$$x_{n+1} = x_n - x_n^2,$$

in which, there appears a constant C depending on the initial value x_1. And they have obtained some properties of Bruijn ([18, Section 8.6], also [7]), who obtained the first six terms of the asymptotic expansion of

$$x_{x+1} = \sin(x_n),$$

in which, there also appears a constant C depending on the initial value x_1.

In many asymptotic expansions, a special constant C often appears, which is related to the initial value and can be regard as a function of the initial value

$$x_1 = x,$$

say $C(x)$. Such constants do not have closed form expressions, but can be expressed in the limiting form of a sequence. Studying $C(x)$ gives an insight into the dependence of the limiting behavior on the initial value. However, it is difficult to study the properties of such $C(x)$ in asymptotic expansions. First, $C(x)$ is very sensitive to these iterated functions. Second, $C(x)$ has no analytic closed-form expression. As far as we know, there is no general method to study it.

In order to discover some properties of these constants in asymptotic expansions, we consider the iterates of the cubic, i.e.,

$$x_{n+1} = f(x_n), \quad f : (0,1) \to (0,1), \quad f(x) = x - x^3. \tag{1.1}$$

The aim of this paper is to find the limiting expression of such $C(x)$ in the asymptotic expansion, to prove the monotonicity, differentiability of $C(x)$ and the functional equation that is satisfied.

The remainder of this paper is structured as follows. Section 2 presents some preliminary results on asymptotic analysis. Section 3 gives the first six terms of the asymptotic expansion of iterates of the cubic function is given, where $C(x)$ does not appear in a closed form, but only in a limiting form. And we prove some properties of $C(x)$. A short conclusion is given in the last section.

2. Preliminaries

To compute the asymptotic expansion, we need the asymptotic estimates of the following sums.

Lemma 2. *For* $n \to \infty$*, we have*

$$\sum_{k=n}^{\infty} \frac{\ln k}{k^2} = \frac{\ln n}{n} + \frac{1}{n} + o\left(\frac{1}{n}\right).$$

Proof. For every $n \geq 1$, let define

$$z_n = \sum_{k=n}^{\infty} \frac{\ln k}{k^2} - \frac{\ln n}{n}.$$

We have

$$z_{n+1} - z_n = -(\frac{\ln(n+1)}{n+1} - \frac{\ln n}{n} + \frac{\ln n}{n^2}) \sim -\frac{1}{n^2}.$$

By the Stolz-Cesáro Lemma, we have

$$\lim_{n\to\infty} \frac{z_n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{z_{n+1} - z_n}{\frac{1}{n+1} - \frac{1}{n}} = 1.$$

It follows that

$$z_n = \sum_{k=n}^{\infty} \frac{\ln k}{k^2} - \frac{\ln n}{n} \sim \frac{1}{n}.$$

This completes the proof. □

Lemma 3. *For $n \to \infty$, we have*

$$\sum_{k=n}^{\infty} \frac{1}{k^2} \sim \frac{1}{n}.$$

Proof. The result follows immediately from the Euler Maclaurin formula

$$\begin{aligned}\sum_{k=n}^{\infty} \frac{1}{k^2} &= \int_n^{\infty} x^{-2} dx + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \\ &= \frac{1}{n} + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right).\end{aligned}$$

This completes the proof. □

The following lemma gives the first two terms of the asymptotic expansion of x_n for Eq (1.1).

Lemma 4. *Let*

$$x_{n+1} = f(x_n)$$

for all $n \in \mathbb{N}$, where

$$f : (0,1) \to (0,1), \quad f(x) = x - x^3.$$

Then the following results hold:

(i) $\lim_{n\to\infty} \sqrt{n} x_n = \frac{1}{\sqrt{2}}$.

(ii) $\lim_{n\to\infty} \frac{n^{3/2}}{\ln n}(x_n - \frac{1}{\sqrt{2n}}) = -\frac{3}{8\sqrt{2}}$.

Proof. (i) It directly follows from Lemma 1.

(ii) Since $x_n > 0$, the equation

$$x_{n+1} = x_n - x_n^3$$

is rewritten into

$$\frac{1}{x_{n+1}} = \frac{1}{x_n - x_n^3} = \frac{1}{x_n} + \frac{x_n}{1 - x_n^2} = \frac{1}{x_n} + \frac{1/x_n}{(1/x_n^2) - 1}.$$

Let $a_n = x_n^{-1}$. Then

$$a_{n+1} = a_n + \frac{a_n}{a_n^2 - 1}.$$

Putting

$$b_n = \frac{1}{2}a_n^2,$$

we have

$$\begin{aligned} b_{n+1} - b_n &= \frac{1}{2}a_{n+1}^2 - \frac{1}{2}a_n^2 \\ &= 1 + \frac{3}{2}\frac{1}{(a_n^2 - 1)} + \frac{1}{2\,(a_n^2 - 1)^2} \\ &= 1 + \frac{3}{2}\frac{1}{(2b_n - 1)} + \frac{1}{2}\frac{1}{(2b_n - 1)^2}. \end{aligned}$$

Thus

$$b_{n+1} - b_n = 1 + \frac{3}{2}(2b_n - 1)^{-1} + \frac{1}{2}(2b_n - 1)^{-2}. \tag{2.1}$$

Since

$$b_n = \frac{1}{2x_n^2},$$

one can see that $b_n \to \infty$ for $n \to \infty$. It follows that

$$b_{n+1} - b_n \geq \frac{1}{2}$$

as $n \geq n_0$. So there exists a positive integer m such that $b_n \geq mn$. Then

$$b_n^{-1} = O\left(n^{-1}\right)$$

and

$$b_n = n + O(\log n).$$

Substituting

$$b_n = n + O(\log n)$$

into (2.1), we have

$$b_n = n + \frac{3}{4}\log n + r_n, \tag{2.2}$$

where r_n satisfies

$$r_{n+1} - r_n = O\left(\frac{\log n}{n^2}\right), \quad n \to \infty.$$

Consequently,

$$x_n = \frac{1}{\sqrt{2}}(b_n)^{-\frac{1}{2}} = \frac{1}{\sqrt{2n}}\left(1 - \frac{3}{8}\frac{\log n}{n} + O\left(\frac{\log n}{n^2}\right)\right) \sim \frac{1}{\sqrt{2n}} - \frac{3}{8\sqrt{2}}\frac{\log n}{n^{3/2}}.$$

This completes the proof. □

The following result comes from [19, Corollary 2.2.3].

Lemma 5. *Let $\left\{a_j\right\}$ be a sequence of non-negative real numbers. Then the series $\sum_{j=1}^{\infty} a_j$ and the product $\prod_{j=1}^{\infty}\left(1-a_j\right)$ either both converge or both diverge.*

3. Main results

First, we give the first six terms of the asymptotic expansion of x_n for Eq (1.1).

Theorem 1. *Let*

$$x_{n+1} = f(x_n)$$

for all integers $n \geq 1$ where

$$f : (0,1) \to (0,1), \quad f(x) = x - x^3.$$

Then there exists a constant $C \in \mathbb{R}$ such that

$$C = \lim_{n\to\infty}\left(2n - \frac{3\log n}{4} - 2n\sqrt{2n}x_n\right),$$

where C depends on the initial value $x_1 = x$. Moreover

$$x_n = \frac{1}{\sqrt{2n}}\left(1 - \frac{3}{8}\frac{\log n}{n} - \frac{C}{2n} + \frac{\alpha\log^2 n + \beta\log n + \gamma}{n^2} + O\left(\frac{\log^3 n}{n^3}\right)\right), \quad n \to \infty,$$

where

$$\alpha = \frac{27}{128}, \quad \beta = \frac{9}{16}C - \frac{9}{32}, \quad \gamma = \frac{3}{8}C^2 - \frac{3}{8}C + \frac{5}{32}.$$

Proof. We will continue to use those symbols and expressions in the proof of Lemma 4. Since the series $\sum_{k=1}^{+\infty}(r_{k+1} - r_k)$ is convergent, say C. It follows from Lemma 2 that

$$\begin{aligned} r_n &= \sum_{k=1}^{n-1}(r_{k+1} - r_k) + r_1 \\ &= \sum_{k=1}^{\infty}(r_{k+1} - r_k) - \sum_{k=n}^{\infty}(r_{k+1} - r_k) \\ &= C - \sum_{k=n}^{\infty}(r_{k+1} - r_k) \\ &= C + O(1)\sum_{k=n}^{\infty}\frac{\log k}{k^2} \\ &= C + O\left(\frac{\log n}{n}\right). \end{aligned}$$

According to (2.2), let

$$b_n = n + \frac{3}{4}\log n + C + \lambda_n.$$

It follows from (2.1) that

$$\begin{aligned}\lambda_{n+1}-\lambda_n =& -\frac{3}{4}\log\left(1+\frac{1}{n}\right)+\frac{3}{2}\left(2n+\frac{3}{2}\log n+2C-1+2\lambda_n\right)^{-1}\\ &+\frac{1}{2}\left(2n+\frac{3}{2}\log n+2C-1+2\lambda_n\right)^{-2}+O\left(\frac{1}{n^3}\right)\\ =& -\frac{3}{4}\log\left(1+\frac{1}{n}\right)+\frac{3}{4n}\left(1+\frac{3}{4}\frac{\log n}{n}+\frac{2C-1}{2n}+\frac{\lambda_n}{n}\right)^{-1}\\ &+\frac{1}{8n^2}\left(1+\frac{3}{4}\frac{\log n}{n}+\frac{2C-1}{2n}+\frac{\lambda_n}{n}\right)^{-2}+O\left(\frac{1}{n^3}\right)\\ =& -\frac{3}{4}\left(\frac{1}{n}-\frac{1}{2n^2}+O\left(\frac{1}{n^3}\right)\right)+\frac{3}{4n}\left(1-\frac{3}{4}\frac{\log n}{n}-\frac{2C-1}{2n}+O\left(\frac{\log^2 n}{n^2}\right)\right)\\ &+\frac{1}{8n^2}\left(1+O\left(\frac{\log n}{n}\right)\right)\\ =& -\frac{9}{16}\frac{\log n}{n^2}+\frac{7-6C}{8n^2}+O\left(\frac{\log^2 n}{n^3}\right).\end{aligned}$$

Thus

$$\begin{aligned}\lambda_{n+1}-\lambda_n &= -\frac{9}{16}\frac{\log n}{n^2}+\frac{7-6C}{8n^2}+O\left(\frac{\log^2 n}{n^3}\right),\\ \lambda_n-\lambda_{n-1} &= -\frac{9}{16}\frac{\log(n-1)}{(n-1)^2}+\frac{7-6C}{8(n-1)^2}+O\left(\frac{\log^2(n-1)}{(n-1)^3}\right),\\ &\cdots\\ \lambda_2-\lambda_1 &= -\frac{9}{16}\frac{\log 1}{1^2}+\frac{7-6C}{8}+O\left(\frac{\log^2 1}{1^3}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\lambda_n-\lambda_1 &= -\frac{9}{16}\sum_{k=1}^{n-1}\frac{\log k}{k^2}+\frac{(7-6C)}{8}\sum_{k=1}^{n-1}\frac{1}{k^2}+O\left(\frac{\log^2 n}{n^2}\right)\\ &= -\frac{9}{16}\left(\sum_{k=1}^{\infty}\frac{\log k}{k^2}-\sum_{k=n}^{\infty}\frac{\log k}{k^2}\right)+\frac{(7-6C)}{8}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}-\sum_{k=n}^{\infty}\frac{1}{k^2}\right)+O\left(\frac{\log^2 n}{n^2}\right).\end{aligned}$$

It follows from Lemmas 2 and 3, and

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$$

that

$$\lambda_n=\frac{9}{16}\frac{\log n}{n}+\frac{-5+12C}{16n}+O\left(\frac{\log^2 n}{n^2}\right).$$

Then

$$b_n=n+\frac{3}{4}\log n+C+\frac{9}{16}\frac{\log n}{n}+\frac{-5+12C}{16n}+O\left(\frac{\log^2 n}{n^2}\right).$$

Therefore,

$$x_n = \frac{1}{\sqrt{2n}}(b_n)^{-\frac{1}{2}} = \frac{1}{\sqrt{2n}}\left(1 - \frac{3}{8}\frac{\log n}{n} - \frac{C}{2n} + \frac{\alpha\log^2 n + \beta\log n + \gamma}{n^2} + O\left(\frac{\log^3 n}{n^3}\right)\right),$$

where C depends on the initial value, and

$$\alpha = \frac{27}{128}, \quad \beta = \frac{9}{16}C - \frac{9}{32}, \quad \gamma = \frac{3}{8}C^2 - \frac{3}{8}C + \frac{5}{32}.$$

Let the initial value $x_1 = x$. From the asymptotic expansion of x_n, one can see that C: $(0,1) \to \mathbb{R}$ is given by

$$C(x) = \lim_{n\to\infty}\left(2n - \frac{3\log n}{4} - 2n\sqrt{2n}x_n\right).$$

□

Next, we will discuss the monotonicity and smoothness of $C(x)$.

Theorem 2. *The function $C(x)$ is strictly decreasing on $(0, \frac{\sqrt{3}}{3})$, strictly increasing on $(\frac{\sqrt{3}}{3}, 1)$, and its minimum value*

$$C\left(\frac{\sqrt{3}}{3}\right) \approx 1.5739.$$

Proof. By the proof of Lemma 4,

$$b_n = \frac{1}{2x_n^2}, \quad n \geq 1,$$

satisfies the recurrence relation

$$b_{n+1} = h(b_n),$$

where

$$h(x) = x + 1 + \frac{3}{2}\cdot\frac{1}{(2x-1)} + \frac{1}{2}\cdot\frac{1}{(2x-1)^2}, \quad x \in (1/2, +\infty).$$

It follows that

$$h^n(x) = x + n + \frac{3}{2}\sum_{k=0}^{n-1}\frac{1}{(2h^k(x)-1)} + \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{(2h^k(x)-1)^2}, \quad n \geq 2. \tag{3.1}$$

And $h^n(x)$ is a strictly increasing sequence with respect to n. Furthermore, one can see that

$$h^n(x) > n + 1/2$$

for all $x \in (1/2, +\infty)$ and every $n \in \mathbb{N}$.

According to the proof of Theorem 1, let us define

$$g(x) = \lim_{n\to\infty}\left(h^{n-1}(x) - n - \frac{3}{4}\log n\right).$$

Obviously,

$$g(\frac{1}{2x^2}) = C(x).$$

Thus we can consider $g(x)$ instead of $C(x)$. One can obtain

$$g'(x) = \left(1 - \frac{3}{(2x-1)^2} - \frac{2}{(2x-1)^3}\right) \prod_{k=1}^{+\infty} \left(1 - \frac{3}{(2h^k(x)-1)^2} - \frac{2}{(2h^k(x)-1)^3}\right),$$

since

$$h'(x) = 1 - \frac{3}{(2x-1)^2} - \frac{2}{(2x-1)^3}$$

and

$$(h^n)'(x) = h'\left(h^{n-1}(x)\right) h'\left(h^{n-2}(x)\right) \cdots h'(x), \quad n \geq 1.$$

Therefore $g'(x) > 0$ for $x > \frac{3}{2}$ and $g'(x) < 0$ for

$$\frac{1}{2} < x < \frac{3}{2}.$$

It follows from

$$g(x) = C\left(\frac{1}{\sqrt{2x}}\right)$$

that $C(x)$ is strictly decreasing on $(0, \frac{\sqrt{3}}{3})$, strictly increasing on $(\frac{\sqrt{3}}{3}, 1)$, and $C(x)$ takes the minimum value at $\frac{\sqrt{3}}{3}$. □

See Figures 1 and 2, which are plotted with Matlab. Figure 2 is a magnified view of the central section shown in Figure 1.

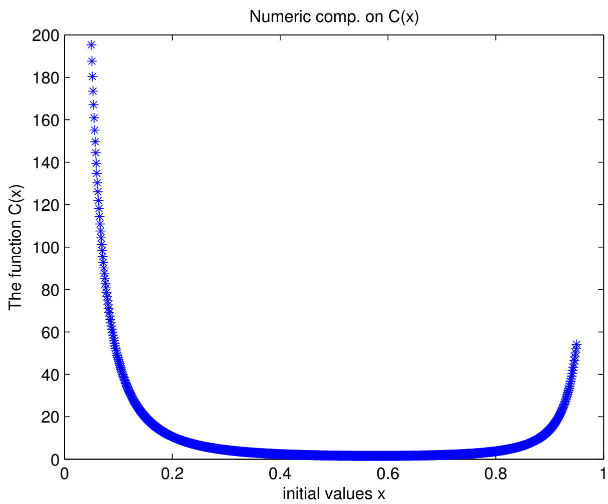

Figure 1. $C(x)$ where $x \in (0.05, 0.950)$.

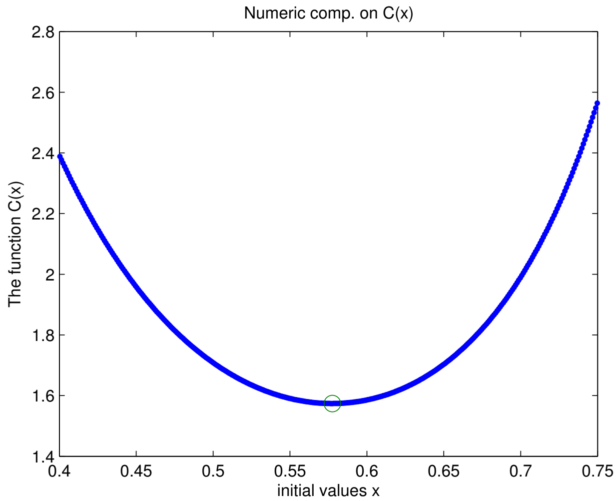

Figure 2. $C(x)$ where $x \in (0.40, 0.75)$.

Theorem 3. *Let*

$$h(x) = x + 1 + \frac{3}{2} \cdot \frac{1}{(2x-1)} + \frac{1}{2} \cdot \frac{1}{(2x-1)^2}$$

for $x \in (1/2, \infty)$ *and*

$$g(x) = \lim_{n\to\infty} \left(h^{n-1}(x) - n - \frac{3}{4} \log n \right).$$

Then g *is continuously differentiable and*

$$g(h(x)) = g(x) + 1, \quad x \in (1/2, \infty).$$

Proof. From the previous discussion, one can see that

$$(h^{n-1}(x))' = \prod_{k=0}^{n-2} \left(1 - \frac{3}{(2h^k(x)-1)^2} - \frac{2}{(2h^k(x)-1)^3} \right), \quad n \geq 2.$$

According to the inequality

$$h^n(x) > n + \frac{1}{2}$$

and Lemma 5, the sequence of $(h^{n-1})'(x)$ is absolutely convergent. Therefore the sequence $h^{n-1}(1) + \int_1^x (h^{n-1})'(t)dt$ converges to

$$g(x) = g(1) + \int_1^x \prod_{k=0}^{\infty} (1 - \frac{3}{(2h^k(t)-1)^2} - \frac{2}{(2h^k(t)-1)^3})dt.$$

This fact shows that g is continuously differentiable.

The functional equation

$$g(h(x)) = g(x) + 1,$$

immediately, follows from

$$
\begin{aligned}
g\left(h(x)\right) &= \lim_{n\to\infty}\left(h(h^{n-1}(x)) - n - \frac{3}{4}\log n\right)\\
&= \lim_{n\to\infty}\left(h^{n-1}(x) + 1 + \frac{3}{2}\frac{1}{(2h^{n-1}(x)-1)} + \frac{1}{2}\frac{1}{(2h^{n-1}(x)-1)^2} - n - \frac{3}{4}\log n\right)\\
&= g(x) + 1.
\end{aligned}
$$

This completes the proof. □

Corollary 1. *The function* $C(x)$ *is continuously differentiable on the interval* $(0, 1)$.

Proof. The result follows directly from the composition

$$C(x) = g\left(\frac{1}{2x^2}\right).$$

This completes the proof. □

4. Conclusions

Our analysis gives the first six terms of the asymptotic expansion of iterates of the cubic function

$$x_n = f(x_{n-1}) = \frac{1}{\sqrt{2n}}\left(1 - \frac{3}{8}\frac{\log n}{n} - \frac{C}{2n} + \frac{\alpha\log^2 n + \beta\log n + \gamma}{n^2} + O\left(\frac{\log^3 n}{n^3}\right)\right), \quad n \to \infty,$$

where the constant C with respect to the initial value $x = x_1$

$$C(x) = \lim_{n\to\infty}\left(2n - \frac{3\log n}{4} - 2n\sqrt{2n}x_n\right).$$

One can see that $C(x)$ does not have a closed-form expression, but only a limiting form. We study the properties of $C(x)$ through the iterated function. This is a natural and tractable approach. It is proved that $C(x)$ is continuously differentiable on the interval $(0, 1)$, and strictly decreasing on $(0, \frac{\sqrt{3}}{3})$, strictly increasing on $(\frac{\sqrt{3}}{3}, 1)$, with minimum value $C(\frac{\sqrt{3}}{3})$.

Our approach to studying $C(x)$ for the cubic function provides an example for studying and understanding some properties of constants in the other asymptotic expansions.

Author contributions

Xiaoyu Luo: writing original draft, methodology, proof of conclusions; Yong-Guo Shi: writing original draft, methodology, proof of conclusions; Kelin Li: validation, writing review, editing, proof of conclusions; Pingping Zhang: validation, writing review, editing, proof of conclusions. The authors contributed equally to this work. All the authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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